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Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients

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Abstract

This paper aims at giving an overview of estimates in general Besov spaces for the Cauchy problem on $t = 0$ related to the vector field $\partial_t + v \cdot \nabla$. The emphasis is on the conservation or loss of regularity for the initial data.

When ∇v belongs to $L^1(0, T; L^\infty)$ (plus some convenient conditions depending on the functional space considered for the data), the initial regularity is preserved. On the other hand, if ∇v is slightly less regular (e.g. ∇v belongs to some limit space for which the embedding in L^∞ fails), the regularity may coarsen with time. Different scenarios are possible going from linear to arbitrarily small loss of regularity. This latter result will be used in a forthcoming paper to prove global well-posedness for two-dimensional incompressible density-dependent viscous fluids (see [11]).

Besides, our techniques enable us to get estimates uniformly in $\nu \geq 0$ when adding a diffusion term $-\nu\Delta u$ to the transport equation.

Introduction

This paper is concerned with estimates in Besov spaces for transport-diffusion equations:

$$\begin{cases} \partial_t f + v \cdot \nabla f - \nu \Delta f = g, \\ f_{t=0} = f_0, \end{cases} \qquad (\mathcal{T}_\nu)$$

where $\nu \geq 0$ stands for a constant diffusion parameter.

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Such equations appear as the result of linearization in a number of PDE's coming from fluid mechanics and have been extensively studied. In the case $\nu=0$, existence and uniqueness theory in L^∞ has been studied under very weak assumptions on v (roughly $v\in W^{1,1}$ (or even BV) with $\operatorname{div} v\in L^\infty$, see e.g [13], [7], [8] and the references therein).

As our work is motivated by the study of nonlinear models, we aim at estimating the fractional derivatives of the solutions to $(\mathcal{T}_\nu)$. It is well known that such estimates are available when v has enough regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as ∇v belongs to $L^1(0,T;L^\infty)$ (plus some convenient assumption depending on the number of derivatives to be transported). When $\nu=0$, this qualitative result has been proved in a number of functional frameworks:

- Sobolev spaces H^s with $0\leq s<\frac{N}{2}$ provided that $\nabla v\in L^1(0,T;H^{\frac{N}{2}}\cap L^\infty)$,
- Hölder spaces C^r if $|r|\leq 1$, $\nabla v\in L^1(0,T;L^\infty)$ and $\operatorname{div} v=0$ (see e.g. [3]),
- General Besov spaces $B^s_{p,r}$ with $1\leq p,r\leq+\infty$ and $-\frac{N}{p}<s<\frac{N}{p}+1$ (or $|s|<\frac{N}{p}+1$ if $\operatorname{div} v=0$) if $\nabla v\in L^1(0,T;B^{\frac{N}{p}}_{p,r}\cap L^\infty)$ (see e.g [10]).

In section 2 of the present paper, we state estimates in $B^\sigma_{p,r}$ for $(\mathcal{T}_\nu)$ whereas ∇v belongs to some different Besov space $B^{\sigma'}_{p_2,r_2}$ and is bounded. The main novelty is that p_2 may differ from p and that the estimates do not depend on ν. The reader is referred to proposition 2.1 for more details.

On the other hand, when ∇v fails to be in $L^1(0,T;L^\infty)$, the initial regularity is unlikely to be preserved. Nevertheless, if v is "almost" lipschitz with respect to the space variables then the solution f may be estimated in spaces whose regularity index coarsens with time. This fact has been observed several times by different authors.

In the case of Besov spaces with third index $r=+\infty$, H. Bahouri and J.-Y. Chemin proved a linear loss of regularity under the hypothesis that v is log-lipschitz:

$$\|v\|_{LL}\stackrel{\text{def}}{=}\sup_{0<|x-y|<e^{-1}}\frac{|v(y)-v(x)|}{|y-x|(1-\log|y-x|)}<+\infty.$$

Assuming that $|\sigma|<1$, they prove the following inequality in [1]:

$$\|f(t)\|_{B^{\sigma_t}_{p,\infty}}\leq 2\Big(\|f_0\|_{B^{\sigma}_{p,r}}+\int_0^t\|g(\tau)\|_{B^{\sigma_\tau}_{p,\infty}}\,d\tau\Big)$$

whenever $\sigma_t\stackrel{\text{def}}{=}\sigma-C\displaystyle\int_0^t\|v(\tau)\|_{LL}\,d\tau>-1.$

This result has been improved in [6]. Moreover the regularity assumption on the right-hand side g may be somewhat weakened (see [9]).

Let us mention in passing that results in the same spirit have been used by M. Vishik for solving incompressible Euler equations in limit Besov spaces (see [19]), and by F. Planchon to improve the Beale-Kato-Majda blow-up criterion ([17]).

Section 3.2 of the present paper is devoted to the proof of similar estimates for $(\mathcal{T}_\nu)$ uniformly in ν in general Besov spaces (see theorems 3.2, 3.4, 3.9 and 3.10 below.)

It has also been observed that if v is better than log-lipschitz, e.g

$$\sup_{0<|x-y|<e^{-1}} \frac{|v(y)-v(x)|}{|y-x|(1-(\log|y-x|)^\alpha)} < +\infty. \tag{0.1}$$

for some $\alpha \in (0,1)$, then the loss of regularity is arbitrarily small.

In [5], J.-Y. Chemin and N. Lerner noticed that the flow associated to a vector-field v whose gradient belongs to a space slightly larger than $L^1(0,T;H^{\frac{N}{2}})$ remains in $C^{1-\epsilon}$ on $[0,T]$ for ϵ arbitrarily small despite the fact that $H^{\frac{N}{2}}$ is not embedded in L^∞. The proof lies on the fact that v satisfies an inequality of type (0.1).

In [15], B. Desjardins stated an even more accurate result in the framework of the two-dimensional torus $\mathbb{T}^2$. Here the flow to a vector-field $v \in L^2(0,T;H^2)$ is shown to remain in $W^{2,p}$ for all $p<2$. In the case of a bounded N-dimensional domain, if the symmetric part of ∇v belongs to $L^p(0,T;W^{\frac{N}{p},p})$ with $1<p<+\infty$, and $f_0 \in W^{1,r}$ then $f \in C([0,T];W^{1,q})$ for all $q<r$ (see [14]).

Theorem 3.12 of the present paper implies the following result:

Theorem 0.1 *Assume that* $\nabla v \in L^1(0,T;B^{\frac{N}{p_2}}_{p_2,r_2})$ *for some* $1 \le p_2 \le +\infty$ *and* $r_2 \in (1,+\infty)$*. Let* $1 \le p,r \le +\infty$*. Let* p' *be the conjugate exponent of* p*, and* σ *be such that*

$$\sigma > -N\min\left(\frac{1}{p_2},\frac{1}{p'}\right) \quad \left(\sigma > -1-N\min\left(\frac{1}{p_2},\frac{1}{p'}\right) \quad \textit{if} \quad \operatorname{div} v = 0\right)$$

$$\textit{and} \quad \sigma < 1+\frac{N}{p_2}.$$

Let $\epsilon>0$ *and let* f *solve* $(\mathcal{T}_\nu)$*. There exists* $C=C(N,\epsilon,p,p_2,r_2,\sigma)$ *such that the following inequality holds true on* $[0,T]$ *uniformly in* ν*:*

$$\|f(t)\|_{B^{\sigma-\epsilon}_{p,r}} \le C\left(\|f_0\|_{B^\sigma_{p,r}} + \int_0^t \|g(\tau)\|_{B^\sigma_{p,r}}d\tau\right)\exp\left\{C\left(\int_0^t \|\nabla v(\tau)\|_{B^{\frac{N}{p_2}}_{p_2,r_2}}d\tau\right)^{r_2}\right\}.$$

We intend to use the above result to prove global well-posedness for the inhomogeneous incompressible Navier-Stokes equations in $\mathbb{T}^2$ or $\mathbb{R}^2$ (see [11]).

Remark 0.2 *As our results strongly rely on the use of Fourier analysis (namely the Littlewood-Paley decomposition), we restricted ourselves to the case of* $\mathbb{T}^N$ *or* $\mathbb{R}^N$ *with* $N \geq 1$. *In the case* $\nu = 0$ *however, we expect our results to be true in any smooth domain* Ω *if* v *is tangent to the boundary. As a matter of fact, in the case of a Lipschitz vector-field* v*, estimates in certain Besov spaces have been proved in [16].*

Remark 0.3 *In order to simplify the statements, we did not track systematically the gain of derivatives induced by the diffusion operator* $-\nu\Delta$. *This work has been done in proposition 2.1 and leads to uniform estimates for* νf *in a space very close to* $L^1(0,t;B^{\sigma+2}_{p,r})$ *whereas* f_0 *is in* $B^\sigma_{p,r}$. *A careful reading of our proofs should enable us to get a similar gain of two derivatives in the theorems of section 3.*

Our paper is structured as follows. The first section is devoted to a basic presentation of Littlewood-Paley decomposition and Besov spaces. In section 2, we state general estimates in Besov spaces in the case when v is Lipschitz. Section 3 is devoted to the study of the case when v *is not* Lipschitz. We focus on the study of "abstract" coupled inequalities from which general uniform (with respect to ν) estimates for $(\mathcal{T}_\nu)$ may be inferred. We then give several examples with either linear loss of derivatives or arbitrarily small loss of derivatives depending on the assumption made on ∇v. An appendix is devoted to the proof of technical estimates pertaining to a commutator.

Notation: In order to have more concise statements, we shall adopt the convention that $\sigma > [-1] - \alpha_1$ means that $\sigma > -\alpha_1$ has to be satisfied for a general vector-field v, and that this condition may be weakened into $\sigma > -\alpha_1 - 1$ if $\operatorname{div} v = 0$.

1. Besov spaces and Littlewood-Paley decomposition

The proof of the results presented in the paper is based on a dyadic decomposition in Fourier variables, the so-called *(inhomogeneous) Littlewood-Paley decomposition*. For the sake of conciseness, we only treat the case of $\mathbb{R}^N$. The reader is referred to [10] for a similar construction in $\mathbb{T}^N$.

