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Minimal Slant Submanifolds of the smallest dimension in S-manifolds

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Abstract

We study slant submanifolds of S-manifolds with the smallest dimension, specially minimal submanifolds and establish some relations between them and anti-invariant submanifolds in S-manifolds, similar to those ones proved by B.-Y. Chen for slant surfaces and totally real surfaces in Kaehler manifolds.

1. Introduction

Slant immersions in complex geometry were defined by B.-Y. Chen as a natural generalization of both holomorphic and totally real immersions [4, 6]. Recently, A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [8]. Slant submanifolds of Sasakian manifolds have been studied in [2] and a general view about slant immersions can be found in [3].

On the other hand, for manifolds with an f-structure, D.E. Blair has introduced S-manifolds as the analogue of the Kaehler structure in the almost complex case and of Sasakian structure in the almost contact case [1].

The purpose of the present paper is to study slant submanifolds of S-manifolds with the smallest dimension, specially, minimal slant submanifolds. After recalling, in Section 2, some basic ideas of Riemannian geometry, we review, in Section 3, formulas and definitions for metric f-manifolds and their submanifolds, which we shall use later. In Section 4 we prove that the smallest dimension of a slant submanifols in an S-manifold is $2 + s$, where s is denoting the number of structure vector fields of the ambient S-manifold (note that $s = 0$ for Kaehler manifolds and $s = 1$ for Sasakian manifolds) and we give some characterization theorems for these submanifolds in terms

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of the covariant derivatives of the f-structure projection operators on the submanifold. Finally, in Section 5 we study minimal slant submanifolds of the smallest dimension. In particular, we establish some relations between minimal slant $(2+s)$-dimensional submanifolds and anti-invariant submanifolds in S-manifolds, which correspond, in same sense, to those ones proved by B.-Y. Chen in [4,6].

2. Preliminaries

In this section, we will recall some fundamental results and formulas concerning Riemannian submanifolds for later use (see, e.g. [5] as a general reference).

Let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $\widetilde{M}$. Let g denote the metric tensor of $\widetilde{M}$ as well as the induced metric tensor on M. Let $\mathcal{X}(\widetilde{M})$ be de Lie algebra of tangent vector fields on $\widetilde{M}$, $\mathcal{X}(M)$ the Lie algebra of tangent vector fields on M and $T^{\perp}M$ the set of vector fields on $\widetilde{M}$ which are normal to M, that is, $\mathcal{X}(\widetilde{M}) = \mathcal{X}(M) \oplus T^{\perp}M$.

If ∇ y $\widetilde{\nabla}$ denote the Levi-Civita connections of M and $\widetilde{M}$, respectively, the Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),\ \ \widetilde{\nabla}_X V = -A_V X + D_X V,$$

for any $X, Y \in \mathcal{X}(M)$ and any $V \in T^{\perp}M$, where D is the normal connection, σ is the second fundamental form of the immersion and A_V is the Weingarten endomorphism associated with V. The endomorphisms A_V and σ are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V), \tag{2.1}$$

for any $X, Y \in \mathcal{X}(M)$ and any $V \in T^{\perp}M$.

The mean curvature vector H is defined by

$$H = \frac{1}{m}\text{trace}\ \sigma = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i),$$

where $\dim M = m$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of $\mathcal{X}(M)$. M is said to be *minimal* if H vanishes identically or, equivalently, if

$$\text{trace}\, A_V = 0, \qquad \text{for any } V \in T^{\perp}M.$$

If $\dim(\widetilde{M}) = \widetilde{m}$, a local orthonormal basis of $\mathcal{X}(\widetilde{M})$

$$\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{\widetilde{m}}\}$$

can be chosen such that, restricted to M, the vector fields $e_1, \ldots, e_m$ are tangent to M and so, $e_{m+1}, \ldots, e_{\widetilde{m}}$ are normal to M.

Then, for any $X \in \mathcal{X}(M)$, it can be written that

$$\widetilde{\nabla}_X e_i = \sum_{j=1}^{m} w_i^j(X)e_j + \sum_{k=m+1}^{\widetilde{m}} w_i^k(X)e_k, \tag{2.2}$$

$$\widetilde{\nabla}_X e_r = \sum_{j=1}^{m} w_r^j(X)e_j + \sum_{k=m+1}^{\widetilde{m}} w_r^k(X)e_k, \tag{2.3}$$

for $i \in \{1, \dots, m\}$ and $r \in \{m+1, \dots, \widetilde{m}\}$. The 1-forms w_i^j, w_i^k, $w_{\widetilde{r}}^k$ given by equations (2.1) and (2.2) are called *connection forms* of M in $\widetilde{M}$. It is easy to show that

$$w_j^i + w_i^j = 0, \qquad \text{for any } i, j \in \{1, \dots, m\}. \tag{2.4}$$

3. Slant submanifolds of S-manifolds

Let $(\widetilde{M}, g)$ be a $(2m+s)$-dimensional Riemannian manifold. Then, it is said to be a *metric f-manifold* if there exist on $\widetilde{M}$ an f-structure f, that is, a tensor field f of type (1,1) satisfying $f^3 + f = 0$ (see [9]), of rank $2m$ and s global vector fields $\xi_1, \dots, \xi_s$ (called *structure vector fields*) such that, if $\eta_1, \dots, \eta_s$ are the dual 1-forms of $\xi_1, \dots, \xi_s$, then

$$f\xi_\alpha = 0; \qquad \eta_\alpha \circ f = 0; \qquad f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X,Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{3.1}$$

for any $X, Y \in \mathcal{X}(\widetilde{M})$ and $\alpha = 1, \dots, s$.

The f-structure f is normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes \mathrm{d}\eta_\alpha = 0,$$

where $[f, f]$ is the Nijenhuis tensor of f. Let F be the fundamental 2-form defined by $F(X,Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(\widetilde{M})$. Then, $\widetilde{M}$ is said to be an *S-manifold* if the f-structure is normal and

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge (\mathrm{d}\eta_\alpha)^n \neq 0, \; F = \mathrm{d}\eta_a,$$

for any $\alpha = 1, \dots, s$. In this case, the structure vector fields are Killing vector fields. When $s = 1$, S-manifolds are Sasakian manifolds.

