A note on the existence of H-bubbles via perturbation methods

Veronica Felli

Abstract

We study the problem of existence of surfaces in $\mathbb{R}^3$ parametrized on the sphere $\mathbb{S}^2$ with prescribed mean curvature H in the perturbative case, i.e. for $H = H_0 + \varepsilon H_1$, where H_0 is a nonzero constant, H_1 is a C^2 function and ε is a small perturbation parameter.

1. Introduction

In this paper we are interested in the existence of H-bubbles, namely of $\mathbb{S}^2$-type parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature H. This geometrical problem is motivated by some models describing capillarity phenomena and has the following analytical formulation: given a function $H \in C^1(\mathbb{R}^3)$, find a smooth nonconstant function $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ which is conformal as a map on $\mathbb{S}^2$ and solves the problem

$$(P_H) \qquad \begin{cases} \Delta \omega = 2H(\omega)\,\omega_x \wedge \omega_y, & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla \omega|^2 < +\infty, \end{cases}$$

where

$$\omega_x = \Big(\frac{\partial \omega_1}{\partial x}, \frac{\partial \omega_2}{\partial x}, \frac{\partial \omega_3}{\partial x}\Big), \qquad \omega_y = \Big(\frac{\partial \omega_1}{\partial y}, \frac{\partial \omega_2}{\partial y}, \frac{\partial \omega_3}{\partial y}\Big),$$
$$\Delta \omega = \omega_{xx} + \omega_{yy}, \qquad \nabla \omega = (\omega_x, \omega_y),$$

and $\wedge$ denotes the exterior product in $\mathbb{R}^3$.

Brezis and Coron [4] proved that for constant nonzero mean curvature $H(u) \equiv H_0$ the only nonconstant solutions are spheres of radius $|H_0|^{-1}$.

While the Plateau and the Dirichlet problems have been largely studied both for H constant and for H nonconstant (see [3, 4, 10, 12, 13, 14, 15, 16]), problem (P_H) in the case of nonconstant H has been investigated only recently, see [5, 6, 7]. In [5] Caldiroli and Musina proved the existence of H-bubbles with minimal energy under the assumptions that $H \in C^1(\mathbb{R}^3)$ satisfies

$$(i) \quad \sup_{u \in \mathbb{R}^3} |\nabla H(u + \xi) \cdot u\,u| < 1 \quad \text{for some } \xi \in \mathbb{R}^3,$$

$$(ii) \quad H(u) \to H_\infty \quad \text{as } |u| \to \infty \quad \text{for some } H_\infty \in \mathbb{R},$$

$$(iii) \quad c_H = \inf_{\substack{u \in C_c^1(\mathbb{R}^2, \mathbb{R}^3) \\ u \neq 0}} \sup_{s>0} \mathcal{E}_H(su) < \frac{4\pi}{3H_\infty^2}$$

where

$$\mathcal{E}_H(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + 2 \int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$$

and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $\operatorname{div} Q = H$.

The perturbative method introduced by Ambrosetti and Badiale [1, 2] was used in [7] to treat the case in which H is a small perturbation of a constant, namely

$$H(u) = H_\varepsilon(u) = H_0 + \varepsilon H_1(u),$$

where $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$, and ε is a small real parameter. This method allows to find critical points of a functional f_ε of the type $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ in a Banach space by studying a finite dimensional problem. More precisely, if the unperturbed functional f_0 has a finite dimensional manifold of critical points Z which satisfies a nondegeneracy condition, it is possible to prove, for $|\varepsilon|$ sufficiently small, the existence of a smooth function $\eta_\varepsilon(z) : Z \to (T_z Z)^\perp$ such that any critical point $\bar{z} \in Z$ of the function

$$\Phi_\varepsilon : Z \to \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon\big(z + \eta_\varepsilon(z)\big)$$

gives rise to a critical point $u_\varepsilon = \bar{z} + \eta_\varepsilon(\bar{z})$ of f_ε, i.e. the perturbed manifold $Z_\varepsilon := \{z + \eta_\varepsilon(z) : z \in Z\}$ is a natural constraint for f_ε. Furthermore Φ_ε can be expanded as

$$(1.1) \qquad \Phi_\varepsilon(z) = b - \varepsilon \Gamma(z) + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

where $b = f_0(z)$ and Γ is the Melnikov function defined as the restriction of the perturbation G on Z, namely $\Gamma = G\big|_Z$. For problem (P_{H_ε}), Γ is given by

$$\Gamma : \mathbb{R}^3 \to \mathbb{R}, \quad \Gamma(p) = \int_{|p-q| < \frac{1}{|H_0|}} H_1(q)\,dq.$$

In [7] Caldiroli and Musina studied the functional Γ giving some existence results in the perturbative setting for problem (P_{H_ε}). They prove that for $|\varepsilon|$ small there exists a smooth H_ε-bubble if one of the following conditions holds

1) H_1 has a nondegenerate stationary point and $|H_0|$ is large;

2) $\max\limits_{p\in\partial K} H_1(p) < \max\limits_{p\in K} H_1(p)$ or $\min\limits_{p\in\partial K} H_1(p) > \min\limits_{p\in K} H_1(p)$ for some non-empty compact set $K \subset \mathbb{R}^3$ and $|H_0|$ is large;

3) $H_1 \in L^r(\mathbb{R}^3)$ for some $r \in [1,2]$.