Let (χ, φ) be a couple of C^∞ functions with

$$\operatorname{Supp}\chi \subset \left\{|\xi| \leq \frac{4}{3}\right\}, \quad \operatorname{Supp}\varphi \subset \left\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\},$$
$$\forall \xi \in \mathbb{R}^N,\ \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1.$$

Denoting $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$ and $\check{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks as

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if} \quad q \leq -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^N} \check{h}(y)u(x-y)\,dy,$$
$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y)\,dy \quad \text{if} \quad q \geq 0.$$

We shall also use the following low-frequency cut-off:

$$S_q u \stackrel{\text{def}}{=} \sum_{k \leq q-1} \Delta_k u = \chi(2^{-q}D)u.$$

One can easily prove that

$$\forall u \in \mathcal{S}'(\mathbb{R}^N), \quad u = \sum_{q \in \mathbb{Z}} \Delta_q u\,. \tag{1.1}$$

Littlewood-Paley decomposition has nice properties of quasi-orthogonality:

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k-q| \geq 2 \ \text{ and } \ \Delta_k(S_{q-1}u\Delta_q u) \equiv 0 \quad \text{if} \quad |k-q| \geq 5. \tag{1.2}$$

Let us now define the (non-homogeneous) Besov spaces:

Definition 1.1 *For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \left(\sum_{q \geq -1} 2^{rsq} \|\Delta_q u\|^r_{L^p} \right)^{\frac{1}{r}} \textit{ if } r < +\infty \textit{ and } \|u\|_{B^s_{p,\infty}} \stackrel{\text{def}}{=} \sup_{q \geq -1} 2^{sq} \|\Delta_q u\|_{L^p}.$$

We then define the Besov space $B^s_{p,r}$ as the set of temperate distributions with finite $\|\cdot\|_{B^s_{p,r}}$ norm.

The definition of $B^s_{p,r}$ does not depend on the choice of the couple (χ,φ). One can further remark that H^s coincide with $B^s_{2,2}$, and that $C^r = B^r_{\infty,\infty}$ if $r \in \mathbb{R}^+ \backslash \mathbb{N}$.

The reader is referred to [18] for a complete study of Besov spaces. Let us just recall some of their most basic properties.

Proposition 1.2 *The following properties hold:*

i) *Derivatives: we have $\|\nabla u\|_{B^{s-1}_{p,r}} \lesssim \|u\|_{B^s_{p,r}}$.*

ii) *Sobolev embeddings:*

- *If $p_1 \leq p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

- *If* $s_1 > s_2$ *and* $1 \le p, r_1, r_2 \le +\infty$*, then* $B^{s_1}_{p,r_1} \hookrightarrow B^{s_2}_{p,r_2}$.
- *If* $1 \le p \le +\infty$ *then* $B^{\frac{N}{p}}_{p,1} \hookrightarrow L^\infty$.

iii) *Algebraic properties: for* $s > 0$*,* $B^s_{p,r} \cap L^\infty$ *is an algebra.*

iv) *Real interpolation:* $(B^{s_1}_{p,r}, B^{s_2}_{p,r})_{\theta,r'} = B^{\theta s_2 + (1-\theta)s_1}_{p,r'}$.

We aim at proving estimates for $(\mathcal{T}_\nu)$ in spaces $L^\rho(0,T;B^\sigma_{p,r})$. Taking into account the definition of Besov spaces, it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces which are not of type $L^\rho(0,T;B^s_{p,r})$. That remark naturally leads to the following definition (introduced in [4]):

Definition 1.3 *Let* $s \in \mathbb{R}$*,* $1 \le p, r, \rho \le +\infty$ *and* $T \in [0, +\infty]$*. We set*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \stackrel{\text{def}}{=} \left(\left(\int_0^T \|\Delta_{-1}u(t)\|^\rho_{L^p}\,dt\right)^{\frac{r}{\rho}} + \sum_{q\in\mathbb{N}} 2^{rqs}\left(\int_0^T \|\Delta_q u(t)\|^\rho_{L^p}\,dt\right)^{\frac{r}{\rho}}\right)^{\frac{1}{r}}$$

and denote by $\widetilde{L}^\rho_T(B^s_{p,r})$ *the set of distributions of* $\mathcal{S}'(0,T\times\mathbb{R}^N)$ *with finite* $\|\cdot\|_{\widetilde{L}^\rho_T(B^s_{p,r})}$ *norm.*

Let us remark that by virtue of Minkowski inequality, we have

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \le \|u\|_{L^\rho_T(B^s_{p,r})} \quad \text{if } \rho \le r$$

and

$$\|u\|_{L^\rho_T(B^s_{p,r})} \le \|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \quad \text{if } \rho \ge r.$$

2. A priori estimates with no loss of regularity

We first concentrate on the case when ∇v belongs to $L^1(0,T;L^\infty)$. In this case, the initial Besov regularity is conserved:

Proposition 2.1 *Let* $1 \le p, p_2, r \le +\infty$ *and* $p' \stackrel{\text{def}}{=} (1-1/p)^{-1}$*. Assume that*

$$\sigma > [-1] - N\min\left(\frac{1}{p_2}, \frac{1}{p'}\right).$$

Denote $f^{HF} \stackrel{\text{def}}{=} f - \Delta_{-1}f$*. There exists a constant* C *depending on* N*,* p*,* p_2 *and* σ *but not on* ν *and a universal constant* $\kappa > 0$ *such that the following* *estimates hold:*

$$\|f\|_{\widetilde{L}_t^\infty(B^\sigma_{p,r})} + \kappa\nu\left(\frac{p-1}{p^2}\right)\|f^{HF}\|_{\widetilde{L}_t^1(B^{\sigma+2}_{p,r})} \leq \left(\|f_0\|_{B^\sigma_{p,r}} + \int_0^t e^{-CZ(\tau)}\|g(\tau)\|_{B^\sigma_{p,r}}\,d\tau\right)e^{CZ(t)}, \tag{2.1}$$

$$\|f\|_{\widetilde{L}_t^\infty(B^\sigma_{p,r})} + \kappa\nu\left(\frac{p-1}{p^2}\right)\|f^{HF}\|_{\widetilde{L}_t^1(B^{\sigma+2}_{p,r})} \leq \left(\|f_0\|_{B^\sigma_{p,r}} + \|g\|_{\widetilde{L}_t^1(B^\sigma_{p,r})}\right)e^{CZ(t)}, \tag{2.2}$$

with

$$\begin{cases} Z(t) = \displaystyle\int_0^t \|\nabla v(\tau)\|_{B^{\frac{N}{p_2}}_{p_2,\infty}\cap L^\infty}\,d\tau \quad \text{if} \quad \sigma < 1 + \frac{N}{p_2}, \\ Z(t) = \displaystyle\int_0^t \|\nabla v(\tau)\|_{B^{\sigma-1}_{p_2,r}}\,d\tau \quad \text{if } \sigma > 1 + \frac{N}{p_2} \text{ or } \left\{\sigma = 1 + \frac{N}{p_2} \text{ and } r = 1\right\}. \end{cases}$$

If $f = v$ then for all $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$) estimates (2.1) and (2.2) hold with

$$Z(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}\,d\tau.$$

Proof: Applying the operator Δ_q to $(\mathcal{T}_\nu)$ yields

$$\begin{cases} (\partial_t + S_{q+1}v\cdot\nabla)\Delta_q f - \nu\Delta\Delta_q f = \Delta_q F + R_q, \\ \Delta_q f_{|t=0} = \Delta_q f_0, \end{cases} \tag{$\mathcal{T}_q$}$$

with $R_q \stackrel{\text{def}}{=} S_{q+1}v\cdot\nabla\Delta_q f - \Delta_q(v\cdot\nabla f)$.

As $S_{q+1}v$ is smooth, one readily gets in the case $\nu = 0$:

$$\|\Delta_q f(t)\|_{L^p} \leq \|\Delta_q f_0\|_{L^p} + \int_0^t\left(\|R_q(\tau)\|_{L^p} + \frac{1}{p}\|\operatorname{div} S_{q+1}v(\tau)\|_{L^\infty}\|\Delta_q f(\tau)\|_{L^p}\right)d\tau$$
$$+ \int_0^t \|\Delta_q F(\tau)\|_{L^p}\,d\tau\,, \tag{2.3}$$

where it is understood that $1/p = 0$ if $p = +\infty$.

Let us now focus on the case $\nu > 0$ and $q \geq 0$.

1. Case $1 < p < +\infty$.

   After multiplying $(\mathcal{T}_q)$ by $|\Delta_q f|^{p-1}\operatorname{sgn}(\Delta_q f)$, the new term we have to deal with is
$$-\nu\int \Delta\Delta_q f\,|\Delta_q f|^{p-1}\operatorname{sgn}(\Delta_q f)\,dx.$$

As $q \geq 0$ (hence $\mathcal{F}\Delta_q f$ is supported in $C(0, \frac{3}{4}2^q, \frac{8}{3}2^q)$), lemma A.5 in [12] ensures that, for some universal constant $\kappa > 0$,

$$-\int \Delta\Delta_q f |\, \Delta_q f|^{p-1} \operatorname{sgn}(\Delta_q f)\, dx \geq \kappa\left(\frac{p-1}{p^2}\right) 2^{2q} \int |\Delta_q f|^p\, dx, \tag{2.4}$$

hence

$$\begin{aligned} \|\Delta_q f(t)\|_{L^p} + \kappa\nu\left(\frac{p-1}{p^2}\right)2^{2q}\int_0^t \|\Delta_q f(\tau)\|_{L^p}\, d\tau \leq \\ \leq \|\Delta_q f_0\|_{L^p} + \int_0^t \Big(\|R_q(\tau)\|_{L^p} + \frac{1}{p}\, \|\operatorname{div} S_{q+1} v(\tau)\|_{L^\infty} \|\Delta_q f(\tau)\|_{L^p}\Big)\, d\tau \\ + \int_0^t \|\Delta_q F(\tau)\|_{L^p}\, d\tau. \end{aligned} \tag{2.5}$$

2. Case $p = 1$.

   The same estimate holds true. Indeed, introduce the function

$$T_\epsilon(x) \stackrel{\text{def}}{=} x/\sqrt{\epsilon^2 + x^2}.$$

   From a straightforward integration by parts, the following inequality is inferred:

$$-\int \Delta\Delta_q u\, T'_\epsilon(\Delta_q u)\, dx = \int \frac{\epsilon^2 |\nabla \Delta_q u|^2}{\epsilon^2 + |\Delta_q u|^2}\, dx \geq 0.$$

   Hence, multiplying equation $(\mathcal{T}_q)$ by $T_\epsilon(\Delta_q f)$, integrating in space and in time and using Lebesgue dominated convergence theorem to pass to the limit $\epsilon \to 0$ eventually yields inequality (2.5).