The Riemannian connection $\widetilde{\nabla}$ of an S-manifold satisfies ([1])

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{3.2}$$

and

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X), \tag{3.3}$$

for any $X, Y \in \mathcal{X}(\widetilde{M})$ and any $\alpha = 1, \ldots, s$.

Next, let M be a isometrically immersed submanifold of a metric f-manifold $\widetilde{M}$. For any $X \in \mathcal{X}(M)$ we write

$$fX = TX + NX, \tag{3.4}$$

where TX and NX are the tangential and normal components of fX, respectively. Similarly, for any $V \in T^\perp M$, we have

$$fV = tV + nV, \tag{3.5}$$

where tV (resp., nV) is the tangential component (resp., the normal component) of fV. Since, from (3.1), the metric g satisfies that $g(fX, Y) = -g(X, fY)$, for any $X, Y \in \mathcal{X}(\widetilde{M})$, by using (3.4) and (3.5), we get

$$g(TX, Y) = -g(X, TY), \tag{3.6}$$

$$g(nV, U) = -g(V, nU), \tag{3.7}$$

$$g(NX, V) = -g(X, TV), \tag{3.8}$$

for any $X, Y \in \mathcal{X}(M)$, $U, V \in T^\perp M$ and, by using (3.5), if the structure vector fields are tangent to M,

$$NTX + nNX = 0, \tag{3.9}$$

for any $X \in \mathcal{X}(M)$. Moreover, in this last case, if $\widetilde{M}$ is an S-manifold, from (3.2) and (3.4) it is easy to show that

$$\sigma(X, \xi_\alpha) = -NX, \tag{3.10}$$

for any $X \in \mathcal{X}(M)$, $\alpha = 1, \ldots, s$ and, consequently $\sigma(\xi_\alpha, \xi_\beta) = 0$, for any $\alpha, \beta = 1, \ldots, s$.

The covariant derivatives of T and N are given by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{3.11}$$

$$(\nabla_X N)Y = D_X NY - N\nabla_X Y, \tag{3.12}$$

for any $X, Y \in \mathcal{X}(M)$.

Then, by using (3.3), (3.11), (3.12) and Gauss-Weingarten formulas, it can be obtained that

$$(3.13)\quad (\nabla_X T)Y = t\sigma(X,Y) + A_{NY}X + \sum_{\alpha=1}^{s}(g(fX,fY)\xi_\alpha + \eta_\alpha(Y)f^2X),$$

$$(3.14)\quad (\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY),$$

for any $X, Y \in \mathcal{X}(M)$.

Now, for later use, we establish two general lemmas for submanifolds of S-manifolds which can be proved from (2.1) and (3.6)-(3.8) by a straightforward computation:

Lemma 3.1 *Let M be a submanifold of an S-manifold, tangent to the structure vector fields. Then, there exists a differentiable function λ such that*

$$(\nabla_X T)Y = \lambda\sum_{\alpha=1}^{s}(g(fX,fY)\xi_\alpha + \eta_\alpha(Y)f^2X),$$

for any $X, Y \in \mathcal{X}(M)$, if and only if:

$$A_{NY}X - A_{NX}Y = (\lambda - 1)\sum_{\alpha=1}^{s}(\eta_\alpha(Y)f^2X - \eta_\alpha(X)f^2Y).$$

Lemma 3.2 *Let M be a submanifold of an S-manifold, tangent to the structure vector fields. Then,*

$$(\nabla_X N)Y = \sum_{\alpha=1}^{s}(2\eta_\alpha(X)NTY + \eta_\alpha(Y)NTX),$$

for any $X, Y \in \mathcal{X}(M)$, if and only if:

$$A_V TY + A_{nV}Y = \sum_{\alpha=1}^{s}(2g(Y,tnV)\xi_\alpha + \eta_\alpha(Y)tnV),$$

for any $Y \in \mathcal{X}(M)$ and any $V \in T^\perp M$.

The submanifold M is said to be *invariant* if N is identically zero, that is, if $fX \in \mathcal{X}(M)$, for any $X \in \mathcal{X}(M)$. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if $fX \in T^\perp M$, for any $X \in \mathcal{X}(M)$.

From now on, we suppose that all the structure vector fields are tangent to the submanifold M. Then, M is said to be a *slant* submanifold if for any $x \in M$ and any $X \in T_xM$, linearly independent on $\xi_1, \dots, \xi_s$, the Wirtinger angle between fX and T_xM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$. Note that this definition generalizes that one given by B.-Y. Chen ([6]) for Complex Geometry and that one given by A. Lotta ([8]) for Contact Geometry.

Furthermore, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is not invariant nor anti-invariant is called a *proper* slant immersion. Observe that, for invariant submanifolds, $T = f$ and, so

$$T^2 = f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha,$$

while for anti-invariant submanifolds, $T^2 = 0$. In fact, we have the following general result whose proof can be obtained by following the same steps as in the case $s = 1$ (see [2]):

Theorem 3.1 *Let M be a submanifold of a metric f-manifold $\widetilde{M}$, tangent to the structure vector fields. Then, M is a slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that:*

$$T^2 = -\lambda I + \lambda \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha = \lambda f^2.$$

Furthermore, in such case, if θ is the slant angle of M, it satisfies that $\lambda = \cos^2\theta$.

