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Resolution of a family of Galois embedding problems with cyclic kernel

Montserrat Vela

Abstract

In this paper we compute the obstruction and the solutions of cyclic embedding problems given by

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \Gamma = \mathbb{Z}/n\mathbb{Z}\times \overset{m)}{\cdots} \times\mathbb{Z}/n\mathbb{Z} \to 0\,,$$

with $\mathbb{Z}/n\mathbb{Z}$ trivial Γ-modulo, finding adequate representations of Γ in the automorphisms group of a generalized Clifford algebra.

1. Introduction

In [7] we have studied Galois embedding problems given by central extensions with cyclic kernel. In particular, we have computed an expression for the obstruction to the solvability of these embedding problems in terms of Galois symbols, generalizing the formula given by Fröhlich in [3]. To compute this obstruction, we associate to the embedding problem a representation t of Γ in the group of graded automorphisms of an adequate generalized Clifford algebra with an admissible norm.

We have too given a method for constructing the solutions when these problems are solvable, generalizing the results obtained by Crespo in [1] and [2]. We obtain a way to compute a solution of the solvable embedding problems of the form $L(\sqrt[n]{\gamma})$ where γ is a coordinate of the norm of an adequate element in a generalized Clifford algebra.

Using this study we solve now a complete family of problems.

We recall in short the notations and results used in [7] that we need to do this study.

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Let $n \geq 2$ be an integer, K be a field of characteristic not dividing n containing the group μ_n of nth roots of unity, and fix ω a primitive nth root of unity. We denote by lg the group homomorphism $lg : \mu_n \to \mathbb{Z}/n\mathbb{Z}$ where $lg(\omega^i) = i$, the separable closure of K is denoted by K^{sep} and the absolute Galois group by $G_K = Gal(K^{sep}/K)$.

We consider cyclic embedding problems given by

$$L = K(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_m})/K, \qquad \Gamma \simeq \mathbb{Z}/n\mathbb{Z} \times \overset{m)}{\cdots} \times \mathbb{Z}/n\mathbb{Z},$$

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \Gamma \to 0,$$

with $\mathbb{Z}/n\mathbb{Z}$ trivial Γ-module. Let $j : G_K \to \Gamma$ be the surjective homomorphism corresponding to L/K. We consider the homomorphism between the cohomology groups $j_2^* : H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to H^2(G_K, \mathbb{Z}/n\mathbb{Z})$ induced by j and let ε be the element in $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ corresponding to (E).

The embedding problem given by $L/K, \Gamma, (E)$ is solvable if and only if $j_2^*(\varepsilon) = 0$. The element $j_2^*(\varepsilon)$ is called the obstruction to the solvability of the embedding problem.

We use the term *generalized Clifford algebra* to refer to a finite dimensional K-algebra generated by elements $\{e_1, \ldots, e_m\}$ with relations $e_i^n = a_i \in K^*$ and $e_i e_j = \omega e_j e_i$ if $i < j$. We denote this algebra as $C(V)$ for $V = < e_1, \ldots, e_m >_K$. We can make $C(V)$ into a $\mathbb{Z}/n\mathbb{Z}$-graded K-algebra

$$C(V) = C(V)_0 \oplus \cdots \oplus C(V)_{n-1},$$

by setting

$$C(V)_l = < e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m} \mid \varepsilon_1 + \cdots + \varepsilon_m \equiv l \pmod{n} >_K .$$

If m is odd the invariant of $C(V)$ is

$$a = (-1)^{\frac{(n-1)(m-1)}{2}} a_1 a_2^{-1} \cdots a_{m-1}^{-1} a_m$$

and if m is even the invariant is

$$a = (-1)^{\frac{(n-1)m}{2}} a_1^{-1} a_2 \cdots a_{m-1}^{-1} a_m.$$

The Brauer invariant of $C(V)$ defined in [7, Definition 4.2] is the class in the Brauer group of $C(V)$ if m is even or of $C(V)_0$ if m is odd.

We call the norm of the generalized Clifford algebra $C(V)$ the map $N : C(V) \longrightarrow C(V)$ given by $N(z) := \beta(z)z$ where

$$\beta\Big(\sum \alpha_{\varepsilon_1, \cdots, \varepsilon_m} e_1^{\varepsilon_1} \cdots e_m^{\varepsilon_m}\Big) = \sum \alpha_{\varepsilon_1, \cdots, \varepsilon_m}^{n-1} e_m^{\varepsilon_m(n-1)} \cdots e_1^{\varepsilon_1(n-1)}, \quad \alpha_{\varepsilon_1, \cdots, \varepsilon_m} \in K .$$

The norm N restricted to the subgroup

$$F(C(V)) := \{x \in C(V)^* \text{ homog. s.t. } N(x) \in K^* \\ \text{and } \beta(xy) = \beta(y)\beta(x)\, \forall y \in C(V)\}$$

is multiplicative.

Let A be a subgroup of $F(C(V))$ such that $K^* \subset A$. An admissible norm in A is a map $\mathcal{N} : A \longrightarrow K^*$ such that for $a, a_1, a_2 \in A$,

$$\mathcal{N}(a) \in K^*, \quad \mathcal{N}(a_1a_2) = \mathcal{N}(a_1)\mathcal{N}(a_2) \quad \text{and} \quad \mathcal{N}(\lambda) = \lambda^n \text{ if } \lambda \in K.$$

In particular, for $A = F(C(V))$, the norm N is an admissible norm.

We recall that, given a profinite group Γ, a representation of Γ over K is the pair given by

(1) A generalized Clifford algebra $C_t = C(V_t)$ over K with an admissible norm $(\mathcal{N}, A)$ together with

(2) A continuous homomorphism $t : \Gamma \to O(C_t) \subset \mathrm{Autgr}(C_t)$, where $O(C_t) = \varphi(A)$ with the homomorphism $\{\text{Homogeneous elements of } C(V)^*\} \xrightarrow{\varphi} \mathrm{Autgr}(C(V))$ where, for all $x \in C(V)$ homogeneous,

 (a) $\varphi(s)(x) = sxs^{-1}$ if $\dim(V)$ is even,

 (b) $\varphi(s)(x) = \omega^{\partial(s)\partial(x)}sxs^{-1}$ if $\dim(V)$ is odd (where ∂ denotes the degree).

Given a representation $t : \Gamma \to O(C_t)$, we use the term twisted algebra of C_t by t to refer to the K-algebra $\mathfrak{C}_t$ corresponding to the element

$$\begin{aligned} \alpha : \Gamma &\xrightarrow{t} O(C_t) \to O(C_t \otimes L) \subset \mathrm{Autgr}(C_t \otimes L)\,, \\ \sigma &\mapsto t(\sigma)_L(x \otimes \lambda) = t(\sigma)(x) \otimes \lambda \end{aligned}$$

of $H^1(\Gamma, \mathrm{Aut}(C_t \otimes L))$ by the bijection given in [3, III.2] (see too [6, Chap. X]). We have $\mathfrak{C}_t \simeq (C_t \otimes L)^\Gamma$ where Γ acts on $C_t \otimes L$ via $t \otimes gal$, i.e., $\sigma(x \otimes \lambda) = t(\sigma)(x) \otimes \sigma(\lambda)$ and the isomorphism $g : \mathfrak{C}_t \otimes L \xrightarrow{\sim} C_t \otimes L$, where Γ acts on $C_t \otimes L$ via $1 \otimes gal$, that is, $\sigma(x \otimes \lambda) = x \otimes \sigma(\lambda)$. This morphism satisfies $g^{-1}g^\sigma = t(\sigma)$ for each $\sigma \in \Gamma$.

We have obtained, from the diagram of [7, Proposition 5.3] the exact sequence

$$1 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ker}\mathcal{N}_{K^{sep}} \to O(C_t \otimes_K K^{sep}) \to 1$$

where $\mathcal{N}_{K^{sep}}$ denotes the extension of $\mathcal{N}$ to the subgroup $A_{K^{sep}}$ of

$$F(C(V) \otimes_K K^{sep})$$

generated by A.

Composing the representation t with

$$O(C_t) \to O(C_t \otimes_K K^{sep}) \subset \text{Autgr}(C_t \otimes_K K^{sep})$$

we obtain

$$\begin{array}{ccccccccc} & & & & & & \Gamma & & \\ & & & & & & \downarrow t & & \\ & & & & & & O(C_t) & & \\ & & & & & & \downarrow & & \\ 1 & \to & \mathbb{Z}/n\mathbb{Z} & \to & \text{Ker}\mathcal{N}_{K^{sep}} & \to & O(C_t \otimes_K K^{sep}) & \to & 1 \end{array}$$

that is an extension of Γ by $\mathbb{Z}/n\mathbb{Z}$ and so, an element of $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$. The analogue to the second Stiefel-Whitney class, denoted by s_t, is this element of $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$. If $\Gamma = G_K$, the element s_t belongs to

$$H^2(G_K, \mathbb{Z}/n\mathbb{Z}) = \text{Br}_n(K).$$

To compute the obstruction to the embedding problem, we need to find a representation t of Γ so that $\varepsilon = s_t$. From such a representation we have the Formula [7, 6.7]:

Formula 1.1. *The obstruction j_2^* is*

(a) $j_2^*(s_t) = [\mathfrak{C}_t] - [C_t] + P\mathcal{N}_2[t \circ j]$ *if $dim(V_t)$ is even,*

(b) $j_2^*(s_t) = [\mathfrak{C}_t] - [C_t] - (a_t, b_{t\circ j}) + P\mathcal{N}_2[t \circ j]$ *if $dim(V_t)$ is odd,*

where $\mathfrak{C}_t$ is the twisted algebra of C_t by t, a_t is the invariant of C_t, $b_t \in K^*/K^{*^n}$ is the element corresponding to $d(t) \in Hom(\Gamma, \mathbb{Z}/n\mathbb{Z})$ defined by $\sigma \mapsto d(t)(\sigma) = \partial(s(\sigma))$ (where $s(\sigma) \in \varphi^{-1}(t(\sigma))$) by Kummer theory and $P\mathcal{N}_2[t] \in Br_n(K)$ is the class of the cocycle

$$(\sigma, \tau) \mapsto lg \frac{\sigma(\sqrt[n]{\mathcal{N}(s(\tau))})}{\sqrt[n]{\mathcal{N}(s(\tau))}}$$

and can be expressed as a sum of Galois symbols ([7, Proposition 5.6]).