They prove that critical points of Γ give rise to solutions to (P_{H_ε}) for ε sufficiently small. Precisely the assumption that H_0 is large required in cases 1) and 2) ensures that if H_1 is not constant then Γ is not identically constant. If we let this assumption drop, it may happen that Γ is constant even if H_1 is not. This fact produces some loss of information because the first order expansion (1.1) is not sufficient to deduce the existence of critical points of Φ_ε from the existence of critical points of Γ. Instead of studying Γ we perform a direct study of Φ_ε which allows us to prove some new results. In the first one, we assume that H_1 vanishes at ∞ and has bounded gradient, and prove the existence of a solution without branch points. Let us recall that a branch point for a solution ω to (P_H) is a point where $\nabla\omega = 0$, i.e. a point where the surface parametrized by ω fails to be immersed.

Theorem 1.1 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that*

$$(H1) \quad \lim_{|p|\to\infty} H_1(p) = 0;$$
$$(H2) \quad \nabla H_1 \in L^\infty(\mathbb{R}^3, \mathbb{R}^3).$$

Let $H_\varepsilon = H_0 + \varepsilon H_1$. Then for $|\varepsilon|$ sufficiently small there exists a smooth H_ε-bubble without branch points.

With respect to case 1) of [7] we require neither nondegeneracy of critical points of H_1 nor largeness of H_0. With respect to case 2) we do not assume that H_0 is large; on the other hand our assumption $(H1)$ implies 2). Moreover we do not assume any integrability condition of type 3). With respect to the result proved in [5], we have the same kind of behavior of H_1 at ∞ (see (ii) and assumption $(H1)$) but we do not need any assumption of type (iii); on the other hand in [5] it is not required that the prescribed curvature is a small perturbation of a constant.

The following results give some conditions on H_1 in order to have two or three solutions.

Theorem 1.2 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$, $(H2)$,*

> $(H3)$ Hess $H_1(p)$ *is positive definite for any $p \in B_{1/|H_0|}(0)$,*
>
> $(H4)$ $H_1(p) > 0$ *in $B_{1/|H_0|}(0)$,*

hold. Then for $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.

Remark 1.3 *If we assume $(H1)$, $(H2)$, and, instead of $(H3) - (H4)$, that $H_1(0) > 0$ and $\mathrm{Hess}\, H_1(0)$ is positive definite, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.*

Theorem 1.4 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$ and $(H2)$ hold. Assume that there exist $p_1, p_2 \in \mathbb{R}^3$ such that*

$$(H5) \quad \int_{B(p_1, 1/|H_0|)} H_1(\xi)\, d\xi > 0 \qquad and \qquad \int_{B(p_2, 1/|H_0|)} H_1(\xi)\, d\xi < 0.$$

Then for $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.

Remark 1.5 *If we assume $(H1)$, $(H2)$, and, instead of $(H5)$, that there exist $p_1, p_2 \in \mathbb{R}^3$ such that $H_1(p_1) > 0$ and $H_1(p_2) < 0$, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.*

The present paper is organized as follows. In Section 2 we introduce some notation and recall some known facts whereas Section 3 is devoted to the proof of Theorems 1.1, 1.2, and 1.4.

2. Notation and known facts

In the sequel we will take $H_0 = 1$; this is not restrictive since we can do the change $H_1(u) = H_0 \widetilde{H}_1(H_0 u)$. Hence we will always write

$$H_\varepsilon(u) = 1 + \varepsilon H(u),$$

where $H \in C^2(\mathbb{R}^3)$. Let us denote by ω the function $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ defined as

$$\omega(x, y) = \big(\mu(x,y)x, \mu(x,y)y, 1 - \mu(x,y)\big), \quad \text{where} \quad \mu(x, y) = \frac{2}{1 + x^2 + y^2}.$$

Note that ω is a conformal parametrization of the unit sphere and solves

$$(2.1) \qquad \begin{cases} \Delta \omega = 2\, \omega_x \wedge \omega_y & \text{on } \mathbb{R}^2 \\ \displaystyle \int_{\mathbb{R}^2} |\nabla \omega|^2 < +\infty. \end{cases}$$

Problem (2.1) has in fact a family of solutions of the form $\omega \circ \phi + p$ where $p \in \mathbb{R}^3$ and ϕ is any conformal diffeomorphism of $\mathbb{R}^2 \cup \{\infty\}$. For $s \in (1, +\infty)$, we will set $L^s := L^s(\mathbb{S}^2, \mathbb{R}^3)$, where any map $v \in L^s$ is identified with the map $v \circ \omega : \mathbb{R}^2 \to \mathbb{R}^3$ which satisfies

$$\int_{\mathbb{R}^2} |v \circ \omega|^s \mu^2 = \int_{\mathbb{S}^2} |v|^s.$$

We will use the same notation for v and $v \circ \omega$. By $W^{1,s}$ we denote the Sobolev space $W^{1,s}(\mathbb{S}^2, \mathbb{R}^3)$ endowed (according to the above identification) with the norm

$$\|v\|_{W^{1,s}} = \left[\int_{\mathbb{R}^2} |\nabla v|^s \mu^{2-s} \right]^{1/s} + \left[\int_{\mathbb{R}^2} |v|^s \mu^2 \right]^{1/s}.$$

