

Completeness in $L^1(\mathbb{R})$ of discrete translates

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Abstract

We characterize, in terms of the Beurling-Malliavin density, the discrete spectra $\Lambda \subset \mathbb{R}$ for which a generator exists, that is a function $\varphi \in L^1(\mathbb{R})$ such that its Λ -translates $\varphi(x - \lambda), \lambda \in \Lambda$, span $L^1(\mathbb{R})$. It is shown that these spectra coincide with the uniqueness sets for certain analytic classes. We also present examples of discrete spectra $\Lambda \subset \mathbb{R}$ which do not admit a single generator while they admit a pair of generators.

1. Introduction

1.1. The famous Wiener Tauberian theorem states that a function $f \in L^1(\mathbb{R})$ spans $L^1(\mathbb{R})$ by translations, in the sense that the linear combinations of translates $(\tau_\lambda f)(t) = f(t - \lambda)$ of f , $\lambda \in \mathbb{R}$, are dense in $L^1(\mathbb{R})$, if and only if $\hat{f}(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}$. The corresponding result for $L^2(\mathbb{R})$ is that a function $f \in L^2(\mathbb{R})$ spans $L^2(\mathbb{R})$ by translations if and only if $\hat{f}(\zeta) \neq 0$ almost everywhere in \mathbb{R} .

Let X be some translation-invariant function space on \mathbb{R} (that is, $\tau_\lambda f \in X$ for $f \in X$, $\lambda \in \mathbb{R}$). A function $\varphi \in X$ may have the property that only a certain set of translates $\tau_\lambda \varphi$ where λ belong to some set Λ , suffice to span X :

Definition 1 *Let $\varphi \in X$ and $\Lambda \subset \mathbb{R}$. We say that φ is a Λ -generator for X if the linear span $T(\varphi, \Lambda)$ of the translates $\tau_\lambda \varphi$, with $\lambda \in \Lambda$, is dense in X .*

It is natural to ask which spectra Λ admit a generator φ in a fixed space X . This question is most interesting when Λ is discrete, which we will assume from now on.

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In case Λ is the set of integers \mathbb{Z} , that is, we are dealing with *integer translates* of a fixed function, it is well-known that no \mathbb{Z} -generators exist in $L^p(\mathbb{R})$, $1 \leq p \leq 2$. In $L^2(\mathbb{R})$, this easily follows from Plancherel's theorem and the fact that $T(\varphi, \Lambda)$ has Fourier transform

$$(1.1) \quad T(\varphi, \Lambda)^\wedge = \hat{\varphi} \mathcal{E}(\Lambda),$$

where $\mathcal{E}(\Lambda)$ denotes the linear span of the exponentials $e^{i\lambda\zeta}$ with frequencies $\lambda \in \Lambda$. If $\Lambda = \mathbb{Z}$ this consists entirely of 2π -periodic functions and hence $\hat{\varphi} \cdot \mathcal{E}(\Lambda)$ cannot be dense in $L^2(\mathbb{R})$. A similar argument works in $L^p(\mathbb{R})$, $1 \leq p < 2$.

However, surprisingly enough, in $L^p(\mathbb{R})$, $p > 2$, there do exist \mathbb{Z} -generators. This result was established in [1], and another proof can be obtained from results of [7] (see also [3] for a particular case).

In the space $L^2(\mathbb{R})$, using a certain construction based on small divisors, Olevskii [8] showed that an arbitrary perturbation of \mathbb{Z} of the form

$$(1.2) \quad \Lambda = \{n + a_n, a_n \neq 0, a_n \rightarrow 0\}$$

admits a generator $\varphi \in L^2(\mathbb{R})$.

It is immediate to see using Plancherel's theorem that $T(\varphi, \Lambda)$ is dense in $L^2(\mathbb{R})$ if and only if $\hat{\varphi}(\zeta) \neq 0$ almost everywhere and $\mathcal{E}(\Lambda)$ is dense in the weighted L^2 -space $L^2(\mathbb{R}, \omega)$, with $\omega = |\hat{\varphi}|^2$, an a.e. positive weight. In particular, if E_ε denotes the set $E_\varepsilon = \{\omega \geq \varepsilon\}$, $\mathcal{E}(\Lambda)$ will be dense in $L^2(E_\varepsilon)$, and $|E_\varepsilon| \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Thus, if Λ has a generator in $L^2(\mathbb{R})$, $\mathcal{E}(\Lambda)$ is dense in L^2 in sets of arbitrarily large measure.

This shows the connection of these questions with the subject of *density of exponentials* $\mathcal{E}(\Lambda)$ in function spaces and, in particular, with Landau's results. Landau [5] constructed certain perturbations of the integers $\Lambda = \{n + a_n\}$ where a_n are bounded, such that $\mathcal{E}(\Lambda)$ is dense in L^2 on any finite union of the intervals $(2\pi(k-1) + \varepsilon, 2\pi k - \varepsilon)$, $\varepsilon > 0$, in particular on sets with arbitrarily large measure. In [11] Landau's result was extended to every sequence Λ as in (1.2), where a_n have an exponential decay. We mention here that if $\mathcal{E}(\Lambda)$ is complete in L^2 on 'Landau sets', then one can construct a Λ -generator for $L^2(\mathbb{R})$ which belongs to the Schwartz class $S(\mathbb{R})$. Such generators are presented in [9] for sequences (1.2) with exponentially small a_n . It is also shown in [9] that the exponential decay is in a sense necessary, since a slower decay of a_n cannot guarantee existence of a generator even from $L^1(\mathbb{R})$.

