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Random walks on graphs with volume and time doubling

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Abstract

This paper studies the on- and off-diagonal upper estimate and the two-sided transition probability estimate of random walks on weighted graphs.

1. Introduction

Let us consider a countable infinite connected graph Γ. A weight function $\mu_{x,y} = \mu_{y,x} > 0$ is given on the edges $x \sim y$. This weight induces a measure $\mu(x)$

$$\mu(x) = \sum_{y \sim x} \mu_{x,y},\ \ \mu(A) = \sum_{y \in A} \mu(y)$$

on the vertex set $A \subset \Gamma$ and defines a reversible Markov chain $X_n \in \Gamma$, i.e. a random walk on the weighted graph (Γ, μ) with transition probabilities

$$P(x,y) = \frac{\mu_{x,y}}{\mu(x)},$$
$$P_n(x,y) = \mathbb{P}(X_n = y | X_0 = x).$$

For a set $A \subset \Gamma$ the killed random walk is defined by the transition operator restricted to $c_0(A)$, and the corresponding transition probability is denoted by $P_n^A(x,y)$.

The graph is equipped with the usual (shortest path length) graph distance $d(x,y)$ and open metric balls are defined for $x \in \Gamma$, $R > 0$ as $B(x,R) = \{y \in \Gamma : d(x,y) < R\}$ and its $\mu-$measure is denoted by

$$V(x,R) = \mu\left(B\left(x,R\right)\right). \tag{1.1}$$

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If $\Gamma = \mathbb{Z}^d$ and $\mu_{x,y} = 1$ if $d(x,y) = 1$ we get back the classical, simple symmetric nearest neighbor random walk on $\mathbb{Z}^d$. This random walk serves as a discrete approximation and model for the diffusion in continuous space and time. It is widely accepted that the interesting phenomena and results found on continuous space and time have their random walks counterparts (and vice versa) (c.f. just as example [13], [3] of the link between the two frameworks). The first rigorously studied fractal type graph was the Sierpiński triangle (see Figure 1).

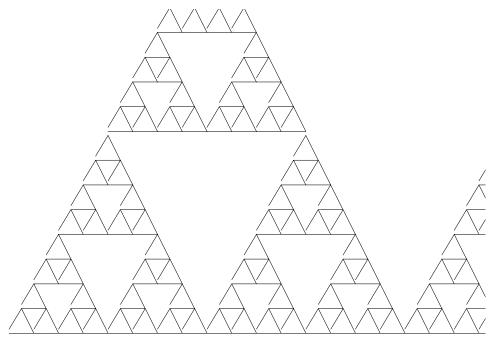

Figure 1 The Sierpiński graph

On this graph the volume growth is polynomial

$$V(x,R) \simeq R^\alpha$$

with exponent $\alpha = \frac{\log 3}{\log 2}$. Here $\simeq$ means that the ratio of the two functions of r is uniformly separated from zero and infinity. On this infinite graph the transition probability estimate has the form (c.f. [12])

$$\begin{aligned} p_n(x,y) &\leq \frac{C}{n^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right), \\ p_n(x,y) + p_{n+1}(x,y) &\geq \frac{c}{n^{\alpha/\beta}} \exp\left(-C\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right), \end{aligned}$$

where $C, c > 0$, and the walk dimension is $\beta = \frac{\log 5}{\log 2} > 2$. This walk moves slower than the classical one due to the big holes and narrow connections. This is reflected in the exponent $\beta > 2$. In the classical $\mathbb{Z}^d$ case the mean exit time $E(x,R) \simeq R^2$, which is the expected value of the time needed by the walk to leave the ball $B(x,R)$. For the Sierpiński graph it is $E(x,R) \simeq R^\beta$ with $\beta = \frac{\log 5}{\log 2} > 2$. Many efforts have been devoted to the investigation of other particular fractals and general understanding what kind of structural properties are responsible for the leading and exponential term of the upper and lower estimate (for further background and literature please see [1], [10]). The next challenge was to obtain such kind of "heat kernel" estimates on a larger class of graphs and drop the Ahlfors regularity: $V(x,R) \simeq R^{\alpha}$. An easy example for such a graph is described in [8]. The Vicsek tree is considered, which is built in a recursive way. If the weights assigned to the edges are slightly increasing by the distance to the root, the resulted weighted graph is not Ahlfors regular any more (see Figure 2),

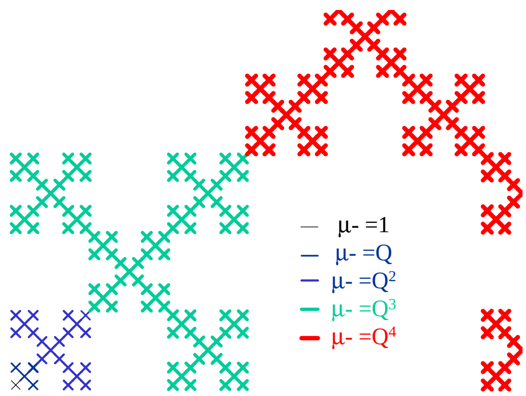

Figure 2 The Vicsek tree with increasing weights

but satisfies the volume doubling condition (see Definition 1.1.) In [8] the authors gave necessary and sufficient conditions for two-sided sub-Gaussian estimates of the following form,

$$p_n(x,y) \leq \frac{C}{V(x,n^{1/\beta})}\exp\left(-c\left(\frac{d^{\beta}}{n}\right)^{\frac{1}{\beta-1}}\right), \tag{1.2}$$

$$p_n(x,y)+p_{n+1}(x,y) \geq \frac{c}{V(x,n\alpha^{1/\beta})}\exp\left(-C\left(\frac{d^{\beta}}{n}\right)^{\frac{1}{\beta-1}}\right) \tag{1.3}$$

which is local in the volume $V(x,R)$ but the mean exit time is uniform with respect to the space, $E(x,R) \simeq R^{\beta}$. (See [1], [7], [8] or [16] for further remarks and history of the heat kernel estimates.) One can rise the next natural question:

> What can be said if the mean exit time is not polynomial, and what if it depends on the center of the test ball?

The present paper answers both questions. Before we explain the results let us see an example based again on the Vicsek tree. Now the edges have been replaced with paths of slowly increasing length (as we depart from the root) (see Figure 3). Let us consider a vertex x in a middle of a distant path.

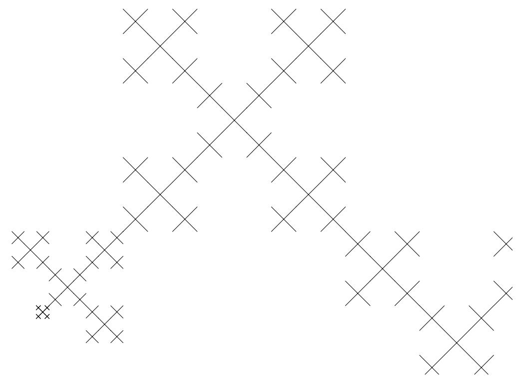

Figure 3 The Vicsek tree with path of increasing lenght

Let be the radius r of the test ball is smaller (or comparable) to the half of the length of the path, then we have a classical one dimensional walk in that ball, consequently $E(x,r) \simeq r^2$. While for big R we have the large scale behavior of the Vicsek tree, hence $E(x,R) \simeq R^{\frac{\log 15}{\log 3}} >> R^2$ for large R. On the other hand all balls centered at the root has the usual behavior of the Vicsek tree, $E(x,R) \simeq R^{\frac{\log 15}{\log 3}}$. We shall see that this stretched Vicsek tree has all the properties needed to obtain an upper bound for the heat kernel. The details of this example will be given in Section 5. Several further graphs can be constructed in a similar way. For instance we consider a graph which possesses some nice properties and replace the edges (or well-defined subgraphs) with elements of a class of graphs (again with increasing size as we depart from a reference vertex) connecting them on a subset of prescribed vertices. Such a construction is shown in Figure 4.

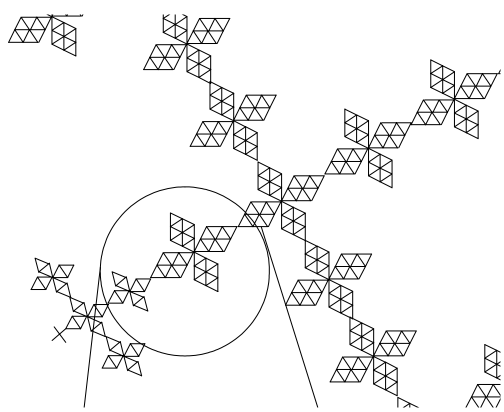

Figure 4 Vicsek tree with embedded triangles

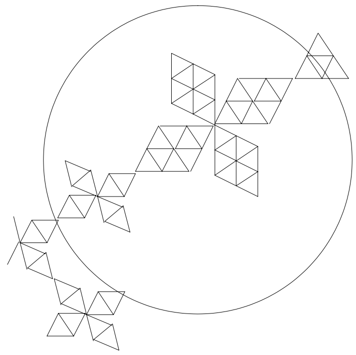

Figure 5 Enlarged part of the tree

Here we replace the edges of the Vicsek tree with diamonds formed by two Sierpiński triangles As the distance grows from the root, bigger Sierpiński triangles are inserted. (To keep the needed properties of the graph the increase of the size of the triangles should be slow.)

In order to present the main results we have to introduce the essential notions of the paper.