If we know that the embedding problem is solvable, according [7, Theorem 8.2], the steps to be followed in order to obtain an element γ such that $L(\sqrt[n]{\gamma})$ is a solution are:

1. Find a "good" representation of degree 0 such that $s_t = \varepsilon$.
2. Write down explicitly the isomorphism $g : C_L \to \mathfrak{C}_L$ over L such that $g^{-1}g^{\sigma} = t(\sigma)$ $\forall \sigma \in \Gamma$.

3. Determine an isomorphism $f : C \to \mathfrak{C}$ over K such that the element z defined in [7, Theorem 8.2] be different from 0.
4. Compute the expression of the element z in the basis of $\mathfrak{C}_L$.
5. Compute the norm $N(z) \in \mathfrak{C}_L$ and consider a non-zero coordinate α of it.
6. Find $\eta \in L$ such that $\eta^{-\sigma}\eta = n_\sigma^{-1},\ \forall \sigma \in \Gamma$.
7. Compute the element $\gamma = \eta\alpha^{-1}$.

   (In fact, here we compute γ^{-1} because of the difficulty to compute α^{-1} from α and the fact that the field $K(\sqrt[n]{\gamma^{-1}})$ is equal to $K(\sqrt[n]{\gamma})$).

In this way we obtain all solutions of the embedding problem because if $L(\sqrt[n]{\gamma})$ is a solution of the solvable embedding problem $L/K, \Gamma, E$, the set $\{\sqrt[n]{r\gamma} \mid r \in K\}$ contains all proper and improper solutions.

2. Representations for a family of problems

Let n be an odd integer and $L = K(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_m})$ be a Galois extension of K with Galois group

$$\Gamma = Gal(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \overset{m)}{\cdots} \times \mathbb{Z}/n\mathbb{Z}.$$

In particular we suppose $a_i \notin K^d$ for all $d \mid n$, $d \neq 1$ and the a_i are independent in K^*/K^{*^n}.

We compute the obstruction to the solvability of embedding problems given by

$$L/K, \Gamma, (E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \Gamma \to 0\,,$$

with $\mathbb{Z}/n\mathbb{Z}$ trivial Γ-module, that is, where E is a central extension of Γ by $\mathbb{Z}/n\mathbb{Z}$. Equivalently, we can consider the problem given by $L/K, \Gamma, \varepsilon$, where $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ is the element corresponding to (E). Let σ_i $(1 \leq i \leq m)$ be the automorphisms of L given by

$$\sigma_i(\sqrt[n]{a_j}) = \omega^{\delta_{ij}} \sqrt[n]{a_j}$$

such that $\Gamma = <\sigma_1, \ldots, \sigma_m>$.

We look for a representation t of Γ over K such that

$$\varepsilon = s_t \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}),$$

where s_t is the analogue to the second Stiefel-Whitney class.

We begin by defining a generalization of Galois symbols.

Definition 2.1. *Let L/K be a Galois extension and $\Gamma = Gal(L/K)$. For $a \in L^{*^n} \cap K^*$ we denote by*

$$\chi_a \in Hom(\Gamma, \mathbb{Z}/n\mathbb{Z}) = H^1(\Gamma, \mathbb{Z}/n\mathbb{Z})$$

the corresponding element by Kummer isomorphism relative to L, that is

$$\chi_a(\sigma) = lg(\frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}).$$

For $a, b \in L^{*^n} \cap K^*$ we define

$$(a, b)_L := \chi_a \cup \chi_b$$

where the cup-product is considered via the multiplication in $\mathbb{Z}$.

Proposition 2.2. *These symbols have the next properties:*

1. *They are bilineal, that is,*
$$(aa', b)_L = (a, b)_L + (a', b)_L \text{ and } (a, bb')_L = (a, b)_L + (a, b')_L.$$
2. *If n is odd $(a, a)_L = 0$ and $(a, b)_L + (b, a)_L = 0$.*

Proof. *1.* As $\chi_{aa'}(\sigma) = \chi_a(\sigma) + \chi_{a'}(\sigma)$ for all $\sigma \in \Gamma$, they are bilineal.
2. For the first equality our goal is to prove that $\chi_a \cup \chi_a$ is a coboundary, that is, that there exists a map $h : \Gamma \to \mathbb{Z}/n\mathbb{Z}$ such that

$$\chi_a(\sigma)\chi_a(\tau) = -h(\sigma\tau) + h(\sigma) + h(\tau) \quad \forall \sigma, \tau \in \Gamma.$$

If we consider the morphism

$$\psi_a : G_K \to \mathbb{Z}/n\mathbb{Z}, \ \sigma \mapsto \psi_a(\sigma) = \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}},$$

by properties of Galois symbols ([6, XIV.2, Prop. 4] and [7, Lemma 4.1]), $\psi_a \cup \psi_a$ is a coboundary. Therefore, there exists a map $\eta : G_K \to \mathbb{Z}/n\mathbb{Z}$ such that

$$\psi_a(\sigma)\psi_a(\tau) = -\eta(\sigma\tau) + \eta(\sigma) + \eta(\tau) \quad \text{for all } \sigma, \tau \in G_K.$$

Let $j : G_K \to \Gamma$ be the surjective morphism and we fix u a section of j. We define the map $h(\sigma) := \eta(u(\sigma))$ for $\sigma \in \Gamma$. This map satisfies for $\sigma, \tau \in \Gamma$,

$$\psi_a(u(\sigma))\psi_a(u(\tau)) = -\eta(u(\sigma\tau)) + \eta(u(\sigma)) + \eta(u(\tau)) = -h(\sigma\tau) + h(\sigma) + h(\tau)$$

and, since $\psi_a(u(\sigma)) = \chi_a(\sigma)$ for all $\sigma \in \Gamma$ because $a \in L$, we get the relation $(a, a)_L = 0$.

From the previous results, we have

$$0 = (ab, ab)_L = (a, a)_L + (a, b)_L + (b, a)_L + (b, b)_L = (a, b)_L + (b, a)_L. \quad \blacksquare$$

We now define some representations:

Definition 2.3. *1. For the following representations we consider the generalized Clifford algebra $C(e)$ such that $e^n = 1$ and $\mathcal{N} = N$ (norm of the algebra):*

(a) For $i \in \{1,\dots,m\}$ we define

$$\begin{aligned} \phi_i : \Gamma &\longrightarrow Autgr(C(e)) \\ \sigma_i &\longrightarrow \varphi(e) \\ \sigma_r &\longrightarrow 1 \qquad \text{if } r \neq i \end{aligned}$$

and

$$\begin{aligned} \phi_i^k : \Gamma &\longrightarrow Autgr(C(e)) \\ \sigma_i &\longrightarrow \varphi(e^k) \\ \sigma_r &\longrightarrow 1 \quad \text{if } r \neq i. \end{aligned}$$

(b) For $i_1, \cdots, i_k \in \{1,\dots,m\}$ all different we define

$$\begin{aligned} \phi_{i_1} \cdots \phi_{i_k} : \Gamma &\longrightarrow Autgr(C(e)) \\ \sigma_{i_j} &\longrightarrow \varphi(e) \quad \text{for } j = 1, \dots k \\ \sigma_r &\longrightarrow 1 \qquad \text{if } r \neq i_j, \end{aligned}$$

and in the same way the representation

$$\phi_{i_1}^{m_{i_1}} \cdots \phi_{i_k}^{m_{i_k}}.$$

2. We now consider the generalized Clifford algebra $C(v)$ with $v^n = \omega$, and define the representation:

$$\begin{aligned} \psi_i : \Gamma &\longrightarrow Autgr(C(v)) \\ \sigma_i &\longrightarrow \varphi(v) \\ \sigma_r &\longrightarrow 1 \qquad \text{if } r \neq i, \end{aligned}$$

with

$$A_{\psi_i} = \{\lambda v^r \mid \lambda \in K, r \in \{0,\dots,n-1\}\}$$

and the admissible norm $\mathcal{N}(v) = \omega^{-1} N(v) = 1$.