In general there is a sort of balance between the size of Λ and the 'smallness' of $\hat{\varphi}$. The faster $\hat{\varphi}$ tends to zero at $\pm\infty$ the sparser spectra Λ may serve as translation sets, the "denser" Λ is the more general φ may work

as a generator. The spectra Λ considered in [8] and [9] are "sparse" in the sense that they all have density one; a number of results for spectra Λ with infinite density can be found in [12] and [10].

In connection with these questions, we point out that no Riesz bases (nor a frame) exists in $L^2(\mathbb{R})$ consisting of translates of a fixed function φ ([2]). On the other hand, to the best of our knowledge, it is not known whether a Schauder bases of translates exists in $L^2(\mathbb{R})$.

1.2. This paper deals with the case $X = L^1(\mathbb{R})$. Our main result (Theorem 1 below) gives a characterization of the translation sets $\Lambda \subset \mathbb{R}$ admitting a generator φ . The L^1 -case is in a sense easier than the L^2 -case because now $\hat{\varphi}$ must be a non-vanishing continuous function and what is involved is the question of density of exponentials $\mathcal{E}(\Lambda)$ in *intervals*. It is therefore not surprising that the *Beurling-Malliavin spectral radius formula*, which we now recall, appears in this setting and in the statement of the main result. The spectral radius $R(\Lambda)$ of a set $\Lambda \subset \mathbb{R}$ is defined

$$R(\Lambda) = \sup\{\rho > 0; \mathcal{E}(\Lambda) \text{ is complete in } C[-\rho, \rho]\},$$

where $C(I)$ denotes the space of continuous functions on the interval I . One sets $R(\Lambda) = 0$ when $\mathcal{E}(\Lambda)$ is not complete in $C[-\rho, \rho]$ for any positive ρ . Thus if $0 < R(\Lambda) < \infty$, we have that $\mathcal{E}(\Lambda)$ is complete in $C[-\rho, \rho]$ if $\rho < R(\Lambda)$ and incomplete if $\rho > R(\Lambda)$. The Fourier transform of the dual space, the space of finite complex Borel measures supported in $[-\rho, \rho]$, is a (proper) subspace of the Bernstein space

$$B_\rho = \{F \text{ entire: } |F(x + iy)| \leq C_F e^{\rho|y|}, x + iy \in \mathbb{C}\},$$

where $C_F > 0$ is a constant depending on F . If Λ is not complete in $C[-\rho, \rho]$, then there is a function $F \in B_\rho$ which vanishes on Λ . On the other hand, if a function $G \in B_\rho$ vanishes on Λ , the function $F(t) = G(t)(\sin \varepsilon t / \varepsilon t)^2$ is the Fourier transform of a finite (absolute continuous) measure concentrated on $[-\rho - 2\varepsilon, \rho + 2\varepsilon]$ which is orthogonal to $\mathcal{E}(\Lambda)$. Hence, for $R(\Lambda)$ finite and positive, we have

$$\begin{aligned} (1.3) \quad R(\Lambda) &= \sup\{\rho > 0 : \Lambda \text{ is a uniqueness set for } B_\rho\} \\ &= \inf\{\rho > 0 : \exists F \in B_\rho, F \neq 0, \text{ such that } F|_\Lambda = 0\}. \end{aligned}$$

Here and in the sequel, Λ being a *uniqueness set for a certain class* Y means that $F \in Y$ and $F|_\Lambda = 0$ imply $F \equiv 0$.

In the beginning of the sixties Beurling and Malliavin (see [4, vol. 2]) established that $R(\Lambda) = \pi D_{BM}(\Lambda)$, where $D_{BM}(\Lambda)$ is a certain exterior density whose definition will be recalled in section 3.

1.3. With this notation we can now formulate our main result:

Theorem 1 *A discrete set $\Lambda \subset \mathbb{R}$ admits a generator for $L^1(\mathbb{R})$ if and only if $R(\Lambda) = +\infty$.*

The necessity of the condition $R(\Lambda) = +\infty$ follows essentially from the remarks in paragraph 1.1. and will be detailed in section 2. We will give two proofs of the sufficiency. The first one, also in section 2, is constructive, based solely on $R(\Lambda) = +\infty$ without using any density. The second one, in section 3, uses the geometrical information $D_{BM}(\Lambda) = +\infty$.

This second proof develops the following natural approach to the problem, relating it with uniqueness sets for *generalized Bernstein classes*. Let N be the class of all functions $f \in L^1(\mathbb{R})$ such that \hat{f} does not vanish:

$$(1.4) \quad N = \{f \in L^1(\mathbb{R}) : \hat{f}(\zeta) \neq 0, \zeta \in \mathbb{R}\}.$$

By duality, a function φ is a Λ -generator for $L^1(\mathbb{R})$ if and only if $h \in L^\infty(\mathbb{R})$ and $(h \star \check{\varphi})(\lambda) = \int_{-\infty}^{+\infty} h(t)\varphi(t - \lambda) dt = 0$, $\lambda \in \Lambda$, implies $h = 0$ (here $\check{\varphi}(t) = \varphi(-t)$). Applying Wiener's theorem, we see that φ is a Λ -generator if and only if $\varphi \in N$, and $h \in L^\infty(\mathbb{R})$ and $(h \star \check{\varphi})(\lambda) = 0$, $\lambda \in \Lambda$, imply $h \star \check{\varphi} = 0$. This can be restated by saying that φ is a Λ -generator in $L^1(\mathbb{R})$ if and only if it satisfies: (i) $\varphi \in N$, and (ii) Λ is a uniqueness set for the class $L^\infty(\mathbb{R}) \star \check{\varphi}$.