Using (3.1), (3.4), (3.6) and Theorem 3.1, a direct computation gives:

Corollary 3.1 *Let M be a slant submanifold of a metric f-manifold $\widetilde{M}$, with slant angle θ. Then, for any $X, Y \in \mathcal{X}(M)$, we have:*

$$\begin{aligned} g(TX,TY) &= \cos^2\theta(g(X,Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)), \\ g(NX,NY) &= \sin^2\theta(g(X,Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)). \end{aligned}$$

We also have:

Corollary 3.2 *Let M be a non-invariant slant $(m+s)$-dimensional submanifold of a $(2m+s)$-dimensional metric f-manifold $\widetilde{M}$ with slant angle θ and let $\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_s\}$ be a local orthonormal basis of $\mathcal{X}(M)$. Then,*

$$\{(\csc\theta)Ne_1, \ldots, (\csc\theta)Ne_m\}$$

is a local orthonormal basis of $T^\perp M$.

Proof. It is easy to show that $\{(\csc\theta)Ne_1, \ldots, (\csc\theta)Ne_m\}$ is a set of m linearly independent vector fields of $T^\perp M$, that is, a local basis of $T^\perp M$. Moreover, from Corollary 3.1, we obtain that:

$$g((\csc\theta)Ne_i, (\csc\theta)Ne_j) = \csc^2\theta \sin^2\theta g(e_i, e_j) = \delta_{ij}. \qquad \blacksquare$$

In a similar way, by using Theorem 3.1 and Corollary 3.1, we get:

Corollary 3.3 *Let M be a non anti-invariant $(2+s)$-dimensional slant submanifold of a metric f-manifold with slant angle θ. Let e_1 be a unit vector field, tangent to M and normal to the structure vector fields and define $e_2 = (\sec\theta)Te_1$. Then $e_1 = -(\sec\theta)Te_2$ and $\{e_1, e_2, \xi_1, \dots, \xi_s\}$ is a local orthonormal basis of $\mathcal{X}(M)$.*

Finally, combining Corollary 3.2 and Corollary 3.3 and using Theorem 3.1 again, we obtain:

Corollary 3.4 *Let M be a proper $(2+s)$-dimensional slant submanifold of a $(4+s)$-dimensional metric f-manifold with slant angle θ. Let e_1 be a unit vector field, tangent to M and normal to the structure vector fields and define:*

$$e_2 = (\sec\theta)Te_1, \ e_3 = (\csc\theta)Ne_1 \text{ and } e_4 = (\csc\theta)Ne_2.$$

Then, $e_1 = -(\sec\theta)Te_2$ and $\{e_1, e_2, e_3, e_4, \xi_1, \dots, \xi_s\}$ is a local orthonormal basis of $\mathcal{X}(\widetilde{M})$ such that $e_1, e_2, \xi_1, \dots, \xi_s$ are tangent to M and e_3, e_4 are normal to M. Moreover:

$$te_3 = -\sin\theta e_1, \ ne_3 = -\cos\theta e_4, \ te_4 = -\sin\theta e_2, \ ne_4 = \cos\theta e_3.$$

The basis $\{e_1, e_2, e_3, e_4, \xi_1, \dots, \xi_s\}$ is said to be an **adapted slant basis**.

4. Slant submanifolds of the smallest dimension

Observe that $2+s$ is the smallest dimension of a proper slant submanifold in a metric f-manifold. Indeed, if we denote $Q = T^2$ and consider the orthogonal decomposition

$$\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M},$$

where $\mathcal{M}$ is the distribution spanned by the structure vector fields and $\mathcal{L}$ is its complementary orthogonal distribution, then, since $T\mathcal{L} \subseteq \mathcal{L}$, $Q|_{\mathcal{L}}$ is an endomorphism on $\mathcal{L}$. Furthermore, it is a symmetric endomorphism because, from (3.6),

$$g(QX, Y) = g(T^2X, Y) = -g(TX, TY) = g(X, T^2Y) = g(X, QY),$$

for any $X, Y \in \mathcal{X}(M)$. Consequently, for each $x \in M$, the subspace $\mathcal{L}_x$ of T_xM admits a decomposition of the form

$$\mathcal{L}_x = \mathcal{L}_x^1 \oplus \mathcal{L}_x^2 \oplus \dots \oplus \mathcal{L}_x^{k(x)},$$

where $\mathcal{L}_x^i$ is the proper subspace of eigenvectors associated with an eigenvalue λ_i of $Q|_{\mathcal{L}}$. Then, we can easily prove:

Proposition 4.1 *Let M be a submanifold of a metric f-manifold, tangent to the structure vector fields. Then, at each point of M, we have the following properties:*

1. *$\lambda_i \in [-1, 0]$, for any eigenvalue λ_i of $Q|_{\mathcal{L}}$.*
2. *$TX \in \mathcal{L}^i$, for any $X \in \mathcal{L}^i$.*
3. *If $\lambda_i \neq 0$, $\mathcal{L}^i$ is of even dimension and $T(\mathcal{L}^i) = \mathcal{L}^i$.*

Corollary 4.1 *Let M be a $(1+s)$-dimensional submanifold of a metric f-manifold, tangent to the structure vector fields. Then, M is an anti-invariant submanifold.*

Proof. Since $\mathcal{L}$ is of odd dimension (equal to 1), from Proposition 4.1 we get $\lambda = 0$ and M is an anti-invariant submanifold. ■

From this corollary, we deduce that there are not proper slant submanifolds of a metric f-manifold of dimension smaller than $2+s$. Now, we are going to study submanifolds of such dimension when the ambient manifold is an S-manifold. First, by using Theorem 3.1, if M is a slant submanifold with slant angle θ, a direct calculation gives

$$(\nabla_X Q)Y = \cos^2\theta \sum_{\alpha=1}^{s} (g(X, TY)\xi_\alpha - \eta_\alpha(Y)TX), \tag{4.1}$$

for any $X, Y \in \mathcal{X}(M)$, where we recall that

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y.$$

Next, we have the following general characterization:

Theorem 4.1 *Let M be a submanifold of an S-manifold, tangent to the structure vector fields. Then, M is a slant submanifold if and only if the following conditions are satisfied:*

1. *The endomorphism $Q|_{\mathcal{L}}$ has only one eigenvalue at any point of M.*
2. *There exists a function $\lambda : M \longrightarrow [0,1]$ such that*

$$(\nabla_X Q)Y = \lambda \sum_{\alpha=1}^{s} (g(X, TY)\xi_\alpha - \eta_\alpha(Y)TX),$$

   for any $X, Y \in \mathcal{X}(M)$.