Definition 1.1 *The weighted graph has the* volume doubling (**VD**) *property if there is a constant $D_V > 0$ such that for all $x \in \Gamma$ and $R > 0$*

$$V(x, 2R) \leq D_V V(x, R). \tag{1.4}$$

Notation 1 *For convenience we introduce a short notation for the volume of the annulus; $v = v(x, r, R) = V(x, R) - V(x, r)$.*

Notation 2 *For two real sequences a_ξ, b_x we will write*

$$a_\xi \simeq b_\xi$$

if there is a $C \geq 1$ such that for all ξ

$$\frac{1}{C} a_\xi \leq b_\xi \leq C a_\xi.$$

Now let us consider the exit time

$$T_{B(x,R)} = \min\{k : X_k \notin B(x, R)\}$$

from the ball $B(x, R)$ and its mean value

$$E_z(x, R) = \mathbb{E}(T_{B(x,R)} | X_0 = z)$$

and let us use the

$$E(x, R) = E_x(x, R)$$

short notation.

Definition 1.2 *We will say that the weighted graph* (Γ, μ) *satisfies the time comparison principle* (**TC**) *if there is a constant* $C_T > 1$ *such that for all* $x \in \Gamma$ *and* $R > 0, y \in B(x, R)$

$$\frac{E(x, 2R)}{E(y, R)} \leq C_T. \tag{1.5}$$

Definition 1.3 *We will say that* (Γ, μ) *has* time doubling *property* (**TD**) *if there is a* $D_T > 0$ *such that for all* $x \in \Gamma$ *and* $R \geq 0$

$$E(x, 2R) \leq D_T \, E(x, R). \tag{1.6}$$

Remark 1.1 *It is clear that from the time doubling property* (TD) *it follows that there are* $C > 0$ *and* $\beta > 0$ *such that for all* $x \in \Gamma$ *and* $R > r > 0$

$$\frac{E(x, R)}{E(x, r)} \leq C \left(\frac{R}{r}\right)^{\beta}. \tag{1.7}$$

Basically the volume doubling condition (1.4) and the time comparison principle (1.5) specify the local framework for our study. It is clear that (TC) implies (TD), the time doubling property, while the reverse seems to be not true even if (VD) is assumed.

The goal of the present paper is twofold. First we would like to give characterization of graphs which have on- and off-diagonal upper estimates if neither the volume nor the mean exit time is uniform in the space like in the above examples. Secondly we give characterization of graphs which have two-sided heat kernel estimates. For that we consider graphs with the volume doubling property and it is assumed that the mean exit time is uniform in the space, more precisely satisfies (**E**):

$$E(x, R) \simeq E(y, R) \tag{1.8}$$

holds, i.e. the mean exit time does not depend on the center of the ball. The *semi-local framework* will be the determined by the conditions (VD) and (E).

In [16], it was shown that for strongly recurrent graphs upper estimates can be obtained in the local framework and two-sided estimates in the semi-local framework. Here we present similar results dropping the condition of strong recurrence and generalize them in many respect.

Condition 1 *In several statements we assume that condition* $(\mathbf{p}_0)$ *holds, that is, there is a universal* $p_0 > 0$ *such that for all* $x, y \in \Gamma, x \sim y$

$$\frac{\mu_{x,y}}{\mu(x)} \geq p_0. \tag{1.9}$$

Notation 3 *For a set $A \subset \Gamma$ denote the closure by*

$$\overline{A} = \{y \in \Gamma : \text{there is an } x \in A \text{ such that } x \sim y\}.$$

The external boundary is defined as $\partial A = \overline{A} \backslash A$.

Definition 1.4 *A function h is harmonic on a set $A \subset \Gamma$ if it is defined on $\overline{A}$ and*

$$Ph(x) = \sum_y P(x,y)\, h(y) = h(x)$$

for all $x \in A$.

Theorem 1.1 *For any weighted graph (Γ, μ), if $(p_0), (VD)$ and (TC) hold, then the following statements are equivalent:*

1. *The mean value inequality* (**MV**) *holds: there is a $C > 0$ such that for all $x \in \Gamma, R > 0$ and for all $u \geq 0$ harmonic functions on $B = B(x,R)$*

$$u(x) \leq \frac{C}{V(x,R)} \sum_{y \in B} u(y)\, \mu(y). \tag{1.10}$$

2. *The local diagonal upper estimate* (**DUE**) *holds: there is a $C > 0$ such that for all $x \in \Gamma, n > 0$*

$$p_n(x,x) \leq \frac{C}{V(x, e(x,n))}, \tag{1.11}$$

   where $e(x,n)$ is the inverse of $E(x,R)$ in the second variable.

3. *The upper estimate* (**UE**) *holds: there are $C, \beta > 1, c > 0$ such that for all $x, y \in \Gamma$, $n > 0$*

$$p_n(x,y) \leq \frac{C}{V(x, e(x,n))} \exp\left[-c\left(\frac{E(x, d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{1.12}$$

The existence of e will be clear from the properties of the mean exit time (c.f. Section 3).

This theorem can be given in a different form if we introduce the skewed version of the parabolic mean value inequality.

Definition 1.5 *We shall say that the skewed parabolic mean value inequality holds if for* $0 < c_1 < c_2$ *constants there is a* $C \geq 1$ *such that for all* $R > 0$, $x \in \Gamma$, $y \in B = B(x,R)$ *for all non-negative Dirichlet solutions* u_k *of the heat equation*

$$P^B u_k = u_{k+1} \tag{1.13}$$

on $[0, c_2 E(x,R)] \times B(x,R)$

$$u_n(x) \leq \frac{C}{V(y,2R)E(y,2R)} \sum_{i=c_1 E}^{c_2 E} \sum_{z \in B(x,R)} u_i(z)\mu(z) \tag{1.14}$$

is satisfied, where $E = E(x,R)$, $n = c_2 E$.

Remark 1.2 *One can see easily with the choice of* $u_i(y) \equiv 1$ *that* (1.14) *implies* (VD) *and* (TC).

Having this condition the above theorem can be restated as follows.

Theorem 1.2 *If* (Γ, μ) *satisfies* (p_0) *then the following conditions are equivalent:*

1. *the skewed parabolic mean value inequality* (1.14) *holds;*
2. (VD), (TC) *and* (MV) *holds;*
3. (VD), (TC) *and the diagonal upper estimate* (DUE) *holds;*
4. (VD), (TC) *and the upper estimate* (UE) *holds.*

The above results deal with graphs which satisfy the volume doubling property and time comparison principle. Let us observe that the exponent in (1.12) depends on x, not only on the distance between x and y. To find matching exponents for the upper and lower off-diagonal estimates it seems plausible to assume that the mean exit time is (up to a constant) is uniform in the space, that is it satisfies (E) :

$$E(x,R) \simeq E(y,R).$$

It is convenient to specify a function $F(R)$ for $R \geq 0$

$$F(R) = \inf_{x \in \Gamma} E(x,R),$$

for which from (E) it follows that there is a $C_0 > 1$ such that for all $x \in \Gamma$ and $R \geq 0$

$$F(R) \leq E(x,R) \leq C_0 F(R). \tag{1.15}$$

This function inherits certain properties of $E(x,R)$, first of all from the time doubling property it follows that F also has doubling property:

$$F(2R) \leq D_E F(R). \tag{1.16}$$

We shall say that F is a (very) *proper space time scale function* if it has certain properties which will be defined in Section 4 (c.f. Definition 4.3).

The function F with the inherited properties will take over the role of R^β (or R^2). The inverse function of F, $f(.) = F^{-1}(.)$ takes over the role of $(R^{\frac{1}{\beta}})$ $R^{\frac{1}{2}}$ in the (sub-) Gaussian estimates. The existence of f will be shown in Section 4.

Definition 1.6 *The sub-Gaussian kernel function with respect to a function F is $k = k(n,R) \geq 1$, defined as the maximal integer for which*

$$\frac{n}{k} \leq F\Big(\Big\lfloor \frac{R}{k} \Big\rfloor\Big) \tag{1.17}$$

or $k = 0$ by definition if there is no appropriate k.

Definition 1.7 *The transition probability satisfies* $(\mathbf{UE}_F)$, *the sub-Gaussian upper estimate with respect to F, if there are $c, C > 0$ such that for all $x, y \in \Gamma, n > 0$*

$$p_n(x,y) \leq \frac{C}{V(x,f(n))} \exp\big[-c\,k(n,d(x,y))\big], \tag{1.18}$$

and $(\mathbf{LE}_F)$, *the sub-Gaussian lower estimate, is satisfied if*

$$\widetilde{p}_n(x,y) \geq \frac{c}{V(x,f(n))} \exp\big[-Ck(n,d(x,y))\big], \tag{1.19}$$

where $\widetilde{p_n} = p_n + p_{n+1}$.

In the semi-local framework we have the following result.

Theorem 1.3 *If a weighted graph (Γ,μ) satisfies (p_0) then the following statements are equivalent:*

1. *for a very proper F, (UE_F) and (LE_F) hold;*
2. *for a very proper F the F-parabolic Harnack inequality holds;*
3. *$(VD), (E)$ and the elliptic Harnack inequality hold.*

The definition of the elliptic and parabolic Harnack inequality and some other definitions are given in Sections 2 and 4.

In Section 4 characterization of graphs satisfying separately the upper estimate (UE_F) will also be given

Let us mention that in this generality Hebisch and Saloff-Coste in [11] proved the equivalence of 1. and 2. of Theorem 1.3.

The complete characterization of graphs which have two-sided heat kernel Gaussian estimates (c.f. (1.2), (1.3) with $\beta = 2$) was given by Delmotte [5]. This characterization has been extended to two-sided sub-Gaussian estimates ($\beta \geq 2$) in [8].

