Some Remarks on
the Weak Maximum Principle

Marco Rigoli, Maura Salvatori and Marco Vignati

Abstract

We obtain a maximum principle at infinity for solutions of a class of nonlinear singular elliptic differential inequalities on Riemannian manifolds under the sole geometrical assumptions of volume growth conditions. In the case of the Laplace-Beltrami operator we relate our results to stochastic completeness and parabolicity of the manifold.

Dedicated to the memory of Franca Burrone Rigoli.

1. Introduction

In what follows we let $(M, \langle \cdot, \cdot \rangle)$ denote a non-compact, connected Riemannian manifold, and $r(x) = dist_{(M, \langle \cdot, \cdot \rangle)}(x, o)$ for some fixed reference point $o \in M$, so that B_R is the geodesic ball of radius R centered at o. A famous result of H. Omori and S. T. Yau asserts that for each $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence $\{x_k\} \subset M$ with the following properties: for each $k \in \mathbb{N}$,

$$(1.1) \qquad i) \ u(x_k) \to u^* \qquad ii) \ \Delta u(x_k) < \frac{1}{k} \qquad iii) \ |\nabla u|(x_k) < \frac{1}{k}$$

provided the metric is complete and some curvature assumptions are satisfied [2].

This result is known in the literature as the Omori-Yau Maximum Principle, and it has been used in the solution of a variety of geometrical problems since its appearance in the beautiful proof of the Schwarz's lemma for holomorphic maps between Kähler manifolds due to Yau himself [18].

In some situation this principle is not applied in its full strength, in the sense that often the information (1.1)*iii)* does not need to be used. This is for instance the case when, according to an old question of Calabi, we study the existence of complete minimal surfaces in $\mathbb{R}^3$ with bounded image. Thus, the following definition seems to be appropriate:

Definition 1 *Let $(M, \langle \cdot, \cdot \rangle)$ be a (not necessarily complete) Riemannian manifold. We say that $(M, \langle \cdot, \cdot \rangle)$ satisfies the weak maximum principle if for each $u \in \mathcal{C}^2(M)$, $u^* < +\infty$, there exists a sequence $\{x_k\} \subset M$ with the properties (1.1) i); ii) above.*

This seemingly simple minded definition is in fact surprisingly deep; indeed, it unexpectedly relates to stochastic completeness. As a matter of fact, in a recent paper, [9], the authors proved the equivalence of the following statements:

(1.2) $(M, \langle \cdot, \cdot \rangle)$ *is stochastically complete;*

(1.3) $\forall \lambda > 0$, *the differential equation $\Delta u = \lambda u$ has no non-negative, non-identically null bounded solutions;*

(1.4) $\forall u \in \mathcal{C}^2(M)$, *u non-constant, $u^* < +\infty$ and $\forall \gamma < u^*$,*

$$\inf_{\Omega_\gamma} \Delta u \leq 0, \quad with \quad \Omega_\gamma = \{x \in M : u(x) > \gamma\};$$

(1.5) $(M, \langle \cdot, \cdot \rangle)$ *satisfies the weak maximum principle.*

We stress that the metric has not been assumed to be complete. Indeed, neither the weak maximum principle nor any of the above equivalent formulations require it.

The equivalence of (1.2) and (1.3) is well known, and due to Grigor'yan, see for instance [3].

We observe that, for $\lambda > 0$, any positive constant is a super-solution of $\Delta u = \lambda u$ thus, using the Monotone Iteration Scheme on an appropriate exhaustion of M, (1.3) can be equivalently substituted by

(1.3′) $\forall \lambda > 0$, *the differential inequality $\Delta u \geq \lambda u$ has no non-negative, non-identically null bounded solutions.*

This can also be seen by observing that the statement

(1.6) $\forall f \in \mathcal{C}^0(\mathbb{R})$ *and $\forall u \in \mathcal{C}^2(M)$ with $u^* < +\infty$ satisfying $\Delta u \geq f(u)$ we have $f(u^*) \leq 0$.*

is implied by (1.5), and it implies (1.3), so that it is in fact equivalent to the above.

To analyse (1.3′) a step further, we let u be a subharmonic function on M with $u^* < +\infty$. Without loss of generality we can suppose that $u^* > 0$ and therefore, since $\sup\{0, u\}$ is still a subharmonic function, we can assume that u is a non-negative, bounded subharmonic function. This shows that we can interpret (1.3′) with $\lambda = 0$ just as the parabolicity of the manifold $(M, \langle \cdot, \cdot \rangle)$. To be completely precise, the following statements are equivalent:

(1.7) $(M, \langle \cdot, \cdot \rangle)$ *is parabolic;*

(1.8)   *the differential inequality $\Delta u \geq 0$ has no non-constant, non-negative bounded solutions;*

(1.9) $\forall u \in \mathcal{C}^2(M)$, *u non-constant, $u^* < +\infty$ and $\forall \gamma < u^*$, $\inf_{\Omega_\gamma} \Delta u \leq 0$,, with $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$;*

(1.10) $\forall u \in \mathcal{C}^2(M)$, *u non-constant, $u^* < +\infty$, there exists a sequence $\{x_k\} \subset M$ such that, $\forall k \in \mathbb{N}$,*

$$u(x_k) > u^* - \frac{1}{k} \; ; \quad \Delta u(x_k) < 0 \,.$$

The only implication that needs some explanation is $(1.7) \implies (1.9)$. (The equivalence of (1.7) with (1.8) having been pointed out above.)

Towards this aim let u be as in (1.9) and assume that, for some $\gamma < u^*$, $\Delta u \geq 0$ on Ω_γ. For $0 < \varepsilon < u^* - \gamma$ we define

$$v = \begin{cases} \gamma + \dfrac{\varepsilon}{2} & \text{on } M \setminus \overline{\Omega_{\gamma + \varepsilon/2}} \\ u & \text{on } \overline{\Omega_{\gamma + \varepsilon}} \end{cases}$$

Then $v = \sup\{u, \gamma + \varepsilon/2\}$ on $\Omega_\gamma \setminus \overline{\Omega_{\gamma+\varepsilon}}$ and therefore v is subharmonic on M. Because of (1.7) it follows that $v \equiv \gamma + \varepsilon/2$ contradicting $u^* > \gamma + \varepsilon$.

Thus, $(M, \langle \cdot, \cdot \rangle)$ is parabolic if and only if it satisfies the somewhat stronger version of the weak maximum principle given in (1.10).

The above discussion shows the relevance of the study of appropriate formulations of the weak maximum principle. Motivated by the aim of investigating differential inequalities of the type (see Corollary 1.3 below)

$$\Delta u \geq b(x) f(u)$$

we look for a weak maximum principle in the following form:

(1.11)   *given $\Lambda, \sigma \geq 0$, $\mu \in \mathbb{R}$, for each $u \in \mathcal{C}^2(M)$, u non-constant with $\hat{u} = \limsup_{r(x) \to +\infty} \dfrac{u(x)}{r(x)^\sigma} < +\infty$, if $\Omega_\gamma = \{x \in M : u(x) > \gamma\} \neq \emptyset$ then:*

$$\inf_{\Omega_\gamma} (1 + r(x))^\mu \Delta u(x) \leq \Lambda \max\{\hat{u}, 0\} \,.$$

We note that (1.11) can indeed be considered as some form of the weak maximum principle, in the sense that it reduces exactly to it in case $\mu = 0$, $\sigma = 0$ and $\Lambda = 0$. However, we are now in general allowed to consider situations where $u^* = +\infty$.