Moreover, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. If M is a slant submanifold with slant angle θ, from Theorem 3.1, we have

$$T^2X = QX = \cos^2\theta f^2 X,$$

for any $X \in \mathcal{X}(M)$. Then, $Q|_{\mathcal{L}} = -\cos^2\theta I$ and $\lambda_1 = -\cos^2\theta$ is the only eigenvalue of $Q|_{\mathcal{L}}$ at any point of M. Furthermore, Condition 2 is (4.1).

Conversely, let $\lambda_1(x)$ be the only eigenvalue of $Q|_{\mathcal{L}}$ at any point $x \in M$. Thus, by using Condition 2 we get that λ_1 is a constant. Now, let $X \in \mathcal{X}(M)$. If we put

$$X = \widetilde{X} + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_a,$$

where $\widetilde{X} \in \mathcal{L}$, then $QX = Q\widetilde{X} = \lambda_1\widetilde{X}$ and, so:

$$QX = \lambda_1 X - \lambda_1 \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_a.$$

By applying Theorem 3.1 we obtain that M is a slant submanifold and, by (4.1), $\lambda = -\lambda_1 = \cos^2\theta$. ■

Corollary 4.2 *Let M be a $(2+s)$-dimensional submanifold of an S-manifold tangent to the structure vector fields. Then, M is a slant submanifold if and only if there exists a function $\lambda : M \longrightarrow [0,1]$ such that*

$$(\nabla_X Q)Y = \lambda \sum_{\alpha=1}^{s}(g(X,TY)\xi_\alpha - \eta_\alpha(Y)TX), \tag{4.2}$$

for any $X, Y \in \mathcal{X}(M)$. Moreover, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. We only have to prove that $Q|_{\mathcal{L}}$ has only one eigenvalue at any point of M. But it is a direct consequence of 3. of Proposition 4.1. ■

Theorem 4.2 *Let M be a $(2+s)$-dimensional submanifold of an S-manifold, tangent to the structure vector fields. Then, M is a slant submanifold if and only if there exists a function $\lambda : M \longrightarrow [0,1]$ such that*

$$(\nabla_X T)Y = \lambda \sum_{\alpha=1}^{s}(g(fX,fY)\xi_\alpha + \eta_\alpha(Y)f^2X), \tag{4.3}$$

for any $X, Y \in \mathcal{X}(M)$. Moreover, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. First, it is easy to show that (4.3) implies (4.2). Then, we only have to apply Corollary 4.2 to get that M is a slant submanifold. Conversely, we can suppose that M is a proper slant submanifold because if M is an invariant or an anti-invariant submanifold, we obtain (4.3) directly. Now, since $\dim(M) = 2+s$, from Corollary 3.3, we can choose a local orthonormal basis of $\mathcal{X}(M)$, $\{e_1, e_2, \xi_1, \ldots, \xi_s\}$, such that $e_2 = (\sec\theta)Te_1$ and $e_1 = -(\sec\theta)Te_2$. Thus, for any $X \in \mathcal{X}(M)$, we have

$$(\nabla_X T)e_1 = \cos\theta \sum_{\alpha=1}^{s} w_2^{\alpha}(X)\xi_\alpha,$$

because $w_i^i(X) = 0$ and $w_i^j(X) = -w_j^i(X)$. But, by using (3.2) and (3.4), $w_2^\alpha(X) = g(e_2, TX)$, for any $\alpha = 1, \ldots, s$ and so:

$$(\nabla_X T)e_1 = \cos\theta \sum_{\alpha=1}^{s} g(e_2, TX)\xi_\alpha = \cos^2\theta \sum_{\alpha=1}^{s} g(X, e_1)\xi_\alpha. \tag{4.4}$$

Similarly:

$$(\nabla_X T)e_2 = \cos^2\theta \sum_{\alpha=1}^{s} g(X, e_2)\xi_\alpha. \tag{4.5}$$

On the other hand, for any $\alpha = 1, \ldots, s$:

$$(\nabla_X T)\xi_\alpha = \cos^2\theta f^2 X. \tag{4.6}$$

Now, given any $Y \in \mathcal{X}(M)$, since locally

$$Y = Y_1 e_1 + Y_2 e_2 + \sum_{\alpha=1}^{s} \eta_\alpha(Y)\xi_\alpha,$$

we obtain that:

$$(\nabla_X T)Y = Y_1(\nabla_X T)e_1 + Y_2(\nabla_X T)e_2 + \sum_{\alpha=1}^{s} \eta_\alpha(Y)(\nabla_X T)\xi_\alpha. \tag{4.7}$$

Substituting (4.4)-(4.6) into (4.7) we conclude the proof. ■

From Lemma 3.1 we get:

Corollary 4.3 *Let M be a submanifold of dimension $2+s$ in an S-manifold, tangent to the structure vector fields. Then, M is a slant submanifold if and only if there exists a differentiable function $\mu : M \longrightarrow [0, 1]$ such that*

$$A_{NY}X - A_{NX}Y = \mu \sum_{\alpha=1}^{s} (\eta_\alpha(X) f^2 Y - \eta_\alpha(Y) f^2 X),$$

for any $X, Y \in \mathcal{X}(M)$. Moreover, in this case, if θ is the slant angle of M, then $\mu = \sin^2\theta$.