In a recent work Li and Wang [14] proved in the context of complete Riemannian manifolds that if the *volume doubling property* holds then the following particular upper bound for the Green kernel,

$$g^B(x,y) = \int_0^\infty p_t^B(x,y)\,dt,$$

for $B = B(x,R)$:

$$g^B(p,x) \leq C \int_{d^2(x,p)}^{(CR)^2} \frac{dt}{V\left(p,\sqrt{t}\right)}, \tag{1.20}$$

implies the mean value inequality

$$u(x) \leq \frac{C}{V(x,R)} \int_{B(x,R)} u(y)\,dy \tag{1.21}$$

for $u \geq 0$ sub-harmonic functions. The opposite direction remained open question. Here we show that for weighted graphs in the semi-local framework, namely under conditions $(VD), (TD)$ and (E), the mean value inequality (MV) implies a Green's function bound equivalent to (1.20).

The organization of the paper is the following. In Section 2 we collect the basic definitions and preliminary observations. In Section 3 we discuss the local framework and present a much more detailed version of Theorem 1.1. In Section 4 we study the semi-local setup and prove Theorem 1.3.

Acknowledgment. The author is deeply indebted to Professor Alexander Grigor'yan. Neither this nor recent other papers of the author would exists without his ideas and friendly support. Particularly he proposed to study what is the necessary and sufficient condition of the off-diagonal upper estimate.

2. Basic definitions and preliminaries

In this section we recall basic definitions and observations (mainly from [16], but we warn the reader that there are minor deviations from the conventions that have been used there).

Definition 2.1 *The random walk defined on* (Γ,μ) *will be denoted by* (X_n)*. It is a reversible Markov chain on the state space* Γ*, reversible with respect to the measure* μ *and has one step transition probability*

$$\mathbb{P}(X_{n+1}=y|X_n=x)=P(x,y)=\frac{\mu_{x,y}}{\mu(x)}.$$

2.1. Volume doubling

The volume function V has been already defined in (1.1).

Remark 2.1 *It is evident that on weighted graphs the volume doubling property* (VD) *is equivalent with the volume comparison principle, namely there is a constant* $C_V>1$ *such that for all* $x\in\Gamma$ *and* $R>0, y\in B(x,R)$

$$\frac{V(x,2R)}{V(y,R)}\leq C_V. \tag{2.1}$$

Proposition 2.1 *If* (p_0) *holds, then, for all* $x,y\in\Gamma$ *and* $R>0$ *and for some* $C>1$*,*

$$V(x,R) \leq C^R\mu(x), \tag{2.2}$$

$$p_0^{d(x,y)}\mu(y) \leq \mu(x), \tag{2.3}$$

and for any $x\in\Gamma$

$$|\{y:y\sim x\}|\leq\frac{1}{p_0}. \tag{2.4}$$

Proof. (c.f. [7, Proposition 3.1]) ■

Remark 2.2 *It follows from the inequality* (2.2) *that, for a fixed* R_0*, for all* $R<R_0$*,*

$$V(x,R)\simeq\mu(x).$$

Remark 2.3 *It is easy to show (c.f. [4]) that the volume doubling property implies an anti-doubling property: there is an* $A_V > 1$ *such that for all* $x \in \Gamma, R > 0$

$$2V(x,R) \le V(x, A_V R). \tag{2.5}$$

One can also show that (VD) *is equivalent with*

$$\frac{V(x,R)}{V(y,S)} \le C\left(\frac{R}{S}\right)^{\alpha},$$

where $\alpha = \log_2 D_V$ *and* $d(x,y) \le R$*, which is the original form of Gromov's volume comparison inequality (c.f. [9]). (For the proof see again [4].)*

Remark 2.4 *Another direct consequence of* (p_0) *and* (VD) *is that*

$$v\left(x,R,2R\right) = V(x,2R) - V(x,R) \simeq V\left(x,R\right). \tag{2.6}$$

2.2. The resistance

Definition 2.2 *For any two disjoint sets,* $A, B \subset \Gamma$*, the resistance,* $\rho(A,B)$*, is defined as*

$$\rho(A,B) = \left(\inf\left\{\left(\left(I-P\right)f,f\right)_\mu : f|_A = 1, f|_B = 0\right\}\right)^{-1}$$

and we introduce

$$\rho(x,S,R) = \rho(B(x,S), \Gamma\backslash B(x,R))$$

for the resistance of the annulus around $x \in \Gamma$*, with* $R > S \ge 0$*.*

Definition 2.3 *We say that the product of the resistance and volume of the annulus is uniform in the space if for all* $x, y \in \Gamma, R \ge 0$

$$\rho(x,R,2R)v(x,R,2R) \simeq \rho(y,R,2R)v(y,R,2R). \tag{2.7}$$

Corollary 2.2 *For all weighted graphs,* $x \in \Gamma, r \ge s \ge 0$

$$\rho(x,s,r)v(x,s,r) \ge (r-s)^2, \tag{2.8}$$

Proof. The idea of the proof has been taken from [15], for the details see [17]. ■

2.3. The mean exit time

Let us introduce the exit time T_A.

Definition 2.4 *The exit time from a set A is defined as*

$$T_A = \min\{k : X_k \in \Gamma \backslash A\},$$

its expected value is denoted by

$$E_x(A) = \mathbb{E}(T_A | X_0 = x)$$

and let us use the $E = E(x,R) = E_x(x,R)$ short notation.

In this section we introduce several properties of the mean exit time which will play crucial role in the whole sequel.

The time comparison principle evidently implies the following weaker inequality: there is a $C > 0$ such that

$$\frac{E(x,R)}{E(y,R)} \le C \tag{2.9}$$

for all $x \in \Gamma, R > 0, y \in B(x,R)$. One can observe that (2.9) is the difference between (TC) and (TD). It is easy to see that

$$(TC) \Longleftrightarrow (TD) + (2.9)\,.$$

It also follows easily that (TC) is equivalent with the existence of a $C, \beta \ge 1$ constants for which

$$\frac{E(x,R)}{E(y,S)} \le C\left(\frac{R}{S}\right)^{\beta}, \tag{2.10}$$

for all $y \in B(x,R), R \ge S > 0$.

Remark 2.5 *It is easy to see that condition (p_0) implies that*

$$E(x,R) \le \left(\frac{1}{p_0}\right)^{R} \qquad \text{for all } x \in \Gamma,\ R \ge 1.$$

Definition 2.5 *The maximal mean exit time is defined as*

$$\overline{E}(A) = \max_{x \in A} E_x(A)$$

and particularly the $\overline{E}(x,R) = \overline{E}(B(x,R))$ notation will be used.

Definition 2.6 *The local kernel function $\underline{k}$, $\underline{k} = \underline{k}(n,x,R) \ge 1$, is defined as the maximal integer for which*

$$\frac{n}{k} \le \min_{y \in B(x,R)} E\Big(y, \left\lfloor \frac{R}{k} \right\rfloor\Big) \tag{2.11}$$

or $\underline{k} = 0$ by definition if there is no appropriate k.

2.4. Mean value inequalities

Definition 2.7 *The random walk on the weighted graph is a reversible Markov chain and the Markov operator P is naturally defined by*

$$Pf(x) = \sum P(x,y) f(y).$$

Definition 2.8 *The Laplace operator on the weighted graph (Γ, μ) is defined simply as*

$$\Delta = P - I.$$

Definition 2.9 *For $A \subset \Gamma$ consider the Markov operator P^A restricted to A. This operator is the Markov operator of the killed Markov chain, which is killed on leaving A, also corresponds to the Dirichlet boundary condition on A. Its iterates are denoted by P_k^A.*

Definition 2.10 *The Laplace operator with Dirichlet boundary conditions on a finite set $A \subset \Gamma$ is defined as*

$$\Delta^A f(x) = \begin{cases} \Delta f(x) & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}.$$

The smallest eigenvalue of $-\Delta^A$ is denoted in general by $\lambda(A)$ and for $A = B(x,R)$ it is denoted by $\lambda = \lambda(x,R) = \lambda(B(x,R))$.

Definition 2.11 *The energy or Dirichlet form $\mathcal{E}(f,f)$ associated to the electric network can be defined as*

$$\mathcal{E}(f,f) = -(\Delta f, f) = \frac{1}{2} \sum_{x,y \in \Gamma} \mu_{x,y} (f(x) - f(y))^2.$$

Using this notation, the smallest eigenvalue of $-\Delta^A$ can be defined by

$$\lambda(A) = \inf \left\{ \frac{\mathcal{E}(f,f)}{(f,f)} : f \in c_0(A), f \neq 0 \right\} \tag{2.12}$$

as well.

Definition 2.12 *We introduce*

$$G^A(y,z) = \sum_{k=0}^{\infty} P_k^A(y,z)$$

the local Green function, the Green function of the killed walk and the corresponding Green's kernel as

$$g^A(y,z) = \frac{1}{\mu(z)} G^A(y,z).$$

Definition 2.13 *We say that the parabolic mean value inequality holds on* (Γ,μ) *if for fixed constants* $0 \leq c_1 < c_2$ *there is a* $C > 1$ *such that for arbitrary* $x \in \Gamma$, $n \in \mathbb{N}$ *and* $R > 0$, *using the notations* $E = E(x,R), B = B(x,R), n = c_2 E, \Psi = [0,n] \times B$ *for any non-negative Dirichlet solution of the heat equation*

$$P^B u_i = u_{i+1}$$

on Ψ, *the inequality*

$$u_n(x) \leq \frac{C}{V(x,R)E(x,R)} \sum_{i=c_1 E}^{n} \sum_{y \in B(x,R)} u_i(y)\mu(y) \tag{2.13}$$

holds.