If s' is the conjugate exponent of s, i.e. $1/s + 1/s' = 1$, the duality product between $W^{1,s}$ and $W^{1,s'}$ is given by

$$\langle v, \varphi \rangle = \int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi + \int_{\mathbb{R}^2} v \cdot \varphi \, \mu^2 \quad \text{for any } v \in W^{1,s} \text{ and } \varphi \in W^{1,s'}.$$

Let Q be any smooth vector field on $\mathbb{R}^3$ such that $\operatorname{div} Q = H$. The energy functional associated to problem

$$(P_\varepsilon) \qquad \begin{cases} \Delta u = 2\big(1 + \varepsilon H(u)\big) u_x \wedge u_y, & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty, \end{cases}$$

is given by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + 2\mathcal{V}_1(u) + 2\varepsilon \mathcal{V}_H(u), \quad u \in W^{1,3},$$

where

$$\mathcal{V}_H(u) = \int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$$

has the meaning of an algebraic volume enclosed by the surface parametrized by u with weight H (it is independent of the choice of Q); in particular

$$\mathcal{V}_1(u) = \frac{1}{3} \int_{\mathbb{R}^2} u \cdot u_x \wedge u_y.$$

In [7], Caldiroli and Musina studied some regularity properties of $\mathcal{V}_H$ on the space $W^{1,3}$. In particular they proved the following properties.

a) For $H \in C^1(\mathbb{R}^3)$, the functional $\mathcal{V}_H$ is of class C^1 on $W^{1,3}$ and the Fréchet differential of $\mathcal{V}_H$ at $u \in W^{1,3}$ is given by

$$(2.2) \qquad d\mathcal{V}_H(u)\varphi = \int_{\mathbb{R}^2} H(u)\,\varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3}$$

and admits a unique continuous and linear extension on $W^{1,3/2}$ defined by (2.2). Moreover for every $u \in W^{1,3}$ there exists $\mathcal{V}'_H(u) \in W^{1,3}$ such that

$$(2.3) \qquad \langle \mathcal{V}'_H(u), \varphi \rangle = \int_{\mathbb{R}^2} H(u)\,\varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3/2}.$$

b) For $H \in C^2(\mathbb{R}^3)$, the map $\mathcal{V}'_H : W^{1,3} \to W^{1,3}$ is of class C^1 and

$$\langle \mathcal{V}''_H(u) \cdot \eta, \varphi \rangle = \int_{\mathbb{R}^2} H(u)\varphi \cdot (\eta_x \wedge u_y + u_x \wedge \eta_y) + \int_{\mathbb{R}^2} (\nabla H(u) \cdot \eta)\varphi \cdot (u_x \wedge u_y)$$
$$(2.4) \qquad\qquad\qquad\qquad \text{for any } u, \eta \in W^{1,3} \text{ and } \varphi \in W^{1,3/2}.$$

Hence for all $u \in W^{1,3}$, $\mathcal{E}'_\varepsilon(u) \in W^{1,3}$ and for any $\varphi \in W^{1,3/2}$

$$\langle \mathcal{E}'_\varepsilon(u), \varphi \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} \varphi \cdot u_x \wedge u_y + 2\varepsilon \int_{\mathbb{R}^2} H(u)\,\varphi \cdot u_x \wedge u_y.$$

As remarked in [7], critical points of $\mathcal{E}_\varepsilon$ in $W^{1,3}$ give rise to bounded weak solutions to (P_ε) and hence by the regularity theory for H-systems (see [9]) to classical conformal solutions which are $C^{3,\alpha}$ as maps on $\mathbb{S}^2$.

The unperturbed problem, i.e. (P_ε) for $\varepsilon = 0$, has a 9-dimensional manifold of solutions given by

$$Z = \{R\omega \circ L_{\lambda,\xi} + p : \ R \in SO(3), \ \lambda > 0, \ \xi \in \mathbb{R}^2, \ p \in \mathbb{R}^3\}$$

where $L_{\lambda,\xi} z = \lambda(z - \xi)$ (see [11]). In [11] the nondegeneracy condition $T_u Z = \ker \mathcal{E}''_0(u)$ for any $u \in Z$ (where $T_u Z$ denotes the tangent space of Z at u) is proved (see also [8]).

As observed in [7], in performing the finite dimensional reduction, the dependence on the 6-dimensional conformal group can be neglected since any H-system is conformally invariant. Hence we look for critical points of $\mathcal{E}_\varepsilon$ constrained on a three-dimensional manifold Z_ε just depending on the translation variable $p \in \mathbb{R}^3$.

3. Proof of Theorem 1.1

We start by constructing a perturbed manifold which is a natural constraint for $\mathcal{E}_\varepsilon$.

Lemma 3.1 *Assume $H \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$, and a C^1 map $\eta : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^3 \to W^{1,3}$ such that for any $p \in \mathbb{R}^3$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*

$$(3.1) \qquad \mathcal{E}_\varepsilon'\big(\omega + p + \eta(\varepsilon, p)\big) \in T_\omega Z,$$

$$(3.2) \qquad \eta(\varepsilon, p) \in (T_\omega Z)^\perp,$$

$$(3.3) \qquad \int_{\mathbb{S}^2} \eta(\varepsilon, p) = 0,$$

$$(3.4) \qquad \|\eta(\varepsilon, p)\|_{W^{1,3}} \leq C_1 |\varepsilon|.$$

Moreover if we assume that the limit of H at ∞ exists and

$$(3.5) \qquad \lim_{|p| \to \infty} H(p) = 0$$

we have that $\eta(\varepsilon, p)$ converges to 0 in $W^{1,3}$ as $|p| \to \infty$ uniformly with respect to $|\varepsilon| < \varepsilon_0$.