5. Minimal slant submanifolds of the smallest dimension

For later use, we are going to prove the following lemmas:

Lemma 5.1 *Let M be a proper slant, $(2+s)$-dimensional submanifold of an S-manifold $\widetilde{M}$ with $\dim(\widetilde{M}) = 4+s$. If θ is the slant angle,*

$$\{e_1, \ldots, e_4, e_5 = \xi_1, \ldots, e_{4+s} = \xi_s\}$$

is an adapted slant basis and if we put

$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r), \qquad \text{for any } i, j = 1, 2, 5, \ldots, 4+s \text{ and } r = 3, 4,$$

then:

$$\sigma^3_{12} = \sigma^4_{11}, \ \sigma^3_{22} = \sigma^4_{12}, \tag{5.1}$$

$$\sigma^3_{1(4+\alpha)} = \sigma^4_{2(4+\alpha)} = -\sin\theta, \ \alpha = 1, \ldots, s \tag{5.2}$$

$$\sigma^3_{2(4+\alpha)} = \sigma^4_{1(4+\alpha)} = \sigma^3_{(4+\alpha)(4+\beta)} = \sigma^4_{(4+\alpha)(4+\beta)} = 0, \ \alpha, \beta = 1, \ldots, s. \tag{5.3}$$

Proof. We obtain (5.1) by virtue of Corollary 4.3 while (5.2) and (5.3) hold because $\widetilde{M}$ is an S-manifold. ■

Lemma 5.2 *Let M be a $(2+s)$-dimensional slant submanifold of an S-manifold $\widetilde{M}$ with $\dim(\widetilde{M}) = 4+s$. Then, $\nabla N = 0$ if and only if M is either an invariant or an anti-invariant submanifold.*

Proof. If $\nabla N = 0$, then, by applying (3.14) we get, for any $X, Y \in \mathcal{X}(M)$, $V \in T^\perp M$:

$$-g(\sigma(X, TY), V) = g(\sigma(X, Y), nV). \tag{5.4}$$

If we suppose that M is a proper slant submanifold with slant angle θ and choose an adapted slant basis

$$\{e_1, \ldots, e_4, e_5 = \xi_1, \ldots, e_{4+s} = \xi_s\},$$

then, from (5.4), since $Te_{4+\alpha} = T\xi_\alpha = 0$, for any $\alpha = 1, \ldots, s$ and $ne_4 = \cos\theta e_3$,

$$\begin{aligned} 0 = g(\sigma(e_1, e_{4+\alpha}), ne_4) = \cos\theta g(\sigma(e_1, e_{4+\alpha}), e_3) = \\ = \cos\theta \sigma^3_{1(4+\alpha)} = -\cos\theta \sin\theta, \end{aligned}$$

where we have used (5.2). But this contradicts the fact of M being a proper slant submanifold.

Conversely, if M is an invariant submanifold, then $N = 0$ and so, $\nabla N = 0$. Finally, if M is anti-invariant submanifold, then $n = 0$ and we only need to apply (3.14). ■

Theorem 5.1 *Let M be a $(2+s)$-dimensional submanifold of a $(4+s)$-dimensional S-manifold $\widetilde{M}$, tangent to the structure vector fields.*

1. *If M is a minimal proper slant submanifold of $\widetilde{M}$, then*

$$(\nabla_X N)Y = \sum_{\alpha=1}^{s}(2\eta_\alpha(X)NTY + \eta_\alpha(Y)NTX). \tag{5.5}$$

   for any $X, Y \in \mathcal{X}(M)$.

2. *Conversely, suppose that there is an eigenvalue λ of $Q|_{\mathcal{L}}$ at each point of M such that $\lambda \in (-1, 0)$. In this case, if (5.5) holds, M is a minimal proper slant submanifold of $\widetilde{M}$.*

Proof. To prove statement 1, we choose an adapted slant basis:

$$\{e_1, \dots, e_4, e_5 = \xi_1, \dots, e_{4+s} = \xi_s\}.$$

Then, we can show that

$$n\sigma(e_i, e_j) = \cos\theta\sigma_{ij}^4 e_3 - \cos\theta\sigma_{ij}^3 e_4, \tag{5.6}$$

for any $i, j = 1, 2, 5, \dots, 4+s$. Moreover, since M is minimal, by using $\sigma(\xi_a, \xi_\alpha) = 0$ for any $\alpha = 1, \dots, s$, we have:

$$\sigma_{11}^3 = -\sigma_{22}^3,\ \sigma_{11}^4 = -\sigma_{22}^4. \tag{5.7}$$

Next, writing $X, Y \in \mathcal{X}(M)$ in terms of the adapted slant basis and taking into account (5.1)-(5.3), (5.6) and (5.7), we obtain (5.5) from (3.14) and (3.9).

To prove statement 2, we can choose a unit local vector field e_1 in $\mathcal{L}$, such that

$$T^2 e_1 = -\cos^2\theta_1 e_1,$$

where $\theta_1 = \theta(e_1) \in (0, \pi/2)$ denotes the Wirtinger angle of e_1. Now, we define e_2, e_3, e_4 by

$$e_2(\sec\theta_1)Te_1,\ e_3 = (\csc\theta_1)Ne_1,\ e_4 = (\csc\theta_1)Ne_2 \tag{5.8}$$

and $e_{4+\alpha} = \xi_\alpha$, $\alpha = 1, \dots, s$. It is easy to show that $\{e_1, \dots, e_{4+s}\}$ is a local orthonormal basis of $\widetilde{M}$ such that:

$$te_3 = -\sin\theta_1 e_1,\ te_4 = -\sin\theta_1 e_2,\ ne_3 = -\cos\theta_1 e_4,\ ne_4 = \cos\theta_1 e_3.$$

Next, from (5.8) and by using Lemma 3.2, we get:

$$A_{Ne_1}e_2 = \sec\theta_1 \sin\theta_1 A_{e_3}Te_1 = \sin\theta_1 A_{e_4}e_1 = A_{Ne_2}e_1.$$