We also define the representation

$$\begin{aligned} \psi_i^k : \Gamma &\longrightarrow Autgr(C(v)) \\ \sigma_i &\longrightarrow \varphi(v^k) \\ \sigma_r &\longrightarrow 1 \qquad \text{if } r \neq i. \end{aligned}$$

As in the previous case we define the representations:

$$\psi_{i_1} \cdots \psi_{i_k} \qquad \text{and} \qquad \psi_{i_1}^{m_{i_1}} \cdots \psi_{i_k}^{m_{i_k}}.$$

Proposition 2.4. *For these representations we have:*

1. $s_{\phi_i} = 0$ *and* $b_{\phi_i} = a_i$.
   $s_{\phi_i^k} = 0$ *and* $b_{\phi_i^k} = a_i^k$.
   $s_{\phi_{i_1}\cdots\phi_{i_k}} = 0$ *and* $b_{\phi_{i_1}\cdots\phi_{i_k}} = a_{i_1}\cdots a_{i_k}$.
   $s_{\phi_{i_1}^{m_{i_1}}\cdots\phi_{i_k}^{m_{i_k}}} = 0$ *and* $b_{\phi_{i_1}^{m_{i_1}}\cdots\phi_{i_k}^{m_{i_k}}} = a_{i_1}^{m_{i_1}}\cdots a_{i_k}^{m_{i_k}}$.
2. *We denote* $s_{\psi_i} = (a_i) \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ *(see Remark 2.5).Then*

   $b_{\psi_i} = a_i$.
   $s_{\psi_i^k} = (a_i^k)$ *and* $b_{\psi_i^k} = a_i^k$.
   $s_{\psi_{i_1}\cdots\psi_{i_k}} = (a_{i_1}\cdots a_{i_k})$ *and* $b_{\psi_{i_1}\cdots\psi_{i_k}} = a_{i_1}\cdots a_{i_k}$.
   $s_{\psi_{i_1}^{m_{i_1}}\cdots\psi_{i_k}^{m_{i_k}}} = (a_{i_1}^{m_{i_1}}\cdots a_{i_k}^{m_{i_k}})$ *and* $b_{\psi_{i_1}^{m_{i_1}}\cdots\psi_{i_k}^{m_{i_k}}} = a_{i_1}^{m_{i_1}}\cdots a_{i_k}^{m_{i_k}}$.

Proof. To prove these results it is only necessary to apply the definition of analogue to Stiefel-Whitney class s_t. ■

Remark 2.5. The element (a_i) defined before coincides, if n is a prime number with the element $((a_i))$ defined in [5] because both are defined by the same 2-cocycle.

Proposition 2.6. *Let j be the surjective morphism $j : G_K \to \Gamma$ and j_2^* the homomorphism between the cohomology groups induced by j. For the above representations we have:*

1. $j_2^*(s_{\phi_i}) = 0, \quad j_2^*(s_{\phi_i^k}) = 0.$
   $j_2^*(s_{\phi_i\phi_k}) = 0, \quad j_2^*(s_{\phi_{i_1}\cdots\phi_{i_k}}) = 0, \quad j_2^*(s_{\phi_{i_1}^{m_{i_1}}\cdots\phi_{i_k}^{m_{i_k}}}) = 0.$
2. $j_2^*(s_{\psi_i}) = j_2^*((a_i)) = (a_i, \omega), \quad j_2^*(s_{\psi_i^k}) = j_2^*((a_i^k)) = (a_i^k, \omega).$
   $j_2^*(s_{\psi_i\psi_k}) = j_2^*((a_ia_k)) = (a_ia_k, \omega), j_2^*(s_{\psi_{i_1}^{m_{i_1}}\cdots\psi_{i_k}^{m_{i_k}}}) = (a_{i_1}^{m_{i_1}}\cdots a_{i_k}^{m_{i_k}}, \omega).$
3. $j_2^*(s_{\phi_i\hat{\otimes}\phi_j}) = (a_i, a_j)$*, that is* $j_2^*((a_i, a_j)_L) = (a_i, a_j)$.

Proof. If t, t_1, t_2 are representations of Γ, it is known that $s_{t\circ j} = j_2^*(s_t)$ and therefore $j_2^*(s_{t_1\hat{\otimes}t_2}) = j_2^*(s_{t_1}) + j_2^*(s_{t_2}) + (b_{t_1}, b_{t_2})$ ([7, Proposition 5.5]).

1. For t a representation of 1., we have by 1.1 $j_2^*(s_t) = [\mathfrak{C}_t] - [C_t] - (a_t, b_t)$ where $[C_t] = [C(e)] = 0$ and the invariant $a_t = 1$. The twisted algebra $\mathfrak{C}_t$ is generated only by an element, so $[\mathfrak{C}_t] = 0$ and $j_2^*(s_t) = 0$.

2. For t a representation of 2., we have, as before, $j_2^*(s_t) = [\mathfrak{C}_t] - [C_t] - (a_t, b_t)$ where $[C_t] = [C(v)] = 0$ and $[\mathfrak{C}_t] = 0$. But in this case, its invariant $a_t = \omega$. Thus, $j_2^*(s_t) = (b_t, \omega)$.

3. It is a consequence of the above equalities. ■

Theorem 2.7. *A representation t of Γ such that*

$$s_t = \sum_{1\leq i<j\leq m} \lambda_{i,j}(a_i,a_j)_L + \mu(a_{i_1}^{m_{i_1}}\cdots a_{i_k}^{m_{i_k}}), \qquad \mu = 0 \text{ or } 1$$

is $t = \rho\,\hat{\otimes}\mu(\rho'\hat{\otimes}\rho'')$, where:

1. *The representation ρ satisfies*

$$s_\rho = \sum_{1\leq i<j\leq m} \lambda_{i,j}(a_i,a_j)_L = \sum_{1\leq i<j\leq m} (a_i, a_j^{\lambda_{i,j}})_L$$

and it is $\rho = \rho_1\hat{\otimes}\cdots\hat{\otimes}\rho_{m-1}$ for

$$\begin{aligned} \rho_1 &= \phi_1\,\hat{\otimes}\,\phi_2^{\lambda_{1,2}}\cdots\phi_m^{\lambda_{1,m}}, \\ \rho_i &= \phi_i\,\hat{\otimes}\,\phi_{i+1}^{\lambda_{i,i+1}}\cdots\phi_m^{\lambda_{i,m}}\,\hat{\otimes}\,\phi_i\phi_{i+1}^{\lambda_{i,i+1}}\cdots\phi_m^{\lambda_{i,m}}\,\hat{\otimes}\,\phi_1^{d_{i,1}}\cdots\phi_m^{d_{i,m}}, \quad i>1. \end{aligned}$$

with the elements $d_{i,k}$:

$$\begin{aligned} d_{2,1} &= 1, \quad d_{i+1,1} = 1 + \sum_{r=2}^{i} d_{r,1} \\ d_{2,2} &= \lambda_{1,2} \quad d_{i+1,2} = \lambda_{1,2} + 2 + \sum_{r=2}^{i} d_{r,2} \text{ if } i>1 \\ d_{i+1,k} &= \lambda_{1,k} + 2\sum_{r=2}^{i-1}\lambda_{r,k} + 2 + \sum_{r=2}^{i} d_{r,k} \quad \text{ if } 2<k\leq i \\ d_{i+1,k} &= \lambda_{1,k} + 2\sum_{r=2}^{i}\lambda_{r,k} + \sum_{r=2}^{i} d_{r,k} \quad \text{ if } k>i \end{aligned}$$

where $\lambda_{k,k} = 1$ and $\lambda_{i,j} = 0$ if $i>j$.

2. *The representation ρ' is*

$$\rho' = \psi_{i_1}^{m_{i_1}}\cdots\psi_{i_k}^{m_{i_k}}.$$

3. *The representation ρ'' is*

$$\rho'' = \phi_{i_1}^{m_{i_1}}\cdots\phi_{i_k}^{m_{i_k}}\,\hat{\otimes}\,\phi_1^{d_{m,1}}\cdots\phi_m^{d_{m,m}},$$

where the elements $d_{m+1,i}$ are the ones defined above.

Remark 2.8. If n is prime, Theorem 2.7 gives a representation t such that $s_t = \varepsilon$ for all elements $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ (see the decomposition done by Massy [5]).

Theorem 2.9. *The obstruction to the solvability of the embedding problem given by $L/K, \Gamma$ and $s_t \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ for*

$$s_t = \sum_{1\le i<j\le m} \lambda_{i,j}(a_i, a_j)_L + \mu(a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}}), \qquad \text{where } \mu = 0, 1$$

is

$$j_2^*(s_t) = \sum_{1\le i<j\le m} \lambda_{i,j}(a_i, a_j) + \mu(a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}}, \omega).$$

The last theorem is a direct consequence of Theorem 2.7 and Proposition 2.6. We will now prove Theorem 2.7.

Proof of Theorem 2.7.

Step 1: We begin to prove that the representation ρ is such that

$$s_\rho = \sum_{1\le i<j\le m} (a_i, a_j^{\lambda_{i,j}})_L = \sum_{i=1}^{m-1} (a_i, a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}})_L,$$

where we obtain the last equality by properties of Galois symbols.