Given any Λ with infinite spectral radius, it follows from (1.3) that Λ is a uniqueness set for every Bernstein class B_ρ . However, clearly, we have $N \cap B_\rho = \emptyset$, for all ρ , and so no space $L^\infty(\mathbb{R}) \star \check{\varphi}$, $\varphi \in N$, is included in any of the Bernstein spaces. In order to prove Theorem 1, one may try to construct *larger* spaces of analytic functions Y such that Λ is a uniqueness set for Y , and $L^\infty(\mathbb{R}) \star \check{\varphi} \subseteq Y$, for some $\varphi \in N$. It turns out that the 'smallest' spaces Y with this property can be realized as the generalized Bernstein classes B_σ defined as follows. Let σ be a monotone function on $(0, \infty)$ satisfying $\sigma(y) \nearrow \infty, y \rightarrow \infty$. We set:

$$(1.5) \quad B_\sigma = \{F \text{ entire} : |F(x + iy)| \leq C_F e^{|y|\sigma(|y|)}, x + iy \in \mathbb{C}\}.$$

In section 3 we prove:

Theorem 2 *For a discrete set $\Lambda \subset \mathbb{R}$, the following conditions are equivalent:*

- (a) Λ admits a generator φ for $L^1(\mathbb{R})$.
- (b) Λ is a uniqueness set for some generalized Bernstein class B_σ .

When (b) holds, the generator φ can be chosen in B_σ .

If a set Λ has finite spectral radius then, by Theorem 1, Λ does not admit generators in $L^1(\mathbb{R})$. In section 4 we look at the problem of *two* generators and show that there exist spectra Λ which do not admit a single generator while they admit a pair of generators. More specifically, we show that every Λ of form (1.2) with ‘exponentially small’ a_n admits two generators.

2. The spectral radius proof of the main theorem

A basic well know remark concerning the spectral or completeness radius $R(\Lambda)$ is that its value is not affected if the sup-norm on $[-\rho, \rho]$ is replaced by any reasonable norm. This fact is behind both the proof of the necessity and sufficiency of the condition.

2.1. Let us prove first that $R(\Lambda) = +\infty$ is necessary. If $\varphi \in L^1(\mathbb{R})$ is a Λ -generator, it follows from (1.1) and the trivial estimate $\|\hat{f}\|_\infty \leq \|f\|_1$ that an arbitrary \hat{f} , with $f \in L^1(\mathbb{R})$, can be approximated in the sup-norm by functions in $\hat{\varphi} \mathcal{E}(\Lambda)$. Since $\varphi \star f \in L^1(\mathbb{R})$, $\hat{\varphi} \hat{f}$ can be approximated as well. We know too that $\hat{\varphi}$ has no zeros, by Wiener’s theorem. Now fix $\rho > 0$; every test function ψ supported in $(-\rho, \rho)$ serves as \hat{f} , and therefore $\hat{\varphi} \psi$ is well approximated by $\hat{\varphi} \mathcal{E}(\Lambda)$ in the sup-norm. Since $\hat{\varphi}$ is bounded below on $(-\rho, \rho)$ it follows that every ψ is approximated in the sup-norm by $\mathcal{E}(\Lambda)$. The density of such ψ in $C[-\rho, \rho]$ shows that $\mathcal{E}(\Lambda)$ is dense in $C[-\rho, \rho]$ and ρ being arbitrary, one has $R(\Lambda) = +\infty$.

2.2. To prove that $R(\Lambda) = +\infty$ is a sufficient condition we will make use of the remark above and consider instead of the sup-norm the Sobolev norm

$$\|\psi\|_2 + \|\psi'\|_2.$$

More precisely, if I is an interval we consider the space

$$L_1^2(I) = \{h \in L^2(I) : h' \in L^2(I)\}$$

(consisting of absolutely continuous functions) endowed with the norm $\|h\|_I = \|h\|_{L^2(I)} + \|h'\|_{L^2(I)}$. The reason to consider $\|h\|_I$ is that if $\|\hat{f}\|_{\mathbb{R}}$ is finite then $f \in L^1(\mathbb{R})$ and

$$(2.1) \quad \|f\|_1 \leq \|\hat{f}\|_{\mathbb{R}},$$

as it is immediately checked.

The fact that if $R(\Lambda) = +\infty$, $\mathcal{E}(\Lambda)$ is dense in $L_1^2(-\rho, \rho)$ for every $\rho > 0$ can be checked in an elementary way, i.e. by first approximating h' and then integrating.

We consider now the space

$$E = \{ f \in L^1(\mathbb{R}) : \hat{f} \text{ is of class } C^1 \text{ and compactly supported} \}.$$

It is clear that E is dense in $L^1(\mathbb{R})$ and separable. We fix a sequence $(f_k)_{k=1}^\infty$, $f_k \in E$ dense in $L^1(\mathbb{R})$. Fix some sequence $\varepsilon_k \rightarrow 0$, say $\varepsilon_k = 2^{-k}$. Associated to Λ with $R(\Lambda) = +\infty$ and to the sequences $(f_k)_{k=1}^\infty$, $(\varepsilon_k)_{k=1}^\infty$ we will construct a positive $\Phi \in L^2_1(\mathbb{R})$ with the following property:

“For each $k = 2, 3, \dots$ there exists a trigonometric polynomial $P_k(\zeta) = \sum_{\lambda \in \Lambda_k} c_\lambda e^{i\lambda\zeta}$ with frequencies in a finite subset Λ_k of Λ , such that

$$\| \hat{f}_k - P_k \Phi \|_{\mathbb{R}} < \varepsilon_k.”$$

If $\hat{\varphi} = \Phi$ then (2.1) gives $\varphi \in L^1(\mathbb{R})$ such that

$$\| f_k - \sum_{\lambda \in \Lambda_k} c_\lambda \tau_\lambda \varphi \|_1 \leq \varepsilon_k.$$

Since given an arbitrary $g \in L^1(\mathbb{R})$ and $\varepsilon > 0$ there are infinitely many k such that $\| g - f_k \|_1 < \varepsilon$, this will prove that φ is indeed a Λ -generator in $L^1(\mathbb{R})$.