Proof. Let us define the map

$$F = (F_1, F_2) : \mathbb{R} \times \mathbb{R}^3 \times W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

given by

$$\langle F_1(\varepsilon, p, \eta, \lambda, \alpha), \varphi \rangle = \langle \mathcal{E}_\varepsilon'(\omega + p + \eta), \varphi \rangle - \sum_{i=1}^{6} \lambda_i \int_{\mathbb{R}^2} \nabla \varphi \cdot \nabla \tau_i + \alpha \cdot \int_{\mathbb{S}^2} \varphi,$$

for all $\varphi \in W^{1,3/2}$ and

$$F_2(\varepsilon, p, \eta, \lambda, \alpha) = \left(\int_{\mathbb{R}^2} \nabla \eta \cdot \nabla \tau_1, \ldots, \int_{\mathbb{R}^2} \nabla \eta \cdot \nabla \tau_6, \int_{\mathbb{S}^2} \eta \right)$$

where $\tau_1, \ldots, \tau_6$ are chosen in $T_\omega Z$ such that

$$\int_{\mathbb{R}^2} \nabla \tau_i \cdot \nabla \tau_j = \delta_{ij} \quad \text{and} \quad \int_{\mathbb{S}^2} \tau_i = 0 \quad i, j = 1, \ldots, 6$$

so that $T_\omega Z$ is spanned by $\tau_1, \ldots, \tau_6, e_1, e_2, e_3$. It has been proved by Caldiroli and Musina [7] that F is of class C^1 and that the linear continuous operator

$$\mathcal{L} : W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

$$\mathcal{L} = \frac{\partial F}{\partial(\eta, \lambda, \alpha)}(0, p, 0, 0, 0)$$

i.e.

$$\langle \mathcal{L}_1(v, \mu, \beta), \varphi \rangle = \langle \mathcal{E}_0''(\omega) \cdot v, \varphi \rangle - \sum_{i=1}^{6} \mu_i \int_{\mathbb{R}^2} \nabla\varphi \cdot \tau_i - \beta \int_{\mathbb{S}^2} \varphi$$

for all $\varphi \in W^{1,3/2}$ and

$$\mathcal{L}_2(v, \mu, \beta) = \left(\int_{\mathbb{R}^2} \nabla v \cdot \nabla\tau_1, \ldots, \int_{\mathbb{R}^2} \nabla v \cdot \nabla\tau_6, \int_{\mathbb{S}^2} v \right)$$

is invertible. Caldiroli and Musina applied the Implicit Function Theorem to solve the equation $F(\varepsilon, p, \eta, \lambda, \alpha) = 0$ locally with respect to the variables ε, p, thus finding a C^1-function η on a neighborhood $(-\varepsilon_0, \varepsilon_0) \times B_R \subset \mathbb{R} \times \mathbb{R}^3$ satisfying (3.1), (3.2), and (3.3). We will use instead the Contraction Mapping Theorem, which allows to prove the existence of such a function η globally on $\mathbb{R}^3$, thanks to the fact that the operator $\mathcal{L}$ does not depend on p and hence it is invertible uniformly with respect to $p \in \mathbb{R}^3$.

We have that $F(\varepsilon, p, \eta, \lambda, \alpha) = 0$ if and only if (η, λ, α) is a fixed point of the map $T_{\varepsilon,p}$ defined as

$$T_{\varepsilon,p}(\eta, \lambda, \alpha) = -\mathcal{L}^{-1} F(\varepsilon, p, \eta, \lambda, \alpha) + (\eta, \lambda, \alpha).$$

To prove the existence of η satisfying (3.1), (3.2), and (3.3), it is enough to prove that $T_{\varepsilon,p}$ is a contraction in some ball $B_\rho(0)$ with $\rho = \rho(\varepsilon) > 0$ independent of p, whereas the regularity of $\eta(\varepsilon, p)$ follows from the Implicit Function Theorem.

We have that if $\|\eta\|_{W^{1,3}} \le \rho$

$$\|T_{\varepsilon,p}(\eta, \lambda, \alpha)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}$$
$$\le C_2 \|F(\varepsilon, p, \eta, \lambda, \alpha) - \mathcal{L}(\eta, \lambda, \alpha)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}$$
$$\le C_2 \|\mathcal{E}_\varepsilon'(\omega + p + \eta) - \mathcal{E}_0''(\omega)\eta\|_{W^{1,3}}$$
$$\le C_2 \left(\|\mathcal{E}_0'(\omega + \eta) - \mathcal{E}_0''(\omega)\eta\|_{W^{1,3}} + 2|\varepsilon| \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}} \right)$$
$$\le C_2 \left(\int_0^1 \|\mathcal{E}_0''(\omega + t\eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3/2}} \, dt \|\eta\|_{W^{1,3}} \right.$$
$$\left. + 2|\varepsilon| \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}} \right)$$
$$\le C_2 \rho \sup_{\|\eta\|_{W^{1,3}} \le \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3/2}}$$

(3.6)
$$+ 2C_2|\varepsilon| \sup_{\|\eta\|_{W^{1,3}} \le \rho} \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}}$$

where $C_2 = \|\mathcal{L}^{-1}\|_{\mathcal{L}(W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3)}$.