Remark 2.6 *Let us observe the difference between the definitions of the parabolic and skewed parabolic mean value inequality in Definitions 1.5 and 2.13. As it was noted in Remark 1.2 the skewed parabolic mean value inequality implies* (VD) *and* (TC), *which yields in fact the equivalence*

$$(1.14) \Longleftrightarrow (2.13) + (VD) + (TC).$$

Remark 2.7 *The above definitions of the parabolic mean value inequality and mean value inequality can be extended to sub-solutions and the corresponding results remain valid.*

Definition 2.14 *We say that the a mean value property holds for the Green kernels on* (Γ,μ) *if there is a* $C > 1$ *such that for all* $R > 0, x \in \Gamma$, $B = B(x,R)$ *and* $y \in \Gamma, d = d(x,y) > 0$

$$g^B(y,x) \leq \frac{C}{V(x,d)} \sum_{z \in B(x,d)} g^B(y,z)\mu(y). \tag{2.14}$$

Definition 2.15 *We say that the Green kernel satisfy upper bound with respect to a function* F *(c.f. [14]) on* (Γ,μ) *if for a* $C' > 1$ *there is a* $C > 1$ *such that for all* $R > 0$ *and* $y \in \Gamma, d =: d(x,y) > 0$, $B = B(x,R)$

$$g^B(y,x) \leq \sum_{i=F(d)}^{C'F(R)} \frac{C}{V(x,f(i))} \tag{2.15}$$

where $f(.)$ *is the inverse function of* $F(.)$.

Definition 2.16 *The Green kernel is bounded by the ratio of the mean exit time and volume on* (Γ,μ) *if there is a* $C > 1$ *such that for all* $R > 0$ *and* $y \in \Gamma, d =: d(x,y) > 0, B = B(x,R)$

$$g^B(y,x) \leq C\frac{E(x,R)}{V(x,d)}. \tag{2.16}$$

3. The local theory

In this section we shall prove the following theorem, which implies Theorem 1.1 and by Remark 2.6, Theorem 1.2 as well.

Theorem 3.1 *For a weighted graph* (Γ, μ) *if* $(p_0), (VD), (TC)$ *conditions hold, then the following statements are equivalent:*

1. *the local diagonal upper estimate* (**DUE**) *holds; there is a* $C > 0$ *such that for all* $x \in \Gamma$, $n > 0$
$$p_n(x,x) \le \frac{C}{V(x, e(x,n))};$$
2. *the upper estimate* (**UE**) *holds: there are* $C, \beta > 1, c > 0$ *such that for all* $x, y \in \Gamma$, $n > 0$
$$p_n(x,y) \le \frac{C}{V(x, e(x,n))} \exp\left[-c\left(\frac{E(x, d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right];$$
3. *the parabolic mean value inequality* (2.13) *holds;*
4. *the mean value inequality* (MV) *holds;*
5. (2.14) *holds;*
6. (2.16) *holds.*

Proposition 3.2 *For any weighted graph* (Γ, μ) *if the inequality*

$$\overline{E}(x,R) \le CE(x,R) \tag{3.1}$$

holds, then the local diagonal lower estimate

$$P_{2n}(x,x) \ge \frac{c\mu(x)}{V(x, e(x,2n))} \tag{3.2}$$

is true and

$$\mathbb{P}(T_{x,R} < n) \le C\exp\left[-c\underline{k}(x,n,R)\right], \tag{3.3}$$

where $\underline{k}$ *is local sub-Gaussian kernel defined in* (2.11).

The statement (3.3) is given in [16, Theorem 5.1], (3.2) in [16, Propositions 6.4, 6.5].

Remark 3.1 *It is worth to observe that the diagonal lower estimate* (3.2) *and the diagonal upper estimate* (1.11) *matches up to a constant.*

3.1. Properties of the mean exit time

In this section we recall some results from [17] which describe the behavior of the mean exit time in the local framework. The first one is the Einstein relation below in its multiplicative form which plays an important role.

Theorem 3.3 *If* $(p_0), (VD)$ *and* (TC) *hold then* (ER), *the Einstein relation*

$$E(x,2R) \simeq \rho(x,R,2R)v(x,R,2R) \tag{3.4}$$

holds.

For the proof see [17].

Lemma 3.4 *On all* (Γ,μ) *for any* $x\in\Gamma, R>S>0$

$$E(x,R+S) \geq E(x,R) + \min_{y\in S(x,R)} E(y,S).$$

Proof. Let us denote $A=B(x,R), B=B(x,R+S)$. First let us observe that from the triangular inequality it follows that for any $y\in S(x,R)$

$$B(y,S)\subset B(x,R+S).$$

From this and from the strong Markov property one obtains that

$$\begin{aligned} E(x,R+S) &= E_x\left(T_B + E_{X_{T_B}}(x,R+S)\right) \\ &\geq E(x,R) + E_x\left(E_{X_{T_B}}(X_{T_B},S)\right). \end{aligned}$$

But $X_{T_B}\in S(x,R)$ which gives the statement. ■

Corollary 3.5 *The mean exit time is strictly increasing in* $R\in\mathbb{N}$ *and hence* $E(x,R)$ *has an inverse in the second variable*

$$e(x,n) = \min\{R\in\mathbb{N} : E(x,R)\geq n\}.$$

Proof. From Lemma 3.4 and $E(x,1)\geq 1$ it follows that

$$E(x,R+1) \geq E(x,R)+1. \tag{3.5}$$

■

Definition 3.1 *We shall say that the mean exit time has the anti-doubling property if there is an* $A_E>1$ *such that for all* $x\in\Gamma, R>0$

$$E(x,A_ER) \geq 2E(x,R). \tag{3.6}$$

Proposition 3.6 *If* (p_0) *and* (TC) *hold then* (3.6), *anti-doubling for the mean exit time holds.*

Proof. In [17] it is shown that (2.9) implies (3.6), but from (TC) the inequality (2.9) follows. ■

The anti-doubling property of the mean exit time is equivalent with the existence of $c, \beta' > 0$ such that

$$\frac{E(x,R)}{E(x,S)} \geq c\left(\frac{R}{S}\right)^{\beta'} \tag{3.7}$$

for all, $R > S > 0$, $x \in \Gamma$, $y \in B(x,R)$.

Sometimes we will refer to the pair of doubling and anti-doubling property as doubling properties. The properties of the inverse function e of E (which exists by Corollary 3.5) and properties of E are linked as the following evident lemma states.

Lemma 3.7 *The following statements are equivalent:*

1. *There are* $C, c > 0, \beta \geq \beta' > 0$ *such that for all* $x \in \Gamma, R \geq S > 0$, $y \in B(x,R)$

$$c\left(\frac{R}{S}\right)^{\beta'} \leq \frac{E(x,R)}{E(y,S)} \leq C\left(\frac{R}{S}\right)^{\beta}. \tag{3.8}$$

2. *There are* $C, c > 0, \beta \geq \beta' > 0$ *such that for all* $x \in \Gamma, n \geq m > 0$, $y \in B(x, e(x,n))$

$$c\left(\frac{n}{m}\right)^{1/\beta} \leq \frac{e(x,n)}{e(y,m)} \leq C\left(\frac{n}{m}\right)^{1/\beta'}. \tag{3.9}$$

Remark 3.2 *Let us recall that the doubling property and the anti-doubling property of* E *is equivalent with the right and left hand side of* (3.8) *for* $y = x$.

The following corollary is from [17].

Corollary 3.8 *If* (p_0), (VD) *and*

$$\max_{y\in B(x,R)} E_y(x,R) \leq CE(x,R) \tag{3.10}$$

hold, then

$$E(x,R) \geq cR^2, \tag{3.11}$$

i.e.,

$$\beta \geq 2. \tag{3.12}$$

Remark 3.3 *In the present context we need a weaker statement,*

$$(p_0) + (VD) + (TC) \Longrightarrow (3.11)\,, (3.12)\,.$$

This immediately follows from Theorem 3.3 and (2.8) *and the fact that* (TC) *implies* (3.10).

Lemma 3.9 *If* (p_0), (VD) *and* (TC) *hold, then for* $\underline{k} = \underline{k}(x,n,R)$ *defined in* (2.11),

$$\underline{k} + 1 \geq c \left(\frac{E(x,R)}{n} \right)^{\frac{1}{\beta-1}} \quad . \tag{3.13}$$

for all $x \in \Gamma$, $R, n > 0$ *for fixed* $c > 0$, $\beta > 1$.

Proof. The statement follows from (TC) easily, $\beta > 1$ is ensured by $\beta \geq 2$ from Corollary 3.8. ■

3.2. The resolvent

In this section we recall from [8] the key intermediate step to prove the diagonal upper estimate. We introduce for a finite set $A \subset \Gamma$ the $\lambda, m-$resolvent

$$G^A_{\lambda,m} = \left((\lambda + 1)\, I - P^A \right)^{-m}$$

for $\lambda \geq 0, m \geq 0$ and let us define the kernel corresponding to the resolvent as

$$g^A_{\lambda,m}\,(x,y) = \frac{1}{\mu\,(y)}\; G^A_{\lambda,m}\,(x,y)\,.$$

Theorem 3.10 *Assuming* (p_0), (VD) *and* (TC) *the condition* (2.16) *implies, for a large enough* $m > 1$ *and for all* $0 < \lambda < 1$, $x \in \Gamma$, *the inequality*

$$g_{\lambda,m}(x,x) \leq C\; \frac{\lambda^{-m}}{V(x, e(x,\lambda^{-1}))}. \tag{3.14}$$

The proof closely follows the corresponding proof of [8, Theorem 5.7] so we omit it. One should reproduce it simply replacing the space-time scale function R^β by $E\,(x,R)$ and using the doubling properties repeatedly.