Of course, it is essential to depict geometrical conditions to assure the validity of (1.11) on $(M, \langle \cdot, \cdot \rangle)$ for appropriate values of Λ, σ, μ. We recall that stochastic completeness and parabolicity, in case the metric is complete, are guaranteed, respectively, by the following volume growth assumptions:

$$\frac{r}{\log \, vol\,(B_r)} \notin L^1\,(+\infty) \quad ; \quad \frac{1}{vol\,(\partial B_r)} \notin L^1\,(+\infty) \ .$$

The first is, at present, the best general condition in this direction, and it has been proved by Grigor'yan (see for instance [3]). The second has been known for some time, and rediscovered by many researchers.

In the next result we shall require a more demanding type of volume growth assumption that, when $\mu = 0$, is in some sense intermediate between the two above and which, however, is directly related to the value of Λ in (1.11). We prove

Theorem 1.1 *Let* $(M, \langle \cdot, \cdot \rangle)$ *be a complete manifold and let* $\sigma, \mu \in \mathbb{R}$ *satisfy*

$$(1.12) \qquad\qquad i) \ \ \sigma \geq 0 \ \ ; \quad ii) \ \ \sigma + \mu < 2$$

Assume that

$$(1.13) \qquad\qquad \liminf_{r \to +\infty} \frac{\log \, vol\,(B_r)}{r^{2-\sigma-\mu}} = d_0 < +\infty \ .$$

Then, given $u \in \mathcal{C}^2\,(M)$ *such that*

$$(1.14) \qquad\qquad \hat{u} = \limsup_{r(x) \to +\infty} \frac{u\,(x)}{r\,(x)^\sigma} < +\infty$$

and $\gamma \in \mathbb{R}$ *with*

$$(1.15) \qquad\qquad \Omega_\gamma = \{x \in M : u\,(x) > \gamma\} \neq \emptyset$$

we have

$$(1.16) \qquad\qquad \inf_{\Omega_\gamma}[1 + r\,(x)]^\mu \Delta u\,(x) \leq d_0 \max\{\hat{u}, 0\}\, C\,(\sigma, \mu)$$

with

$$C\,(\sigma, \mu) = \begin{cases} 0 & if \ \sigma = 0 \\ (2 - \sigma - \mu)^2 & if \ \sigma > 0, \ 2\,(1 - \sigma) > \mu \\ \sigma\,(2 - \sigma - \mu) & if \ \sigma > 0, \ 2\,(1 - \sigma) \leq \mu \end{cases}$$

Condition (1.13) describes, broadly speaking, a volume growth of at most exponential type. For a growth of at most polynomial type we substitute (1.13) with

$$(1.17) \qquad \liminf_{r \to +\infty} \frac{\log \, vol \, (B_r)}{\log \, r} = d_0 < +\infty .$$

In this case, with the notation of Theorem 1.1, we observe that, if $\sigma + \mu < 2$, (1.17) implies the validity of (1.13) with $d_0 = 0$, so that

$$\inf_{\Omega_\gamma}[1 + r \, (x)]^\mu \Delta u \, (x) \leq 0 .$$

The interesting cases left to analyse are therefore

$$(1.18) \qquad\qquad i) \ \sigma \geq 0 \quad ; \quad ii) \ \sigma + \mu = 2$$

Theorem 1.2 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold and assume (1.17) and (1.18). Then, given $u \in \mathcal{C}^2 \, (M)$ satisfying (1.14) and $\gamma \in \mathbb{R}$ such that $\Omega_\gamma \neq \emptyset$ we have*

$$\inf_{\Omega_\gamma}[1 + r \, (x)]^\mu \Delta u \, (x) \leq \sigma \max \{\hat{u}, 0\} \, C \, (\sigma, d_0)$$

with

$$C \, (\sigma, d_0) = \begin{cases} 0 & if \ \sigma + d_0 < 2 \\ \sigma + d_0 - 2 & if \ \sigma + d_0 \geq 2 \end{cases}$$

We shall consider sharpness of the constants $C \, (\sigma, \mu)$, $C \, (\sigma, d_0)$ and of the volume growth assumptions in Section 3 below.

Corollary 1.3 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold and let $b \in \mathcal{C}^0 \, (M)$ be such that*

$$(1.19) \qquad i) \ \ b \, (x) > 0 \ \ on \ M \quad ; \quad ii) \ \ b \, (x) \geq \frac{B}{r \, (x)^\mu} \ \ on \ M \setminus B_{R_0}$$

for some $R_0, B > 0$ and $\mu \in \mathbb{R}$. Let $f \in \mathcal{C}^0 \, (\mathbb{R})$ with

$$(1.20) \qquad\qquad \liminf_{t \to +\infty} f \, (t) > 0 .$$

Let $u \in \mathcal{C}^2 \, (M)$ be a solution of

$$\Delta u \geq b \, (x) \, f \, (u) \quad on \ M$$

such that, for some $\sigma \geq 0$

$$(1.21) \qquad\qquad u \, (x) = o \, (r \, (x)^\sigma) \quad as \ r \, (x) \to +\infty .$$

Assume that

$$\sigma + \mu < 2 \quad and \quad \liminf_{r \to +\infty} \frac{\log \, vol \, (B_r)}{r^{2-\sigma-\mu}} < +\infty .$$

Then $u^* < +\infty$ and $f \, (u^*) \leq 0$.

It would be interesting to determine conditions on f and the manifold M to guarantee the validity of (1.21). This has been done, for instance, in Theorem 2.6 of [13] where, improving an original result of Cheng and Yau, it is shown that $u^* < +\infty$ under curvature assumptions, $b(x) \equiv 1$ and, basically, f is eventually positive and superlinear. However, under sole volume growth conditions as geometrical requirements, the task is really hard. In the recent paper [10] the authors succeeded in proving an L^∞ bound for non-negative solutions u of

$$\Delta u \geq b(x) u^\gamma - a(x) u, \quad \gamma > 1$$

under volume growth assumptions and conditions on the behaviour of the coefficients $a(x)$ and $b(x)$. The technique devised in that paper can also be used to prove a version of Theorem 1.1 but only, in the present notations, for $\sigma = 0$ and $0 \leq \mu < 2$. We also would like to recall that the first result in this direction is due to L. Karp, [8].

A second aim of this paper is to extend Theorems 1.1 and 1.2 and Corollary 1.3 to a large class of differential operators, defined in Section 2 below, such as, for instance, the $p-$Laplacian and the mean curvature operator. This extension has strong geometrical motivations. To quote a few, we refer to the recent works of Holopainen [6], [7]. Our class of operators essentially coincides with those considered by Serrin and collaborators, see for instance from the seminal paper [16] to the recent [11], [12], and this class has been intensively studied in this last decade. Results in this direction, especially on a generalization of the notion of parabolicity, have been considered in [6] and [14]. Our techniques can also be applied to Markov energy measure spaces and corresponding differential operators, with some assumptions. For this topic we refer to [17], [1], and [4], where beautiful applications to the geometry of minimal surfaces are presented. We shall pursue this direction in a forthcoming paper.