For $(\eta_1, \lambda_1, \alpha_1)$, $(\eta_2, \lambda_2, \alpha_2) \in B_\rho(0) \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ we have

$$\frac{\|T_{\varepsilon,p}(\eta_1, \lambda_1, \alpha_1) - T_{\varepsilon,p}(\eta_2, \lambda_2, \alpha_2)\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3}}{C_2 \|\eta_1 - \eta_2\|_{W^{1,3}}}$$

$$\leq \frac{\|\mathcal{E}'_\varepsilon(\omega + p + \eta_1) - \mathcal{E}'_\varepsilon(\omega + p + \eta_2) - \mathcal{E}''_0(\omega)(\eta_1 - \eta_2)\|_{W^{1,3}}}{C_2 \|\eta_1 - \eta_2\|_{W^{1,3}}}$$

$$\leq \int_0^1 \|\mathcal{E}''_\varepsilon(\omega + p + \eta_2 + t(\eta_1 - \eta_2)) - \mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\, dt$$

$$\leq \int_0^1 \|\mathcal{E}''_0(\omega + p + \eta_2 + t(\eta_1 - \eta_2)) - \mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\, dt$$

$$+ 2|\varepsilon| \int_0^1 \|\mathcal{V}''_H(\omega + p + \eta_2 + t(\eta_1 - \eta_2))\|_{W^{1,3/2}}\, dt$$

$$\leq \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho} \|\mathcal{E}''_0(\omega + \eta) - \mathcal{E}''_0(\omega)\|_{W^{1,3/2}} + 2|\varepsilon| \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho} \|\mathcal{V}''_H(\omega + p + \eta)\|_{W^{1,3/2}}.$$

From (2.3), (2.4), and the Hölder inequality it follows that there exists a positive constant C_3 such that for any $\eta \in W^{1,3}$, $p \in \mathbb{R}^3$

$$(3.7) \quad \|\mathcal{V}'_H(\omega + p + \eta)\|_{W^{1,3}} \leq C_3 \left[\left(\int_{\mathbb{R}^2} |H(\omega + p + \eta)|^{3/2} |\nabla \omega|^3 \mu^{-1} \right)^{2/3} + \|\eta\|^2_{W^{1,3}} \right]$$

and

$$\|\mathcal{V}''_H(\omega + p + \eta)\|_{W^{1,3/2}} \leq C_3 \left[\left(\int_{\mathbb{R}^2} |H(\omega + p + \eta)|^2 |\nabla(\omega + \eta)|^2 \right)^{1/2} \right.$$

$$(3.8) \qquad \left. + \left(\int_{\mathbb{R}^2} |\nabla H(\omega + p + \eta)|^{3/2} |\nabla(\omega + \eta)|^3 \mu^{-1} \right)^{2/3} \right].$$

Choosing $\rho_0 > 0$ such that

$$C_2 \sup_{\|\eta\|_{W^{1,3}} \leq 3\rho_0} \left\| \mathcal{E}''_0(\omega + \eta) - \mathcal{E}''_0(\omega) \right\|_{W^{1,3/2}} < \frac{1}{2}$$

and $\varepsilon_0 > 0$ such that

$$(3.9) \qquad 8 C_2 C_3 \varepsilon_0 \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} < \min \left\{ 1, \rho_0, \frac{1}{8 C_2 C_3 \varepsilon_0} \right\},$$

$$(3.10) \qquad \sup_{\substack{\|\eta\|_{W^{1,3}} \leq \rho_0 \\ p \in \mathbb{R}^3}} \left\| \mathcal{V}'_H(\omega + p + \eta) \right\|_{W^{1,3}} < \frac{\rho_0}{6 \varepsilon_0 C_2},$$

$$(3.11) \qquad \sup_{\substack{\|\eta\|_{W^{1,3}} \leq 3\rho_0 \\ p \in \mathbb{R}^3}} \left\| \mathcal{V}''_H(\omega + p + \eta) \right\|_{W^{1,3/2}} < \frac{1}{8 \varepsilon_0 C_2},$$

we obtain that $T_{\varepsilon,p}$ maps the ball $\overline{B_{\rho_0}(0)}$ into itself for any $|\varepsilon| < \varepsilon_0$, $p \in \mathbb{R}^3$, and is a contraction there.