Furthermore, from (3.2) and Gauss-Weingarten formulas, we have, for any $\alpha = 1, \dots, s$,

$$A_{Ne_1}e_{4+\alpha} = \sin\theta_1 A_{e_3}e_{4+\alpha} = \sin\theta_1 te_3 = -\sin^2\theta_1 e_1$$

and

$$A_{Ne_2}e_{4+\alpha} = \sin\theta_1 A_{e_4}e_{4+\alpha} = \sin\theta_1 te_4 = -\sin^2\theta_1 e_2.$$

Hence, a direct computation gives that

$$A_{NY}X = A_{NX}Y - \sin^2\theta_1 \sum_{\alpha=1}^{s} \Big(\eta_a(Y)f^2X - \eta_a(X)f^2Y\Big),$$

for any $X, Y \in \mathcal{X}(M)$ and so, by applying Corollary 4.3, we know that M is a proper slant submanifold, with slant angle θ_1. Finally, to prove that M is also a minimal submanifold, we only need to show that:

$$\sigma_{11}^3 = -\sigma_{22}^3,\ \sigma_{11}^4 = -\sigma_{22}^4.$$

But,

$$\sigma_{11}^3 = g(\sigma(e_1, e_1), e_3) = (-\sec\theta_1)g(\sigma(e_1, Te_2), e_3)$$

and, from (3.14) y (5.5), $\sigma(e_1, Te_2) = n\sigma(e_1, e_2)$, which together (3.7) implies:

$$\sigma_{11}^3 = -\sigma_{12}^4.$$

Now, since we have already proved that M is a proper slant submanifold and the chosen basis is an adapted slant one, from Lemma 5.1 we conclude the proof. ■

Note that (5.5) holds directly in the invariant and anti-invariant cases, since $\nabla N = 0$. On the other hand, the above theorem is the corresponding one to Theorem 5.5 in [6], proved by B.-Y. Chen for surfaces in 4-dimensional Kaehler manifols.

Next, we want to establish some relations between minimal slant $(2+s)$-dimensional submanifolds and anti-invariant submanifolds in S-manifolds. First, we have the following lemma:

Lemma 5.3 *Let M be a proper slant $(2+s)$-dimensional submanifold in a $(4+s)$-dimensional S-manifold $\widetilde{M}$, with slant angle θ. Then, with respect to an adapted slant basis $\{e_1, \dots, e_{4+s}\}$, we have*

$$(5.9)\quad w_3^4 - w_1^2 = -\cot\theta((\text{trace }\ \sigma^3)w^1 + (\text{trace }\ \sigma^4)w^2 - \sum_{\alpha=1}^{s}(2\sin\theta)\eta_\alpha),$$

where w^1, w^2 are the dual forms of e_1, e_2.

Proof. Since the local basis is an adapted slant one, then, by using (3.14):

$$D_{e_1}e_3 = (\csc\theta)D_{e_1}Ne_1 = (\csc\theta)(N(\nabla_{e_1}e_1)+n\sigma(e_1,e_1)-\sigma(e_1,Te_1)). \tag{5.10}$$

But, from (2.2), (2.4) and applying N, we get:

$$N(\nabla_{e_1}e_1) = w_1^2(e_1)Ne_2 = \sin\theta w_1^2(e_1)e_4. \tag{5.11}$$

On the other hand:

$$n\sigma(e_1,e_1) = \sigma_{11}^3 ne_3 + \sigma_{11}^4 ne_4 = \cos\theta(-\sigma_{11}^3 e_4 + \sigma_{11}^4 e_3), \tag{5.12}$$

$$\sigma(e_1,Te_1) = \cos\theta\sigma(e_1,e_2) = \cos\theta(\sigma_{12}^3 e_3 + \sigma_{12}^4 e_4). \tag{5.13}$$

Substituting (5.11)-(5.13) into (5.10),

$$D_{e_1}e_3 = w_1^2(e_1)e_4 + \cot\theta(-\sigma_{11}^3 e_4 + \sigma_{11}^4 e_3 - \sigma_{12}^3 e_3 - \sigma_{12}^4 e_4),$$

by virtue of Lemma 5.1, since

$$\text{trace } \sigma^3 = \sum_{i=1}^{2} g(\sigma(e_i,e_i),e_3),$$

we have

$$D_{e_1}e_3 = w_1^2(e_1)e_4 - \cot\theta(\text{trace } \sigma^3)e_4$$

and, from (2.3):

$$w_3^4(e_1) - w_1^2(e_1) = -\cot\theta(\text{trace } \sigma^3). \tag{5.14}$$

Similarly:

$$w_3^4(e_2) - w_1^2(e_2) = -\cot\theta(\text{trace } \sigma). \tag{5.15}$$

Moreover, for any $\alpha = 1,\dots,s$,

$$D_{e_{4+\alpha}}e_3 = \csc\theta(N(\nabla_{\xi_\alpha}e_1) + n\sigma(e_1,\xi_\alpha) - \sigma(Te_1,\xi_\alpha)), \tag{5.16}$$

but, by applying (3.9) and (3.10),

$$n\sigma(e_1,\xi_\alpha) - \sigma(Te_1,\xi_\alpha) = -nNe_1 + NTe_1 = 2NTe_1,$$

and, consequently, from Corollary 3.4, we obtain:

$$n\sigma(e_1,\xi_\alpha) - \sigma(Te_1,\xi_\alpha) = 2\sin\theta\cos\theta e_4. \tag{5.17}$$

Furthermore:

$$N(\nabla_{e_{4+\alpha}}e_1) = w_1^2(e_{4+\alpha})Ne_2 = \sin\theta w_1^2(\xi_\alpha)e_4. \tag{5.18}$$

Thus, substituting (5.17) and (5.18) into (5.16) and taking into account that

$$D_{e_{4+\alpha}} e_3 = w_3^4(e_{4+\alpha}) e_4,$$

we get:

$$w_3^4(e_{4+\alpha}) - w_1^2(e_{4+\alpha}) = 2\cos\theta = -\cot\theta(-2\sin\theta). \tag{5.19}$$