We will use at certain points the trivial estimate

$$\| HG \|_I \leq B(G) \| H \|_I, \quad G, H \in L^2_1(\mathbb{R}),$$

where $B(G) = \| G \|_\infty + \| G' \|_\infty$.

Let $(I_k)_{k=1}^\infty$, $(J_k)_{k=1}^\infty$ be open intervals centered at 0, with

$$J_1 \subset I_1 \subset J_2 \subset I_2 \subset \dots \subset J_k \subset I_k \subset J_{k+1} \subset I_{k+1} \subset \dots$$

$I_k \setminus J_k$ consisting in two unit intervals and such that \hat{f}_k is supported in J_k . The function Φ will be an even continuous piecewise linear function that will be built step by step (the condition of piecewise continuity is not essential for the construction). We will exploit the fact that

(2.2) “ $\mathcal{E}(\Lambda)$ is dense in $L^2_1(I)$ for every interval I ”

Let $\delta_k > 0$ be such that $\sum_{n=k}^\infty \delta_n = \varepsilon_k$. We shall now construct even piecewise linear functions G_k , $k = 1, 2, \dots$, and trigonometrical polynomials P_k with frequencies from Λ , each G_k being positive and decreasing on $I_k \cap (0, \infty)$, $G_k = 0$ outside I_k , $B(G_k) < 1$, $\| G_k \|_{\mathbb{R}} \leq 1$, $G_k = G_{k+1}$ in I_{k-1} , $k = 2, 3, \dots$ and such that

$$(2.3) \quad \| \hat{f}_k - P_k G_k \|_{\mathbb{R}} \leq \delta_k, \quad k = 1, 2, \dots,$$

and

$$(2.4) \quad \max\{1, B(P_1), \dots, B(P_k)\} \| G_{k+1} - G_k \|_{\mathbb{R}} \leq \delta_{k+1}, \quad k = 1, 2, \dots$$

To begin, clearly there exists an even piecewise linear function G_1 positive, decreasing on $I_1 \cap (0, \infty)$ and vanishing outside I_1 such that $B(G_1) < 1$ and $\|G_1\|_{\mathbb{R}} < 1$. Recall that \hat{f}_1 is zero outside $J_1 \subset I_1$. By (2.2), there is a trigonometric polynomial P_1 with frequencies from Λ such that $\|\hat{f}_1/G_1 - P_1\|_{I_1} < \delta_1$. Since G_1 vanishes outside I_1 and $B(G_1) < 1$, this implies (2.3) with $k = 1$.

Next, one can choose an even piecewise linear function G_2 positive decreasing on $I_2 \cap (0, \infty)$ and vanishing outside I_2 such that G_2 is so close to G_1 (hence, G_2 is close to zero on $I_2 \setminus I_1$) that $B(G_2) < 1$, $\|G_2\|_{\mathbb{R}} < 1$ and (2.4) holds with $k = 1$. The same argument we used for G_1 shows that there exists P_2 such that (2.3) holds with $k = 1$.

Assume that we have found G_1, \dots, G_n satisfying the properties above. Then, clearly, we may find G_{n+1} which is even, piecewise linear, positive and decreasing on $I_{n+1} \cap (0, \infty)$ and vanishing outside I_{n+1} such that $G_{n+1} = G_n$ on I_{n-1} and G_{n+1} is so close to G_n that $B(G_{n+1}) < 1$, $\|G_{n+1}\|_{\mathbb{R}} < 1$ and (2.4) holds with $k = n$. Again, the argument we used for G_1 shows that there is a trigonometrical polynomial P_{n+1} with frequencies from Λ such that (2.3) holds true with $k = n + 1$.

Now define $\Phi = G_k$ on I_{k-1} . Then,

$$\|\Phi\|_{\mathbb{R}} \leq \lim_{k \rightarrow \infty} \|G_k\|_{\mathbb{R}} \leq 1,$$

which shows that $\varphi \in L^1$. Moreover, by (2.3) and (2.4), since $\hat{f}_k = 0$ outside $J_k \subset I_k$ and $G_n = 0$ outside I_n , we have for every $k = 2, 3, \dots$

$$\begin{aligned} \|\hat{f}_k - P_k \Phi\|_{\mathbb{R}} &= \|\hat{f}_k - P_k G_{k+1}\|_{I_k} + \sum_{n=k}^{\infty} \|P_k G_{n+2}\|_{I_{n+1} \setminus I_n} \\ &\leq \|\hat{f}_k - P_k G_k\|_{\mathbb{R}} + \|P_k G_{k+1} - P_k G_k\|_{\mathbb{R}} + \sum_{n=k}^{\infty} \|P_k G_{n+2} - P_k G_{n+1}\|_{\mathbb{R}} \\ &\leq \delta_k + \delta_{k+1} + \sum_{n=k+1}^{\infty} \delta_{n+1} = \varepsilon_k, \end{aligned}$$

as desired.

3. Uniqueness sets for generalized Bernstein classes

In this section we prove Theorem 2 stated in the Introduction.

3.1. We first verify the ‘easy’ part (b) \Rightarrow (a). We assume that Λ is a uniqueness set for some B_σ with $\sigma(y) \nearrow \infty$ as $y \rightarrow +\infty$, and we want to produce $\varphi \in B_\sigma$ which is a Λ -generator. The proof is based on the fact that for any $\sigma(y) \nearrow \infty$ we have $B_\sigma \cap N \neq \emptyset$, where the class N is defined in (1.4).