Hence it has a unique fixed point $\big(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p)\big) \in \overline{B_{\rho_0}(0)}$. From (3.6) we have that the following property holds

$(*)$ $T_{\varepsilon,p}$ maps a ball $\overline{B_\rho(0)} \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ into itself whenever $\rho \leq \rho_0$

$$\text{and } \rho > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}}\leq\rho} \big\|\mathcal{V}'_H(\omega + p + \eta)\big\|_{W^{1,3}}.$$

In particular let us set

$$(3.12) \qquad \rho_\varepsilon = 5|\varepsilon|C_2 \sup_{\substack{\|\eta\|_{W^{1,3}}\leq\rho_0 \\ p\in\mathbb{R}^3}} \big\|\mathcal{V}'_H(\omega + p + \eta)\big\|_{W^{1,3}}.$$

In view of (3.10) and (3.12), we have that for any $|\varepsilon| < \varepsilon_0$ and for any $p \in \mathbb{R}^3$

$$\rho_\varepsilon \leq \rho_0 \quad \text{and} \quad \rho_\varepsilon > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}}\leq\rho_\varepsilon} \big\|\mathcal{V}'_H(\omega + p + \eta)\big\|_{W^{1,3}}$$

so that, due to $(*)$, $T_{\varepsilon,p}$ maps $\overline{B_{\rho_\varepsilon}(0)}$ into itself. From the uniqueness of the fixed point we have that for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$

$$(3.13) \qquad \|(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p))\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3} \leq \rho_\varepsilon \leq C_1|\varepsilon|$$

for some positive constant C_1 independent of p and hence $\|\eta(\varepsilon,p)\|_{W^{1,3}} \leq \rho_\varepsilon \leq C_1|\varepsilon|$ thus proving (3.4). Assume now (3.5) and set for any $p \in \mathbb{R}^3$

$$\rho_p = 8C_2C_3\varepsilon_0 \bigg(\int_{\mathbb{R}^2} \sup_{|q-p|\leq 1+C_0} |H(q)|^{3/2}|\nabla\omega|^3\mu^{-1} \bigg)^{2/3}$$

where C_0 is a positive constant such that $\|u\|_{L^\infty} \leq C_0\|u\|_{W^{1,3}}$ for any $u \in W^{1,3}$. From (3.9) we have that

$$\rho_p < \min\bigg\{1, \rho_0, \frac{1}{8C_2C_3\varepsilon_0}\bigg\}.$$

Hence, due to (3.7), we have that for $|\varepsilon| < \varepsilon_0$ and $\|\eta\|_{W^{1,3}} \leq \rho_p$

$$4|\varepsilon|C_2\big\|\mathcal{V}'_H(\omega + p + \eta)\big\|_{W^{1,3}}$$
$$\leq 4\varepsilon_0C_2C_3\bigg(\int_{\mathbb{R}^2} \sup_{|q-p|\leq 1+C_0} |H(q)|^{3/2}|\nabla\omega|^3\mu^{-1} \bigg)^{2/3} + 4\varepsilon_0C_2C_3\rho_p^2 < \rho_p.$$

From $(*)$ and the uniqueness of the fixed point, we deduce that

$$\|\eta(\varepsilon,p)\|_{W^{1,3}} \leq \rho_p$$

for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$. On the other hand, since H vanishes at ∞, by the definition of ρ_p we have that $\rho_p \to 0$ as $|p| \to \infty$, hence

$$\lim_{|p|\to\infty} \eta(\varepsilon,p) = 0 \quad \text{in } W^{1,3} \text{ uniformly for } |\varepsilon| < \varepsilon_0.$$

The proof of Lemma 3.1 is now complete.   ∎

Remark 3.2 *The map η given in Lemma 3.1 satisfies*

$$\langle \mathcal{E}'_\varepsilon(\omega+p+\eta(\varepsilon,p)), \varphi \rangle - \sum_{i=1}^{6} \lambda_i(\varepsilon,p) \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i + \alpha(\varepsilon,p) \cdot \int_{\mathbb{S}^2} \varphi, \quad \forall \varphi \in W^{1,3/2}$$

where $\big(\eta(\varepsilon,p), \lambda(\varepsilon,p), \alpha(\varepsilon,p)\big) \in \overline{B_{\rho_\varepsilon}(0)} \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ being ρ_ε given in (3.12), hence

$$\int_{\mathbb{R}^2} \nabla(\omega + \eta(\varepsilon,p)) \cdot \nabla\varphi + 2 \int_{\mathbb{R}^2} \varphi \cdot (\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y$$

$$+ 2\varepsilon \int_{\mathbb{R}^2} H(\omega + p + \eta(\varepsilon,p))\varphi \cdot (\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y$$

$$= \sum_{i=1}^{6} \lambda_i(\varepsilon,p) \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i - \alpha(\varepsilon,p) \cdot \int_{\mathbb{S}^2} \varphi, \quad \forall \varphi \in W^{1,3/2},$$

i.e. $\eta(\varepsilon,p)$ satisfies the equation

$$\Delta\eta(\varepsilon,p) = F(\varepsilon,p)$$

where

$$F(\varepsilon,p) = 2(\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y - 2\omega_x \wedge \omega_y + \lambda(\varepsilon,p) \cdot \Delta\tau$$
$$- \alpha(\varepsilon,p)\mu^2 + 2\varepsilon H(\omega + p + \eta(\varepsilon,p))(\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y$$

in $\mathbb{R}^2$. Since $F(\varepsilon,p) \in L^{3/2}$ and, in view of (3.4) and (3.13), $F(\varepsilon,p) \to 0$ in $L^{3/2}$ as $\varepsilon \to 0$ uniformly with respect to p, by regularity we have that

$$\eta(\varepsilon,p) \in W^{2,3/2} \quad and \quad \eta(\varepsilon,p) \to 0 \ in \ W^{2,3/2}$$

hence, by Sobolev embeddings, $F(\varepsilon,p) \in L^3$ and $F(\varepsilon,p) \to 0$ in L^3 as $\varepsilon \to 0$ uniformly with respect to p. Again by regularity

$$\eta(\varepsilon,p) \in W^{2,3} \quad and \quad \eta(\varepsilon,p) \to 0 \ in \ W^{2,3}$$

hence $\eta(\varepsilon,p) \in C^{1,1/3}$ and

(3.14) $\eta(\varepsilon,p) \to 0 \quad in \ C^{1,1/3} \ as \ \varepsilon \to 0 \ uniformly \ with \ respect \ to \ p.$