2. Proofs and generalizations of the results

Geometrical problems, for instance the study of assigned mean curvature graphs on a complete manifold, or non-linear potential theory, point out the interesting goal of extending the analysis to a larger class of differential operators in divergence form, other than the Laplacian. Towards this aim we let $\varphi \in \mathcal{C}^1((0, +\infty)) \cap \mathcal{C}^0([0, +\infty))$ satisfy the following structural conditions:

$$(2.1) \quad i)\ \varphi(0) = 0\ ; \quad ii)\ \varphi(t) > 0\ \ if\ t > 0\ ; \quad iii)\ \varphi(t) \leq At^\delta\ \ if\ t \geq 0$$

for some positive constants A and δ.

We shall focus our attention on the differential operator defined, for $u \in \mathcal{C}^1(M)$, by

$$(2.2) \qquad div\,(Tu) \equiv div\left(|\nabla u|^{-1}\varphi\left(|\nabla u|\right)\nabla u\right)$$

and which we shall refer to as the $\varphi - Laplacian$. Of course, if the vector field in brackets is not $\mathcal{C}^1$ the divergence in (2.2) must be considered in distributional sense. Note that the vector field in consideration may fail to be $\mathcal{C}^1$ at the points where $\nabla u = 0$ even if u is assumed to be $\mathcal{C}^2$.

As important natural examples we mention:

i) the *Laplace-Beltrami* operator Δu, corresponding to $\varphi(t) = t$;

ii) more generally, the *p-Laplacian* operator $div\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, corresponding to $\varphi(t) = t^{p-1}$;

iii) the *generalized mean curvature* operator $div\left(\dfrac{\nabla u}{\left(1 + |\nabla u|^2\right)^\alpha}\right)$, $\alpha > 0$,

corresponding to $\varphi(t) = \dfrac{t}{(1 + t^2)^\alpha}$.

We mention that the usual maximum principle and the Compact Support Property have been recently studied (under some further assumptions on φ) for instance by Pucci and Serrin, [11].

In the next statements we shall always assume that $\varphi \in \mathcal{C}^1\left((0, +\infty)\right) \cap \mathcal{C}^0\left([0, +\infty)\right)$ satisfies the structural conditions (2.1).

Theorem 2.1 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, and let $\sigma, \mu \in \mathbb{R}$ satisfy*

$$\sigma > 0 \;\; ; \;\; \delta\sigma + \mu < 1 + \delta \ .$$

Assume that

$$(2.3) \qquad \liminf_{r \to +\infty} \frac{\log\,vol\,(B_r)}{r^{1+\delta-\delta\sigma-\mu}} = d_0 < +\infty \ .$$

Then, given $u \in \mathcal{C}^2(M)$ for which $Tu \equiv |\nabla u|^{-1}\varphi\left(|\nabla u|\right)\nabla u$ is at least of class $\mathcal{C}^1$, and such that

$$(2.4) \qquad \hat{u} = \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^\sigma} < +\infty$$

and given $\gamma \in \mathbb{R}$ with

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} \neq \emptyset$$

we have

(2.5) $\displaystyle\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div\,(Tu)\,(x) \le A \left(\frac{2}{1+\delta}\right)^{1+\delta} \delta^\delta d_0 \, [\max{(\hat{u}; 0)}]^\delta \, C(\sigma, \mu, \delta)$

with

$$C(\sigma, \mu, \delta) = \begin{cases} (1 + \delta - \delta\sigma - \mu)\,\sigma^\delta & \text{if } (1+\delta)(1-\sigma) \le \mu \\ (1 + \delta - \delta\sigma - \mu)^{1+\delta} & \text{if } (1+\delta)(1-\sigma) > \mu \end{cases}$$

Proof: It will be convenient to use the notation

(2.6) $\qquad\qquad \beta \equiv (1+\delta)(1-\sigma) - \mu = (1 + \delta - \delta\sigma - \mu) - \sigma;$

in terms of this notation, recall that we are working with

$$\beta \in \mathbb{R} \;\; ; \;\; \sigma > 0 \;\; ; \;\; \beta + \sigma > 0\,.$$

Moreover, we note that (2.5) is an estimate from above of the quantity

(2.7) $\qquad\qquad K \equiv \displaystyle\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div\,(Tu)\,(x)$

and the proof is needed only in case $K > 0$. Hence, we assume that

(2.8) $\qquad\qquad [1 + r(x)]^\mu \, div\,(Tu)\,(x) \ge K > 0 \quad \text{on } \Omega_\gamma.$

Let $b > \max{(\hat{u}; 0)}$; since $\sigma > 0$, we may translate the function u by a constant, so that $\hat{u}$ remains unchanged, and

$$\frac{u(x)}{[1 + r(x)]^\sigma} < b \;\; \text{on } M \;\; ; \;\; u(x_0) > 0$$

for some $x_0 \in M$. Moreover, $div\,(Tu)$ is invariant under translation, the left hand side of (2.5) is increasing with γ, and the right hand side does not depend on γ; hence we may assume $\gamma \ge 0$.

Now, fix $\vartheta \in \left(\frac{1}{2}, 1\right)$; for any $R > 0$ such that $B_R \cap \Omega_\gamma \ne \emptyset$ let $\psi \in \mathcal{C}^\infty(M)$ be a cut-off function such that

(2.9)
$\qquad i)\; 0 \le \psi \le 1 \;; \qquad\qquad ii)\; \psi \equiv 1 \text{ on } B_{\vartheta R}\;;$
$\qquad iii)\; \psi \equiv 0 \text{ on } M \setminus B_R \;; \quad iv)\; |\nabla\psi| \le \dfrac{c}{R(1-\vartheta)}$

for some absolute constant $c > 0$.

Next, we choose $\lambda \in \mathcal{C}^\infty(\mathbb{R})$ with the following properties

(2.10)
$\qquad i)\; 0 \le \lambda \le 1 \;; \qquad\qquad ii)\; \lambda \equiv 0 \text{ on } (-\infty, \gamma] \;;$
$\qquad iii)\; \lambda > 0 \text{ on } (\gamma, +\infty) \;; \quad iv)\; \lambda' \ge 0 \text{ on } (\gamma, +\infty)\,.$

We introduce the auxiliary function

(2.11)
$$v = 2b(1+r)^\sigma - u$$

and notice that it satisfies

(2.12)
$$b(1+r)^\sigma \le v \le 2b(1+r)^\sigma \quad \text{on } \Omega_\gamma.$$

With all these ingredients, we define the vector field

(2.13)
$$W = \psi^{1+\delta}\lambda(u)F(v,r)Tu$$

where the function $F : [0,+\infty) \times [0,+\infty) \to (0,+\infty)$ is given by

(2.14)
$$
\begin{aligned}
&i) \quad F(v,r) = F(v) = \exp\left[-qv^{1+(\beta/\sigma)}\right] \quad &&\text{if } \beta \le 0 \\
&ii) \quad F(v,r) = \exp\left[-q(1+r)^\beta v\right] \quad &&\text{if } \beta > 0
\end{aligned}
$$

where $q > 0$ is a constant (possibly different in the two cases) to be fixed later.