To show it, we prove by induction on r that if $t_r = \rho_1 \hat{\otimes} \cdots \hat{\otimes} \rho_r$, then

$$s_{t_r} = \sum\nolimits_{i=1}^{r} (a_i, a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}})_L.$$

- If $r = 1$,

  $$t_1 = \rho_1 = \phi_1 \hat{\otimes} \phi_2^{\lambda_{1,2}} \cdots \phi_m^{\lambda_{1,m}}$$

  and

  $$s_{\rho_1} = (b_{\phi_1}, b_{\phi_2^{\lambda_{1,2}} \cdots \phi_m^{\lambda_{1,m}}})_L = (a_1, a_2^{\lambda_{1,2}} \cdots a_m^{\lambda_{1,m}})_L$$

  by [7, Proposition 5.5]. Moreover,

  $$b_{\rho_1} = a_1 a_2^{\lambda_{1,2}} \cdots a_m^{\lambda_{1,m}}.$$

- We suppose the result is true for r and we will compute $s_{t_{r+1}}$. We have, for any i, that

  $$s_{\rho_i} = (a_i, a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}})_L + 2(a_i a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}}, a_1^{d_{i,1}} \cdots a_m^{d_{i,m}})_L,$$

  and

  $$b_{\rho_i} = a_1^{d_{i,1}} \cdots a_{i-1}^{d_{i,i-1}} a_i^{2+d_{i,i}} a_{i+1}^{2\lambda_{i,i+1}+d_{i,i+1}} \cdots a_m^{2\lambda_{i,m}+d_{i,m}}.$$

Lemma 2.10. $b_{t_r} = a_1^{d_{r+1,1}} \cdots a_m^{d_{r+1,m}}$.

By applying the lemma and properties of symbols, we obtain the result:

$$\begin{aligned} s_{t_{r+1}} &= s_{t_r} + s_{\rho_{r+1}} + (b_{t_r}, b_{\rho_{r+1}})_L \\ &= \sum_{i=1}^{r+1} (a_i, a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}})_L + (a_{r+1}^2 a_{r+2}^{2\lambda_{r+1,r+2}} \cdots a_m^{2\lambda_{r+1,m}}, a_1^{d_{r+1,1}} \cdots a_m^{d_{r+1,m}})_L \\ &\qquad + (a_1^{d_{r+1,1}} \cdots a_m^{d_{r+1,m}}, a_{r+1}^2 a_{r+2}^{2\lambda_{r+1,r+2}} \cdots a_m^{2\lambda_{r+1,m}})_L \\ &= \sum_{i=1}^{r+1} (a_i, a_{i+1}^{\lambda_{i,i+1}} \cdots a_m^{\lambda_{i,m}})_L. \end{aligned}$$

To prove the lemma we prove by induction the formula:

$$b_{t_r} = a_1^{1+d_{2,1}+\cdots+d_{r,1}} \prod_{2\le k<r} a_k^{\lambda_{1,k}+2\lambda_{2,k}+d_{2,k}+\cdots+2\lambda_{k-1,k}+d_{k-1,k}+2+d_{k,k}+d_{k+1,k}+\cdots d_{r,k}} \cdot$$
$$a_r^{\lambda_{1,r}+2\lambda_{2,r}+d_{2,r}+\cdots+2\lambda_{r-1,r}+d_{r-1,r}+2+d_{r,r}} \cdot \prod_{k>r} a_k^{\lambda_{1,k}+2\lambda_{2,k}+d_{2,k}+2\lambda_{3,k}+d_{3,k}+\cdots+2\lambda_{r,k}+d_{r,k}}.$$

It gives the result for the values of the exponents $d_{r+1,k}$.

If $\mu = 0$ we have proved the theorem. Now, we suppose $\mu = 1$.

Step 2: We compute $s_{\rho\hat{\otimes}\rho'}$. By lemma 2.10, $b_\rho = a_1^{d_{m,m}} \cdots a_m^{d_{m,m}}$. Then

$$s_{\rho\hat{\otimes}\rho'} = s_\rho + s_{\psi_{i_1}^{m_{i_1}} \cdots \psi_{i_k}^{m_{i_k}}} + (b_\rho, a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}})_L.$$

Step 3: In addition,

$$s_{\rho''} = (a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}}, a_1^{d_{m,1}} \cdots a_m^{d_{m,m}})_L,$$

and then, by properties of symbols,

$$s_t = s_{\rho\hat{\otimes}\rho'} + s_{\rho''} + (b_{\rho\hat{\otimes}\rho'}, b_{\rho''}) = \sum_{1\le i<j\le m} \lambda_{i,j}(a_i, a_j)_L + (a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}}).$$ ■

By considering the representation of degree 0 associated to the given in Theorem 2.7 following [7, Section 6] and with the method of [7, Section 8] we can compute the solutions of the corresponding embedding problem.

3. An example: The exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to G \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to 0$$

3.1. Computation of the obstruction

We consider n odd and K as before and the field extension

$$L = K(\sqrt[n]{a_1}, \sqrt[n]{a_2}, \sqrt[n]{a_3})$$

such that $[K(\sqrt{a_i}) : K] = n$, $i = 1, 2, 3$ and $[L : K] = n^3$ with

$$\Gamma = Gal(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Let σ_1, σ_2 and σ_3 be generators of Γ determined by $\sigma_i(\sqrt[n]{a_j}) = \omega^{\delta_{ij}} \sqrt[n]{a_j}$.

Problem 3.1. *Let us consider the embedding problem given by*

$$L = K(\sqrt[n]{a_1}, \sqrt[n]{a_2}, \sqrt[n]{a_3})/K \text{ with } \Gamma = Gal(L/K) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

and the exact sequence

$$(E): \quad 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} G \xrightarrow{h} \Gamma \to 0$$

where G is the group

$$\begin{aligned} G := \ & < g_1, g_2, g_3 \mid g_1, g_2 \text{ have order } n, g_3 \text{ has order } n^2, \\ & g_1 g_2 = g_2 g_1 g_3^n, g_1 g_3 = g_3 g_1, g_2 g_3 = g_3 g_2 >, \end{aligned}$$

and the morphisms are $h(g_1) = \sigma_1, h(g_2) = \sigma_2, h(g_3) = \sigma_3$ and $i(lg(\omega)) = g_3^n$.

Remark 3.2. G is a non commutative group of order n^4 extension of Γ.
Other groups with these properties are $E_1 \times \mathbb{Z}/n\mathbb{Z}$ where

$$E_1 =< \sigma, \tau \mid \sigma \text{ has order } n, \tau \text{ has order } n^2, \sigma\tau = \tau^{n+1}\sigma >,$$

and $E_2 \times \mathbb{Z}/n\mathbb{Z}$ where

$$E_2 =< \sigma, \tau, \rho \mid \sigma, \tau, \rho \text{ have order } n, \ \sigma\tau\sigma^{-1}\tau^{-1} = \rho, \sigma\rho = \rho\sigma, \tau\rho = \rho\tau > .$$

If $n = p$ prime integer, it is the Heisenberg group of degree p.

The obstruction to the solvability of the problems given by them is $(\omega^{-1}a_1, a_2)$ and (a_1, a_2) respectively. These obstructions are computed in [7, Theorems 7.2 and 7.4]). If $n = 2$ there are not other noncommutative groups of order n^4 ([4, sec. 4]).

Proposition 3.3. *The element* $\varepsilon \in H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ *corresponding to the exact sequence of the previous problem* (E) *is* $\varepsilon = (a_1, a_2)_L + (a_3)$.

Proof. We compute the cocycles $a, a', a'' : \Gamma \times \Gamma \to \mathbb{Z}/n\mathbb{Z}$ associated to (E), $(a_1, a_2)_L$ and (a_3) respectively.

The one associated to (E) is

$$a_{\sigma_1^r\sigma_2^k\sigma_3^l, \sigma_1^\alpha\sigma_2^b\sigma_3^c} = \begin{cases} -k\alpha & \text{if } l+c<n \\ -k\alpha+1 & \text{if } l+c \geq n. \end{cases}$$

The cocycle associated to $(a_1, a_2)_L = -(a_2, a_1)_L$ is

$$a'_{\sigma_1^r\sigma_2^k\sigma_3^l, \sigma_1^\alpha\sigma_2^b\sigma_3^c} = -k\alpha,$$

and the cocycle associated to (a_3) is

$$a''_{\sigma_1^r\sigma_2^k\sigma_3^l, \sigma_1^\alpha\sigma_2^b\sigma_3^c} = \begin{cases} 0 & \text{if } l+c<n \\ 1 & \text{if } l+c \geq n. \end{cases}$$

So, $a = a' + a''$ and we get the result. ■

Applying Theorem 2.7, the representation corresponding to the embedding problem 3.1 is

$$t = \rho\hat{\otimes}(\rho'\hat{\otimes}\rho'')$$

where $\rho = \rho_1\hat{\otimes}\rho_2$ for $\rho_1 = \phi_1\hat{\otimes}\phi_2$ and $\rho_2 = \phi_2\hat{\otimes}\phi_2\hat{\otimes}\phi_1\phi_2$, $\rho' = \psi_3$ and $\rho'' = \phi_3\hat{\otimes}\phi_1^2\phi_2^4$.