3. Case $p = +\infty$. It stems from the maximum principle.

That the term induced by $-\nu\Delta\Delta_q f$ is also non-negative in the case $q = -1$ is left to the reader. This entails that

$$\begin{aligned} \|\Delta_{-1} f(t)\|_{L^p} \leq \|\Delta_{-1} f_0\|_{L^p} + \int_0^t \Big(\|R_{-1}(\tau)\|_{L^p} + \frac{1}{p}\, \|\operatorname{div} S_0 v(\tau)\|_{L^\infty} \|\Delta_{-1} f(\tau)\|_{L^p}\Big)\, d\tau \\ + \int_0^t \|\Delta_{-1} F(\tau)\|_{L^p}\, d\tau. \end{aligned} \tag{2.6}$$

Let us admit the following lemma, the proof of which is postponed in appendix:

Lemma 2.2 *Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq p_2 \leq +\infty$. Let $p_1 = (1/p - 1/p_2)^{-1}$. There exists a constant $K = K(N)$ such that*

(2.7)
$$
\begin{aligned}
2^{q\sigma}\|R_q\|_{L^p} \leq 4^{|\sigma|}K\Bigg(&\sum_{|q'-q|\leq 4}\Big(\|\Delta_{-1}\nabla v\|_{L^\infty} + \|S_{q'-1}\nabla v\|_{L^\infty}\Big)2^{q'\sigma}\|\Delta_{q'}f\|_{L^p}\\
&+ \sum_{q'\geq q-3} 2^{q-q'}2^{q\sigma}\|\Delta_q f\|_{L^p}\|\Delta_{q'}\nabla v\|_{L^\infty}\\
&+ \sum_{\substack{|q'-q|\leq 4\\ q''\leq q'-2}} 2^{(q-q'')(\sigma-1-\frac{N}{p_2})}2^{q''\sigma}\|\Delta_{q''}f\|_{L^p}\,2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}\\
&+ \sum_{\substack{q'\geq q-3\\ |q'-q''|\leq 1}} 2^{(q-q')\left(\sigma+N\min(\frac{1}{p_2},\frac{1}{p'})\right)}\times\\
&\qquad\times 2^{q'\frac{N}{p_2}}\Big(2^{q-q'}\|\Delta_{q'}\nabla v\|_{L^{p_2}} + \|\Delta_{q'}\operatorname{div} v\|_{L^{p_2}}\Big)2^{q''\sigma}\|\Delta_{q''}f\|_{L^p}\Bigg).
\end{aligned}
$$

Besides, the third term in the right-hand side may be replaced by

$$
16^{|\sigma|}K\|\nabla f\|_{L^{p_1}}\sum_{|q'-q|\leq 4} 2^{q'(\sigma-1)}\|\Delta_{q'}\nabla v\|_{L^{p_2}}. \tag{2.8}
$$

Let $\kappa_p \stackrel{\text{def}}{=} \kappa(p-1)/p^2$, $f_q \stackrel{\text{def}}{=} 2^{q\sigma}\|\Delta_q f\|_{L^p}$ and $g_q \stackrel{\text{def}}{=} 2^{q\sigma}\|\Delta_q g\|_{L^p}$. Assume that $\sigma < 1 + N/p_2$ and denote for a suitably large $K = K(N)$

$$
\begin{cases}
v_q \stackrel{\text{def}}{=} K\Big(\|\Delta_{-1}\nabla v\|_{L^\infty} + \|S_q\nabla v\|_{L^\infty}\\
\qquad\qquad + \sum_{i\in\mathbb{N}} 2^{-i}\|\Delta_{q+i}\nabla v\|_{L^\infty} + \sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}\Big),\\
w_q \stackrel{\text{def}}{=} z_q \stackrel{\text{def}}{=} K\sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}.
\end{cases} \tag{2.9}
$$

Further denote

$$
\sigma_2 \stackrel{\text{def}}{=} 1+\frac{N}{p_2}-\sigma \quad\text{and}\quad \sigma_1 \stackrel{\text{def}}{=} \begin{cases} \sigma+\min\left(\dfrac{N}{p_2},\dfrac{N}{p'}\right) & \text{if } \operatorname{div} v \neq 0,\\[2mm] 1+\sigma+\min\left(\dfrac{N}{p_2},\dfrac{N}{p'}\right) & \text{if } \operatorname{div} v = 0.\end{cases} \tag{2.10}
$$

Then inserting inequality (2.7) into (2.5) and (2.6) yields

$$
\begin{aligned}
f_q(t) + \kappa_p\,\delta_q\nu 2^{2q}\int_0^t f_q(\tau)\,d\tau \leq f_q(0) + \int_0^t g_q(\tau)\,d\tau + \sum_{|q'-q|\leq 4}\int_0^t v_{q'}(\tau)f_{q'}(\tau)\,d\tau\\
+ \sum_{q'>q+4}\int_0^t 2^{\sigma_1(q-q')}w_{q'}(\tau)f_{q'}(\tau)\,d\tau + \sum_{q'<q-4}\int_0^t 2^{\sigma_2(q'-q)}z_q(\tau)f_{q'}(\tau)\,d\tau,
\end{aligned} \tag{2.11}
$$

with $\delta_{-1} = 0$ and $\delta_q = 1$ if $q \geq 0$.

With our assumptions on ∇v, we have for a suitably large K:

$$\max(v_q, w_q, z_q) \le a \stackrel{\mathrm{def}}{=} K\|\nabla v\|_{B^{\frac{N}{p_2}}_{p_2,\infty}\cap L^\infty}.$$

Moreover $f_q(t)$ may be replaced by $\sup_{\tau\in[0,t]} f_q(\tau)$ in the left-hand side.

Hence, taking advantage of $\sigma_1, \sigma_2 > 0$ for using convolution inequalities and applying, where needed, Minkowski inequality, one ends up with

$$\|f\|_{\widetilde{L}^\infty_t(B^\sigma_{p,r})} + \kappa_p\nu\|f^{HF}\|_{\widetilde{L}^1_t(B^{\sigma+2}_{p,r})} \le$$
$$\le \|f_0\|_{B^\sigma_{p,r}} + \int_0^t a(\tau)\|f\|_{\widetilde{L}^\infty_\tau(B^\sigma_{p,r})}\,d\tau + \begin{cases} \|g\|_{\widetilde{L}^1_t(B^\sigma_{p,r})} \\ \int_0^t \|g(\tau)\|_{B^\sigma_{p,r}}\,d\tau \end{cases}$$

so that Gronwall lemma completes the proof of proposition 2.1 in the case $\sigma < 1 + N/p_2$.

The case $\sigma > 1+N/p_2$ or $\{\sigma = 1+N/p_2$ and $r = 1\}$ is left to the reader. It is based on inequality (2.8) and embeddings $B^{\sigma-1}_{p,r} \hookrightarrow B^{\frac{N}{p_2}}_{p,1} \hookrightarrow L^{p_1}$. The additional estimate in the case $f = v$ is inferred from remark A.1. ∎

3. Losing estimates

The key point in the proof of proposition 2.1 is that ∇v belongs to

$$L^1(0,T; B^{\frac{N}{p_2}}_{p_2,\infty} \cap L^\infty).$$

In the present section, we make slightly weaker assumptions on v. The price to pay is a possible loss of derivatives in the estimates.

Throughout this section, the regularity index σ for f satisfies (2.10) (hence in particular $|\sigma| \le 1 + N$).

3.1. A general statement

Note that no regularity assumptions on v are needed to get (2.11). This induces us to study the following type of coupled inequalities (we set $\kappa_p\nu$ to 0 to simplify the presentation):

$$f_q(t) \le f_q(0) + \int_0^t g_q(\tau)\,d\tau + \int_0^t \sum_{|q'-q|\le N_0} v_{q'}(\tau) f_{q'}(\tau)\,d\tau$$
$$+ \int_0^t \Big(\sum_{q'>q+M} 2^{\sigma_1(q-q')} w_{q'}(\tau) f_{q'}(\tau)\Big)\,d\tau + \int_0^t z_q(\tau)\Big(\sum_{q'<q-P} 2^{\sigma_2(q'-q)} f_{q'}(\tau)\Big)\,d\tau, \tag{3.1}$$

whenever $q \in \mathbb{Z}$ and $t \in [0,T]$. Above, N_0, M and P belong to $\mathbb{N}$.

We further assume that

$$v_q(t) \le v_q^1(t) + v_q^2(t), \quad w_q(t) \le w_q^1(t) + w_q^2(t), \quad z_q(t) \le z_q^1(t) + z_q^2(t)$$

for some measurable and nonnegative functions v_q^1, w_q^1, etc. on $[0,T]$.

Before stating estimates pertaining to (3.1), let us introduce a few notation:

$$V_q^1(t) \stackrel{\text{def}}{=} \int_0^t v_q^1(\tau)\,d\tau, \quad W_q^1(t) \stackrel{\text{def}}{=} \int_0^t w_q^1(\tau)\,d\tau, \quad Z_q^1(t) \stackrel{\text{def}}{=} \int_0^t z_q^1(\tau)\,d\tau.$$

For $R \in \mathbb{Z}$, let $v^R(t) \stackrel{\text{def}}{=} \sup_{q\le R} v_q^2(t)$ and $V^R(t) \stackrel{\text{def}}{=} \int_0^t v^R(\tau)\,d\tau$. Further denote $v^\infty(t) \stackrel{\text{def}}{=} \sup_{q\in\mathbb{Z}} v_q^2(t)$ and $V^\infty(t) \stackrel{\text{def}}{=} \int_0^t v^\infty(\tau)\,d\tau$.

Define in the same way the functions w^R and W^R (resp. z^R and Z^R) pertaining to w^2 (resp. z^2). Finally, let $a \wedge b$ stand for $\max(a,b)$.