More precisely, we need the following

Lemma 3.1 *For any $\sigma(y) \nearrow \infty, y \rightarrow \infty$, there is a positive function on $(0, \infty)$, $\omega(s) \nearrow \infty$, such that*

$$\int_0^\infty e^{ys-\omega(s)} ds \leq e^{y\sigma(y)}, \quad y \geq 0.$$

This is a simple and well-known fact, and so we omit the proof.

The proof of (a) \Rightarrow (b) is as follows. First we apply Lemma 3.1 to $\sigma(y) - 2\epsilon$, with some fixed number $\epsilon > 0$. Define g as $\hat{g}(s) = e^{-\omega(|s|)}$, then $g \in B_{\sigma-2\epsilon}$. Set $\varphi(t) = g(t) \operatorname{sinc}^2(\epsilon t)$, where $\operatorname{sinc} t = \sin t/t$. The Fourier transform of $\operatorname{sinc}(\epsilon t)$ is $\chi_\epsilon(s)$, the characteristic function of the interval $[-\epsilon, \epsilon]$. Hence, $\hat{\varphi} = \hat{g} \star \chi_\epsilon \star \chi_\epsilon$. Since \hat{g} is everywhere positive, the same is true for $\hat{\varphi}$, which gives $\varphi \in N$.

We shall use the trivial estimate:

$$(3.1) \quad |\operatorname{sinc}^2 \epsilon(x + iy)| \leq C \frac{e^{2\epsilon|y|}}{1 + x^2 + y^2}, \quad x + iy \in \mathbb{C},$$

where C is some constant. This estimate shows that $\varphi \in B_\sigma$ and that $\varphi(x + iy)$ is in L^1 on any line $z = x + iy, -\infty < x < \infty$. The latter implies that for every $f \in L^\infty(\mathbb{R})$ the convolution $f \star \check{\varphi}$ is defined at every complex point $x + iy$. Using (3.1) and $g \in B_{\sigma-2\epsilon}$ we obtain:

$$\begin{aligned} |f \star \check{\varphi}(x + iy)| &= \left| \int_{-\infty}^\infty \varphi(x + iy - s) f(s) ds \right| \leq \|f\|_\infty \int_{-\infty}^\infty |\varphi(x + iy)| dx \\ &\leq \|f\|_\infty \max_x |g(x + iy)| \int_{-\infty}^\infty C \frac{e^{2\epsilon|y|}}{1 + x^2 + y^2} dx \\ &\leq C_1 e^{|y|\sigma(|y|)}, \quad \exists C_1 > 0, \quad x + iy \in \mathbb{C}. \end{aligned}$$

Hence, $L^\infty(\mathbb{R}) \star \varphi \subseteq B_\sigma$. As explained in paragraph 1.3 this shows that φ is a Λ -generator for $L^1(\mathbb{R})$.

3.2. For the proof of (a) \Rightarrow (b) we start recalling the definition of the Beurling-Malliavin exterior density. There are several equivalent definitions of this density D_{BM} . One suitable for us here is as follows (see [4, vol. 2]). Suppose $\Lambda \subset (0, \infty)$ and let D be a positive number. A family of disjoint intervals $I_k = (a_k, b_k)$, $0 < a_1 < b_1 < \dots, a_k \nearrow +\infty$ is called *substantial for D* if

$$\frac{n_\Lambda(a_k, b_k)}{b_k - a_k} > D, \quad k = 1, 2, \dots, \quad \sum_k \left(\frac{b_k - a_k}{b_k} \right)^2 = +\infty.$$

Here $n_\Lambda(I)$ is the number of points of Λ in an interval I .

The density $D_{BM}(\Lambda)$ is then defined,

$$D_{BM}(\Lambda) = \sup \{ D > 0 : \text{there exists a substantial family for } D \}.$$

If no $D > 0$ admits a substantial family, one sets $D_{BM}(\Lambda) = 0$. For a general Λ , one defines $D_{BM}(\Lambda) = \max\{D_{BM}(\Lambda^+), D_{BM}(\Lambda^-)\}$ where $\Lambda^+ = \Lambda \cap \mathbb{R}^+$, $\Lambda^- = (-\Lambda) \cap \mathbb{R}^+$. From now on we may assume $\Lambda \subset (0, \infty)$. Observe that traditionally in the definition of substantial family, one writes $\sum_k (b_k - a_k)^2 a_k^{-2} = \infty$, while for our purposes it is more convenient to use $\sum_k (b_k - a_k)^2 b_k^{-2} = \infty$. However, these definitions are equivalent, since as remarked in [4], the divergence of the first series implies the divergence of the other.

According to the definitions, $D_{BM}(\Lambda) = +\infty$ means that for every $D > 0$ there exists a substantial family of intervals for D . In order to quantify this we introduce the following definition.

Definition Suppose $\Lambda \subset (0, \infty)$. If $\Psi(s)$ is an increasing function in $(0, +\infty)$, we call a family of disjoint intervals $I_k = (a_k, b_k) \subset (0, \infty)$ substantial for Ψ if

$$\frac{\Lambda(a_k, b_k)}{b_k - a_k} > \Psi(b_k - a_k) \quad \text{and} \quad \sum_k \left(\frac{b_k - a_k}{b_k} \right)^2 = +\infty.$$

By a diagonal procedure it is immediate to prove:

Lemma 3.2 If $\Lambda \subset (0, \infty)$ and $D_{BM}(\Lambda) = +\infty$, there exists an increasing function $\Psi(s) \nearrow +\infty$ which admits a substantial family of intervals.