We notice that, in both cases, $\partial F/\partial v \equiv F_v < 0$; moreover, the vector field W vanishes on $\partial(B_R \cap \Omega_\gamma)$, so that, by the Divergence Theorem,

(2.15)
$$\int_{B_R \cap \Omega_\gamma} div\, W = 0 .$$

Computing the divergence of W we obtain:

$$
\begin{aligned}
div\, W = {}&(1+\delta)\psi^\delta\lambda(u)F(v,r)\langle\nabla\psi,Tu\rangle + \psi^{1+\delta}\{\lambda'(u)F(v,r)\langle\nabla u,Tu\rangle \\
&+ \lambda(u)[F_v\langle\nabla v,Tu\rangle + F_r\langle\nabla r,Tu\rangle] + \lambda(u)F(v,r)div(Tu)\}.
\end{aligned}
$$

The Cauchy-Schwarz inequality, properties (2.10), (2.11) and (2.14) lead to

(2.16)

$$
\begin{aligned}
div\, W \ge {}&-(1+\delta)\psi^\delta\lambda(u)F(v,r)\varphi(|\nabla u|)|\nabla\psi| + \\
&+ \psi^{1+\delta}\lambda(u)|F_v|\left\{\frac{F}{|F_v|}K(1+r)^{-\mu} + |\nabla u|\varphi(|\nabla u|) + \right. \\
&\left. + \left[\frac{F_r}{|F_v|} - 2b\sigma(1+r)^{\sigma-1}\right]|\nabla u|^{-1}\varphi(|\nabla u|)\langle\nabla r,\nabla u\rangle\right\}.
\end{aligned}
$$

Using (2.14) and (2.12) we estimate in both cases the quantity within the square brackets

(2.17)
$$
\left[\frac{F_r}{|F_v|} - 2b\sigma(1+r)^{\sigma-1}\right] \ge
\begin{cases}
(-\sigma)2b(1+r)^{\sigma-1} & \text{if } \beta \le 0 \\
-(\beta+\sigma)2b(1+r)^{\sigma-1} & \text{if } \beta > 0
\end{cases}
,
$$

and the quantity

(2.18)
$$\frac{F}{|F_v|} \geq \begin{cases} \dfrac{\sigma}{q\,(\beta+\sigma)} b^{-\beta/\sigma}\,(1+r)^{-\beta} & \text{if } \beta \leq 0 \\[2mm] \dfrac{(1+r)^{-\beta}}{q} & \text{if } \beta > 0 \end{cases}$$

and we note that from the structural condition $(2.1)iii)$ we have the inequality

(2.19)
$$|\nabla u|\,\varphi\,(|\nabla u|) \geq A^{-1/\delta}\varphi\,(|\nabla u|)^{(\delta+1)/\delta}\ .$$

Hence, from (2.16)–(2.19) we obtain

$$div\,W \geq - (1+\delta)\,\psi^\delta \lambda\,(u)\,F\,(v,r)\,\varphi\,(|\nabla u|)\,|\nabla \psi|$$
$$+ \psi^{1+\delta}\lambda\,(u)\,|F_v|\left\{ A^{-\frac{1}{\delta}}\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}} - 2b\sigma\,(1+r)^{\sigma-1}\,\varphi\,(|\nabla u|) \right.$$

(2.20)
$$\left. + \frac{K\sigma}{q\,(\beta+\sigma)} b^{-\beta/\sigma}\,(1+r)^{-\beta-\mu} \right\}$$

if $\beta \leq 0$, and

$$div\,W \geq - (1+\delta)\,\psi^\delta \lambda\,(u)\,F\,(v,r)\,\varphi\,(|\nabla u|)\,|\nabla \psi|$$
$$+ \psi^{1+\delta}\lambda\,(u)\,|F_v|\left\{ A^{-\frac{1}{\delta}}\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}} - 2b\,(\beta+\sigma)\,(1+r)^{\sigma-1}\,\varphi\,(|\nabla u|) \right.$$

(2.21)
$$\left. + \frac{K}{q}\,(1+r)^{-\beta-\mu} \right\}$$

if $\beta > 0$. Moreover, we can estimate from below the quantities within the braces in (2.20) and (2.21).

Claim: *For $0 < \tau < 1$ and for any $x \in M$:*
i)   *the choices*

(2.22)
$$0 < \Lambda \leq \frac{1-\tau}{2A^{1/\delta}}\ ;\qquad q = \frac{\tau^\delta K\,(1+\delta)^{1+\delta}\,\sigma b^{-\beta/\sigma}}{A\delta^\delta(2b\sigma)^{1+\delta}\,(\beta+\sigma)}$$

imply

$$A^{-\frac{1}{\delta}}\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}} - 2b\sigma\,(1+r)^{\sigma-1}\varphi(|\nabla u|) + \frac{K\sigma b^{-\beta/\sigma}}{q\,(\beta+\sigma)}\,(1+r)^{-\beta-\mu} \geq \Lambda\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}$$

ii)   *the choices*

(2.23)
$$0 < \Lambda \leq \frac{1-\tau}{2A^{\frac{1}{\delta}}}\ ;\qquad q = \frac{\tau^\delta K\,(1+\delta)^{1+\delta}}{A\delta^\delta\,[2b\,(\beta+\sigma)]^{1+\delta}}$$

imply

$$A^{-\frac{1}{\delta}}\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}} - 2b(\beta+\sigma)\,(1+r)^{\sigma-1}\varphi(|\nabla u|) + \frac{K}{q}\,(1+r)^{-\beta-\mu} \geq \Lambda\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}\ .$$

Proof of the Claim: It is an easy Calculus exercise to show that if M, N, P are positive constants, and if

$$0 < \Lambda \leq M - \frac{\delta N^{1+\frac{1}{\delta}}}{(1+\delta)^{1+\frac{1}{\delta}} P^{\frac{1}{\delta}}}$$

then

$$Mt^{1+\frac{1}{\delta}} - Nt + P \geq \Lambda t^{1+\frac{1}{\delta}}$$

for any $t \geq 0$.

Hence, the use of the Claim and the choices (2.22), (2.23) allow to obtain, from (2.20) and (2.21), the common estimate

$$div\ W \geq - (1+\delta)\,\psi^\delta \lambda\,(u)\,F\,(v,r)\,\varphi\,(|\nabla u|)\,|\nabla \psi| + \Lambda \psi^{1+\delta} \lambda\,(u)\,|F_v|\,\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}$$

and from (2.15) it follows

$$(2.24) \qquad \Lambda \int_{B_R \cap \Omega_\gamma} \psi^{1+\delta} \lambda\,(u)\,|F_v|\,\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}$$

$$\leq (1+\delta) \int_{B_R \cap \Omega_\gamma} \psi^\delta \lambda\,(u)\,F\,(v,r)\,\varphi\,(|\nabla u|)\,|\nabla \psi|.$$

We now replace the correct expressions for F_v obtained from (2.14), and make use of Hölder's inequality with conjugate exponents $1+\delta$, $1+1/\delta$ to obtain the chain of inequalities

$$\int_{B_R \cap \Omega_\gamma} \psi^{1+\delta} \lambda\,(u)\,F\,(v,r)\,(1+r)^\beta\,\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}$$

$$\leq c \int_{B_R \cap \Omega_\gamma} \psi^\delta \lambda\,(u)\,F\,(v,r)\,\varphi\,(|\nabla u|)\,|\nabla \psi|$$

$$\leq c \left(\int_{B_R \cap \Omega_\gamma} \psi^{1+\delta} \lambda\,(u)\,F\,(v,r)\,(1+r)^\beta\,\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}} \right)^{\delta/(1+\delta)} \times$$

$$\times \left(\int_{B_R \cap \Omega_\gamma} \lambda\,(u)\,F\,(v,r)\,(1+r)^{-\beta\delta}\,|\nabla \psi|^{1+\delta} \right)^{1/(1+\delta)}$$

whence

$$(2.25) \qquad \int_{B_R \cap \Omega_\gamma} \psi^{1+\delta} \lambda\,(u)\,F\,(v,r)\,(1+r)^\beta\,\varphi\,(|\nabla u|)^{1+\frac{1}{\delta}}$$

$$\leq c \int_{B_R \cap \Omega_\gamma} \lambda\,(u)\,F\,(v,r)\,(1+r)^{-\beta\delta}\,|\nabla \psi|^{1+\delta}.$$