Writing the representation in terms of generalized Clifford algebras and morphism it is

$$C_t = C(e_1, e_2, e_3, e_4, e_5, v, e_6, e_7) \text{ with } e_i^n = 1 \text{ for } i = 1, \ldots, 7, \text{ and } v^n = \omega$$

and

$$\begin{aligned} t : \Gamma &\longrightarrow Autgr(C_t) \\ \sigma_1 &\longrightarrow \varphi(e_1e_5e_7^2) \\ \sigma_2 &\longrightarrow \varphi(e_2e_3e_4e_5e_7^4) \\ \sigma_3 &\longrightarrow \varphi(ve_6). \end{aligned}$$

By putting $u_1 := e_1e_5e_7^2$, $u_2 := e_2e_3e_4e_5e_7^4$ and $u_3 := ve_6$, they satisfy the next relations

$$\begin{aligned} u_1^k &= \omega^{-5\frac{k(k-1)}{2}} e_1^k e_5^k e_7^{2k}, \quad \text{so } u_1^n = 1, \\ u_2^k &= \omega^{-22\frac{k(k-1)}{2}} e_2^k e_3^k e_4^k e_5^k e_7^{4k}, \quad \text{so } u_2^n = 1, \\ u_3^k &= \omega^{-\frac{k(k-1)}{2}} v^k e_6^k = \omega, \quad \text{so } u_3^n = \omega \end{aligned}$$

and moreover the relations

$$u_1u_2 = \omega u_2u_1, \qquad u_1u_3 = u_3u_1, \qquad \text{and} \qquad u_2u_3 = u_3u_2.$$

Then, we have checked again that $\varepsilon = s_t$.

The twisted algebra by the representation t is

$$\begin{aligned}\mathfrak{C}_t = & \; C(e_1 \otimes \sqrt[n]{a_1^3 a_2^8 a_3^2}, e_2 \otimes \sqrt[n]{a_1^2 a_2^7 a_3^2}, e_3 \otimes \sqrt[n]{a_1^2 a_2^5 a_3^2}, e_4 \otimes \sqrt[n]{a_1^2 a_2^3 a_3^2}, \\ & e_5 \otimes \sqrt[n]{a_1 a_2 a_3^2}, v \otimes \sqrt[n]{a_3}, e_6 \otimes \sqrt[n]{a_3^{-1}}, e_7 \otimes \sqrt[n]{a_1^{-2} a_2^{-4} a_3^{-2}}) \\ = & \; C(a_1^3 a_2^8 a_3^2, a_1^2 a_2^7 a_3^2, a_1^2 a_2^5 a_3^2, a_1^2 a_2^3 a_3^2, a_1 a_2 a_3^2, \omega a_3, a_3^{-1}, a_1^{-2} a_2^{-4} a_3^{-2})\end{aligned}$$

because the generating elements are fixed by the action $t \otimes gal$ of Γ. The element $b_t = a_1^4 a_2^8 a_3^2$.

Corollary 3.4. *The obstruction to the solvability is*

$$j_2^*(s_t) = [\mathfrak{C}_t] = (a_1, a_2) + (a_3, \omega).$$

For instance, for $n = 3, K = \mathbb{Q}(\omega)$ (where ω is a third root of unity) and $a_1 = 2, a_2 = 5$ and $a_3 = 3$ the embedding problem is solvable because $(2,5) = 0$, $(3,\omega) = 0$ and therefore $j_2^*(s_t)$.

The representation of degree 0 corresponding to t is

$$\begin{aligned} t_0 : \Gamma & \longrightarrow Autgr(C_t) \\ \sigma_1 & \longrightarrow \varphi(e_1 e_5 e_7^2 e_8^{-4}) \\ \sigma_2 & \longrightarrow \varphi(e_2 e_3 e_4 e_5 e_7^4 e_8^{-8}) \\ \sigma_3 & \longrightarrow \varphi(v e_6 e_8^{-2}) \end{aligned}$$

where

$C_{t_0} = C(e_1, e_2, e_3, e_4, e_5, v, e_6, e_7, e_8)$ with $e_i^n = 1$ for $i = 1, \ldots, 8$, and $v^n = \omega$.

The twisted algebra by t_0 is

$$\begin{aligned}\mathfrak{C}_{t_0} = & \; C(e_1 \otimes \sqrt[n]{a_1^{-1}}, e_2 \otimes \sqrt[n]{a_1^{-2} a_2^{-1}}, e_3 \otimes \sqrt[n]{a_1^{-2} a_2^{-3}}, e_4 \otimes \sqrt[n]{a_1^{-2} a_2^{-5}}, \\ & e_5 \otimes \sqrt[n]{a_1^{-3} a_2^{-7}}, v \otimes \sqrt[n]{a_1^{-4} a_2^{-8} a_3^{-1}}, e_6 \otimes \sqrt[n]{a_1^{-4} a_2^{-8} a_3^{-3}}, \\ & e_7 \otimes \sqrt[n]{a_1^{-6} a_2^{-12} a_3^{-4}}, e_8 \otimes \sqrt[n]{a_1^{-4} a_2^{-8} a_3^{-2}}) \\ = & \; C(a_1^{-1}, a_1^{-2} a_2^{-1}, a_1^{-2} a_2^{-3}, a_1^{-2} a_2^{-5}, a_1^{-3} a_2^{-7}, \omega a_1^{-4} a_2^{-8} a_3^{-1}, a_1^{-4} a_2^{-8} a_3^{-3}, \\ & a_1^{-6} a_2^{-12} a_3^{-4}, a_1^{-4} a_2^{-8} a_3^{-2}).\end{aligned}$$

To find the solution of the embedding problem it is only necessary determine an isomorphism over K from C_{t_0} into $\mathfrak{C}_{t_0}$ and compute the elements $z \in \mathfrak{C}_{t_0}$ and $\gamma \in L$ using the method given in [7, Section 8].

Remark 3.5. Given a embedding problem may be there is a representation simpler than the given in Theorem 2.7, but this theorem assure the existence of at least one.

In order to compute an element $\gamma \in L$ such that $L(\sqrt[n]{\gamma})$ is a solution of the embedding problem and to simplify the computations, we will consider another simpler representation T of degree 0 of Γ which allows us to determine the isomorphism over K of C_T into $\mathfrak{C}_T$ more easily and then, compute the elements z and γ.

We consider the representation (C, T) where

$$C = C(e_1, e_2, e_3, e_4) = C(1, 1, 1, \omega^{-1})$$

and the representation

$$T : \Gamma \to O(C), \quad \sigma_1 \mapsto \varphi(e_1 e_2^{-1}),\ \sigma_2 \mapsto \varphi(e_1 e_3^{-1}),\ \sigma_3 \mapsto \varphi(e_1 e_2^{-1} e_3 e_4^{-1}).$$

The twisted algebra by T is

$$\mathfrak{C} = C(v_1, v_2, v_3, v_4) = C(a_1^{-1}a_2^{-1}a_3^{-1}, a_1^{-1}a_2^{-2}a_3^{-1}, a_2^{-1}a_3^{-1}, \omega^{-1}a_3^{-1})$$

because the elements

$$\begin{aligned} v_1 &= e_1 \otimes \sqrt[n]{a_1^{-1}a_2^{-1}a_3^{-1}}, \\ v_2 &= e_2 \otimes \sqrt[n]{a_1^{-1}a_2^{-2}a_3^{-1}} \\ v_3 &= e_3 \otimes \sqrt[n]{a_2^{-1}a_3^{-1}} \\ v_4 &= e_4 \otimes \sqrt[n]{a_3^{-1}}, \end{aligned}$$

are fixed by the action $T \otimes gal$. Clearly this representation is of degree 0 and it is straightforward to check that $\varepsilon = s_T$.

The obstruction to the solvability of the considered embedding problem is $(a_1, a_2) + (a_3, \omega)$. We suppose the problem is solvable, that is $(a_1, a_2) + (a_3, \omega) = 0$, and we want to find an element $\gamma \in L^*$ such that $M = L(\sqrt[n]{\gamma})$ is a solution.

Moreover, to simplify the computations we suppose, that the symbols

$$(a_1, a_2) = 0, (a_3, \omega) = 0 \text{ and } (a_2, a_3) = 0.$$

For the values $a_1 = 2, a_2 = 5, a_3 = 3$ and $n = 3$ it is true.

3.2. The isomorphism $g : C_L \to \mathfrak{C}_L$ over L

Proposition 3.6. *The isomorphism $g : C_L \to \mathfrak{C}_L$ given by:*

$$e_1 \mapsto \sqrt[n]{a_1a_2a_3}\ v_1, \quad e_2 \mapsto \sqrt[n]{a_1a_2^2a_3}\ v_2, \quad e_3 \mapsto \sqrt[n]{a_2a_3}\ v_3, \quad e_4 \mapsto \sqrt[n]{a_3}\ v_4$$

is a graded isomorphism which satisfies $g^{-1}g^{\tau} = T(\tau)$ for each $\tau \in \Gamma$.

The proof is a simple computation.