This lemma gives all we will use on Λ . To prove the implication (a) \Rightarrow (b), for a given Λ with $D_{BM}(\Lambda) = \infty$, we have to find a function $\sigma(y) \nearrow \infty$ such that Λ is a uniqueness set for B_σ . This is the subject of the following

Theorem 3.3 Assume $\Lambda \subset (0, \infty)$, and a family of disjoint intervals $I_k = (a_k, b_k)$ is substantial for some function $\Psi(s) \nearrow \infty$. Then Λ is a uniqueness set for B_σ , whenever σ satisfies:

(a) For all $x \in \mathbb{R}$

$$(3.2) \quad \sigma(x) \leq \frac{1}{2e} \Psi\left(\frac{x}{2e}\right).$$

(b) There exists a sequence of integers $n_j \rightarrow \infty$ such that

$$(3.3) \quad \frac{1}{\sigma(2b_{n_j})} \sum_{k=1}^{n_j} \Psi(b_k - a_k) \left(\frac{b_k - a_k}{b_k} \right)^2 \rightarrow \infty, \quad j \rightarrow \infty.$$

It is clear that for any function $\Psi(s) \nearrow \infty$ there is a function $\sigma(s) \nearrow \infty$ which satisfies both assumptions (3.2) and (3.3). Hence, the implication (a) \Rightarrow (b) of Theorem 2 is a consequence of Theorem 3.3.

3.3. The proof of Theorem 3.3 will be a consequence of a series of lemmas. Denote by B_σ^0 the subclass of B_σ of functions

$$(3.4) \quad |F(x + iy)| \leq e^{|y|^\sigma(|y|)}, \quad x + iy \in \mathbb{C}.$$

Lemma 3.4 *Assume a function $F \in B_\sigma^0$ has n zeros on some interval $[a, b]$. Then*

$$(3.5) \quad |F(x)| \leq (b - a)^n \min_{y>0} \frac{e^{y^\sigma(y)}}{y^n}, \quad \text{for every } x \in [a, b].$$

Proof. Indeed, set $Q(x) = \prod_{k=1}^n (x - x_k)$ where x_k are the zeros of F on (a, b) . The following interpolation formula is well known: for every $x \in [a, b]$ there is a number $t \in (a, b)$ such that

$$F(x) = \sum_{k=1}^n \frac{F(x_k)Q(x)}{Q'(x_k)(x - x_k)} + \frac{F^{(n)}(t)Q(x)}{n!} = \frac{F^{(n)}(t)Q(x)}{n!}.$$

By Cauchy's inequality, we have

$$\begin{aligned} |F(x)| &\leq |Q(x)| \max_{a \leq t \leq b} \frac{|F^{(n)}(t)|}{n!} \leq (b - a)^n \max_{a \leq t \leq b} \min_{R>0} \frac{\max_{0 \leq \theta < 2\pi} |F(t + Re^{i\theta})|}{R^n} \\ &\leq (b - a)^n \min_{y>0} \frac{e^{y^\sigma(y)}}{y^n}. \end{aligned} \quad \blacksquare$$

Lemma 3.5 *Suppose Ψ and σ satisfy (3.2). If $F \in B_\sigma^0$ has*

$$n \geq (b - a) \Psi(b - a)$$

zeros on an interval (a, b) , $a > 0$, then

$$(3.6) \quad \int_a^b \frac{\log |F(x)|}{x^2} dx \leq -(\log 2) \left(\frac{b - a}{b} \right)^2 \Psi(b - a).$$

Proof. Let y_n be the number defined by $y_n \sigma(y_n) = n$. Then, by (3.5),

$$|F(x)| \leq (b - a)^n \frac{e^{y_n \sigma(y_n)}}{y_n^n} = (b - a)^n e^{n - n \log y_n}, \quad \text{for every } x \in [a, b].$$

Hence,

$$\begin{aligned} \int_a^b \frac{\log |F(x)|}{x^2} dx &\leq \frac{b - a}{ab} (n \log(b - a) + n - n \log y_n) \\ &= -\frac{b - a}{ab} \left(n \log \frac{y_n}{e(b - a)} \right). \end{aligned}$$

Since $n \geq (b - a) \Psi(b - a)$, it follows from (3.2) and the definition of y_n that $y_n \geq 2e(b - a)$, and this gives (3.6). \blacksquare

Lemma 3.5. relates the size of the logarithmic integral to the number of zeros. One can obtain a slightly better estimate by using Jensen's formula for ellipses (see [4] for the case of Bernstein classes). On the other hand, the next lemma establishes that the logarithmic integral of $F \in B_\sigma$ cannot be too large negative.

Lemma 3.6 *If $F \in B_\sigma^0$, $F \not\equiv 0$ then*

$$(3.7) \quad \int_{1 \leq |x| \leq R} \frac{\log |F(x)|}{x^2} dx \geq -\frac{16}{3} \sigma(2R) + o(1), \quad R \rightarrow \infty.$$

Proof. We use Carleman's formula [6, Ch. 5]; for $R > 1$:

$$\int_1^R \left(\frac{1}{x^2} - \frac{1}{R^2} \right) \log |F(x) F(-x)| dx + \frac{2}{R} \int_0^\pi \log |F(Re^{i\theta})| \sin \theta d\theta \geq C$$

where C depends only on F . By (3.4), $\log |F(x)| \leq 0$. Using that

$$x^{-2} \leq \frac{4}{3}(x^{-2} - R^{-2}) \quad \text{for } |x| \leq R/2,$$

we get

$$\frac{3}{4} \int_{1 \leq |x| \leq R/2} \frac{\log |F(x)|}{x^2} \geq C - 4\sigma(R).$$

■

In fact the proof of the lemma shows that for F satisfying

$$|F(z)| = O(e^{\sigma(|z|)|z|}),$$

$\log^- |F(x)|$ controls $\log^+ |F(x)|$.