For $(f_q)_{q\in\mathbb{Z}}$ satisfying the coupled inequalities (3.1), we have:

Proposition 3.1 *Let $\kappa \ge 0$ and $\lambda > \lambda_0 > 0$. Assume that the following conditions are fulfilled for all $t \in [0,T]$:*

1. *There exist R_1, R_2 and R_3 in $\mathbb{Z} \cup \{+\infty\}$ such that*

$$\begin{cases} \text{if} \quad R_1 < +\infty, & \int_0^t v_q^2(\tau)\,d\tau \le \lambda^{-1} \quad \text{for} \quad q > R_1, \\ \text{if} \quad R_1 = +\infty, & V^\infty(t) < +\infty, \end{cases}$$

and similar conditions for w_q^2 with R_2, and for z_q^2 with R_3.

2. *There exist Q_1, Q_2 and Q_3 in $\mathbb{Z}$ such that*

$$\begin{aligned} v_q^1(t) &\le v_{Q_1}^1(t) \quad \text{for} \quad q \le Q_1, \\ w_q^1(t) &\le w_{Q_2}^1(t) \quad \text{for} \quad q \le Q_2, \\ z_q^1(t) &\le z_{Q_3}^1(t) \quad \text{for} \quad q \le Q_3. \end{aligned}$$

3. *Let $A_m \stackrel{\text{def}}{=} V^1_{m\wedge Q_1} + W^1_{m\wedge Q_2} + Z^1_{m\wedge Q_3}$, $A^\lambda_{q,q'}(t) \stackrel{\text{def}}{=} \sup_{\tau\in[0,t]}\Big(e^{\lambda(A_{q'}-A_q)(\tau)}\Big)$,*

$$K_\lambda^1(t) = \max\Big(\sup_q \sum_{q'>q+M} 2^{\sigma_1(q-q')} A^\lambda_{q,q'}(t),\ \sup_{q'} \sum_{q<q'-M} 2^{\sigma_1(q-q')} A^\lambda_{q,q'}(t)\Big),$$

$$K_\lambda^2(t) = \max\Big(\sup_q \sum_{q'<q-P} 2^{\sigma_2(q'-q)} A^\lambda_{q,q'}(t),\ \sup_{q'} \sum_{q>q'+P} 2^{\sigma_2(q'-q)} A^\lambda_{q,q'}(t)\Big).$$

Then we assume that

$$2\Big(K_\lambda^1(t) + K_\lambda^2(t) + 2^\kappa(2N_0+1)\Big) \le \lambda_0. \tag{3.2}$$

4. For all (q,q') such that $|q-q'|\leq N_0$, we have

$$\lambda|A_{q'}(t)-A_q(t)|\leq\kappa\log 2.$$

Let

$$B\stackrel{\text{def}}{=}V^{R_1}+W^{R_2}+Z^{R_3},\qquad \widetilde{f}_q^\lambda(\tau)\stackrel{\text{def}}{=}e^{-\lambda(A_q+B)(\tau)}f_q(\tau)$$

and

$$\widetilde{g}_q^\lambda(\tau)\stackrel{\text{def}}{=}e^{-\lambda(A_q+B)(\tau)}g_q(\tau).$$

The following estimate holds true for all $r\in[1,+\infty]$ and $t\in[0,T]$:

$$\Big(\sum_q\Big(\sup_{\tau\in[0,t]}\widetilde{f}_q^\lambda(\tau)\Big)^r\Big)^{\frac1r}\leq\frac{\lambda}{\lambda-\lambda_0}\bigg(\Big(\sum_q\Big(f_q(0)\Big)^r\Big)^{\frac1r}+\Big(\sum_q\Big(\int_0^t\widetilde{g}_q^\lambda(\tau)\,d\tau\Big)^r\Big)^{\frac1r}\bigg).$$

Proof: According to (3.1), we have for all $t\in[0,T]$:

$$e^{\lambda(A_q+B)(t)}\widetilde{f}_q^\lambda(t)\leq f_q(0)+\int_0^t e^{\lambda(A_q+B)(\tau)}\widetilde{g}_q^\lambda(\tau)\,d\tau \tag{3.3}$$
$$+\sum_{|q'-q|\leq N_0}\Big(\int_0^t v_{q'}^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau+\int_0^t v_{q'}^2(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau\Big)$$
$$+\sum_{q'>q+M}2^{\sigma_1(q-q')}\Big(\int_0^t w_{q'}^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau+\int_0^t w_{q'}^2(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau\Big)$$
$$+\sum_{q'<q-P}2^{\sigma_2(q'-q)}\Big(\int_0^t z_q^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau+\int_0^t z_q^2(\tau)e^{\lambda(A_{q'}+B)(\tau)}\widetilde{f}_{q'}^\lambda(\tau)\,d\tau\Big).$$

According to assumption 2, we have for all $q\in\mathbb{Z}$ and $t\in[0,T]$,

$$v_q^1(t)\leq v_{q\wedge Q_1}^1(t),\quad w_q^1(t)\leq w_{q\wedge Q_2}^1(t)\quad\text{and}\quad z_q^1(t)\leq z_{q\wedge Q_3}^1(t),$$

whence

$$\left\{\begin{aligned}&\int_0^t v_{q'}^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\,d\tau\leq\frac{e^{\lambda(A_{q'}+B)(t)}}{\lambda},\\ &\int_0^t w_{q'}^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\,d\tau\leq\frac{e^{\lambda(A_{q'}+B)(t)}}{\lambda}.\end{aligned}\right. \tag{3.4}$$

On the other hand, we have for $q'<q-P$,

$$\begin{aligned}\int_0^t z_q^1(\tau)e^{\lambda(A_{q'}+B)(\tau)}\,d\tau&\leq\int_0^t z_{q\wedge Q_3}^1(\tau)e^{\lambda(A_q+B)(\tau)}e^{\lambda(A_{q'}-A_q)(\tau)}\,d\tau,\\ &\leq\lambda^{-1}e^{\lambda(A_q+B)(t)}A_{q,q'}^\lambda(t).\end{aligned} \tag{3.5}$$

By mean of explicit integration, we get

$$(3.6)\quad \int_0^t v_{q'}^2(\tau) e^{\lambda\left(A_{q'}(\tau)+B(\tau)\right)}\,d\tau \le \begin{cases} \lambda^{-1} e^{\lambda\left(A_{q'}(t)+B(t)\right)} & \text{if } q' \le R_1, \\ \left(\int_0^t v_{q'}^2(\tau)\,d\tau\right) e^{\lambda\left(A_{q'}(t)+B(t)\right)} & \text{if } q' > R_1. \end{cases}$$

Similar inequalities hold for $\displaystyle\int_0^t w_{q'}^2(\tau) \exp\left(\lambda\left(A_q(\tau)+B(\tau)\right)\right) d\tau$, and

$$(3.7)\quad \int_0^t z_q^2(\tau) e^{\lambda\left(A_{q'}(\tau)+B(\tau)\right)}\,d\tau \le \begin{cases} \lambda^{-1} e^{\lambda\left(A_{q'}(t)+B(t)\right)} & \text{if } q \le R_3, \\ \left(\int_0^t z_q^2(\tau)\,d\tau\right) e^{\lambda\left(A_{q'}(t)+B(t)\right)} & \text{if } q > R_3. \end{cases}$$

Let $\widetilde{F}_q^\lambda(t) \stackrel{\text{def}}{=} \sup_{\tau\in[0,t]} \widetilde{f}_q^\lambda(\tau)$. Plugging (3.4), (3.5), (3.6) and (3.7) in (3.3) and using 1 yields

$$\widetilde{f}_q^\lambda(t) \le f_q(0) + \int_0^t \widetilde{g}_q^\lambda(\tau)\,d\tau + \frac{2}{\lambda} \sum_{|q'-q|\le N_0} A_{q,q'}^\lambda(t) \widetilde{F}_{q'}^\lambda(t)$$
$$+\frac{2}{\lambda} \sum_{q'>q+M} 2^{\sigma_1(q-q')} A_{q,q'}^\lambda(t) \widetilde{F}_{q'}^\lambda(t) + \frac{2}{\lambda} \sum_{q'<q-P} 2^{\sigma_2(q'-q)} \widetilde{F}_{q'}^\lambda(t) A_{q,q'}^\lambda(t).$$

Clearly, t may be replaced by any $t' \in [0,t]$ in the left-hand side so that $\widetilde{f}_q^\lambda(t)$ may be replaced by $\widetilde{F}_q^\lambda(t)$. Now, Schur's lemma yields

$$\left(\sum_q \left(\widetilde{F}_q^\lambda(t)\right)^r\right)^{\frac{1}{r}} \le \left(\sum_q \left(f_q(0)\right)^r\right)^{\frac{1}{r}} + \left(\sum_q \left(\int_0^t \widetilde{g}_q^\lambda(\tau)\,d\tau\right)^r\right)^{\frac{1}{r}} +$$
$$\frac{2}{\lambda}\left((2N_0+1)2^\kappa + K_\lambda^1(t) + K_\lambda^2(t)\right)\left(\sum_q \left(\widetilde{F}_q^\lambda(t)\right)^r\right)^{\frac{1}{r}},$$

which entails the desired inequality. ∎

Proposition 3.1 may be applied to (2.11). We end up with the following general losing estimates:

Theorem 3.2 *Let* $1 \le p, p_2, r \le +\infty$ *and*

$$\sigma \in \left([-1] - N\min\left(\frac{1}{p_2}, \frac{1}{p'}\right), 1 + \frac{N}{p_2}\right).$$

There exists a $\lambda_0 > 0$ *and* $K = K(N)$ *such that if* v_q, w_q *and* z_q *are defined as in* (2.9) *and satisfy*

$$v_q \le v_q^1 + v_q^2, \quad w_q \le w_q^1 + w_q^2, \quad z_q \le z_q^1 + z_q^2,$$

for some sequences of functions

$$(v_q^i)_{q\in\mathbb{Z}},\quad (w_q^i)_{q\in\mathbb{Z}}\quad \textit{and}\quad (z_q^i)_{q\in\mathbb{Z}}\qquad (i\in\{1,2\})$$

verifying conditions 1, 2, 3 *and* 4 *of proposition* 3.1 *with* $M=N_0=P=4$, σ_1, σ_2 *defined in* (2.10), *some* $\kappa\geq 0$ *and some* $\lambda>\lambda_0$, *then we have*

$$\left[\sum_q\left(\sup_{\tau\in[0,t]}\left(e^{-\lambda(A_q(\tau)+B(\tau))}2^{q\sigma}\left\|\Delta_q f(\tau)\right\|_{L^p}\right)\right)^r\right]^{\frac{1}{r}}$$
$$\leq\frac{\lambda}{\lambda-\lambda_0}\left(\|f_0\|_{B^\sigma_{p,r}}+\left[\sum_q\left(\int_0^t e^{-\lambda(A_q(\tau)+B(\tau))}2^{q\sigma}\left\|\Delta_q g(\tau)\right\|_{L^p}\,d\tau\right)^r\right]^{\frac{1}{r}}\right).$$

Remark 3.3 *Let us mention in passing that in the case where*

$$\nabla v\in L^1(0,T;B^{\frac{N}{p_2}}_{p_2,\infty}\cap L^\infty)\quad \textit{and}\quad \sigma<1+N/p_2$$

then theorem 3.2 leads back to the inequalities of proposition 2.1 (up to a multiplicative constant in the right-hand side).