We express the isomorphism g as the composition of the three isomorphisms g_1,g_2 and g_3

$$C_L \overset{g_1}{\simeq} C_2 = C(1,1,a_3^{-1},\omega^{-1}a_3^{-1})_L \overset{g_2}{\simeq} C_3 = C(1,a_2^{-1},a_2^{-1}a_3^{-1},\omega^{-1}a_3^{-1})_L \overset{g_3}{\simeq} \mathfrak{C}_L$$

given by

$$\begin{array}{llll} g_1(e_1) = u_1, & g_1(e_2) = u_2, & g_1(e_3) = \sqrt[n]{a_3}\ u_3, & g_1(e_4) = \sqrt[n]{a_3}\ u_4 \\ g_2(u_1) = w_1, & g_2(u_2) = \sqrt[n]{a_2}\ w_2, & g_2(u_3) = \sqrt[n]{a_2}\ w_3, & g_2(u_4) = w_4 \\ g_3(w_1) = \sqrt[n]{a}\ v_1, & g_3(w_2) = \sqrt[n]{a}\ v_2, & g_3(w_3) = v_3, & g_3(w_4) = v_4 \end{array}$$

where $a = a_1a_2a_3$ and the elements u_1, u_2, u_3, u_4 and w_1, w_2, w_3, w_4 generate the algebras $C_2 = C(1,1,a_3^{-1},\omega^{-1}a_3^{-1})$ and $C_3 = C(1,a_2^{-1},a_2^{-1}a_3^{-1},\omega^{-1}a_3^{-1})$ respectively.

Clearly $g = g_3 \circ g_2 \circ g_1$.

3.3. The isomorphism $f : C \to \mathfrak{C}$ over K

We know that there exists an isomorphism f defined over K from C into $\mathfrak{C}$. To determine f, we express this isomorphism in terms of three easier isomorphisms between generalized Clifford algebras generated by two elements. We consider the isomorphisms

$$C \overset{f_1}{\simeq} C_2 = C(1,1,a_3^{-1},\omega^{-1}a_3^{-1}) \overset{f_2}{\simeq} C_3 = C(1,a_2^{-1},a_2^{-1}a_3^{-1},\omega^{-1}a_3^{-1}) \overset{f_3}{\simeq} \mathfrak{C}$$

where they are obtained from the isomorphism $C(1,\beta) \simeq C(\alpha,\alpha\beta)$, valid if the symbol $(\alpha,\beta) = 0$, $\alpha,\beta \in K$. For a generalized Clifford algebra $C = C(\nu_1,\nu_2)$, the vector subspace of the elements of degree i of C is denoted by C_i. We consider the following basis of the subspaces C_i for $i = 0,\dots,n-1$:

$$\omega^{c(k)-(i-1)k} e_1^{n+i-k} e_2^k, \qquad k = 0,\dots,n-1$$

where $c(0) = 0$ and $c(k) = kx + \frac{k(k-1)}{2}$ for $k > 0$, where x is an arbitrary integer.

Lemma 3.7. *We consider the algebras $C(\nu_1,\nu_2) = C(1,\beta)$ and $C(\upsilon_1,\upsilon_2) = C(\alpha,\alpha\beta)$. If the Galois symbol $(\alpha,\beta) = 0$, there exists an isomorphism $F : C(1,\beta) \simeq C(\alpha,\alpha\beta)$ defined over K given by:*

$$\nu_1 \mapsto \frac{1}{\alpha^2}\sum_{k=0}^{n-1} y_k\ \omega^{c(k)}\ \upsilon_1^{n+1-k}\upsilon_2^k,\ \ \nu_2 \mapsto \frac{1}{\alpha^2}\beta y_{n-1}\upsilon^{n+1} + \sum_{k=1}^{n-1} y_{k-1}\ \omega^{c(k)}\ \upsilon_1^{n+1-k}\upsilon_2^k,$$

where $y_0 + y_1\sqrt[n]{\beta} + \cdots + y_{n-1}\sqrt[n]{\beta^{n-1}}$ is an element of $K(\sqrt[n]{\beta})$ with norm α^{n-1}.

The proof of this lemma is similar to the proof of [7, Theorems 10.2 and 11.5].

From the previous lemma, we can compute the isomorphisms f_1, f_2 and f_3 over K. For the first one the values are $\alpha = a_3^{-1}$ and $\beta = \omega^{-1}$ and the isomorphism exists because we have supposed that the symbol

$$(a_3^{-1}, \omega^{-1}) = (a_3, \omega) = 0.$$

For f_2, the values are $\alpha = a_2^{-1}$ and $\beta = a_3^{-1}$ and we have supposed that the symbol $(a_2, a_3) = 0$.

For the last one $\alpha = a_1^{-1}a_2^{-1}a_3^{-1}$ and $\beta = a_2^{-1}$. The symbol

$$(a_1^{-1}a_2^{-1}a_3^{-1}, a_2^{-1}) = (a_1a_2a_3, a_2) = (a_1, a_2) + (a_2, a_2) + (a_3, a_2) = 0$$

because we have supposed $(a_1, a_2) = 0 = (a_2, a_3)$ and $(a_2, a_2) = 0$ since n is odd.

Thus, we have the isomorphism f as composition of these isomorphisms.

In particular, for $n = 3$ and $a_1 = 2, a_2 = 5, a_3 = 3$ we have found the elements to determine these isomorphisms:

For f_1, $\alpha = \frac{1}{3}$, $\beta = \omega^{-1}$ and the element

$$\frac{1}{3}(2 - \sqrt[3]{\beta} - \omega\sqrt[3]{\beta^2}) \in K(\sqrt[3]{\beta})$$

has norm $\frac{1}{9}$.

For f_2, $\alpha = \frac{1}{5}$, $\beta = \frac{1}{3}$ and the element

$$\frac{1}{25}(28 + 36\sqrt[3]{\beta} + 57\sqrt[3]{\beta^2}) \in K(\sqrt[3]{\beta})$$

has norm $\frac{1}{25}$.

For f_3, $\alpha = \frac{1}{30}$, $\beta = \frac{1}{5}$ and the element

$$\frac{1}{12}(1 + \sqrt[3]{\beta} - 3\sqrt[3]{\beta^2}) \in K(\sqrt[3]{\beta})$$

has norm $\frac{1}{900}$.

The isomorphisms are then:

$$\begin{aligned}
f_1 : C \to C_2, \quad & e_1 \mapsto u_1, \quad e_2 \mapsto u_2, e_3 \mapsto 2u_3 - u_4 - 3\omega^2 u_3^2 u_4^2, \\
& e_4 \mapsto -u_3 + 2u_4 - 3u_3^2 u_4^2, \\
f_2 : C_2 \to C_3, \quad & u_1 \mapsto w_1, \quad u_2 \mapsto 5(28w_2 + 36w_3 + 285w_2^2 w_3^2), \\
& u_3 \mapsto 5(19w_2 + 28w_3 + 180w_2^2 w_3^2), \quad u_4 \mapsto w_4, \\
f_3 : C_3 \to \mathfrak{C}, \quad & w_1 \mapsto \tfrac{75}{30}(v_1 + v_2 - 90v_1^2 v_2^2), \\
& w_2 \mapsto \tfrac{-3}{2}v_1 + \tfrac{75}{30}(v_2 + 30v_1^2 v_2^2), \quad w_3 \mapsto v_3 \quad w_4 \mapsto v_4.
\end{aligned}$$

3.4. The element z

From the isomorphisms f_1, g_1, f_2, g_2 and f_3, g_3, following [7, Theorem 8.2] and its proof, we can compute elements $z_1 \in (C_2)_L$, $z_2 \in (C_3)_L$ and $z_3 \in \mathfrak{C}_L$ such that $g_i(x) = z_i f_i(x) z_i^{-1}$ for $i = 1, 2, 3$ and x any element of the corresponding Clifford algebra. The element $z = g_3(g_2(z_1)\, z_2)\, z_3 \in \mathfrak{C}_L$ satisfies $g(x) = z f(x) z^{-1}$ for all $x \in C_L$.

We compute now the elements z_1, z_2 and z_3. As $f_1(e_1) = g_1(e_1)$ and $f_1(e_2) = g_1(e_2)$, we can express

$$z_1 = \sum_{\epsilon_i \in \{0,\ldots,n-1\}} g_1(e_3)^{\epsilon_3} g_1(e_4)^{\epsilon_4} f_1(e_4)^{-\epsilon_4} f_1(e_3)^{-\epsilon_3},$$

as $f_2(u_1) = g_2(u_1)$ and $f_2(u_4) = g_2(u_4)$,

$$z_2 = \sum_{\epsilon_i \in \{0,\ldots,n-1\}} g_2(u_2)^{\epsilon_2} g_2(u_3)^{\epsilon_3} f_2(u_3)^{-\epsilon_3} f_2(u_2)^{-\epsilon_2}$$

and, as $f_3(w_3) = g_3(w_3)$ and $f_3(w_4) = g_3(w_4)$,

$$z_3 = \sum_{\epsilon_i \in \{0,\ldots,n-1\}} g_3(w_1)^{\epsilon_1} g_3(w_2)^{\epsilon_2} f_3(w_2)^{-\epsilon_2} f_3(u_1)^{-\epsilon_1}.$$

We compute these elements using the package `n-Clifford`.