3.3. The local diagonal upper estimate

We start with the application of the $\lambda, m-$resolvent bound to obtain the local diagonal upper estimate.

Theorem 3.11 *The conditions* (p_0) , (VD) , (TC) *and* (3.14) *imply* (DUE) ,

$$p_n(x,x) \le \frac{C}{V(x,e(x,n))}.$$

Again the proof is easy modification of [8, Theorem 6.1] therefore we skip it.

Lemma 3.12 *If* (p_0) , (VD) *and* (TC) *hold then*

$$(2.14) \Leftrightarrow (MV) \Longrightarrow (2.16)\,.$$

Proof. First we show (2.14) $\Longrightarrow (MV)$. Denote $B=B(x,R)$, $U=B(x,2R)$.

Let $u \ge 0$ a harmonic function on $B(x,R)$ and consider it's representation:

$$u\left(z\right) = \sum_{w\in U} g^{U}\left(z,w\right)\nu\left(w\right)$$

which always exists with a $\nu \ge 0, \nu \in c_0\left(U\right)$ charge (the standard construction can be reproduced following the proof of [7, Lemma 10.2]). Applying this decomposition and (2.14) to $u\left(x\right)$ the mean value inequality follows.

$$\begin{aligned} u\left(x\right) &= \sum_{w\in U} g^{U}\left(x,w\right)\nu\left(w\right) \le \frac{C}{V\left(x,R\right)} \sum_{w\in U}\sum_{z\in B} g^{U}\left(z,w\right)\nu\left(w\right)\mu\left(z\right) \\ &= \frac{C}{V\left(x,R/2\right)} \sum_{z\in B}\sum_{w\in U} g^{U}\left(z,w\right)\nu\left(w\right)\mu\left(z\right) \le \frac{C}{V\left(x,R\right)} \sum_{z\in B} u\left(z\right)\mu\left(z\right). \end{aligned}$$

The opposite implication $(MV) \Longrightarrow (2.14)$ follows simply applying (MV) to $u\left(x\right) = g^{U}\left(x,w\right)$. Finally $(MV) \Longrightarrow (2.16)$ is immediate. If $d = d(x,y) > R$ then $g^{B(x,R)}(x,y) = 0$ and there is nothing to prove. Otherwise, consider the function $u(z) = g^{B(x,2R)}(y,z)$. This function is non-negative and harmonic in the ball $B(x,d)$. Hence, by (MV), (VD) and (TC)

$$u(x) \le \frac{C}{V(x,d)} \sum_{z\in B(x,d)} u(z)\mu(z) \le \frac{C}{V(x,d)}\overline{E}(x,2R) \le C\frac{E(x,R)}{V(x,d)}.$$

Finally (2.16) follows from $g^{B(x,R)} \le g^{B(x,2R)}$. ■

3.4. From (DUE) to (UE)

The proof is easy modification of the nice argument given in [6] for the corresponding implication.

Lemma 3.13 *Let* $r = \frac{1}{2}d(x,y)$ *then*

$$p_{2n}(x,y) \le P_x(T_{x,r} < n) \max_{\substack{n \le k \le 2n \\ v \in \partial B(x,r)}} p_k(v,y) + P_y(T_{y,r} < n) \max_{\substack{n \le k \le 2n \\ z \in \partial B(y,r)}} p_k(z,x). \tag{3.15}$$

Proof. The statement follows from the first exit decompositions staring from x (and from y respectively) and from the Markov property as in [6]. ■

Theorem 3.14 $(p_0) + (VD) + (TC) + (DUE) \Longrightarrow (UE)$.

Lemma 3.15 *If* $(p_0), (VD)$ *and* (TC) *hold then for all* $\varepsilon > 0$ *there are* $C_\varepsilon, C > 0$ *such that for all* $k > 0, y, z \in \Gamma, r = d(y,z)$

$$\sqrt{\frac{V(y,e(y,k))}{V(v,e(z,k))}} \le C_\varepsilon \exp\left[\varepsilon C \left(\frac{E(y,r)}{k}\right)^{\frac{1}{(\beta-1)}}\right].$$

Proof. Let us consider the minimal m for which $e(y,m) \ge r$,

$$e(y,k) \le e(y,k+m)$$

and use the anti doubling property with $\beta' > 0$ to obtain

$$\begin{aligned}\sqrt{\frac{V(y,e(y,k))}{V(z,e(z,k))}} &\le \sqrt{\frac{V(y,e(y,k+m))}{V(z,e(z,k))}} \\ &\le C\left(\frac{e(y,k+m)}{e(z,k)}\right)^{\alpha/2} \le C\left(\frac{k+m}{k}\right)^{\frac{\alpha}{2\beta'}} = C\left(1+\frac{m-1+1}{k}\right)^{\frac{\alpha}{2\beta'}} \\ &\le C\left(1+\frac{E(y,r)+1}{k}\right)^{\frac{\alpha}{2\beta'}} \le C_\varepsilon \exp\left[\varepsilon C\left(\frac{E(y,r)}{k}\right)^{\frac{1}{(\beta-1)}}\right].\end{aligned}$$

Here we have to note that by Remark 3.3 it follows that $\beta > 1$ furthermore from the conditions that $\alpha, \beta' > 0$. The manipulation of the exponents used the trivial estimate $1 + x^{\frac{a}{a}} \le \left(1 + x^{\frac{1}{a}}\right)^a$, where $x, a > 0$. As a result we obtain by repeated application of (TC) that

$$\sqrt{\frac{V(y,e(y,k)+r)}{V(z,e(z,k))}} \le C_\varepsilon \exp\left[\varepsilon C\left(\frac{E(y,r)}{k}\right)^{\frac{1}{(\beta-1)}}\right].$$

■

Proof of Theorem 3.14. If $d(x,y) \leq 2$ then the statement follows from (p_0) according to Remark 2.5. We use (3.15) with $r=\frac{1}{2}d(x,y)$ and start to handle the first term in

$$P_x(T_{x,r}<n)\max_{\substack{n\leq k\leq 2n\\ v\in\partial B(x,r)}} p_k(v,y).$$

Let us recall that from (TC) it follows that

$$\mathbb{P}(T_{x,r}<n)\leq C\exp\left[-c\underline{k}(x,n,r)\right], \tag{3.16}$$

and use $r\leq d(v,y)\leq 3r$ furthermore (3.13) to get

$$P_x(T_{x,r}<n)\leq C\exp\left[-c\left(\frac{E(x,r)}{n}\right)^{\frac{1}{\beta-1}}\right].$$

Let us treat the other term. First we observe that

$$\begin{aligned} p_{2k+1}(y,v) &\leq \sum_{z\sim y} P_{2k}(y,z)\,P(z,v)\frac{1}{\mu(v)} \\ &= \sum_{z\sim y} P_{2k}(y,z)\,P(v,z)\frac{1}{\mu(z)} \\ &\leq \max_{z\sim y} p_{2k}(y,z)\sum_{z\sim v} P(v,z) \\ &= \max_{z\sim y} p_{2k}(y,z). \end{aligned} \tag{3.17}$$

and recall that

$$p_{2k}(x,y)\leq\sqrt{p_{2k}(x,x)\,p_{2k}(y,y)},$$

which yields using the doubling properties of V, E and for $w\sim v$ $d(y,v)\simeq d(y,w)$ (provided $v,w\neq y$) that

$$\max_{\substack{n\leq k\leq 2n\\ w\in\partial B(x,r)}} p_k(w,y) \tag{3.18}$$

$$\leq \max_{\substack{n\leq 2k\leq 2n\\ v\sim w\in\partial B(x,r)}} p_{2k}(v,y) \tag{3.19}$$

$$\leq \max_{\substack{n\leq 2k\leq 2n\\ v\sim w\in\partial B(x,r)}} \frac{C}{\sqrt{V(y,e(y,2k))\,V(v,e(v,2k))}} \tag{3.20}$$

$$\leq \max_{\substack{n\leq 2k\leq 2n\\ v\sim w\in\partial B(x,r)}} \frac{C}{V(y,e(y,n))}\sqrt{\frac{V(y,e(y,n))}{V(v,e(v,n))}},$$

Let us observe that $d(v,y)\leq 3r+2\leq 5r$ if $r\geq 1$ and apply Lemma 3.15 to proceed with

$$\begin{aligned}&\max_{\substack{n\leq 2k\leq 2n\\ v\sim w\in\partial B(x,r)}} p_{2k}(v,y)\,P_x(T_{x,r}<n)\\ &\qquad\leq \frac{C}{V(y,e(y,n))}C_\varepsilon \exp\left[\varepsilon C\left(\frac{E(y,5r)}{n}\right)^{\frac{1}{(\beta-1)}}-c\left(\frac{E(x,r)}{n}\right)^{\frac{1}{(\beta-1)}}\right],\end{aligned}$$

choosing ε small enough and applying (TC) we have the inequality

$$\begin{aligned}&\max_{\substack{n\leq k\leq 2n\\ v\in\partial B(x,r)}} p_k(v,y)\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right]\\ &\qquad\leq \frac{C}{V(y,e(y,n))}\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right].\end{aligned}$$

By symmetry one gets

$$\begin{aligned}p_{2n}(x,y)&\leq C\left(\frac{1}{V(x,e(x,n))}+\frac{1}{V(y,e(y,n))}\right)\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right]\\ &=\frac{C}{V(x,e(x,n))}\left(1+\frac{V(x,e(x,n))}{V(y,e(y,n))}\right)\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right].\end{aligned}$$