Having fixed $d > d_0$, according to (2.3) there exists a sequence $\{R_k\} \uparrow +\infty$ with $\vartheta R_{k+1} > R_k$ such that

$$(2.26) \qquad\qquad \log \, vol \, (B_{R_k}) \leq dR_k^{\beta+\sigma}.$$

Using (2.9), (2.10) and (2.25) with $R = R_k$ we have

$$c \int_{B_{R_1} \cap \Omega_\gamma} \lambda(u) \, F(v, r) \, \varphi \, (|\nabla u|)^{1+\frac{1}{\delta}}$$

$$(2.27) \qquad\qquad \leq R_k^{-1-\delta} \int_{\left(B_{R_k} \backslash B_{\vartheta R_k}\right) \cap \Omega_\gamma} F(v, r) \, (1+r)^{-\beta\delta} \, .$$

The condition $K > 0$ implies that u is not constant on the connected components of $B_{R_1} \cap \Omega_\gamma$, thus the left hand side of this inequality is strictly positive.

Hence, in case $\beta \leq 0$ from $(2.14)i)$ and (2.26) one gets

$$0 < \liminf_{k \to +\infty} (1 + R_k)^{-1-\delta-\beta\delta} \exp \left\{ dR_k^{\beta+\sigma} - qb^{1+(\beta/\sigma)} \, (\vartheta R_k)^{\beta+\sigma} \right\}$$

where q is taken as in (2.22); this forces

$$d > qb^{1+(\beta/\sigma)} \vartheta^{\beta+\sigma}.$$

Letting $\vartheta, \tau \to 1^-$, $d \to d_0^+$ we obtain

$$d \geq \frac{(1+\delta)^{1+\delta} \, b\sigma K}{A\delta^\delta \, (2b\sigma)^{1+\delta} \, (\beta + \sigma)} \, ,$$

and when $b \downarrow \max(\hat{u}; 0)$ this implies, recalling (2.6),

$$K \leq A \left(\frac{2}{1+\delta} \right)^{1+\delta} \delta^\delta d_0 \, [\max(\hat{u}; 0)]^\delta \, (1 + \delta - \delta\sigma - \mu) \, \sigma^\delta$$

thus ending the proof for the case $\beta \leq 0$.

In case $\beta > 0$ we use $(2.14)ii)$ and (2.26) to obtain, from (2.27),

$$0 < \liminf_{k \to +\infty} (1 + R_k)^{-1-\delta-\beta\delta} \exp \left\{ dR_k^{\beta+\sigma} - qb \, (\vartheta R_k)^{\beta+\sigma} \right\}$$

with q as in (2.23). As above, we get

$$K \leq A \left(\frac{2}{1+\delta} \right)^{1+\delta} \delta^\delta d_0 \, [\max(\hat{u}; 0)]^\delta \, (1 + \delta - \delta\sigma - \mu)^{1+\delta} \, . \qquad \blacksquare$$

Our next result deals with the same kind of estimate, and covers the case when, in the assumptions of Theorem 2.1, σ is null. Part of the proof resembles the previous one, and we shall avoid repetitions, when obvious.

Theorem 2.2 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, and $\mu < 1 + \delta$. Assume that*

$$\liminf_{r \to +\infty} \frac{\log \, vol \, (B_r)}{r^{1+\delta-\mu}} = d_0 < +\infty \; .$$

Then, given $u \in \mathcal{C}^2 \, (M)$ for which $Tu \equiv |\nabla u|^{-1} \varphi \, (|\nabla u|) \, \nabla u$ is at least of class $\mathcal{C}^1$, and such that

$$u^* = \sup_M u < +\infty \; ,$$

and given $\gamma < u^*$ with $\Omega_\gamma = \{x \in M : u \, (x) > \gamma\}$ we have

$$(2.28) \qquad \inf_{\Omega_\gamma} \, [1 + r \, (x)]^\mu \, div \, (Tu) \, (x) \leq 0 \; .$$

Proof: Let

$$\beta = 1 + \delta - \mu > 0 \; .$$

Since $div \, (Tu)$ is translation invariant, we may assume $u^* = 0$. For any $\gamma < 0$, the set $\Omega_\gamma = \{x \in M : u \, (x) > \gamma\}$ is non-empty, and if, as in (2.7) and (2.8)

$$(2.29) \qquad 0 < K \equiv \inf_{\Omega_\gamma} \, [1 + r \, (x)]^\mu \, div \, (Tu) \, (x)$$

the function u is not constant on Ω_γ. We shall prove that this leads to a contradiction.

Again, fix $\vartheta \in \left(\frac{1}{2}, 1 \right)$, and choose $R > 0$ so that $B_R \cap \Omega_\gamma \neq \emptyset$; let ψ and λ be cut-off functions as in (2.9) and (2.10) and define, for any $\eta > 0$, the auxiliary function

$$v \, (x) = \eta - u \, (x)$$

that satisfies

$$\eta \leq v \, (x) \leq \eta - \gamma \quad \text{on } \Omega_\gamma \; .$$

For

$$F \, (v, r) = \exp \left[-q \, (1 + r)^\beta \, v \right] \; , \quad q > 0$$

as in $(2.14) ii)$ the same vector field used in (2.13)

$$W = \psi^{1+\delta} \lambda \, (u) \, F \, (v, r) \, Tu$$

vanishes on $\partial \, (B_R \cap \Omega_\gamma)$, so that, by the Divergence Theorem,

$$\int_{B_R \cap \Omega_\gamma} div \, W = 0 \; .$$

As in the previous proof, the computation of $div\ W$ leads to

$$div\ W \geq -\left(1+\delta\right)\psi^{\delta}\lambda\left(u\right)F\left(v,r\right)\varphi\left(|\nabla u|\right)|\nabla\psi|$$
$$+\psi^{1+\delta}\lambda\left(u\right)|F_{v}|\left\{A^{-\frac{1}{\delta}}\varphi\left(|\nabla u|\right)^{1+\frac{1}{\delta}}-\beta\left(\eta-\gamma\right)\left(1+r\right)^{-1}\varphi\left(|\nabla u|\right)+\right.$$
$$\left.+\frac{K}{q}\left(1+r\right)^{-1-\delta}\right\}$$

which is the analogue of (2.21).

Using part ii) of the Claim, the choices

$$(2.30) \qquad i)\quad 0<\Lambda\leq\frac{1-\tau}{2A^{1/\delta}} \quad;\quad ii)\quad q=\frac{\tau^{\delta}K\left(1+\delta\right)^{1+\delta}}{A\delta^{\delta}\left[\beta\left(\eta-\gamma\right)\right]^{1+\delta}}$$

imply

$$A^{-\frac{1}{\delta}}\varphi\left(|\nabla u|\right)^{1+\frac{1}{\delta}}-\beta\left(\eta-\gamma\right)\left(1+r\right)^{-1}\varphi\left(|\nabla u|\right)+\frac{K}{q}\left(1+r\right)^{-1-\delta}\geq\Lambda\varphi\left(|\nabla u|\right)^{1+\frac{1}{\delta}}$$

whence

$$\Lambda\int_{B_{R}\cap\Omega_{\gamma}}\psi^{1+\delta}\lambda\left(u\right)|F_{v}|\varphi\left(|\nabla u|\right)^{1+\frac{1}{\delta}}$$
$$\leq\left(1+\delta\right)\int_{B_{R}\cap\Omega_{\gamma}}\psi^{\delta}\lambda\left(u\right)F\left(v,r\right)\varphi\left(|\nabla u|\right)|\nabla\psi|$$

which is the same relation contained in (2.24).