For instance, for $n = 3$ the coordinates of the elements z_1, z_2 and z_3 in the fixed basis are:

$$\begin{aligned} z_1 \;=\; & \Big\{\omega\, y_0^2 + \frac{\omega\, y_0}{a_3^{2/3}} + \frac{\omega}{a_3^{4/3}} - y_1\, y_2, y_2^2 + \Big(\frac{1}{a_3^{2/3}} - \omega\, y_0\Big)\, y_1, \\ & \frac{1}{a_3^{2/3}}\big(a_3^{2/3}\, \omega\, y_1^2 - (a_3^{2/3}\, y_0\, \omega + \omega + 1)\, y_2\big)\Big\} \end{aligned}$$

where $y_0 + y_1\sqrt[3]{\omega^{-1}} + y_2\sqrt[3]{\omega^{-2}} \in K(\sqrt[3]{\omega^{-1}})$ such that $N_{K(\sqrt[3]{\omega^{-1}})/K}(y) = a_3^{-2}$,

$$\begin{aligned} z_2 \;=\; & \Big\{a_3\, y_0^2 + \frac{a_3\, y_0}{a_2^{2/3}} + \frac{a_3}{a_2^{4/3}} - y_1\, y_2, \frac{1}{a_2^{2/3}}(a_2^{2/3}\, y_2^2 - a_3\, (\omega + a_2^{2/3}\, y_0 + 1)\, y_1), \\ & a_3\, \Big(y_1^2 + \Big(\frac{\omega}{a_2^{2/3}} - y_0\Big)\, y_2\Big)\Big\} \end{aligned}$$

where $y_0 + y_1\sqrt[3]{a_3^{-1}} + y_2\sqrt[3]{a_3^{-2}} \in K(\sqrt[3]{a_3^{-1}})$ such that $N_{K(\sqrt[3]{a_3^{-1}})/K}(y) = a_2^{-2}$

and

$$
\begin{aligned}
z_3 \quad = \quad & \Big\{ (a_1\ a_2\ a_3)^{4/3}\ y_0^2 + (a_1\ a_2\ a_3)^{2/3}\ y_0 - a_1 a_3\ \sqrt[3]{a_1\ a_2\ a_3}\ y_1\ y_2 + 1, \\
& \sqrt[3]{a_1^2\ a_2\ a_3^2}\ \Big(\sqrt[3]{a_1^2\ a_3^2}\ y_2^2 - \sqrt[3]{a_2}\Big(\omega + \sqrt[3]{a_1^2\ a_2^2\ a_3^2} y_0 + 1\Big)\ y_1 \Big), \\
& (a_1\ a_2\ a_3)^{4/3}\ y_1^2 + \sqrt[3]{a_1^2\ a_2^2\ a_3^2}\ \Big(\omega - \sqrt[3]{a_1^2\ a_2^2\ a_3^2} y_0\Big)\ y_2 \Big\}
\end{aligned}
$$

where $y_0 + y_1 \sqrt[3]{a_2^{-1}} + y_2 \sqrt[3]{a_2^{-2}} \in K(\sqrt[3]{a_2^{-1}})$ such that

$$N_{K(\sqrt[3]{a_2^{-1}})/K}(y) = (a_1 a_2 a_3)^{-1}.$$

For $a_1 = 2, a_2 = 5, a_3 = 3$ the values are:

$$
\begin{aligned}
z_1 = \quad & \{\omega(3 + 3\sqrt[3]{3} + 2\sqrt[3]{9}), \sqrt[3]{3}(\omega - 1 - \sqrt[3]{3}), \sqrt[3]{3}(-\omega - 2 - \sqrt[3]{3})\}, \\
z_2 = \quad & \{25 + 20\sqrt[3]{5} + 28\sqrt[3]{25}, 3(5\sqrt[3]{5} + 12\omega^2\sqrt[3]{25}), 3\sqrt[3]{5}(-20 + 19\omega\sqrt[3]{5})\}, \\
z_3 = \quad & \{12 + 4\sqrt[3]{30} + \sqrt[3]{900}, \sqrt[3]{30}(2 + \omega^2\sqrt[3]{30}), \sqrt[3]{30}(10 - 3\omega\sqrt[3]{30})\}.
\end{aligned}
$$

3.5. The element γ

We look now for an element $\gamma \in L$ such that $M = L(\sqrt[n]{\gamma})$ is a solution to the embedding problem. We compute $\gamma = \gamma_1\,\gamma_2\,\gamma_3$ where $\gamma_1 \in L$ is the element corresponding to z_1, $\gamma_2 \in L$ is the element corresponding to z_2 and $\gamma_3 \in L$ is the element corresponding to z_3. We know (see the proof of [7, Theorem 8.2]) that the element $b_\sigma = zz^{-\sigma} g(u_\sigma) \in L$ (where the element $u_\sigma \in F(C_L)$ is such that $\varphi_{C_L}(u_\sigma) = t(\sigma)$) satisfies $b_\sigma b_\tau^\sigma = a_{\sigma,\tau} b_{\sigma\tau}$ (where the cocycle $a_{\sigma,\tau}$ represents ε) and the element γ that satisfies $\gamma^\sigma \gamma^{-1} = b_\sigma^n$ gives a solution to the embedding problem.

We consider the associated representations to T, T_i such that $g_i^{-1} g_i^\sigma = T_i(\sigma)$ for $i = 1, 2, 3$. They are:

$$
\begin{aligned}
& T_1 : \Gamma \to O(C), && \sigma_1 \mapsto Id, \quad \sigma_2 \mapsto Id, \quad \sigma_3 \mapsto \varphi_C(e_3^{-1} e_4), \\
& T_2 : \Gamma \to O(C_2), && \sigma_1 \mapsto Id, \quad \sigma_2 \mapsto \varphi_{C_2}(\sqrt[n]{a_3^{-1}} u_2 u_3^{-1}), \qquad \sigma_3 \mapsto Id, \\
& T_3 : \Gamma \to O(C_3), && \sigma_1 \mapsto \varphi_{C_3}(\sqrt[n]{a_2^{-1}} w_1 w_2^{-1}),\ \sigma_2 \mapsto \varphi_{C_3}(\sqrt[n]{a_2^{-1}} w_1 w_2^{-1}), \\
& && \sigma_3 \mapsto \varphi_{C_3}(\sqrt[n]{a_2^{-1}} w_1 w_2^{-1}).
\end{aligned}
$$

For T_1, we consider the elements

$$M_{\sigma_1}^{T_1} = M_{\sigma_2}^{T_1} = 1,\ M_{\sigma_3}^{T_1} = g_1(e_3 e_4^{-1}) = u_3 u_4^{-1}$$

$$N_{\sigma_1}^{T_1} = N_{\sigma_2}^{T_1} = N(M_{\sigma_1}^{T_1}) = 1,\ N_{\sigma_3}^{T_1} = N(M_{\sigma_3}^{T_1}) = \omega.$$

For $\sigma \in \Gamma$, we consider the corresponding

$$M_\sigma^{T_1} = g_1(u_\sigma^{T_1}), \quad N_\sigma^{T_1} = N(M_\sigma^{T_1}) \quad \text{and} \quad B_\sigma^{T_1} = z_1 z_1^{-\sigma} M_\sigma^{T_1} \in L.$$

They satisfy $N(z_1) = (B_\sigma^{T_1})^n (N_\sigma^{T_1})^{-1} N(z_1)^\sigma$. Let α_1 be a coordinate of $N(z_1)$. The element $\eta_1 = \sqrt[n]{a_3}$ satisfies $\eta_1^{-\sigma}\eta_1 = (N_\sigma^{T_1})^{-1}$. Then, the element $\gamma_1 = \eta_1 \alpha_1^{-1}$ satisfies the relation $\gamma_1^\sigma = (B_\sigma^{T_1})^n \gamma_1$ for each $\sigma \in \Gamma$.

Similarly, we consider for T_2,

$$M_{\sigma_1}^{T_2} = M_{\sigma_3}^{T_2} = 1,\ M_{\sigma_2}^{T_2} = g_2(\sqrt[n]{a_3^{-1}} u_2 u_3^{-1}) = \sqrt[n]{a_3^{-1}} w_2 w_3^{-1},\ M_\sigma^{T_2} = g_2(u_\sigma^{T_2}),$$

$$N_\sigma^{T_2} = N(M_\sigma^{T2}) = 1 \ \text{ and } \ B_\sigma^{T_2} = z_2 z_2^{-\sigma} M_\sigma^{T_2} \in L,\ \forall \sigma \in \Gamma.$$

They satisfy $N(z_2) = (B_\sigma^{T_2})^n (N_\sigma^{T_2})^{-1} N(z_2)^\sigma$. In this case $\eta_2 = 1$ and, if α_2 is a coordinate of $N(z_2)$, the element $\gamma_2 = \alpha_2^{-1}$ satisfies the relation $\gamma_2^\sigma = (B_\sigma^{T_2})^n \gamma_2$ for each $\sigma \in \Gamma$.