Indeed, it is only a matter of taking

$$v_q^1=w_q^1=z_q^1=0,\quad v_q^2=K\|\nabla v\|_{B^{\frac{N}{p_2}}_{p_2,\infty}\cap L^\infty},\quad \text{and}\quad w_q^2=z_q^2=K\|\nabla v\|_{B^{\frac{N}{p_2}}_{p_2,\infty}}.$$

Assumption 1 is fulfilled with $R_1=R_2=R_3=+\infty$, one can take $\kappa=0$ in 4 and we clearly have $K^1_\lambda(t)=K^2_\lambda(t)=1$.

3.2. Linear loss of regularity

In this part, we aim at extending the results [1] and [6] to general Besov spaces. In the former paper, the vector field v is only log-lipschitz in the space variable (which amounts to assuming that $\|S_q\nabla v(t)\|_{L^\infty}\leq(q+2)u(t)$ for some $u\in L^1(0,T)$) and estimates in Hölder spaces C^σ with $\sigma\in(0,1)$ are investigated. The authors point out a loss of derivatives of order $\int_0^t u(\tau)d\tau$ at time t.

In the latter paper, the functional framework is more general: Besov spaces $B^\sigma_{p,\infty}$ with $\sigma\in(0,1)$, and the assumption on v is somewhat weaker:

$$\exists\, C\geq 0,\ \forall q\in\mathbb{N},\ \frac{1}{q+2}\int_0^T\|S_q\nabla v(\tau)\|_{L^\infty}\,d\tau<C.$$

We here aim at getting estimates in the same spirit in a more general framework. Our most general result reads:

Theorem 3.4 *Let* $\alpha,\beta > 0$ *and* $\sigma \in ([-1] - \min(N/p_2, N/p') + \alpha, 1 + N/p_2 - \beta)$. *Let* σ_1, σ_2 *be defined as in* (2.10). *Assume that the following two conditions are satisfied on* $[0,T]$ *for some nondecreasing bounded functions* V *and* W *and all* $q, q' \geq -1$*:*

1. $\left|\sum_{i=0}^{+\infty} 2^{-i}\int_0^t \Big(\|S_{q+i}\nabla v(\tau)\|_{L^\infty} - \|S_{q'+i}\nabla v(\tau)\|_{L^\infty}\Big)\,d\tau\right| \leq |q-q'|V(t),$
2. $\left|\int_0^t \Big(2^{\frac{qN}{p_2}}\|\Delta_q\nabla v(\tau)\|_{L^{p_2}} - 2^{\frac{q'N}{p_2}}\|\Delta_{q'}\nabla v(\tau)\|_{L^{p_2}}\Big)\,d\tau\right| \leq |q-q'|W(t).$

There exists $K = K(N)$ *and* $\lambda_0 = \lambda_0(\alpha,\beta,N,p,p_2)$ *such that the following inequality holds true for* $(\mathcal{T}_\nu)$ *uniformly in* ν *with* $\sigma_t \stackrel{\text{def}}{=} \sigma - K\lambda(V+W)(t)$*:*

$$\Big(\sum_q\Big(\sup_{\tau\in[0,t]} 2^{q\sigma_\tau}\|\Delta_q f(\tau)\|_{L^p}\Big)^r\Big)^{\frac{1}{r}} \leq \tag{3.8}$$
$$\leq \frac{\lambda}{\lambda-\lambda_0}\Big[\|f_0\|_{B^\sigma_{p,r}} + \Big(\sum_q\Big(\int_0^t 2^{q\sigma_\tau}\|\Delta_q g(\tau)\|_{L^p}\,d\tau\Big)^r\Big)^{\frac{1}{r}}\Big]$$

whenever

$$\sigma_1 - \lambda K(V+W)(t) \geq \alpha \quad \textit{and} \quad \sigma_2 - \lambda K(V+W)(t) \geq \beta. \tag{3.9}$$

Proof: It stems from theorem 3.2 with an appropriate choice of v_q^i, w_q^i and z_q^i: let

$$v_q^1 \stackrel{\text{def}}{=} K\Big(\sum_{i=0}^{+\infty} 2^{-i}\|S_{q+i}\nabla v\|_{L^\infty} + \sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}\Big),$$
$$w_q^1 \stackrel{\text{def}}{=} z_q^1 \stackrel{\text{def}}{=} K\sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}} \quad \text{and} \quad v_q^2 = w_q^2 = z_q^2 = 0.$$

Choosing large enough $K = K(N)$ yields $v_q \leq v_q^1 + v_q^2$, $w_q \leq w_q^1 + w_q^2$ and $z_q \leq z_q^1 + z_q^2$.

Now, condition 1 of proposition 3.1 is trivially satisfied. So does 2 with $Q_1 = Q_2 = Q_3 = -1$. Next, defining A_q as in proposition 3.1 and taking advantage of hypotheses 1 and 2, we have (up to a change of K),

$$|A_q(t) - A_{q'}(t)| \leq K\log 2\,|q-q'|(V(t)+W(t)).$$

Hence,

$$K_\lambda^1(t) \leq \sum_{m>4} 2^{-m(\sigma_1-\lambda K(V+W)(t))} \quad \text{and} \quad K_\lambda^2(t) \leq \sum_{m>4} 2^{-m(\sigma_2-\lambda K(V+W)(t))}.$$

Choosing $\kappa = 4\min(\sigma_1-\alpha, \sigma_2-\alpha)/\log 2$, hypothesis (3.9) ensures that conditions 3 and 4 of proposition 3.1 are satisfied for some $\lambda_0 = \lambda_0(\alpha,\beta,N,p,p_2)$.

Then theorem 3.2 yields the desired inequality. ■

Remark 3.5 *Hypothesis 1 above may be replaced by the stronger following one:*

$$\left|\int_0^t \Big(\|S_q\nabla v(\tau)\|_{L^\infty} - \|S_{q'}\nabla v(\tau)\|_{L^\infty}\Big)\,d\tau\right| \le |q'-q|V(t).$$

Corollary 3.6 *Assume that* $\nabla v \in \widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})$. *Then inequality* (3.8) *holds true with*

$$\sigma_t = \sigma - K\lambda\|\nabla v\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,\infty})}$$

whenever

$$\sigma_1 - \lambda K\|\nabla v\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})} \ge \alpha \quad \textit{and} \quad \sigma_2 - \lambda K\|\nabla v\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})} \ge \beta.$$

Proof: We just have to use the following inequalities (if $q' > q$):

$$\begin{aligned}
&\left|\sum_{i=0}^{+\infty} 2^{-i}\int_0^t \Big(\|S_{q+i}\nabla v(\tau)\|_{L^\infty} - \|S_{q'+i}\nabla v(\tau)\|_{L^\infty}\Big)\,d\tau\right| \\
&\quad\le \sum_{i=0}^{+\infty} 2^{-i}\int_0^t \sum_{p=q+i}^{q'+i-1} \|\Delta_p\nabla v(\tau)\|_{L^\infty}\,d\tau \le \sum_{i=0}^{+\infty} 2^{-i}(q'-q)\|\nabla v\|_{\widetilde{L}^1_t(B^0_{\infty,\infty})}, \\
&\quad\le 2(q'-q)\|\nabla v\|_{\widetilde{L}^1_t(B^0_{\infty,\infty})},
\end{aligned}$$

and, of course, for all $q, q' \ge -1$,

$$\left|\int_0^t \Big(2^{\frac{qN}{p_2}}\|\Delta_q\nabla v(\tau)\|_{L^{p_2}} - 2^{\frac{q'N}{p_2}}\|\Delta_{q'}\nabla v(\tau)\|_{L^{p_2}}\Big)\,d\tau\right| \le \|\nabla v\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,\infty})}.$$

■

Remark 3.7 *In the case* $p_2 = +\infty$, *theorem* 3.4 *entails the result stated in [6].*

Remark 3.8 *One can also allow for a linear growth of the dyadic blocks of the source term g (like in* [9], *lemma* 2.5*). This growth will entail an additional (linear) loss of regularity. The details are left to the reader.*

Actually, in the case where ∇v belongs to $\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})$ and

$$2^{q\frac{N}{p_2}}\|\Delta_q\nabla v\|_{L^1_T(L^{p_2})}$$

is suitably small for large q, only the growth of $\|S_q\nabla v\|_{L^\infty}$ is responsible for the loss of regularity:

Theorem 3.9 *Let α, β and σ satisfy the assumptions of theorem 3.4. Assume that ∇v belongs to $\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})$. There exists $\lambda_0=\lambda_0(\alpha,\beta,N,p,p_2)$ and $K=K(N)$ such that if $\lambda>\lambda_0$ and if there exists some $R\in\mathbb{N}$ such that*

$$K2^{q\frac{N}{p_2}}\|\Delta_q\nabla v\|_{L^1_T(L^{p_2})}\leq\lambda^{-1}\quad for\quad q>R, \tag{3.10}$$

then the following inequality holds true for $(\mathcal{T}_\nu)$ uniformly in ν:

$$\Bigl(\sum_q\Bigl(\sup_{\tau\in[0,t]}e^{-\lambda W^R(\tau)}2^{q\sigma_\tau}\|\Delta_q f(\tau)\|_{L^p}\Bigr)^r\Bigr)^{\frac{1}{r}}$$
$$\leq\frac{\lambda}{\lambda-\lambda_0}\Bigl(\|f_0\|_{B^\sigma_{p,r}}+\Bigl(\sum_q\Bigl(\int_0^t e^{-\lambda W^R(\tau)}2^{q\sigma_\tau}\|\Delta_q g(\tau)\|_{L^p}\,d\tau\Bigr)^r\Bigr)^{\frac{1}{r}}\Bigr),$$

with

$$W^R(t)\stackrel{\text{def}}{=}\int_0^t\sup_{q\leq R+3}2^{q\frac{N}{p_2}}\|\Delta_q\nabla v(\tau)\|_{L^1_T(L^{p_2})}d\tau$$

and

$$\sigma_t\stackrel{\text{def}}{=}\sigma-K\lambda\|\nabla v\|_{\widetilde{L}^1_t(B^0_{\infty,\infty})}\geq\alpha$$

whenever

$$\sigma_1-\lambda K\|\nabla v\|_{\widetilde{L}^1_T(B^0_{\infty,\infty})}\geq\alpha\quad and\quad\sigma_2-\lambda K\|\nabla v\|_{\widetilde{L}^1_T(B^0_{\infty,\infty})}\geq\beta. \tag{3.11}$$

If $\nabla v\in L^1(0,T;B^{\frac{N}{p_2}}_{p_2,\infty})$ then condition (3.10) is useless provided that W^R has been replaced by $\|\nabla v\|_{L^1_t(B^{\frac{N}{p_2}}_{p_2,\infty})}$.

Proof: Choose $w^1_q=z^1_q=0$,

$$v^1_q=K\Bigl(\sum_{q'\leq q}\|\Delta_{q'}\nabla v\|_{L^\infty}+\sum_{i\in\mathbb{N}}2^{-i}\|\Delta_{q+i}\nabla v\|_{L^\infty}\Bigr)$$

and

$$v^2_q=w^2_q=z^2_q=K\sum_{|q'-q|\leq 4}2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}.$$

Taking advantage of the above assumptions and choosing $K=K(N)$ large enough, one can easily check that

$$v_q\leq v^1_q+v^2_q,\quad w_q\leq w^1_q+w^2_q\quad\text{and}\quad z_q\leq z^1_q+z^2_q.$$

Besides, condition 1 of proposition 3.1 is satisfied with $R_1 = R_2 = R_3 = R+3$ (with the convention that $R = +\infty$ if $\nabla v \in L^1(0,T;B^{\frac{N}{p2}}_{p2,\infty})$). So does condition 2 with $Q_1 = Q_2 = Q_3 = -1$. Moreover,

$$\forall q \geq q',\ \int_0^t (v^1_q - v^1_{q'})(\tau)\,d\tau \leq K(q-q')\|\nabla v\|_{\widetilde{L}^1_t(B^0_{\infty,\infty})}.$$

so that choosing $\kappa = 4\min(\sigma_1-\alpha,\sigma_2-\beta)/\log 2$ and using (3.11) ensures conditions 3 and 4. Applying theorem 3.2 completes the proof. ■

In the case where the growth assumptions on the blocks are made *before* time integration, one can exhibit more explicit sufficient conditions for having losing estimates:

Theorem 3.10 *Let $\alpha > 0$ and $\sigma \in ([-1]-\min(N/p_2, N/p')+\alpha, 1+N/p_2)$. Assume that there exist two integrable functions u and w such that the following conditions are satisfied on $[0,T]$:*

1. $\forall q \geq -1,\ \|S_q \nabla v(t)\|_{L^\infty} \leq (q+1)u(t),$
2. $\forall q \geq -1,\ 2^{q\frac{N}{p_2}}\|\Delta_q \nabla v(t)\|_{L^{p_2}} \leq (q+2)w(t),$

Let $V = \int_0^t u(\tau)\,d\tau$ and $W = \int_0^t w(\tau)\,d\tau$. There exist $\lambda_0 = \lambda_0(\alpha,\sigma,N,p,p_2)$ and $K = K(N)$ such that whenever $\lambda > \lambda_0$ and $\sigma_1 - \lambda K(V+W)(t) \geq \alpha$, we have:

$$\Big(\sum_q\Big(\sup_{\tau\in[0,t]} 2^{q\sigma_\tau}\|\Delta_q f(\tau)\|_{L^p}\Big)^r\Big)^{\frac1r} \leq \frac{\lambda}{\lambda-\lambda_0}\Big[\|f_0\|_{B^\sigma_{p,r}} + \Big(\sum_q\Big(\int_0^t 2^{q\sigma_\tau}\|\Delta_q g(\tau)\|_{L^p}\,d\tau\Big)^r\Big)^{\frac1r}\Big],$$

with $\sigma_t = \sigma - K\lambda(V+W)(t)$.

Proof: Take $v^2_q = w^2_q = z^2_q = 0$ and, for conveniently large $K = K(N)$,

$$v^1_q = K(q+2)(u+w) \quad\text{and}\quad w^1_q = z^1_q = K(q+2)w.$$

Then one can easily check that with the hypotheses above, we have $v_q \leq v^1_q + v^2_q$, $w_q \leq w^1_q + w^2_q$ and $z_q \leq z^1_q + z^2_q$. Now, conditions of proposition 3.1 are satisfied with $Q_i = -1$ and $\kappa = 4(\sigma_1-\alpha)/\log 2$ and theorem 3.2 may be applied once again while $\sigma_1 - \lambda K(V+W)(t) \geq \alpha$ (note that A_q is a nondecreasing sequence so that $A^\lambda_{q,q'}(t) \equiv 1$ for $q \geq q'$). ■

Remark 3.11 *Note that the case $p = p_2 = r = +\infty$ leads back to result treated in [1].*

3.3. Limited loss of regularity

We now make the additional assumption that $\nabla v \in L^1(0,T;B^0_{\infty,r_2})$ for some $r_2 \in (1,+\infty)$. The new result we get in this case is the following one:

Theorem 3.12 *Assume that* ∇v *belongs to* $\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty}) \cap L^1(0,T;B^0_{\infty,r_2})$ *for some* $r_2 \in (1,+\infty)$. *Let* $\sigma \in ([-1] - \min(N/p_2, N/p'), 1 + N/p_2)$ *and* $\epsilon \in (0,\sigma_1/2)$ *where* σ_1 *has been defined in* (2.10). *There exist* $\lambda_0 = \lambda_0(\sigma_1,\sigma_2)$ *and* $K = K(N)$ *such that if* $\lambda > \lambda_0$ *and if there exists some* $R \in \mathbb{N}$ *such that*

$$K2^{q\frac{N}{p_2}}\|\Delta_q \nabla v\|_{L^1_T(L^{p_2})} \leq \lambda^{-1} \quad \text{for} \quad q > R, \tag{3.12}$$

then the following inequality holds true:

$$\|f\|_{\widetilde{L}^\infty_T(B^{\sigma-\epsilon}_{p,r})} \leq \tag{3.13}$$
$$\leq \frac{4^\epsilon \lambda e^{\lambda K W^R(T)}}{\lambda - \lambda_0}\, e^{\frac{(K\lambda)^{r_2}}{\epsilon^{r_2-1}}\left(\int_0^T \|\nabla v(\tau)\|_{B^0_{\infty,r_2}}\,d\tau\right)^{r_2}} \Big(\|f_0\|_{B^\sigma_{p,r}} + \|g\|_{\widetilde{L}^1_T(B^\sigma_{p,r})}\Big)$$

with $W^R(t) \stackrel{\text{def}}{=} \int_0^t \sup_{q\leq R+3} 2^{q\frac{N}{p_2}}\|\Delta_q \nabla v(\tau)\|_{L^1_T(L^{p_2})}\,d\tau$.

If $\nabla v \in L^1(0,T;B^{\frac{N}{p_2}}_{p_2,\infty})$ *then condition* (3.12) *may be removed and* $W^R(T)$ *has to be replaced by* $\|\nabla v\|_{L^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})}$.

Proof: Once again, this is a corollary of theorem 3.2. Indeed, according to Hölder inequality,

$$\|S_q \nabla v\|_{L^\infty} \leq (q+1)^\eta \|\nabla v\|_{B^0_{\infty,r_2}} \quad \text{with} \quad \eta = 1 - 1/r_2$$

so that one can choose

$$v^1_q = K(q+1)^\eta \|\nabla v\|_{B^0_{\infty,r_2}},\; v^2_q = K\Big(\sum_{q'\geq q} 2^{q-q'}\|\Delta_{q'}\nabla v\|_{L^\infty} + \sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}\Big),$$
$$w^1_q = z^1_q = 0, \quad w^2_q = z^2_q = K\sum_{|q'-q|\leq 4} 2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}.$$

Condition 1 is satisfied with $R_1 = R_2 = R_3 = R+3$. As $(v^1_q)_{q\geq -1}$ is a nondecreasing sequence of functions, condition 2 is fulfilled for any Q_1 (that we shall merely denote by Q). Next, as $\eta \in (0,1)$, for all $Q \in \mathbb{N}$ and $q' \geq q$, the following inequality holds:

$$(q'\wedge Q + 1)^\eta - (q\wedge Q + 1)^\eta \leq \eta(q'-q)(Q+1)^{\eta-1}. \tag{3.14}$$

Let $Y(t)\stackrel{\text{def}}{=}\int_0^t\|\nabla v(\tau)\|_{B^0_{\infty,r_2}}\,d\tau$. Up to a change of K, $A_q(t)=K\log 2\,Y(t)$ $(q\wedge Q+1)^\eta$. Hence, according to inequality (3.14), we have for $q'\geq q$,

$$A^\lambda_{q,q'}(t)\leq 2^{\lambda\eta KY(t)(q'-q)(Q+1)^{\eta-1}},$$

hence

$$K^1_\lambda(t)\leq\sum_{q'>q+4}2^{(q-q')(\sigma_1-\eta\lambda KY(t)(Q+1)^{\eta-1})}. \tag{3.15}$$

From now on, assume that

$$Q+1\geq\left(\frac{2\eta\lambda KY(t)}{\sigma_1}\right)^{\frac{1}{1-\eta}}. \tag{3.16}$$

Then easy computations show that $K^1_\lambda(t)\leq\sum_{m>4}2^{-p\frac{\sigma_1}{2}}=2^{-5\frac{\sigma_1}{2}}/(1-2^{-\frac{\sigma_1}{2}})$. As A_q is nondecreasing, we readily get $K^2_\lambda(t)\leq 2^{-5\sigma_2}/(1-2^{-\sigma_2})$.