Again, having fixed $d>d_{0}$ and having chosen the sequence $\{R_{k}\}$ as above, we obtain

$$0<\liminf_{k\to+\infty}\left(1+R_{k}\right)^{-1-\delta-\beta\delta}\exp\left\{dR_{k}^{\beta}-q\eta\left(\vartheta R_{k}\right)^{\beta}\right\}$$

where q is given in (2.30), and this implies, letting $\vartheta\to1^{-}$,

$$d\geq\eta q=\frac{\tau^{\delta}K\left(1+\delta\right)^{1+\delta}\eta}{A\delta^{\delta}\left[\left(\eta-\gamma\right)\beta\right]^{1+\delta}}$$

or, equivalently (letting $\tau\to1^{-}$),

$$(2.31) \qquad K\leq\frac{Ad\delta^{\delta}\beta^{1+\delta}\left(\eta-\gamma\right)^{1+\delta}}{\left(1+\delta\right)^{1+\delta}\eta}\ .$$

The quantity K in non-decreasing with γ, while the right hand side of (2.31) is non-increasing, and so

$$K\leq\lim_{\gamma\to0^{-}}\frac{Ad\delta^{\delta}\beta^{1+\delta}\left(\eta-\gamma\right)^{1+\delta}}{\left(1+\delta\right)^{1+\delta}\eta}=\frac{Ad\delta^{\delta}\beta^{1+\delta}\eta^{\delta}}{\left(1+\delta\right)^{1+\delta}}\ .$$

It now suffices to let $\eta\to0^{+}$ to contradict (2.29). $\blacksquare$

Now, Theorem 1.1 of the Introduction is a special case of Theorems 2.1 and 2.2 since, in this case, $\varphi(t) = t$, and $A = \delta = 1$.

Theorems 2.1 and 2.2 deal with the case when $(M, \langle \cdot, \cdot \rangle)$ has volume growth of at most exponential type of order $1 + \delta - \delta\sigma - \mu > 0$, meaning with this precisely condition (2.3). We are now concerned with the case when

$$(2.32) \qquad \liminf_{r \to +\infty} \frac{\log vol(B_r)}{\log r} = d_0 < +\infty$$

that is, the manifold has at most polynomial growth (of order d_0).

Given $\mu \in \mathbb{R}$ and $\sigma \geq 0$, suppose $u \in \mathcal{C}^2(M)$ is such that

$$Tu \equiv |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$$

is a vector field of at least of class $\mathcal{C}^1$ and

$$(2.33) \qquad \hat{u} = \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^\sigma} < +\infty .$$

If $\delta\sigma + \mu < 1 + \delta$, (2.32) implies the validity of (2.3) with $d_0 = 0$. It therefore follows from (2.5) and (2.28) that

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div\,(Tu)(x) \leq 0$$

whenever $\Omega_\gamma \neq \emptyset$. The interesting cases left to analyse are therefore

$$\sigma \geq 0 \; ; \; \delta\sigma + \mu = 1 + \delta .$$

Theorem 2.3 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, and let $\sigma, \mu \in \mathbb{R}$ satisfy*

$$(2.34) \qquad \sigma \geq 0 \; ; \; \delta\sigma + \mu = 1 + \delta .$$

Assume (2.32). Then, given $u \in \mathcal{C}^2(M)$ for which $Tu \equiv |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is at least of class $\mathcal{C}^1$, and such that (2.33) holds, and given $\gamma \in \mathbb{R}$ with $\Omega_\gamma \neq \emptyset$, we have

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div\,(Tu)(x) \leq A \left(\frac{2}{1+\delta} \right)^{1+\delta} \sigma^\delta \delta^\delta [\max(\hat{u}; 0)]^\delta C(\sigma, \delta, d_0)$$

with

$$C(\sigma, \delta, d_0) = \begin{cases} \delta(\sigma - 1) + d_0 - 1 & \text{if } \delta(\sigma - 1) + d_0 - 1 \geq 0 \\ 0 & \text{if } \delta(\sigma - 1) + d_0 - 1 < 0 \end{cases}$$

Proof: In case $\sigma > 0$, we follow the reasoning of Theorem 2.1 with the same notations up to (2.12) included. Next, we choose

$$F(v, r) = F(v) = v^{-q}$$

with $q > 0$ to be specified later. Again we have

$$\left[\frac{F_r}{|F'(v)|} - 2b\sigma (1+r)^{\sigma-1} \right] |\nabla u|^{-1} \varphi(|\nabla u|) \langle \nabla r, \nabla u \rangle$$
$$\geq -2b\sigma (1+r)^{\sigma-1} \varphi(|\nabla u|)$$

while, using (2.12)

$$\frac{F(v)}{|F'(v)|} = \frac{v}{q} \geq \frac{b}{q} (1+r)^{\sigma} \ .$$

From (2.16) we thus deduce

$$div\, W \geq - (1+\delta)\psi^{\delta}\lambda(u) F(v,r) \varphi(|\nabla u|) |\nabla \psi|$$
$$+ \psi^{1+\delta}\lambda(u) |F'(v)| \left\{ A^{-1/\delta}\varphi(|\nabla u|)^{1+\frac{1}{\delta}} - 2b\sigma (1+r)^{\sigma-1} \varphi(|\nabla u|) + \right.$$
$$\left. + \frac{Kb}{q} (1+r)^{\sigma-\mu} \right\} \ .$$

We use again the Claim and (2.34) to deduce that, for $0 < \tau < 1$, $0 < \Lambda \leq \dfrac{1-\tau}{2A^{1/\delta}}$ and

$$(2.35) \qquad\qquad q = \frac{\tau^{\delta} Kb (1+\delta)^{1+\delta}}{A\delta^{\delta} (2b\sigma)^{1+\delta}} \ ,$$

on the set Ω_{γ} the inequality

$$A^{-\frac{1}{\delta}}\varphi(|\nabla u|)^{1+\frac{1}{\delta}} - 2b\sigma (1+r)^{\sigma-1} \varphi(|\nabla u|) + \frac{Kb}{q} (1+r)^{\sigma-\mu} \geq \Lambda \varphi(|\nabla u|)^{1+\frac{1}{\delta}}$$

is true, whence

$$div\, W \geq - (1+\delta)\psi^{\delta}\lambda(u) F(v,r) \varphi(|\nabla u|) |\nabla \psi|$$
$$+ \Lambda\psi^{1+\delta}\lambda(u) |F'(v)| \varphi(|\nabla u|)^{1+\frac{1}{\delta}} \ .$$

Having fixed $d > d_0$, according to (2.32) there exists a sequence $\{R_k\} \uparrow +\infty$ with $\vartheta R_{k+1} > R_k$ such that

$$vol\, (B_{R_k}) \leq cR_k^d$$

for some $c > 0$. Proceeding as in Theorem 2.1 we arrive to

$$0 < \liminf_{k\to+\infty} b^{-q} (1+\vartheta R_k)^{-q\sigma} (1+R_k)^{\delta\sigma} R_k^{d-(1+\delta)}$$

and this forces

$$(2.36) \qquad\qquad \delta(\sigma - 1) + d - 1 \geq q\sigma \ .$$

Hence we get a contradiction, and thus $K \leq 0$, if $\delta(\sigma - 1) + d_0 - 1 < 0$. In case $\delta(\sigma - 1) + d_0 - 1 \geq 0$, (2.36) and (2.35) lead to

$$A\delta^\delta (2b\sigma)^{1+\delta} [\delta(\sigma - 1) + d - 1] \geq \tau^\delta K b (1 + \delta)^{1+\delta} \sigma$$

and when $\tau \uparrow 1^-$, $b \downarrow [\max(\hat{u}; 0)]$ we finish the proof of the case $\sigma > 0$.