With Lemmas 3.5. and 3.6. we can now complete the proof of Theorem 3.3. Suppose $G \not\equiv 0 \in B_\sigma$ vanishes on Λ . Set $F = G/C$ where C is a constant such that $F \in B_\sigma^0$. Then Lemma 3.5. applies to every $I_k = [a_k, b_k]$, so adding (3.6) in $k \leq n$

$$\int_1^{b_n} \frac{\log |F(x)|}{x^2} dx \leq -C_1 \sum_{k=1}^n \Psi(b_k - a_k) \left(\frac{b_k - a_k}{b_k} \right)^2,$$

which combined with (3.7) gives

$$\sigma(2b_n) - C_2 \sum_{k=1}^n \Psi(b_k - a_k) \left(\frac{b_k - a_k}{b_k} \right)^2 \geq C_3,$$

with constants C_1 , C_2 and C_3 independent of n . This is in contradiction with hypothesis (3.3). Hence $F \equiv 0$, and Λ is a uniqueness set for B_σ .

4. Pairs of generators

Definition 2 We say that Λ admits a pair of generators if there exist two L^1 -functions φ_1 and φ_2 such that all linear combinations of $\varphi_1(t - \lambda_1)$ and $\varphi_2(t - \lambda_2)$, $\lambda_1, \lambda_2 \in \Lambda$, are dense in $L^1(\mathbb{R})$.

Suppose a sequence Λ has a finite spectral radius. Then by Theorem 1, the Λ -translations of one function cannot be dense in $L^1(\mathbb{R})$. However, Λ may admit a pair of generators. Observe that all perturbations of the integers (1.2) have spectral radius π (that is, $D_{BM}(\Lambda) = 1$). If the a_n in (1.2) are exponentially small then Λ does admit a pair of generators, as shown in the following

Theorem 4.1 Suppose a real sequence $a_n, n \in \mathbb{Z}$, satisfies:

$$(4.1) \quad 0 \neq |a_n| \leq Cr^{|n|}, \quad n \in \mathbb{Z},$$

where $C > 0$ and $0 < r < 1$ are some constants. Then the sequence $\Lambda = \{n + a_n\}_{n \in \mathbb{Z}}$ admits a pair of generators.

It is easy to check that the set of integers itself does not admit a pair (nor any finite number) of generators. Thus the situation with two generators in $L^1(\mathbb{R})$ is somewhat similar to the situation with one generator in $L^2(\mathbb{R})$, where the set of integers does not admit generator while every small perturbation of it does.

We shall need two auxiliary lemmas.

Lemma 4.2 Suppose a_n satisfy (4.1), $\rho < \pi$, and a function $g \in B_\rho$. If there is a constant $C > 0$ such that $|g(n + a_n)| \leq C|a_n|$ for all $n \in \mathbb{Z}$ then $g \equiv 0$.

Proof. Let $b > 0$ satisfy $\rho + b < \pi$. By (3.1), the function $g(z) \operatorname{sinc} bz$ belongs to $B_{\rho+b} \cap L^2(\mathbb{R})$ (that is it belongs to the Paley-Wiener space $PW_{\rho+b}$). Let us estimate the values of g at the integer points:

$$(4.2) \quad |g(n)| \leq |g(n + a_n)| + |g(n + a_n) - g(n)| \leq (C + \|g'\|_\infty)|a_n| = K|a_n|,$$

where $K < \infty$ since by Bernstein's inequality $\|g'\|_\infty \leq (2\rho + b)\|g\|_\infty < \infty$.

Set $G = \hat{g}$. Then $G \in L^2(\mathbb{R})$ and is concentrated on $[-\rho - b, \rho + b]$. Write

$$G(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} g(-n)e^{inx},$$

where the Fourier series converges to G in L^2 norm.

One can easily verify that by (4.1) and (4.2) the Fourier series admits analytic continuation into some strip in the complex plane. However, since $G(x) = 0$ on $(\rho + b, \pi)$, the Fourier series is identically zero, and we conclude that $G(x) = 0$ a.e. \blacksquare

To formulate the next lemma, we introduce two auxiliary functions

$$\varphi_1(t) = \operatorname{sinc}^2 at \sum_{k=-\infty}^{\infty} e^{-|k|+2\pi ikt}, \quad \varphi_2(t) = e^{-i\pi t} \varphi_1(t),$$

where a is a constant to be chosen later.

Lemma 4.3 *Suppose $0 < a < \pi/2$, and a_n satisfy (4.1). Then the set $\Lambda = \{n + a_n\}$ is a uniqueness set for each class $L^\infty \star \check{\varphi}_j, j = 1, 2$.*

Proof. We use an argument similar to the one in the proof of the main result of [9]. Clearly, $\varphi_1 \in L^1(\mathbb{R})$, for all values a . Since $\operatorname{sinc} at$ has as Fourier transform the characteristic function $\chi_a(x)$ of $[-a, a]$, we get

$$\hat{\varphi}_1 = \chi_a \star \chi_a \star \sum_{k=-\infty}^{\infty} e^{-|k|} \delta_{2\pi k},$$

where δ_c is the unite measure concentrated at the point c . This shows that the support of the Fourier transforms of φ_j is as follows:

$$(4.3) \quad \begin{aligned} \operatorname{supp} \hat{\varphi}_1 &= \bigcup_{k=-\infty}^{\infty} [-2a + 2k\pi, 2a + 2k\pi] = [-2a, 2a] + 2\pi\mathbb{Z}, \\ \operatorname{supp} \hat{\varphi}_2 &= \operatorname{supp} \hat{\varphi}_1 + \pi. \end{aligned}$$

Observe that both $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are strictly positive on their support.