For any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let us define the perturbed manifold

$$Z_\varepsilon := \big\{ \omega + p + \eta(\varepsilon,p) : \ p \in \mathbb{R}^3 \big\}.$$

From [7], we have that Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, namely any critical point $p \in \mathbb{R}^3$ of the functional

$$\Phi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}, \qquad \Phi_\varepsilon(p) = \mathcal{E}_\varepsilon\big(\omega + p + \eta(\varepsilon,p)\big)$$

gives rise to a critical point $u_\varepsilon = \omega + p + \eta(\varepsilon,p)$ of $\mathcal{E}_\varepsilon$.

Proposition 3.3 *Assume $H \in C^2(\mathbb{R}^3)$, $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, and*

$$\lim_{|p| \to \infty} H(p) = 0.$$

Then for any $|\varepsilon| < \varepsilon_0$

$$\lim_{|p| \to \infty} \Phi_\varepsilon(p) = \text{const} = \mathcal{E}_0(\omega).$$

Proof.   We have that

$$\begin{aligned}
\Phi_\varepsilon(p) &= \mathcal{E}_\varepsilon\big(\omega + p + \eta(\varepsilon, p)\big) \\
&= \mathcal{E}_0\big(\omega + p + \eta(\varepsilon, p)\big) + 2\varepsilon \mathcal{V}_H\big(\omega + p + \eta(\varepsilon, p)\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon, p)|^2 + \int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\eta(\varepsilon, p) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \big(\omega + p + \eta(\varepsilon, p)\big) \cdot \big(\omega + \eta(\varepsilon, p)\big)_x \wedge \big(\omega + \eta(\varepsilon, p)\big)_y \\
&\quad + 2\varepsilon\big[\mathcal{V}_H(\omega + p) + \langle \mathcal{V}'_H(\omega + p), \eta(\varepsilon, p)\rangle + o\big(\|\eta(\varepsilon, p)\|_{W^{1,3}}\big)\big] \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{2}{3}\int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon, p)|^2 \\
&\quad + \int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\eta(\varepsilon, p) + \frac{2}{3}\int_{\mathbb{R}^2} \omega \cdot (\omega_x \wedge \eta(\varepsilon, p)_y + \eta(\varepsilon, p)_x \wedge \omega_y) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \omega \cdot \eta(\varepsilon, p)_x \wedge \eta(\varepsilon, p)_y \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \eta(\varepsilon, p) \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y \\
&\quad + 2\varepsilon\mathcal{V}_H(\omega + p) + 2\varepsilon\langle\mathcal{V}'_H(\omega + p), \eta(\varepsilon, p)\rangle + 2\varepsilon\, o\big(\|\eta(\varepsilon, p)\|_{W^{1,3}}\big)
\end{aligned} \tag{3.15}$$

where we have used the fact that

$$\int_{\mathbb{R}^2} p \cdot u_x \wedge u_y = 0 \quad \forall\, p \in \mathbb{R}^3,\ u \in W^{1,3},$$

(see [7], Lemma A.3). Notice that from Lemma 3.1 we have that

$$\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon, p)|^2 \le \sqrt[3]{4\pi}\,\|\eta(\varepsilon, p)\|_{W^{1,3}}^2 \underset{|p| \to \infty}{\longrightarrow} 0, \tag{3.16}$$

$$\left|\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\eta(\varepsilon, p)\right| \le \sqrt[6]{4\pi}\left(\int_{\mathbb{R}^2} |\nabla\omega|^2\right)^{1/2}\|\eta(\varepsilon, p)\|_{W^{1,3}} \underset{|p| \to \infty}{\longrightarrow} 0, \tag{3.17}$$

and, by the Hölder inequality and Lemma 3.1,

$$(3.18) \quad \left| \int_{\mathbb{R}^2} \omega \cdot (\omega_x \wedge \eta(\varepsilon, p)_y + \eta(\varepsilon, p)_x \wedge \omega_y) \right| \le 2\|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow[|p| \to \infty]{} 0,$$

$$(3.19) \quad \left| \int_{\mathbb{R}^2} \omega \cdot (\eta(\varepsilon, p)_x \wedge \eta(\varepsilon, p)_y) \right| \le \|\omega\|_{W^{1,3}} \|\eta(\varepsilon, p)\|^2_{W^{1,3}} \xrightarrow[|p| \to \infty]{} 0,$$

and

$$\left| \int_{\mathbb{R}^2} \eta(\varepsilon, p) \cdot (\omega + \eta(\varepsilon, p))_x \wedge (\omega + \eta(\varepsilon, p))_y \right|$$
$$(3.20) \qquad \le \|\omega + \eta(\varepsilon, p)\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow[|p| \to \infty]{} 0.$$