When $\sigma = 0$ we follow the notations and the steps in the proof of Theorem 2.2, working with the vector field $W = \psi^{1+\delta} \lambda(u) F(v, r) Tu$ obtained with the choice

$$F(v, r) = (1 + r)^{-q} v^{-q}, \quad q > 0.$$

The conclusion

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div \, (Tu)(x) \leq 0$$

is then a consequence of this choice and of the reasoning cited above.   ■

Again, Theorem 1.2 is a special case of Theorem 2.3, with $\varphi(t) = t$, and $A = \delta = 1$.

The argument in the proofs of Theorems 2.1, 2.2, 2.3 is an improvement of original ideas in [14] and [15].

Corollary 2.4 *Let* $(M, \langle \cdot, \cdot \rangle)$ *be a complete manifold with*

$$(2.37) \qquad \liminf_{r \to +\infty} \frac{\log \, vol \, (B_r)}{r^{1+\delta-\delta\sigma-\mu}} < +\infty$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ *and*

$$(2.38) \qquad \delta\sigma + \mu < 1 + \delta.$$

Suppose that, for $b(x) \in \mathcal{C}^0(M)$,

$$(2.39) \qquad i) \ b(x) > 0 \quad on \ M; \qquad ii) \ b(x) \geq \frac{B}{r(x)^\mu} \quad on \ M \setminus B_{R_0}$$

for some $R_0, B > 0$. *Let* $f \in \mathcal{C}^0(\mathbb{R})$ *with*

$$(2.40) \qquad \liminf_{t \to +\infty} f(t) > 0.$$

Let $u \in \mathcal{C}^2(M)$ *be such that* $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ *is a vector field at least of class* $\mathcal{C}^1$, *and* u *be a solution of*

$$(2.41) \qquad div \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq b(x) f(u) \quad on \ M$$

satisfying

$$(2.42) \qquad u(x) = o(r(x)^\sigma) \quad as \ r(x) \to +\infty.$$

Then $u^* = \sup_M u < +\infty$ *and* $f(u^*) \leq 0$.

Proof: Without loss of generality we can suppose, using (2.39), that $b(x)$ satisfies

$$(2.43) \qquad\qquad b(x) \geq \frac{\tilde{B}}{[1 + r(x)]^\mu} \quad \text{on } M$$

for some appropriate constant $\tilde{B} > 0$. First we show that $u^* < +\infty$. Because of (2.40) there exist $T > 0$, $c > 0$ such that

$$(2.44) \qquad\qquad f(t) \geq c, \quad t \geq T.$$

If $u^* = +\infty$,

$$\Omega_T = \{x \in M : u(x) > T\} \neq \emptyset$$

and, using (2.41), (2.43) and (2.44)

$$(2.45) \qquad div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \frac{c\tilde{B}}{[1 + r(x)]^\mu} \quad \text{on } \Omega_T.$$

However, conditions (2.37), (2.38), (2.42) allow to apply Theorem 2.1 with $\hat{u} = 0$. Hence, by (2.5),

$$\inf [1 + r]^\mu div(Tu) \leq 0,$$

contradicting (2.45).

Having shown that $u^* < +\infty$, we can apply Theorem 2.2 to deduce the existence of a sequence $\{x_k\} \subset M$ with the following properties

$$(2.46) \qquad u(x_k) > u^* - \frac{1}{k} \; ; \quad [1 + r(x_k)]^\mu div(Tu)(x_k) < \frac{1}{k}.$$

Thus

$$\frac{\tilde{B}}{[1 + r(x_k)]^\mu} f(u(x_k)) \leq b(x_k) f(u(x_k))$$

$$\leq div(Tu)(x_k) < \frac{1}{k}\frac{1}{[1 + r(x_k)]^\mu},$$

that is

$$\tilde{B}f(u(x_k)) < \frac{1}{k}.$$

Continuity of f and (2.46) then imply $f(u^*) \leq 0$. ∎

Corollary 1.3 of the Introduction is a special case of Corollary 2.4.

3. Sharpness of the results and further comments

We recall that Theorems 2.1 and 2.2 state that, under suitable conditions, one has

$$\inf_{\Omega_\gamma} \left[1 + r\left(x\right) \right]^\mu div \left(Tu\right)\left(x\right) \le Ad_0 \left[\max\left(\hat{u};0\right)\right]^\delta C\left(\sigma,\mu,\delta\right)$$

where $C \equiv C\left(\sigma,\mu,\delta\right)$ is equal to

(3.1) $\quad C = \begin{cases} \left(\dfrac{2}{1+\delta}\right)^{1+\delta} \delta^\delta \left(1+\delta-\delta\sigma-\mu\right)\sigma^\delta & \text{if } \sigma > 0,\ \left(1+\delta\right)\left(1-\sigma\right) \le \mu \\[2ex] \left(\dfrac{2}{1+\delta}\right)^{1+\delta} \delta^\delta \left(1+\delta-\delta\sigma-\mu\right)^{1+\delta} & \text{if } \sigma > 0,\ \left(1+\delta\right)\left(1-\sigma\right) > \mu \\[2ex] 0 & \text{if } \sigma = 0 \end{cases}$

We note that if $\sigma > 0$ and $\left(1+\delta\right)\left(1-\sigma\right) > \mu$ the value of $C\left(\sigma,\mu,\delta\right)$ in (3.1)*ii)* is larger than that in (3.1)*i)*. These different values amount to different choices of the test functions $F\left(v,r\right)$ in the proof of Theorem 2.1. The following example will show that, for the Laplace-Beltrami operator, the correct value of the constant C should be, for $\sigma > 0$,

$$C\left(\sigma,\mu,1\right) = \left(2-\sigma-\mu\right)\sigma$$

that is (3.1)*i)*. In case $\sigma = 0$, the constant is clearly sharp.