Then, since $\{e_1, e_2, e_5 \dots, e_{4+s}\}$ is a local orthonormal basis of $\mathcal{X}(M)$, dual of $\{w^1, w^2, \eta_1, \dots, \eta_s\}$, equation (5.9) follows from (5.14), (5.15) and (5.19). ■

Theorem 5.2 *Let M be a proper slant submanifold of an S-manifold*

$$(\widetilde{M}, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g),$$

with $\dim M = 2 + s$, $\dim \widetilde{M} = 4 + s$ *and slant angle θ. Suppose that there exists on $\widetilde{M}$ an f-structure $\overline{f}$ such that*

$$(\widetilde{M}, \overline{f}, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$$

is a metric f-manifold satisfying

$$g((\widetilde{\nabla}_X \overline{f})Y, Z) = 0, \tag{5.20}$$

for any X, Y, Z normal to the structure vector fields. If M is an anti-invariant submanifold with respect to this structure, then M is a minimal submanifold of $\widetilde{M}$.

Proof. Let $\{e_1, \dots, e_{4+s}\}$ be an adapted slant basis in the S-manifold

$$(\widetilde{M}, f, \xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_s, g),$$

being $\{e_3, e_4\}$ a local orthonormal frame of $T^\perp M$. Hence, since M is an anti-invariant submanifold in

$$(\widetilde{M}, \overline{f}, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g),$$

we have that $\{\overline{f}e_1, \overline{f}e_2\}$ is another local orthonormal basis of $T^\perp M$, by virtue of (3.1). Consequently, there exists a function φ in M such that:

$$\begin{array}{rcl} e_3 &=& (\cos\varphi)\overline{f}e_1 + (\sin\varphi)\overline{f}e_2 \\ e_4 &=& -(\sin\varphi)\overline{f}e_1 + (\cos\varphi)\overline{f}e_2. \end{array} \tag{5.21}$$

Consider $X \in \mathcal{L}$. Then, we get:

$$\begin{aligned} w_3^4(X) = g(\widetilde{\nabla}_X e_3, e_4) = {} & X(\cos\varphi) g(\overline{f}e_1, e_4) + X(\sin\varphi) g(\overline{f}e_2, e_4) + \\ & + (\cos\varphi) g(\widetilde{\nabla}_X \overline{f}e_1, e_4) + (\sin\varphi) g(\widetilde{\nabla}_X \overline{f}e_2, e_4). \end{aligned}$$

Now, since $w_1^1(X) = 0$, $\overline{f}\xi_\alpha = 0$ for any $\alpha = 1, \dots, s$ and $g(\overline{f}e_4, e_4) = 0$, by using (5.20) and (5.21), we obtain:

$$w_3^4(X) - w_1^2(X) = X\varphi = \mathrm{d}\varphi(X). \tag{5.22}$$

Now, consider any

$$X = \widetilde{X} + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha \in \mathcal{X}(M),$$

with $\widetilde{X} \in \mathcal{L}$. We find, by using (5.19) and (5.22) that:

$$w_3^4(X) - w_1^2(X) = w_3^4(\widetilde{X}) - w_1^2(\widetilde{X}) + \sum_{\alpha=1}^{s} \eta_\alpha(X)(w_3^4(\xi_\alpha) - w_1^2(\xi_\alpha)) =$$

$$= \mathrm{d}\varphi(\widetilde{X}) + 2\cos\theta \sum_{\alpha=1}^{s} \eta_\alpha(X).$$

But,

$$\mathrm{d}\varphi(\widetilde{X}) = \mathrm{d}\varphi(X - \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha) = \mathrm{d}\varphi(X) - \sum_{\alpha=1}^{s} \xi_\alpha(\varphi)\eta_\alpha(X)$$

and, so:

$$w_3^4 - w_1^2 = \mathrm{d}\varphi + \sum_{\alpha=1}^{s} (2\cos\theta - \xi_\alpha(\varphi))\eta_\alpha.$$

Next, taking into account (5.9) we have:

$$-\cot\theta\{(\text{trace } \sigma^3)w^1 + (\text{trace } \sigma^4)w^2\} = \mathrm{d}\varphi - \sum_{\alpha=1}^{s} \xi_\alpha(\varphi)\eta_\alpha. \tag{5.23}$$

On the other hand,

$$\sigma_{11}^3 = g(\sigma(e_1, e_1), e_3) = g(A_{e_3}e_1, e_1) = -g(\widetilde{\nabla}_{e_1}e_3, e_1)$$

and from (5.20), (5.21) and since $\overline{f}e_1, \overline{f}e_2 \in T^\perp M$, we get:

$$\sigma_{11}^3 = \cos\varphi g(\sigma(e_1, e_1), \overline{f}e_1) + \sin\varphi g(\sigma(e_1, e_2), \overline{f}e_1).$$

However, from (5.21) again:

$$\overline{f}e_1 = \cos\varphi e_3 - \sin\varphi e_4.$$

Consequently:

$$\begin{aligned} \sigma_{11}^3 &= \cos^2\varphi\sigma_{11}^3 - \cos\varphi\sin\varphi\sigma_{11}^4 + \cos\varphi\sin\varphi\sigma_{12}^3 - \sin^2\varphi\sigma_{12}^4 = \\ &= \cos^2\varphi\sigma_{11}^3 - \sin^2\varphi\sigma_{22}^3, \end{aligned}$$

where we have used Lemma 5.1.