Now we use Lemma 3.15 again obtain

$$\frac{V(x,e(x,n))}{V(y,e(y,n))}\leq C_\varepsilon\exp\varepsilon C\left(\frac{E(x,2r)}{n}\right)^{\frac{1}{(\beta-1)}}$$

and ε can be chosen to satisfy $\varepsilon C<\frac{c}{2}$ to get

$$\left(1+\exp\left[\left(\varepsilon C-\frac{c}{2}\right)\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right]\right)\leq 2$$

$$p_{2n}(x,y)\leq\frac{2C}{V(x,e(x,n))}\exp\left[-\frac{c}{2}\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{(\beta-1)}}\right],$$

which is the needed estimate for even n. For odd number of steps the results follows using for $x\neq y$ the trivial inequality (3.17) and $d(x,y)\simeq d(x,z)$ if $z\neq x$, $y\sim z$. In particular if the maximum in (3.17) attained at $x=z$ then the statement follows from (DUE) and (p_0). ■

Remark 3.4 *With a slight modification of the beginning of the proof one can get*

$$p_n(x,y) \leq \frac{C\exp\left[-c\underline{k}(y,n,r)\right]}{V(x,e(x,n))} + \frac{C\exp\left[-c\underline{k}(x,n,r)\right]}{V(y,e(y,n))} \tag{3.21}$$

where $r = \frac{1}{2}d(x,y)$, *which is sharper then the above upper estimate. Let us note that our deduction shows that* (3.21) *is equivalent with the upper estimate.*

Proof of Theorem 3.1. Let us assume that the conditions (p_0), (VD) and (TC) hold. From Theorem 3.10 and 3.11 it follows that (2.16) $\Longrightarrow$ (3.14) $\Longrightarrow$ (DUE), which covers the implication 6. $\Longrightarrow$ 1. From Lemma 3.12 we know that $(MV) \Longleftrightarrow$ (2.14) and $(MV) \Longrightarrow$ (2.16) i.e. 4. $\Longleftrightarrow$ 5. $\Longrightarrow$ 6. In Theorem 3.14 we have shown that $(DUE) \Longrightarrow (UE)$, which means $1 \Longrightarrow 2$ while the reverse implication is trivial. The parabolic mean value inequality, (2.13) implies (MV), i.e. $3 \Longrightarrow 4$. It is left to show that $2 \Longrightarrow 3$. i.e. $(UE) \Longrightarrow$ (2.13). We shall show a little bit more. Let us consider a Dirichlet solution $u_i(w) \geq 0$ on $B(x,R)$ with initial data $u_0 \in c_0(B(x,R))$. Denote $E = E(x,R)$. Consider $v_i(z) = \mu(z)u_i(z)$, $0 < c_1 < c_2 < c_3 < c_4$ and $n \in [c_3E, c_4E]$, $j \in [c_1E, c_2E]$. By definition

$$u_n(w) = \sum_{y\in\Gamma} P_{n-j}^B(w,y)\,u_j(y)$$

and

$$v_n(w) = \sum_{y\in B(x,R)} P_{n-j}^B(y,w)\,v_j(y) \leq \mu(w)\max_{y\in B} p_{n-j}^B(y,w) \sum_{y\in B(x,R)} v_j(y),$$

from which one has by $p^B \leq p$ and (UE) that

$$\begin{aligned} u_n(w) &\leq \max_{y\in B} p_{n-j}^B(w,y) \sum_{y\in B(x,R)} u_j(y)\,\mu(y) \\ &\leq \frac{C}{V(w,e(w,n-j))} \sum_{y\in B(x,R)} u_j(y)\,\mu(y). \end{aligned}$$

Using the doubling properties of e and V it follows that $V(w,e(w,n-j)) \simeq V(x,R)$ and

$$u_n(w) \leq \frac{C}{V(x,R)} \sum_{y\in B(x,R)} u_j(y)\,\mu(y). \tag{3.22}$$

Finally summing (3.22) for $j \in [c_1 E, c_2 E]$ we obtain

$$u_n(w) \le \frac{C}{E(x,R)\, V(x,R)} \sum_{j=c_1 E}^{c_2 E} \sum_{y \in B(x,R)} u_j(y)\, \mu(y).$$

This means that this inequality holds for all

$$(n,w) \in [c_3 E, c_4 E] \times B(x,R) = \Psi,$$

e.g. using the properties of V and E again, for all $y \in V(x,R)$

$$\max_{\Psi} u \le \frac{C}{E(y,2R)\, V(y,2R)} \sum_{j=c_1 E}^{c_2 E} \sum_{y \in B(x,R)} u_j(y)\, \mu(y). \tag{3.23}$$

also satisfied. It is clear that (3.23) implies (1.14) and (2.13) as well which finishes the proof. ■

4. Semi-local theory

This section is split into two parts. In the first part the reformulation and extension of the upper estimates are developed. In this part typically we work under the assumptions of (VD), (TD) and (E). In the second part, in Section 4.4, we discuss the two-sided estimate. There the main assumptions are (VD), (E) and the elliptic Harnack inequality (to be defined there).

4.1. The upper estimate

Let us start with the definition of the F-parabolic mean value inequality.

Definition 4.1 *We shall say that the $F-$parabolic mean value inequality holds if for the function $F(R) = \inf_{x\in\Gamma} E(x,R)$, $c_2 > c_1 > 0$ constants there is a $C > 1$ such that for all $R > 0, x \in \Gamma$ for all non-negative Dirichlet solutions u_n of the discrete heat equation*

$$P^{B(x,R)} u_n = u_{n+1}$$

on $[0, c_2 E(x,R)] \times B(x,R)$

$$u_n(x) \le \frac{C}{V(x,2R) E(x,2R)} \sum_{i=c_1 F}^{c_2 F} \sum_{z \in B(x,R)} u_i(z) \mu(z) \tag{4.1}$$

is satisfied, where $F = F(R)$, $n = c_2 F(R)$.

Remark 4.1 *Let us observe that in this definition the volume doubling property and time comparison principle are "built in", as in the skewed parabolic mean value inequality. The condition* $E \simeq F$ *follows from* 4.1 *as well.*

In this section we prove the following theorems.

Theorem 4.1 *For any weighted graph* (Γ, μ) *if* (p_0), (VD), (TD) *and* (E) *hold then the following statements are equivalent:*

1. *for a proper function* F*, the* F*-based diagonal upper estimate hold, that is, there is a* $C > 0$ *such that for all* $x \in \Gamma, n > 0$

$$P_n(x,x) \leq \frac{C\mu(x)}{V(x, f(n))}; \tag{4.2}$$

2. *the estimate,* (UE_F) *holds for a proper* F*: there are* $C, c > 0$ *such that for all* $x, y \in \Gamma, n > 0$

$$P_n(x,y) \leq \frac{C\mu(y)}{V(x, f(n))} \exp -ck(n, d(x,y)); \tag{4.3}$$

3. *the parabolic mean value inequality* (2.13) *holds,*
4. *the mean value inequality* (MV) *holds;*
5. (2.14) *holds;*
6. (2.15) *holds;*
7. (2.16) *holds.*

For the notion of (very-) proper F see Definition 4.3, and the existence of the inverse of F in the next section. Similarly to Theorem 1.2 the following is true.

Theorem 4.2 *If* (Γ, μ) *satisfies* (p_0) *then the following conditions are equivalent.*

1. *the* $F-$*parabolic mean value inequality* (4.1) *holds for a proper* F*;*
2. (VD), (TD), (E) *and* (MV) *hold;*
3. (VD), (TD), (E) *and* (4.2) *hold;*
4. (VD), (TD), (E) *and* (UE_F) *hold.*

4.2. The properties of the scale function

Let us recall that $(TD)+(E)\Longrightarrow(TC)$ and consequently we can deduce several properties of the space-time scale function easily. First of all, the Einstein relation holds under the standing assumptions of this section.

Corollary 4.3 *If* (Γ,μ) *satisfies* $(p_0),(VD),(TD)$ *and* (E) *then the Einstein relation*

$$E(x,2R)\simeq\rho(x,R,2R)v(x,R,2R)$$

holds.

Proof. The statement follows from Theorem 3.3 since

$$(TD)+(E)\Longrightarrow(TC).$$

■

From the time doubling property it follows that the function

$$F(R)=\inf_{x\in\Gamma}E(x,R) \tag{4.4}$$

also has doubling property:

$$F(2R)\le D_E F(R), \tag{4.5}$$

in particular it is also clear that

$$\frac{F(R)}{F(S)}\le C_F\left(\frac{R}{S}\right)^{\beta} \tag{4.6}$$

holds, where $\beta=\log_2 D_E$.

Corollary 4.4 *If* (E) *holds and* $F(R)=\inf_{x\in\Gamma}E(x,R)$*, then* $F(R)$ *is strictly increasing in* $R\in\mathbb{N}$ *and has an inverse.*

Proof. The statement follows from (3.5); simply choose x for which

$$F(R+1)\ge E(x,R+1)-\frac{1}{2}\ge E(x,R)+1-\frac{1}{2}>F(R).$$

■

Corollary 4.5 *If* (E) *holds and* $F(R)=\inf_{x\in\Gamma}E(x,R)$, *then for all* $L,R,S\in\mathbb{N}$, *and* $R>S>0$

$$F(R+S)\ge F(R)+F(S) \tag{4.7}$$

and

$$F(LR)\ge LF(R). \tag{4.8}$$

Proof. Both statements are immediate from Lemma 3.4 using the same argument as in Corollary 4.4. ■

Definition 4.2 *We shall say that F has the anti-doubling property if there are $A_F, B_F > 1$ such that*

$$F(A_F R) \geq B_F F(R). \tag{4.9}$$

and the strong anti-doubling property if $B_F > A_F$.