Now, if we take $\kappa=2\sigma_1/\log 2$, inequality (3.14) and the above computations show that conditions 3 and 4 are fulfilled for some $\lambda_0=\lambda_0(\sigma_1,\sigma_2)$. Hence, theorem 3.2 yields

$$\left[\sum_q\left(\sup_{[0,t]}\left(e^{-K\lambda W^R(\tau)}2^{q\epsilon-[(q\wedge Q)+1]^\eta K\lambda Y(\tau)}\,2^{q(\sigma-\epsilon)}\|\Delta_q f(\tau)\|_{L^p}\right)\right)^r\right]^{\frac1r}$$
$$\leq\frac{\lambda}{\lambda-\lambda_0}\left(\|f_0\|_{B^\sigma_{p,r}}+\left[\sum_q\left(\int_0^t e^{-K\lambda W^R(\tau)}2^{-[(q\wedge Q)+1]^\eta K\lambda Y(\tau)}2^{q\sigma}\|\Delta_q g(\tau)\|_{L^p}\,d\tau\right)^r\right]^{\frac1r}\right).$$

Now, getting the desired inequality in theorem 3.12 amounts to making a convenient choice of Q. Indeed, denoting $C=\eta\lambda K\epsilon^{-1}Y(T)$, we have

$$\forall x>0,\ \epsilon(x+1)-Cx^\eta\geq\epsilon+\left(\frac{C}{\epsilon^\eta}\right)^{\frac{1}{1-\eta}}\left(\eta^{\frac{1}{1-\eta}}-\eta^{\frac{\eta}{1-\eta}}\right)\geq\epsilon-\left(\frac{C\eta^\eta}{\epsilon^\eta}\right)^{\frac{1}{1-\eta}}.$$

Therefore

$$q\epsilon-K\lambda Y(t)\,(1+q\wedge Q)^\eta\geq \tag{3.17}$$
$$\geq\begin{cases}\epsilon-\left(\dfrac{K\lambda Y(t)\eta^\eta}{\epsilon^\eta}\right)^{\frac{1}{1-\eta}} & \text{if}\quad q\geq Q,\\ -\epsilon-K\lambda Y(t)(1+Q)^\eta & \text{if}\quad -1\leq q<Q.\end{cases}$$

Assuming that $0<\epsilon\leq\sigma_1/2$ and choosing

$$Q\stackrel{\text{def}}{=}\left[\left(\frac{\eta K\lambda Y(T)}{\epsilon}\right)^{\frac{1}{1-\eta}}\right],$$

condition (3.16) is fulfilled on $[0,T]$, and we have, according to (3.17)

$$2^{q\epsilon-[(q\wedge Q)+1]^\eta K\lambda Y(T)} \leq 2^{2\epsilon\left(1+(\lambda\epsilon^{-1}Y(T))^{\frac{1}{1-\eta}}\right)}.$$

This implies inequality (3.13). ∎

Actually, estimates with limited loss of regularity may be proved under the weaker additional assumption that $\nabla v \in \widetilde{L}^1_T(B^0_{\infty,r_2})$ for some $r_2 \in (1,+\infty)$:

Theorem 3.13 *Assume that* ∇v *belongs to* $\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})\cap\widetilde{L}^1_T(B^0_{\infty,r_2})$ *for some* $r_2 \in (1,+\infty)$*. Let* $\sigma \in ([-1]-\min(N/p_2,N/p'),1+N/p_2)$ *and* $\epsilon \in (0,\sigma_1)$*. There exist* $\lambda_0=\lambda_0(\sigma_1,\sigma_2)$ *and* $K=K(N)$ *such that if* $\lambda>\lambda_0$ *and if there exists some* $R\in\mathbb{N}$ *such that* (3.12) *is fulfilled, then the following inequality holds true:*

(3.18)

$$\|f\|_{\widetilde{L}^\infty_T(B^{\sigma-\epsilon}_{p,r})} \leq \frac{2^\epsilon\lambda e^{\lambda K W^R(T)}}{\lambda-\lambda_0}\, e^{K\lambda(Q+1)^{1-\frac{1}{r_2}}\|\nabla v\|_{\widetilde{L}^1_T(B^0_{\infty,r_2})}}\left(\|f_0\|_{B^\sigma_{p,r}}+\|g\|_{\widetilde{L}^1_T(B^\sigma_{p,r})}\right)$$

with Q *such that* $\sup_{q\geq Q}\|\nabla\Delta_q v\|_{L^1_T(L^\infty)} \leq \epsilon/K\lambda$ *and* $W^R(t)$ *defined as in theorem* 3.12 *(with the usual change if* $\nabla v \in L^1(0,T;B^{\frac{N}{p_2}}_{p_2,\infty})$*).*

Proof: Define v_q^2, w_q^2, z_q^2, w_q^1, z_q^1 as previously, and $v_q^1 \overset{\text{def}}{=} K\sum_{q'\leq q-1}\|\Delta_q\nabla v\|_{L^\infty}$.

As for $q'\geq q$, we have (up to a change of K)

$$\int_0^T\left(v^1_{q'}(t)-v^1_q(t)\right)dt \leq K\log 2\;(q'-q)\sup_{q''\geq q}\|\Delta_q\nabla v\|_{L^1_T(L^\infty)},$$

we easily get by choosing $Q_1=Q$,

$$\forall q'\geq q,\;A^\lambda_{q,q'}\leq 2^{\epsilon(q'-q)}.$$

Hence condition 3 is satisfied. So does 4 with $\kappa=2\sigma_1/\log 2$. Therefore the general inequality stated in theorem 3.2 holds true with $B(t)=W^R(t)$ and $A_q(t)=K\int_0^t\sum_{p\leq q\wedge Q-1}\|\Delta_q\nabla v(\tau)\|_{L^\infty}\,d\tau$. Now, according to Hölder inequality and the definition of Q, we have

$$K\log 2\;\lambda\int_0^t\sum_{p\leq q\wedge Q-1}\|\Delta_q\nabla v(\tau)\|_{L^\infty}\,d\tau$$
$$\leq K\log 2\;\lambda(Q+1)^{1-\frac{1}{r_2}}\|\nabla v\|_{\widetilde{L}^1_T(B^0_{\infty,r_2})}+\epsilon\max(0,q-Q),$$

whence for all $t\in[0,T]$ and $q\geq -1$,

$$e^{\lambda A_q(t)}2^{-q\epsilon}\leq e^{K\lambda(Q+1)^{1-\frac{1}{r_2}}\|\nabla v\|_{\widetilde{L}^1_T(B^0_{\infty,r_2})}}.$$

This yields inequality (3.18). ∎

A. Appendix

We here give the proof of lemma 2.2. In order to show that only the gradient part of v is involved in the estimates, it is convenient to split v into low and high frequencies: $v=\Delta_{-1}v+\widetilde{v}$. Obviously, there exists a constant C such that

$$\forall r\in[1,+\infty],\ \|\Delta_{-1}\nabla v\|_{L^r}\leq C\,\|\nabla v\|_{L^r}\quad\text{and}\quad\|\nabla\widetilde{v}\|_{L^r}\leq C\,\|\nabla v\|_{L^r}\,. \tag{A.1}$$

Since there exists a $R>0$ so that for all $t\in[0,T]$, $\operatorname{Supp}\mathcal{F}\widetilde{v}(t)\cap B(0,R)=\emptyset$, Bernstein inequality holds true for $\Delta_q\widetilde{v}$ even for $q=-1$, namely

$$\forall q\geq-1,\ \|\Delta_q\nabla\widetilde{v}\|_{L^p}\approx 2^q\,\|\Delta_q\widetilde{v}\|_{L^p}\,. \tag{A.2}$$

Let us define the paraproduct between two distributions according to J.-M. Bony in [2]:

$$T_fg\stackrel{\text{def}}{=}\sum_{q\in\mathbb{N}}S_{q-1}f\Delta_qg.$$

Denoting

$$R(f,g)\stackrel{\text{def}}{=}\sum_{q\geq-1}\Delta_qf\widetilde{\Delta}_qg\quad\text{with}\quad\widetilde{\Delta_q}g\stackrel{\text{def}}{=}(\Delta_{q-1}+\Delta_q+\Delta_{q+1})g,$$

we have the following so-called Bony's decomposition:

$$fg=T_fg+T_gf+R(f,g).$$

Now, we have

$$\begin{aligned}R_q&=S_{q+1}v\cdot\nabla\Delta_qf-\Delta_q(v\cdot\nabla f),\\&=[\widetilde{v}^j,\Delta_q]\partial_jf+[\Delta_{-1}v^j,\Delta_q]\partial_jf+(S_{q+1}v-v)\cdot\nabla\Delta_qf,\end{aligned}$$

where the summation convention on repeated indices is understood.

Hence, taking advantage of Bony's decomposition, we end up with $R_q=\sum_{i=1}^7R_q^i$ where

$$\begin{aligned}R_q^1&=[T_{\widetilde{v}^j},\Delta_q]\partial_jf,\\R_q^2&=T_{\partial_j\Delta_qf}\widetilde{v}^j,\\R_q^3&=-\Delta_qT_{\partial_jf}\widetilde{v}^j,\\R_q^4&=\partial_jR(\widetilde{v}^j,\Delta_qf)-\partial_j\Delta_qR(\widetilde{v}^j,f),\\R_q^5&=\Delta_qR(\operatorname{div}\widetilde{v},f)-R(\operatorname{div}\widetilde{v},\Delta_qf),\\R_q^6&=(S_{q+1}v-v)\cdot\nabla\Delta_qf,\\R_q^7&=[\Delta_{-1}v^j,\Delta_q]\partial_jf.\end{aligned}$$

In the following computations, the constant K depends only on N.