Moreover the Gauss-Green Theorem yields

$$\mathcal{V}_H(\omega + p) = - \int_{B_1} H(\xi + p)\, d\xi$$

so that by the Dominated Convergence Theorem we have that

$$(3.21) \qquad \lim_{|p| \to \infty} \mathcal{V}_H(\omega + p) = 0.$$

From (2.3), Hölder's inequality, and Lemma 3.1, we have that

$$|\langle \mathcal{V}'_H(\omega + p), \eta(\varepsilon, p) \rangle| = \left| \int_{\mathbb{R}^2} H(\omega + p)\, \eta(\varepsilon, p) \cdot \omega_x \wedge \omega_y \right|$$
$$(3.22) \qquad \le \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \xrightarrow[|p| \to \infty]{} 0.$$

From (3.15)–(3.22), it follows that

$$\lim_{|p| \to \infty} \Phi_\varepsilon(p) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y = \mathcal{E}_0(\omega).$$

The proposition is thereby proved. ∎

Proof of Theorem 1.1.   As already observed at the beginning of Section 2, it is not restrictive to take $H_0 = 1$. From Proposition 3.3 it follows that for $|\varepsilon| < \varepsilon_0$ either Φ_ε is constant (and hence we have infinitely many critical points) or it has a global maximum or minimum point. In any case Φ_ε has a critical point. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of a critical point of $\mathcal{E}_\varepsilon$ for $|\varepsilon| < \varepsilon_0$ and hence of a solution to (P_ε). The H_ε-bubble ω_ε found in this way is of the form $\omega + p^\varepsilon + \eta(\varepsilon, p^\varepsilon)$ for some $p^\varepsilon \in \mathbb{R}^3$ where η is as in Lemma 3.1. Remark 3.2 yields that ω_ε is closed in $C^{1,1/3}(\mathbb{S}^2, \mathbb{R}^3)$-norm to the manifold $\{\omega + p : p \in \mathbb{R}^3\}$ for ε small. Since ω has no branch points, we deduce that ω_ε has no branch points. ∎

To prove Theorems 1.2 and 1.4, we need the following expansion for Φ_ε:

$$(3.23) \qquad \Phi_\varepsilon(p) = \mathcal{E}_0(\omega) - 2\varepsilon\Gamma(p) + O(\varepsilon^2)$$

as $\varepsilon \to 0$ uniformly in $p \in \mathbb{R}^3$ (see [7]).

Proof of Theorem 1.2. Let $\varepsilon > 0$ small. Assumption $(H4)$ implies that $\Gamma(0) > 0$ and hence from (3.23) we have that for ε small

$$\Phi_\varepsilon(0) < \mathcal{E}_0(\omega),$$

whereas from assumption $(H3)$ we have that $\operatorname{Hess}\Gamma(0)$ is positive definite so that Γ has a strict local minimum in 0 and hence from (3.23) Φ_ε has a strict local maximum in $B_r(0)$ for some $r > 0$ such that

$$\Phi_\varepsilon(p) < \Phi_\varepsilon(0) - c_\varepsilon < \mathcal{E}_0(\omega)$$

for $|p| = r$, where c_ε is some positive constant depending on ε. In particular Φ_ε has a mountain pass geometry. Moreover by Theorem 1.1,

$$\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega) \quad \text{as} \quad |p| \to \infty,$$

and so Φ_ε must have a global minimum point. If the minimum point and the mountain pass point coincide then Φ_ε has infinitely many critical points. Otherwise Φ_ε has at least three critical points: a local maximum point, a global minimum point, and a mountain pass. If $\varepsilon < 0$ we find the inverse inequalities and hence we find that Φ_ε has a local minimum point, a global maximum point, and a mountain pass. As a consequence (P_ε) has at least three solutions provided $|\varepsilon|$ is sufficiently small. ∎

As observed in Remark 1.3, if $H_1(0) > 0$ and $\operatorname{Hess} H_1(0)$ is positive definite, by continuity we have that for H_0 sufficiently large $\Gamma(0) > 0$ and $\operatorname{Hess}\Gamma(0)$ is positive definite, so that we can still prove the existence of three solutions arguing as above.

Proof of Theorem 1.4. Assumption $(H5)$ implies that $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$. Since

$$\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) + 2\varepsilon\big(-\Gamma(p) + o(1)\big) \quad \text{as } \varepsilon \to 0,$$

we have for ε sufficiently small

$$\Phi_\varepsilon(p_1) < \mathcal{E}_0(\omega) \quad \text{and} \quad \Phi_\varepsilon(p_2) > \mathcal{E}_0(\omega)$$

if $\varepsilon > 0$ and the inverse inequalities if $\varepsilon < 0$. Since by Theorem 1.1

$$\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega) \quad \text{as} \quad |p| \to \infty,$$

we conclude that Φ_ε must have a global maximum point and a global minimum point in $\mathbb{R}^3$. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of two critical points of $\mathcal{E}_\varepsilon$ for $|\varepsilon|$ sufficiently small and hence of two solutions to (P_ε).   ∎

As observed in Remark 1.5, if $H_1(p_1) > 0$ and $H_1(p_2) < 0$, by continuity we have that for H_0 sufficiently large $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$, so that we can still prove the existence of two solutions arguing as above.

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Veronica Felli
Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca
Via Cozzi 53
20125 Milano, Italy
veronica.felli@unimib.it