Remark 4.2 *Equivalently, the anti-doubling property for F means that there are $c, \beta' > 0$ such that for $R > S > 0$*

$$\frac{F(R)}{F(S)} \geq c\left(\frac{R}{S}\right)^{\beta'} \tag{4.10}$$

and the strong anti-doubling property is equivalent with (4.10) for a $\beta' > 1$.

Proposition 4.6 *If (Γ, μ) satisfies (p_0) and (E), then for the function defined in (4.4) the anti-doubling property (4.9) holds.*

Proof. Since $(E) \Longrightarrow (2.9)$, by Proposition 3.6 we have

$$E(x, AR) \geq 2E(x, R),$$

and it is clear that for any $\varepsilon > 0$,$R > 0$ there is an x for which

$$F(AR) \geq E(x, AR) - \varepsilon \geq 2E(x, R) - \varepsilon \geq 2F(R) - \varepsilon$$

which yields the statement since ε is arbitrarily small. ■

Corollary 4.7 *If (p_0),(VD), (TD) and (E) holds then*

$$E(x, R) \geq cR^2$$

and

$$F(R) \geq cR^2. \tag{4.11}$$

Proof. The statement follows from Remark 3.3. ■

Definition 4.3 *A function $F : \mathbb{N} \to \mathbb{R}$ will be called proper if it is strictly monotone and satisfies (4.5), (4.7), (4.9) and (4.11), and very proper if in addition it satisfies (4.9) with a $B_F > A_F$.*

The above observations can be summarized as follows.

Corollary 4.8 *If (Γ, μ) satisfies (p_0), (VD), (TD) and (E) then F is proper.*

The following lemma provides estimates of the sub-Gaussian kernel function.

Lemma 4.9 *If (E) and (TD) hold, then for $k = k(n, R)$*

$$k+1 \geq c\left(\frac{F(R)}{n}\right)^{\frac{1}{\beta-1}}, \ k+1 \geq c'\left(\frac{R}{f(n)}\right)^{\frac{\beta}{\beta-1}} \text{ and } k \leq C\left(\frac{R^\beta}{n}\right)^{\frac{1}{\beta-1}} \tag{4.12}$$

Proof. The statement follows from (TD) easily. ■

4.3. The diagonal upper estimate

For the proof of Theorem 4.1 and 1.2, our entry point is Theorem 3.1.

Corollary 4.10 *Assume that* (Γ, μ) *satisfies* (p_0), (VD), (TD) *and* (E). *Then for the function* F *defined in* (4.4),

$$(MV) \Longleftrightarrow (4.2) \Longleftrightarrow (UE_F)$$

and

$$(2.13) \Longleftrightarrow (MV) \Longleftrightarrow (2.16) \Longleftrightarrow (2.14)$$

holds as well.

Proof. It is immediate from Theorem 3.1 since $(TD) + (E) \Longrightarrow (TC)$. ■

The next step is to insert (2.15) into the set of the equivalent conditions.

Before we start the proof we give the next statement, which is immediate consequence of Proposition 3.2.

Proposition 4.11 *For any weighted graph* (Γ, μ) *if we assume* (E) *then*

$$P_{2n}(x, x) \geq \frac{c\mu(x)}{V(x, f(2n))}, \tag{4.13}$$

furthermore

$$\mathbb{P}(T_{x,R} < n) \leq C \exp\left[-ck(n, R)\right], \tag{4.14}$$

where k *is the maximal integer* $1 \leq k \leq R \leq n$ *satisfying* (1.17) *and* F *is defined again by* (4.4).

The next step is to show $(2.15) \Longleftrightarrow (UE_F)$. This is done via (MV).

Theorem 4.12 *Let us assume that* (Γ, μ) *satisfies* (p_0), (VD), (TD) *and* (E). *Then the following statements are equivalent:*

1. *for a fixed* $B = B(x, R)$, $y \in B, d = d(x, y)$ *the upper bound for the Green kernel* (2.15) *holds:*

$$g^B(y, x) \leq C \sum_{i=F(d)}^{F(R)} \frac{1}{V(x, f(i))};$$

2. *for all* $u \geq 0$ *on* $\overline{B}(x, R)$ *harmonic function in* $B(x.R)$, *the mean value inequality* (MV) *holds*

$$u(x) \leq \frac{C}{V(x, R)} \sum_{z \in B(x,R)} u(z)\, \mu(z);$$

3. the upper estimate (UE_F) holds

$$p_n(x,y) \le \frac{C}{V(x,f(n))} \exp\left[-ck(n,d)\right].$$

Proof. The combination of Corollary 4.10 and Theorem 3.14 verifies $(MV) \Longleftrightarrow (UE_F)$. The implication $(UE_F) \Longrightarrow (2.15)$ can be shown as follows. Let us assume (p_0),(VD),(TD),(E) and (UE_F). We can start from the definition of the local Green kernel for $B = B(x,R)$, $d := d(x,y) > 0$, $d < R$ and denote $n = F(d) < m = E(x,R)$

$$\begin{aligned} g^B(y,x) &= \sum_{i=1}^{n-1} p_i^B(y,x) + \sum_{i=n}^{m-1} p_i^B(y,x) + \sum_{i=m}^{\infty} p_i^B(y,x) =: S_1 + S_2 + S_3 \\ S_3 &= \sum_{j=0}^{\infty}\sum_{z\in B} \frac{1}{\mu(x)} P_j^B(y,z) P_m^B(z,x) \le \sum_{j=0}^{\infty}\sum_{z\in B} P_j^B(y,z) \max_{z\in B} p_m^B(z,x) \\ &\le E_y(x,R) \max_{z\in B} \frac{C}{\sqrt{V(z,f(m))V(x,f(m))}} \end{aligned}$$

and using $(VD)\,,(TC)$ and $d(x,y)\,,d(x,z) < R < f(m)$ we conclude to

$$\sum_{i=m}^{\infty} p_i^B(y,x) \le C\frac{E(x,R)}{V(x,R)}.$$

The first term can be estimates as follows using (UE_F) :

$$S_1 \le \sum_{i=1}^{n} \frac{C}{V(y,f(i))} \exp\left[-ck(i,d)\right],$$

using Lemma 4.9 it can be bounded, denoting $a = \log_{A_E}(d)$

$$\begin{aligned} &\le C\frac{E(x,d)}{V(x,d)} \sum_{i=1}^{n-1} \frac{V(x,d)}{V(y,f(i))} \frac{1}{E(x,d)} \exp\left(-c\left(\frac{E(x,d)}{i}\right)^{\frac{1}{\beta-1}}\right) \\ &\le C\frac{E(x,R)}{V(x,R)} \sum_{j=1}^{a} D_V \left(\frac{d}{f\left(F\left(\frac{d}{A_F^j}\right)\right)}\right)^{\alpha} \frac{2^{-j+1}F(d)}{E(x,d)} \exp\left(-c\left(2^j\right)^{\frac{1}{\beta-1}}\right) \\ &\le C\frac{E(x,d)}{V(x,d)} \sum_{j=1}^{a} \left(\frac{A_F^{\alpha}}{2}\right)^{j} \exp\left(-c\left(2^j\right)^{\frac{1}{\beta-1}}\right) \end{aligned}$$

and it is clear that the sum is bounded by a constant independent of d and n which results that

$$S_1 = \sum_{i=1}^{n-1} p_i^B(y,x) \le C \frac{E(x,d)}{V(x,d)}.$$

The estimate of the middle term is straightforward from (UE_F):

$$S_2 = \sum_{i=n}^{m-1} p_i^B(y,x) \le \sum_{i=n}^{m} \frac{C}{V(x,f(i))}.$$

Finally a trivial estimate shows that

$$S_3 \le C\frac{E(x,R)}{V(x,R)} \le C\frac{F(R)}{V(x,f(F(R)))} \le C\sum_{i=F(d)}^{F(R)} \frac{1}{V(x,f(i))}, \tag{4.15}$$

$$S_1 \le C\frac{E(x,d)}{V(x,d)} \le C\frac{F(d)}{V(x,f(C'F(d)))} \le \sum_{i=F(d)}^{C'F(d)} \frac{1}{V(x,f(i))} \tag{4.16}$$

which results

$$g^B(y,w) = S_1 + S_2 + S_3 \le C\sum_{i=F(d)}^{C'F(R)} \frac{1}{V(x,f(i))}. \tag{4.17}$$

The next step is to show (2.15) $\Longrightarrow$ (2.16)

$$g^B(y,x) \le C\sum_{i=F(d)}^{C'F(R)} \frac{1}{V(x,f(i))} \le C\frac{C'F(R)-F(d)}{V(x,f(F(d)))} \le C\frac{E(x,R)}{V(x,d)}$$

where in the last step the doubling property of V and F was used. We have seen in Corollary 4.10 that (2.16) implies (4.2) and it implies (UE_F), hence we have shown that (2.15) $\Longrightarrow$ (UE_F). ■

Proof of Theorem 4.1. The result follows from Corollary 4.10 and Theorems 3.14 and 4.12. ■

Proof of Theorem 4.2. The proof is evident from Theorem 4.1 and Remark 4.1. ■

4.4. The two-sided estimate

In this section we prove Theorem 1.3. First we collect several consequences of the elliptic Harnack inequality which enable us to apply Theorem 4.1, particularly to deduce (UE_F).

As we indicated the parabolic and elliptic Harnack inequalities play an important role in the study of two-sided bound of the heat kernel. Here we give their formal definitions.