We construct a model in the sense of Greene and Wu $[GW]$. Let $M = [0,+\infty) \times S^1$. In polar coordinates $\left(r,\vartheta\right)$, we define on $\left(0,+\infty\right) \times S^1$

(3.2) $$\langle \cdot,\cdot \rangle = dr^2 + g\left(r\right)^2 d\vartheta^2$$

where $d\vartheta^2$ is the canonical metric on S^1 and $g\left(r\right) \in \mathcal{C}^\infty\left([0,+\infty)\right)$, is positive on $\left(0,+\infty\right)$ and it satisfies

$$g\left(r\right) = \begin{cases} r & \text{on } [0,1] \\ r^{1-\sigma-\mu} \exp\left(d_0 r^{2-\sigma-\mu}\right) & \text{on } [2,+\infty) \end{cases}$$

for some parameters $\sigma \ge 0$, $\mu \in \mathbb{R}$, $d_0 > 0$ such that

(3.3) $$2 - \sigma - \mu > 0 \,.$$

We note that, because of the behaviour of g near the origin, the metric $\langle \cdot,\cdot \rangle$ in (3.2) extends to a smooth, obviously complete, metric on all of $M = \mathbb{R}^2$ that we shall again denote with $\langle \cdot,\cdot \rangle$. Computations, using (3.3), show

$$\lim_{r \to +\infty} \frac{\log\ vol\left(B_r\right)}{r^{2-\sigma-\mu}} = d_0 \,.$$

For $\rho \in \mathbb{R}$ we define

$$u(x) = \int_0^{r(x)} g(t)^{-1} \left\{ \int_0^t (1+s)^{-\rho} g(s)\, ds \right\} dt \ .$$

Then u is a $\mathcal{C}^2(M)$, non-negative, radial function such that

(3.4) $\Delta u(x) = [1 + r(x)]^{-\rho}$ on M.

Computations, again using (3.3), show that, if $\mu + \sigma - \rho \neq 0$,

$$\frac{u(r)}{r^\sigma} \sim \frac{r^{\mu - \rho}}{\sigma d_0 (2 - \sigma - \mu)}   \text{ as } r \to +\infty \ .$$

Therefore, if $\rho = \mu$, using the notations of Theorem 2.1, we have

$$\hat{u} = \frac{1}{\sigma d_0 (2 - \sigma - \mu)}$$

and

$$[1 + r(x)]^\mu \, \Delta u(x) \equiv 1 \ .$$

This shows that the bound in (1.16) of Theorem 1.1 is sharp if $2(1 - \sigma) \leq \mu$.

If $\mu - \rho > 0$

$$\limsup_{r \to +\infty} \frac{u(x)}{r(x)^\sigma} = +\infty$$

while (2.3) is still valid. However, according to (3.4), having fixed $N > 0$, choosing γ sufficiently large yields $\Omega_\gamma \neq \emptyset$ and

$$\inf_{\Omega_\gamma} [1 + r]^\mu \, \Delta u \geq N \ .$$

This shows that the exponent σ in (2.4) cannot be taken smaller. The same reasoning shows that the exponent in the volume growth (2.3) cannot be taken smaller.

We recall that Theorem 2.3 states that

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu \, div \, (Tu)(x) \leq A \, [\max(\hat{u}; 0)]^\delta \, \sigma^\delta C(\sigma, \delta, d_0)$$

where $C = C(\sigma, \delta, d_0)$ is equal to

(3.5) $C = \begin{cases} \left(\dfrac{2}{1 + \delta} \right)^{1+\delta} \delta^\delta [\delta(\sigma - 1) + d_0 - 1] & \text{if } \delta(\sigma - 1) + d_0 - 1 \geq 0 \\[2em] 0 & \text{if } \delta(\sigma - 1) + d_0 - 1 < 0 \end{cases}$

An analysis similar to the above can be developed in this case too. Let us only point out that, for the Laplacian, estimate $(3.5)i)$ is sharp. Indeed, in $\mathbb{R}^2$ with its canonical metric, the function $u(x) = r(x)^2$ satisfies $\Delta u(x) \equiv 4$ (here $A = \delta = 1$, $\sigma = 2$, $\mu = 0$).

Similarly, $u(x) = r(x)^{2+\varepsilon}$ for some $\varepsilon > 0$ shows that the exponent $\sigma = 2$ in (2.33) is best possible.

Next we show that assumption (2.38) of Corollary 2.4 can not be weakened. For instance, we consider $\mathbb{H}^m$, the hyperbolic m-dimensional space with its canonical metric of constant sectional curvature -1, that we realize as the model $[0, +\infty) \times S^{m-1}$ with metric, in polar coordinates (r, ϑ), $\langle \cdot, \cdot \rangle = dr^2 + (\sinh r)^2 d\vartheta^2$. Note that

$$\lim_{r \to +\infty} \frac{\log \, vol(B_r)}{r} = m - 1 \ .$$

For $a \in (0, m-1]$ let $u \in \mathcal{C}^2(\mathbb{H}^m)$ be the radial function defined by

$$u(x) = \int_0^{r(x)} \frac{a(\sinh t)^{1-m} \int_0^t (\sinh s)^{m-1} \, ds}{\left\{ 1 - (\sinh t)^{2(1-m)} \left[a \int_0^t (\sinh s)^{m-1} \, ds \right]^2 \right\}^{1/2}} \, dt \ .$$

A simple verification shows that u satisfies

$$(3.6) \qquad div\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = a \quad on \ \mathbb{H}^m \ .$$

Furthermore

$$\frac{u(x)}{r(x)} \to 1 \quad as \ r \to +\infty.$$

In this case $\varphi(t) = t(1+t^2)^{-1/2} \leq t$ on $[0, +\infty)$, so that in the structural conditions $(2.1)\,iii)$ we can take $A = \delta = 1$. Choosing $b(x) \equiv a$ on M, $f(t) \equiv 1$, we see that all the assumptions of Corollary 2.4 are satisfied, but (2.42) barely fails to be met.

Substituting $\mathbb{H}^m$ with a manifold $(M, \langle \cdot, \cdot \rangle)$ we can geometrically interpret equation (3.6) as the condition for the graph of

$$\Gamma_u : M \to M \times \mathbb{R}$$

given by $\Gamma_u(x) = (x, u(x))$ to have constant mean curvature a/m. It is an old problem to determine the class of non-admissible functions $b(x)$ for the prescribed mean curvature equation

$$div\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)(x) = b(x) \quad on \ M \ .$$

The above example shows that some restriction on the growth of u need to be imposed.

Applying Corollary 2.4 we have

Proposition 3.1 *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold with*

$$\liminf_{r \to +\infty} \frac{\log vol\,(B_r)}{r^{2-\sigma-\mu}} < +\infty$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\sigma + \mu < 2$. Suppose that, for $b\,(x) \in \mathcal{C}^0\,(M)$,

$$i)\ b\,(x) > 0 \quad on\ M\ ; \qquad ii)\ b\,(x) \geq \frac{B}{r\,(x)^{\mu}} \quad on\ M \setminus B_{R_0}$$

for some $R_0, B > 0$. Then, for any $\varepsilon > 0$ there are no graphs of mean curvature $b(x)/m$ lying between the graphs of

$$-\Gamma_{\sigma-\varepsilon} : x \mapsto \left(x, -\left(r\,(x)\right)^{\sigma-\varepsilon}\right) \quad and \quad \Gamma_{\sigma-\varepsilon} : x \mapsto \left(x, \left(r\,(x)\right)^{\sigma-\varepsilon}\right)\ .$$

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Marco Rigoli
Dipartimento di Matematica
Università di Milano
Via Saldini 50, 20133-Milano, Italy
`marco.rigoli@mat.unimi.it`

Maura Salvatori
Dipartimento di Matematica
Università di Milano
Via Saldini 50, 20133-Milano, Italy
`maura.salvatori@mat.unimi.it`

Marco Vignati
Dipartimento di Matematica
Università di Milano
Via Saldini 50, 20133-Milano, Italy
`marco.vignati@mat.unimi.it`