Thus, since $\sigma^3_{\alpha\alpha} = 0$, for any $\alpha = 1, \ldots, s$:

$$\sin^2 \varphi(\text{trace } \sigma^3) = 0. \tag{5.24}$$

Analogously:

$$\sin^2 \varphi(\text{trace } \sigma^4) = 0. \tag{5.25}$$

Now, let consider the following open subset of M:

$$U = \{x \in M / H(x) \neq 0\}.$$

To conclude the proof, we only need to show that $U \neq \emptyset$. If it is not the case, then, in U,

$$0 \neq H = \frac{1}{2+s}\text{trace } \sigma = \frac{1}{2+s}((\text{trace } \sigma^3)e_3 + (\text{trace } \sigma^4)e_4),$$

and so:

$$\text{trace } \sigma^3 \neq 0 \text{ or trace } \sigma^4 \neq 0. \tag{5.26}$$

This implies, by virtue of (5.24) and (5.25), that $\varphi \equiv 0$ $(mod\ \pi)$ in U. But φ is a continuous function, thus $\varphi \equiv 0$ in U. Hence, $\mathrm{d}\varphi = 0$ and $\xi_\alpha(\varphi) = 0$ in U, for any $\alpha = 1, \ldots, s$. Then, from (5.23),

$$\cot \theta((\text{trace } \sigma^3)w^1 + (\text{trace } \sigma^4)w^2) = 0,$$

and from (5.26), $\cot \theta = 0$, which is a contradiction with the fact of M being a proper slant submanifold. So, $U = \emptyset$ and M is minimal. ■

Note that the above theorem holds, in particular, if

$$(\widetilde{M}, \overline{f}, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

is an S-structure on $\widetilde{M}$ because, in such a case, for any $X, Y, Z \in \mathcal{X}(\widetilde{M})$, from (3.3) we find

$$g((\widetilde{\nabla}_X \overline{f})Y, Z) = \sum_{\alpha=1}^{s}(g(fX, fY)\eta_\alpha(Z) + \eta_\alpha(Y)g(f^2X, Z)),$$

vanishing this expression if Y, Z are normal to the structure vector fields. In fact, this would be the corresponding theorem to Theorem 4.2 of [4] which was proved by B.-Y. Chen in the Kaehlerian case. However, we have the following proposition:

Proposition 5.1 *Let* $(\widetilde{M}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ *be an* S*-manifold. If there exists another* f*-structure* $\overline{f}$ *on* $\widetilde{M}$ *such that*

$$(\widetilde{M}, \overline{f}, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

is a metric f*-manifold with* $F_{\overline{f}} = \mathrm{d}\eta_a$*, for any* $\alpha = 1, \ldots, s$*, then* $f = \overline{f}$*.*

Proof. The two fundamental 2-forms satisfy

$$F_f = \mathrm{d}\eta_a = F_{\overline{f}}, \qquad \text{for any } \alpha = 1, \ldots, s.$$

Then, for any $X, Y \in \mathcal{X}(\widetilde{M})$,

$$g(X, fY) = F_f(X, Y) = F_{\overline{f}}(X, Y) = g(X, \overline{f}Y),$$

which implies $fY = \overline{f}Y$, for any $Y \in \mathcal{X}(\widetilde{M})$. ∎

Consequently, Theorem 5.2 is the best possible version of Chen's Theorem for S-manifolds, because there are not different compatible S-structures on the same manifold.

Finally, let us consider an example. Let

$$(\mathbb{R}^{4+s}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

be the usual S-structure on $\mathbb{R}^{4+s}$ (see [7] for more details) given by the following elements

$$\eta_\alpha = \frac{1}{2}\left(\mathrm{d}z^\alpha - \sum\nolimits_{i=1}^{2} y^i \mathrm{d}x^i\right), \; \xi_\alpha = 2\frac{\partial}{\partial z^\alpha},$$

$$g = \sum\nolimits_{\alpha=1}^{s} \eta_\alpha \otimes \eta_\alpha + \frac{1}{4}\left(\sum\nolimits_{i=1}^{2} (\mathrm{d}x^i \otimes \mathrm{d}x^i + \mathrm{d}y^i \otimes \mathrm{d}y^i)\right),$$

$$f\left(\sum\nolimits_{i=1}^{2}\left(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}\right) + \sum\nolimits_{\alpha=1}^{s} Z_\alpha \frac{\partial}{\partial z^\alpha}\right) =$$

$$= \sum\nolimits_{i=1}^{2}(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i}) + \sum\nolimits_{\alpha=1}^{s} \sum_{i=1}^{2} Y_i y^i \frac{\partial}{\partial z^\alpha},$$

where $(x^1, x^2, y^1, y^2, z^1, \ldots, z^s)$ are denoting the cartesian coordinates on $\mathbb{R}^{4+s}$. Define on $\mathbb{R}^{4+s}$ the (1,1)-tensor field $\overline{f}$ by:

$$\overline{f}\left(\sum_{i=1}^{2}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + \sum_{\alpha=1}^{s} Z_\alpha \frac{\partial}{\partial z^\alpha}\right) =$$

$$= -X_2 \frac{\partial}{\partial x^1} + X_1 \frac{\partial}{\partial x^2} + Y_2 \frac{\partial}{\partial y^1} - Y_1 \frac{\partial}{\partial y^2} + (y^2 X_1 - y^1 X_2) \sum_{\alpha=1}^{s} \frac{\partial}{\partial z^\alpha}.$$

It is easy to prove that

$$(\mathbb{R}^{4+s}, \overline{f}, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$$

is a metric f-manifold. Moreover,

$$(\widetilde{\nabla}_X \overline{f})Y = \sum_{\alpha=1}^{s} (2\eta_\alpha(X)\overline{f}fY + \eta_\alpha(Y)\overline{f}fX + g(X, \overline{f}fY)\xi_\alpha,$$

for any $X, Y \in \mathcal{X}(\widetilde{M})$. Then, we have (5.20).

Now, consider the $(2+s)$-dimensional submanifold M of $\mathbb{R}^{4+s}$ defined by

$$x(u, v, t_1, \dots, t_s) = 2(u\cos\theta, u\sin\theta, v, 0, t_1, \dots, t_s),$$

for any $\theta \in (0, \pi/2)$. Then, M is a minimal proper slant submanifold in

$$(\mathbb{R}^{4+s}, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$$

(see [3]) and an anti-invariant submanifold in

$$(\mathbb{R}^{4+s}, \overline{f}, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g).$$

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