We have to verify that $f \in L^\infty(\mathbb{R})$ and $f \star \check{\varphi}_j(n + a_n) = 0$, for all $n \in \mathbb{Z}$, imply $f \star \check{\varphi}_j = 0$. However, it suffices to check this for $j = 1$ only. Indeed, since $\varphi_2(t) = e^{-i\pi t} \varphi_1(t)$, we have

$$f \star \check{\varphi}_2(t) = e^{\pi it} [(f e^{-\pi is}) \star \check{\varphi}_1](t),$$

and the implication for $j = 2$ follows from the one for $j = 1$. Notice too that φ_1 is even, that is, $\check{\varphi}_1 = \varphi_1$.

Set

$$g_j(t) = \sum_{k=-\infty}^{\infty} e^{-|k|} k^j \int_{-\infty}^{\infty} \operatorname{sinc}^2 a(t-s) e^{-2\pi iks} f(s) ds, \quad j = 0, 1, 2, \dots$$

so that

$$f \star \varphi_1(t) = \sum_{k=-\infty}^{\infty} e^{-|k|+2\pi ikt} \int_{-\infty}^{\infty} \operatorname{sinc}^2 a(t-s) e^{-2\pi iks} f(s) ds = \sum_{j=0}^{\infty} \frac{(2\pi it)^j}{j!} g_j(t).$$

To prove the lemma, we show by induction that $f \star \varphi_1(n + a_n) = 0, n \in \mathbb{Z}$, implies

$$(4.4) \quad g_j \equiv 0, \quad j = 0, 1, 2, \dots$$

By (3.1) we see that $g_j \in B_{2a}$ for all $j = 0, 1, \dots$. Since

$$0 = f \star \check{\varphi}_1(n + a_n) = \sum_{k=-\infty}^{\infty} e^{-|k|+2\pi ika_n} \int_{-\infty}^{\infty} \operatorname{sinc}^2 a(n + a_n - s) e^{-2\pi iks} f(s) ds,$$

we obtain for each $n \in \mathbb{Z}$

$$\begin{aligned} |g_0(n + a_n)| &= \left| \sum_{k=-\infty}^{\infty} e^{-|k|} (e^{2\pi ika_n} - 1) \int_{-\infty}^{\infty} \operatorname{sinc}^2 a(n + a_n - s) e^{-2\pi iks} f(s) ds \right| \\ &\leq C_0 |a_n|, \end{aligned}$$

where

$$C_0 = \|f\|_{\infty} \|\operatorname{sinc}^2 at\|_1 \sum_{k=-\infty}^{\infty} 2\pi |k| e^{-|k|} < \infty.$$

Hence, by Lemma 4.2, we see that (4.4) holds for $j = 0$.

Suppose (4.4) is true for $0 \leq n < l$. Clearly, for each integer l there is a constant K_l such that the inequality

$$\left| e^{i\alpha} - \sum_{j=0}^l \frac{(i\alpha)^j}{j!} \right| \leq K_l |\alpha|^{l+1}$$

holds for every real α . Applying it with $\alpha = 2\pi ka_n$, we obtain:

$$\begin{aligned} \frac{(2\pi a_n)^l}{l!} |g_l(n + a_n)| &= \left| f \star \check{\varphi}_1(n + a_n) - \sum_{k=0}^l \frac{(2\pi ia_n)^k}{k!} g_k(n + a_n) \right| \\ &= \left| \sum_{k=-\infty}^{\infty} e^{-|k|} \left(e^{2\pi ika_n} - \sum_{j=0}^l \frac{(2\pi ika_n)^j}{j!} \right) \int_{-\infty}^{\infty} \operatorname{sinc}^2 a(n + a_n - s) e^{-2\pi iks} f(s) ds \right| \\ &\leq C_l |a_n|^{l+1}, \end{aligned}$$

where

$$C_l = K_l(2\pi)^{l+1} \sum_{k=-\infty}^{\infty} e^{-|k|} |k|^{l+1} \|f\|_{\infty} \|\operatorname{sinc}^2 at\|_1 < \infty.$$

Hence,

$$|g_l(n + a_n)| \leq C_l |a_n|, n \in \mathbb{Z}$$

(note that here is where the assumption $a_n \neq 0$ is used). Lemma 4.2 gives $g_l \equiv 0$, so that (4.4) is true, which proves the lemma. ■

Let us now turn to the proof of Theorem 4.1. We shall now simply check that for every $\pi/4 < a < \pi/2$ the functions φ_1 and φ_2 form a pair of generators for every sequence $\Lambda = \{n + a_n\}$ satisfying the assumptions of Theorem 4.1. Assume a function $f \in L^{\infty}(\mathbb{R})$ satisfies $f \star \check{\varphi}_j(n + a_n) = 0$, $j = 1, 2$, for all $n \in \mathbb{Z}$. To prove Theorem 4.1 we have to show that $f = 0$ a.e. By Lemma 4.3, $f \star \check{\varphi}_j = 0$, $j = 1, 2$. Hence, $f \star (\check{\varphi}_1 + \check{\varphi}_2) = 0$. However, as it easily follows from (4.3), the function $\hat{\varphi}_1 + \hat{\varphi}_2$ is everywhere positive for $a > \pi/4$. We conclude, by Wiener's theorem, that $f = 0$ a.e.

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