In the same way, we consider for T_3,

$$M_{\sigma_1}^{T_3} = M_{\sigma_3}^{T_3} = M_{\sigma_3}^{T_3} = g_3(\sqrt[n]{a_2^{-1}} w_1 w_2^{-1}) = \sqrt[n]{a_2^{-1}} v_1 v_2^{-1},\ M_\sigma^{T_3} = g_3(u_\sigma^{T_3}),$$

$$N_\sigma^{T_3} = N(M_\sigma^{T_3}) = 1 \ \text{ and } \ B_\sigma^{T_3} = z_3 z_3^{-\sigma} M_\sigma^{T_3} \in L,\ \forall \sigma \in \Gamma.$$

We have $N(z_3) = (B_\sigma^{T_3})^n (N_\sigma^{T_3})^{-1} N(z_3)^\sigma$. Now $\eta_3 = 1$ and $\gamma_3 = \alpha_3^{-1}$ (where α_3 is a coordinate of $N(z_3)$) satisfies the relation $\gamma_3^\sigma = (B_\sigma^{T_3})^n \gamma_3$ for each $\sigma \in \Gamma$.

It should be noted that for the representation T, the elements

$$m_\sigma = g(u_\sigma) = g_3(g_2(M_\sigma^{T_1}) M_\sigma^{T_2}) M_\sigma^{T_3} \nu_\sigma$$

where $\nu_\sigma \in \mu_n$. It is not difficult to prove that $b_\sigma = B_\sigma^{T_1} B_\sigma^{T_2} B_\sigma^{T_3} \nu'_\sigma$ ($\nu'_\sigma \in \mu_n$), and, therefore, $b_\sigma^n = (B_\sigma^{T_1} B_\sigma^{T_2} B_\sigma^{T_3})^n$.

Then, the element $\gamma = \gamma_1\gamma_2\gamma_3$ satisfies $\gamma^\sigma = b_\sigma^n \gamma$ and $M = L(\sqrt[n]{\gamma})$ is a solution to the embedding problem.

The general expression of the elements γ_1, γ_2 and γ_3 for $n = 3$ is:

$$\begin{aligned}\gamma_1 = \tfrac{1}{a_3^{19/3}} \Big(& 3\ (a_3^4\ \omega\ y_1^6 - a_3^{10/3}(1+2\omega)\ y_2\ y_1^4 - 3\ a_3^4\ \omega\ y_0\ y_2\ y_1^4 + a_3^4\ y_2^3\ y_1^3 + \\ & 3\ a_3^2\ y_1^3 + a_3^{10/3}\ y_0^2\ y_1^3 + 2\ a_3^{10/3}\ \omega\ y_0^2\ y_1^3 + 3\ a_3^2\ \omega\ y_1^3 + a_3^{8/3}\ y_0\ y_1^3 + \\ & 2\ a_3^{8/3}\ \omega\ y_0\ y_1^3 + 3\ a_3^4\ \omega\ y_0^2\ y_2^2\ y_1^2 + 6\ a_3^{10/3}\ \omega\ y_0\ y_2^2\ y_1^2 - 2\ a_3^{10/3}\ y_2^4\ y_1 - \\ & a_3^{10/3}\ \omega\ y_2^4\ y_1 - 3\ a_3^4\ y_0\ y_2^4\ y_1 - 3\ a_3^{8/3}\ y_0^2\ y2\ y1 - 3\ a_3^{8/3}\ \omega\ y_0^2\ y_2\ y_1 - \\ & 6\ a_3^2\ y_0\ y_2\ y_1 - 6\ a_3^2\ \omega\ y_0\ y_2\ y_1 - a_3^4\ y_2^6 - a_3^4\ \omega\ y_2^6 - a_3^{10/3}\ y_0^2\ y_2^3 - \\ & 2\ a_3^{10/3}\ \omega\ y_0^2\ y_2^3 - 3\ a_3^2\ \omega\ y_2^3 - a_3^{8/3}\ y_0\ y_2^3 - 2\ a_3^{8/3}\ \omega\ y_0\ y_2^3 + \\ & 3\ a_3^{4/3}\ y_0^2 + 3\ a_3^{2/3}\ y_0 + 3)\Big)\end{aligned}$$

where $y_0+y_1\sqrt[3]{\omega^{-1}}+y_2\sqrt[3]{\omega^{-2}} \in K(\sqrt[3]{\omega^{-1}})$ is such that $N_{K(\sqrt[3]{\omega^{-1}})/K}(y) = a_3^{-2}$,

$$\gamma_2 = \frac{1}{a_2^4}\Big(a_2^4\ a_3^2\ y_1^6 - a_2^{10/3}\ a_3^2\ y_2\ y_1^4 + a_2^{10/3}\ a_3^2\ \omega\ y_2\ y_1^4 - 3\ a_2^4\ a_3^2\ y_0\ y_2\ y_1^4 - 3\ a_2^2\ a_3^3\ y_1^3 + a_2^4\ a_3\ y_2^3\ y_1^3 - 2\ a_2^{10/3}\ a_3^3\ y_0^2\ y_1^3 - a_2^{10/3}\ a_3^3\ \omega\ y_0^2\ y_1^3 - 2\ a_2^{8/3}\ a_3^3\ y_0\ y_1^3 - a_2^{8/3}\ a_3^3\ \omega\ y_0\ y_1^3 + 3\ a_2^4\ a_3^2\ y_0^2\ y_2^2\ y_1^2 + 6\ a_2^{10/3}\ a_3^2\ y_0\ y_2^2\ y_1^2 - 2\ a_2^{10/3}\ a_3\ y_2^4\ y_1 - a_2^{10/3}\ a_3\ \omega\ y_2^4\ y_1 - 3\ a_2^4\ a_3\ y_0\ y_2^4\ y_1 + 3\ a_2^{8/3}\ a_3^3\ y_0^2\ y_2\ y_1 + 6\ a_2^2\ a_3^3\ y_0\ y_2\ y_1 + a_2^4\ y_2^6 + 3\ a_3^4 - 3\ a_2^2\ a_3^2\ y_2^3 - a_2^{10/3}\ a_3^2\ y_0^2\ y_2^3 + a_2^{10/3}\ a_3^2\ \omega\ y_0^2\ y_2^3 - a_2^{8/3}\ a_3^2\ y_0\ y_2^3 + a_2^{8/3}\ a_3^2\ \omega\ y_0\ y_2^3 + 3\ a_2^{4/3}\ a_3^4\ y_0^2 + 3\ a_2^{2/3}\ a_3^4\ y_0\Big)$$

where $y_0+y_1\sqrt[3]{a_3^{-1}}+y_2\sqrt[3]{a_3^{-2}} \in K(\sqrt[3]{a_3^{-1}})$ is such that $N_{K(\sqrt[3]{a_3^{-1}})/K}(y) = a_2^{-2}$ and

$$\gamma_3 = a_1^4\ a_2^2\ a_3^4\ y_1^6 + a_1^3a_2a_3^3\ \sqrt[3]{a_1a_2a_3}\ (\omega-1)\ y_2\ y_1^4 + a_1^2\ a_2\ a_3^2\ (a_1^2\ a_3^2\ y_2^3 - 3)\ y_1^3 - a_1^3a_3^3\ \sqrt[3]{a_1a_2a_3}\ (\omega+2)\ y_2^4\ y_1 + a_1^4\ a_3^4\ y_2^6 - 3\ a_1^2\ a_3^2\ y_2^3 + a_1a_2a_3\ \sqrt[3]{a_1a_2a_3}\ y_0^2\ (-a_1^2\ a_2\ a_3^2\ (\omega+2)\ y_1^3 + 3\ a_1^2a_3^2\ \sqrt[3]{a_1^2a_2^2a_3^2}\ y_2^2\ y_1^2 + \ a_1a_3\ \sqrt[3]{a_1a_2a_3}\ y_2\ y_1 + a_1^2\ a_3^2\ (\omega-1)\ y_2^3 + 3) - \sqrt[3]{a_1^2a_2^2a_3^2}\ y_0\ (3\ a_1^3a_3^3\ \sqrt[3]{a_1a_2a_3}\ y_2\ y_1^4 + a_1^2\ a_2\ a_3^2\ (\omega+2)\ y_1^3 - 6\ a_1^2a_3^2\ \sqrt[3]{a_1^2a_2^2a_3^2}\ y_2^2\ y_1^2 + 3\ a_1a_3\ \sqrt[3]{a_1a_2a_3}\ y_2\ (a_1^2\ a_3^2\ y_2^3 - 2)\ y_1 - a_1^2\ a_3^2\ (\omega-1)\ y_2^3 - 3) + 3$$

where $y_0 + y_1\sqrt[3]{a_2^{-1}} + y_2\sqrt[3]{a_2^{-2}} \in K(\sqrt[3]{a_2^{-1}})$ is such that

$$N_{K(\sqrt[3]{a_2^{-1}})/K}(y) = (a_1a_2a_3)^{-1}.$$

For $a_1 = 2, a_2 = 5$ and $a_3 = 3$ the values are:

$$\gamma_1 = 9\ + 6\ \sqrt[3]{3} + 4\ \sqrt[3]{9}$$
$$\gamma_2 = 8059\ + (1340\ \omega + 2040)\sqrt[3]{5} + (1876\ \omega + 2856\ )\sqrt[3]{25}$$
$$\gamma_3 = 379\ + (116\ - 32\ \omega)\sqrt[3]{30} + (29\ - 8\ \omega)\sqrt[3]{900}.$$

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Montserrat Vela
Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
C/ Jordi Girona, 1–3
Edifici Omega, Campus Nord
E-08034 Barcelona, Spain
Montse.Vela@upc.edu

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