Bounds for $2^{q\sigma}\left\|R_q^1\right\|_{L^p}$**:**

By virtue of (1.2), we have

$$R_q^1=\sum_{|q-q'|\leq 4}[S_{q'-1}\widetilde{v}^j,\Delta_q]\partial_j\Delta_{q'}f.$$

On the other hand,

$$[S_{q'-1}\widetilde{v}^j,\Delta_q]\partial_j\Delta_{q'}f(x)=\int h(y)\Big[S_{q'-1}\widetilde{v}^j(x)-S_{q'-1}\widetilde{v}^j(x-2^{-q}y)\Big]\partial_j\Delta_{q'}f(x-2^{-q}y)dy$$

so that applying first order Taylor's formula, convolution inequalities and (A.1) yields

$$2^{q\sigma}\left\|R_q^1\right\|_{L^p}\leq K\sum_{|q'-q|\leq 4}\left\|S_{q'-1}\nabla v\right\|_{L^\infty}2^{q'\sigma}\left\|\Delta_{q'}f\right\|_{L^p}. \tag{A.3}$$

Bounds for $2^{q\sigma}\left\|R_q^2\right\|_{L^p}$**:**

By virtue of (1.2), we have

$$R_q^2=\sum_{q'\geq q-3}S_{q'-1}\partial_j\Delta_q f\,\Delta_{q'}\widetilde{v}^j.$$

Hence, using inequalities (A.1) and (A.2) yields

$$2^{q\sigma}\left\|R_q^2\right\|_{L^p}\leq K\sum_{q'\geq q-3}2^{q-q'}\,2^{q\sigma}\left\|\Delta_q f\right\|_{L^p}\left\|\Delta_{q'}\nabla v\right\|_{L^\infty}. \tag{A.4}$$

Bounds for $2^{q\sigma}\left\|R_q^3\right\|_{L^p}$**:**

One proceeds as follows:

$$R_q^3=-\sum_{|q'-q|\leq 4}\Delta_q\Big(S_{q'-1}\partial_j f\Delta_{q'}\widetilde{v}^j\Big), \tag{A.5}$$

$$=-\sum_{\substack{|q'-q|\leq 4\\ q''\leq q'-2}}\Delta_q\Big(\Delta_{q''}\partial_j f\Delta_{q'}\widetilde{v}^j\Big). \tag{A.6}$$

Therefore, denoting $1/p_1{=}1/p{-}1/p_2$ and taking advantage of (A.1) and (A.2),

$$\begin{aligned}2^{q\sigma}\left\|R_q^3\right\|_{L^p} &\leq K\sum_{\substack{|q'-q|\leq 4\\ q''\leq q'-2}}2^{q\sigma}\left\|\Delta_{q''}\partial_j f\right\|_{L^{p_1}}\left\|\Delta_{q'}\widetilde{v}^j\right\|_{L^{p_2}},\\ &\leq K\sum_{\substack{|q'-q|\leq 4\\ q''\leq q'-2}}2^{(q-q'')(\sigma-1-\frac{N}{p_2})}2^{q''\sigma}\left\|\Delta_{q''}f\right\|_{L^p}2^{q'\frac{N}{p_2}}\left\|\Delta_{q'}\nabla v\right\|_{L^{p_2}}.\end{aligned} \tag{A.7}$$

Note that, starting from (A.5), one can alternately get

$$2^{q\sigma}\left\|R_q^3\right\|_{L^p}\leq 16^{|\sigma|}K\sum_{|q'-q|\leq 4}\left\|\nabla S_{q'-1}f\right\|_{L^{p_1}}2^{q'(\sigma-1)}\left\|\Delta_{q'}\nabla v\right\|_{L^{p_2}}. \tag{A.8}$$

Bounds for $2^{q\sigma}\left\|R_q^4\right\|_{L^p}$**:**

$$R_q^4=\underbrace{\sum_{|q'-q|\leq 2}\partial_j(\Delta_{q'}\widetilde{v}^j\Delta_q\widetilde{\Delta}_{q'}f)}_{R_q^{4,1}}-\underbrace{\sum_{q'\geq q-3}\partial_j\Delta_q(\Delta_{q'}\widetilde{v}^j\widetilde{\Delta}_{q'}f)}_{R_q^{4,2}}.$$

For the first term, we merely have (by virtue of (A.2)),

$$2^{q\sigma}\left\|R_q^{4,1}\right\|_{L^p}\leq 4^{|\sigma|}K\sum_{|q'-q|\leq 2}\|\Delta_{q'}\nabla\widetilde{v}\|_{L^\infty}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p}. \tag{A.9}$$

For $R_q^{4,2}$, we proceed differently depending on the value of $1/p+1/p_2$.

First case: $1/p_3\stackrel{\text{def}}{=}1/p+1/p_2\leq 1$*:*

$$\begin{aligned}2^{q\sigma}\left\|R_q^{4,2}\right\|_{L^p}&\leq K\sum_{q'\geq q-3}2^{q(1+\sigma)}2^{q\left(\frac{N}{p_3}-\frac{N}{p}\right)}\left\|\Delta_{q'}\widetilde{v}\widetilde{\Delta}_{q'}f\right\|_{L^{p_3}},\\&\leq K\sum_{q'\geq q-3}2^{q(1+\sigma)}2^{q\frac{N}{p_2}}\|\Delta_{q'}\widetilde{v}\|_{L^{p_2}}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p},\\&\leq K\sum_{q'\geq q-3}2^{(q-q')\left(1+\frac{N}{p_2}+\sigma\right)}2^{q'\frac{N}{p_2}}\|\nabla\Delta_{q'}\widetilde{v}\|_{L^{p_2}}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p}.\end{aligned}$$

Second case: $1/p+1/p_2>1$*:*

Taking $p_2=p'$ in the above computations yields

$$\begin{aligned}2^{q\sigma}\left\|R_q^{4,2}\right\|_{L^p}&\leq K\sum_{q'\geq q-3}2^{(q-q')\left(1+\frac{N}{p'}+\sigma\right)}2^{q'\frac{N}{p'}}\|\nabla\Delta_{q'}\widetilde{v}\|_{L^{p'}}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p},\\&\leq K\sum_{q'\geq q-3}2^{(q-q')\left(1+\frac{N}{p'}+\sigma\right)}2^{q'\frac{N}{p_2}}\|\nabla\Delta_{q'}\widetilde{v}\|_{L^{p_2}}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p}.\end{aligned}$$

In view of (A.1) and (A.9), we conclude that

$$\begin{aligned}&2^{q\sigma}\left\|R_q^4\right\|_{L^p}\leq\\&\leq 4^{|\sigma|}K\sum_{q'\geq q-3}2^{(q-q')\left(1+\sigma+N\min(\frac{1}{p_2},\frac{1}{p'})\right)}2^{q'\frac{N}{p_2}}\|\Delta_{q'}\nabla v\|_{L^{p_2}}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p}.\end{aligned}\tag{A.10}$$

Bounds for $2^{q\sigma}\left\|R_q^5\right\|_{L^p}$**:**

Similar computations yield

$$\begin{aligned}&2^{q\sigma}\left\|R_q^5\right\|_{L^p}\leq\\&\leq 4^{|\sigma|}K\sum_{q'\geq q-3}2^{(q-q')\left(\sigma+N\min(\frac{1}{p_2},\frac{1}{p'})\right)}2^{q'\frac{N}{p_2}}\|\Delta_{q'}\operatorname{div}v\|_{L^{p_2}}2^{q'\sigma}\left\|\widetilde{\Delta}_{q'}f\right\|_{L^p}.\end{aligned}\tag{A.11}$$

Bounds for $2^{q\sigma}\left\|R_q^6\right\|_{L^p}$**:**

Since $R_q^6 = -\sum_{q'\geq q+1} \Delta_{q'} v\cdot\nabla\Delta_q f$, we have, by virtue of Bernstein inequality (note that $q' \geq 0$ in the summation),

$$2^{q\sigma}\left\|R_q^6\right\|_{L^p} \leq K\sum_{q'\geq q} 2^{q-q'}\left\|\Delta_{q'}\nabla v\right\|_{L^\infty} 2^{q\sigma}\left\|\Delta_q f\right\|_{L^p}. \tag{A.12}$$

Bounds for $2^{q\sigma}\left\|R_q^7\right\|_{L^p}$**:**
As $R_q^7 = \sum_{|q'-q|\leq 1}[\Delta_q, \Delta_{-1} v]\cdot\nabla\Delta_{q'} f$, the first order Taylor formula yields

$$2^{q\sigma}\left\|R_q^7\right\|_{L^p} \leq 2^{|\sigma|}K\sum_{|q'-q|\leq 1}\left\|\nabla\Delta_{-1} v\right\|_{L^\infty} 2^{q'\sigma}\left\|\Delta_{q'} f\right\|_{L^p}. \tag{A.13}$$

Combining inequalities (A.3), (A.4), (A.7) or (A.8), (A.10), (A.11), (A.12), and (A.13), we end up with the desired estimate for R_q. ∎

Remark A.1 *Straightforward modifications in the estimates for* R_q^3, R_q^4, R_q^5 *show that in the special case where* $f = v$*, the following estimate holds true:*

$$\begin{aligned} 2^{q\sigma}\|R_q\|_{L^p} \lesssim & \sum_{|q'-q|\leq 4}\|S_{q'+1}\nabla v\|_{L^\infty} 2^{q'\sigma}\|\Delta_{q'} v\|_{L^p} + \sum_{q'\geq q-3} 2^{q-q'} 2^{q\sigma}\|\Delta_q v\|_{L^p}\|\Delta_{q'}\nabla v\|_{L^\infty} \\ & + \sum_{\substack{q'\geq q-3 \\ |q'-q''|\leq 1}} 2^{(q-q')\sigma}\Big(2^{q-q'}\|\Delta_{q'}\nabla v\|_{L^\infty} + \|\Delta_{q'}\operatorname{div} v\|_{L^\infty}\Big) 2^{q''\sigma}\|\Delta_{q''} v\|_{L^p}. \end{aligned} \tag{A.14}$$

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