Definition 4.4 *The weighted graph* (Γ, μ) *satisfies the (*F*-parabolic or simply) parabolic Harnack inequality if the following condition holds. For a given profile* $\mathcal{C} = \{c_1, c_2, c_3, c_4, \eta\}$, $0 < c_1 < c_2 < c_3 < c_4, 0 < \eta < 1$, *set of constants, there is a* $C_H(\mathcal{C}) > 0$ *constant such that for any solution* $u \geq 0$ *of the heat equation*

$$Pu_n = u_{n+1}$$

on

$$\mathcal{U} = [k, k + F(c_4 R)] \times B(x, R) \quad \text{for } k, R \in \mathbb{N},$$

the following is true. On the smaller cylinders defined by

$$\mathcal{U}^- = [k + F(c_1 R), k + F(c_2 R)] \times B(x, \eta R)$$

and

$$\mathcal{U}^+ = [k + F(c_3 R), k + F(c_4 R)] \times B(x, \eta R)$$

and taking $(n_-, x_-) \in \mathcal{U}^-, (n_+, x_+) \in \mathcal{U}^+, d(x_-, x_+) \leq n_+ - n_-$ *the inequality*

$$u(n_-, x_-) \leq C_H \widetilde{u}(n_+, x_+)$$

holds, where $\widetilde{u}_n = u_n + u_{n+1}$ *short notation was used. Let us remark that* C_H *depends on the constants (including* c_i, η, D_V, D_E, A_V*) involved.*

It is standard knowledge that if the (classical) parabolic Harnack inequality holds for a given profile then it is true for arbitrary profile. We have shown in [16, Subsection 7.1] that the same holds in the general case if F is proper.

Definition 4.5 *The weighted graph* (Γ, μ) *satisfies the elliptic Harnack inequality* $(\mathbf{H})$ *if there is a* $C > 0$ *such that for all* $x \in \Gamma$ *and* $R > 0$ *and for all* $u \geq 0$ *harmonic functions on* $B(x, 2R)$ *the following inequality holds:*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u. \tag{4.18}$$

The elliptic Harnack inequality is a direct consequence of the F-parabolic one as it is true for the classical case.

The main result of this section is the following, which implies Theorem 1.3.

Theorem 4.13 *If a weighted graph* (Γ, μ) *satisfies* (p_0) *then the following statements are equivalent:*

1. *the F-parabolic Harnack inequality hold for a very proper* F*;*
2. (UE_F) *and* (LE_F) *hold for a very proper* F*;*
3. $(VD), (2.7)$ *and* (H) *hold;*
4. $(VD), (E)$ *and* (H) *hold.*

4.4.1. The Einstein relation

The following five observations are taken from [17].

Proposition 4.14 *If* $(p_0), (VD)$ *and* (H) *holds then the resistance has the doubling properties: there are* $C, C' > 1$ *such that*

$$\frac{\rho(x,R,4R)}{\rho(x,R,2R)} \le C \tag{4.19}$$

and

$$\frac{\rho(x,R,4R)}{\rho(x,2R,4R)} \le C'. \tag{4.20}$$

For the proof see [17]. The next corollary is trivial consequence of Proposition 4.14.

Corollary 4.15 *If* $(p_0), (VD)$ *and* (H) *holds then there is a constant* $C > 1$ *such that*

$$\frac{\rho(x,2R,4R)v(x,2R,4R)}{\rho(x,R,2R)v(x,R,2R)} \le C. \tag{4.21}$$

Theorem 4.16 *If for* (Γ, μ) *conditions* (p_0)*,* $(VD), (H)$ *and* (E) *hold then*

$$E(x,2R) \simeq \rho(x,R,2R)v(x,R,2R).$$

Theorem 4.17 *If for the weighted graph* (Γ, μ) *the conditions* (p_0)*,* (VD)*,* (H) *and* (2.7)*, which is*

$$\rho v(x,R,2R)v(x,R,2R) \simeq \rho v(y,R,2R)v(y,R,2R),$$

hold, then

$$E(x,2R) \simeq \rho(x,R,2R)v(x,R,2R).$$

Proposition 4.18 *If* (Γ,μ) *satisfies* $(p_0),(VD),(H)$ *and* (E) *(or* (2.7)*) then the the function* F

$$F(R)=\inf_{x\in\Gamma}\rho(x,R,2R)v(x,R,2R)$$

is proper; furthermore the strong ant-doubling property holds. The latter means that there are $B_F>A_F>1$ *such that*

$$F(A_F R)\geq B_F F(R) \tag{4.22}$$

for all $R>0$. *In short, under the conditions,* F *if very proper.*

Proof. From Corollary 4.8 we know that F is proper and (4.22) is shown in [17] under the conditions. ■

Proof of Theorem 4.13. The implication 4. $\Longrightarrow$ 3. is given in Theorem 4.16 and 3. $\Longrightarrow$ 4 in Theorem 4.17. 3. $\Longrightarrow$ 2. needs the implication

$$(p_0)+(VD)+(TD)+(H)+(E)\Longrightarrow(4.2),(UE_F)$$

which follows from Theorem 4.1 since $(H)\Longrightarrow(MV)$. The proof of the lower estimate works as in [16]. The return route 2. $\Longrightarrow$ 1. $\Longrightarrow$ 3. also has been shown in [16, Theorem 2.22]. The only minor modification is that the condition of annulus resistance doubling ((2.6) there) follows from the doubling property of F and ρv by Corollary 4.15. ■

5. Example

In this section we describe in details of the example of the stretched Vicsek tree mentioned in the introduction. We show that it satisfies the conditions of Theorems 1.1 and 3.1. Let G_i is the subgraph of the Vicsek tree (see Figure 6) (c.f. [8]) which contains the root z_0 and has diameter $D_i=23^i$.

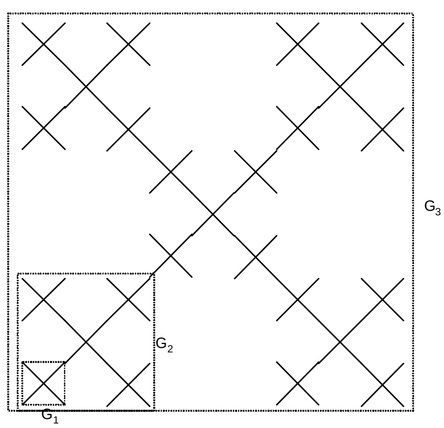

Figure 6 The blocks of the Vicsek tree.

Let us denote by z_i the vertices on the infinite path, $d(z_0, z_i) = D_i$. Denote $G_i' = G_i \backslash G_{i-1} \cup \{z_{i-1}\}$ for $i > 0$, the annulus defined by G-s.

The new graph is defined by stretching the Vicsek tree as follows. Consider the subgraphs G_i' and replace all the edges of them by a path of length $i+1$. Denote the new subgraph by A_i, the new blocks by $\Gamma_i = \cup_{j=0}^{i} A_i$, then the new graphs is $\Gamma = \cup_{j=0}^{\infty} A_j$. We denote by z_i the cut point between A_i and A_{i-1} again. For $x \neq y, x \sim y$ let $\mu_{x,y} = 1$.

One can see that neither the volume nor the mean exit time grows polynomially on Γ and both are not uniform on it. On the other hand Γ is a tree and the resistance grows asymptotically linearly on it. We show that (VD) and (TC) holds on Γ furthermore the elliptic Harnack inequality holds.

Let us recognize some straightforward relations first:

$$
\begin{aligned}
d(z_0, z_n) &= d(z_0, z_{n-1}) + 2n3^n < n3^{n+1} && (5.1)\\
&< (n+2)\,3^{n+1}, && (5.2)\\
\mu(\Gamma_n) &= C\left(4 + \sum_{i=1}^{n} 2(i+1)\,4^i\right) \simeq n4^n \simeq \mu(A_n),\\
\rho(\{x\}, B(x,R)^c) &\simeq \rho(x, R, 2R) \simeq R,\\
E(x,R) &\leq CRV(x,R).
\end{aligned}
$$

Lemma 5.1 *The tree Γ satisfies (VD).*

Proof. Denote $L_i = d(z_0, z_i), d_i = \frac{1}{2}(L_i - L_{i-1})$ and recognize that $L_{n-1} \simeq L_n \simeq d_n$. Let us consider a ball $B(x, 2R)$ and an $N > 0$ such that $x \in A_N$ and k:

$$d_{k-1} \leq R < d_k.$$

First we assume that the ball is large relative to the position of the centre, which means that it captures basically the large scale property of the graphs.

Case 1. $k \geq N$.

For convenience we introduce a notation. Denote Ω_n one of the blocks of A_k of diameter d_k. There is a block Ω_{k-2} which contains x. It is clear that $\Omega_{k-2} \subset B(x,R)$ and

$$V(x,R) \geq \mu(\Omega_{k-2}) \simeq \mu(\Gamma_{k+1}).$$

On the other hand $R < L_k$ which results that $B(x,2R) \subset \Gamma_{k+1}$ and from

$$\mu(\Gamma_{k+1}) \simeq \mu(\Omega_{k-2}),$$

(VD) follows.

Case 1. $k < N$.

Now we have to separate sub-cases. Again let us fix that $x \in \Omega_N$. Denote $d = d(x, z_{N-1})$. If x is not in the central block of A_N then, the $B(x,R) \subset A_N \cup A_{N+1}$ and since these parts of the graph contain only paths of length of $N+1$ or $N+2$ volume doubling follows from the fact that it holds for the original Vicsek tree. The same applies if x is in the central block but $B(x,2R) \subset A_N$. Finally if $B(x,2R) \cap \Gamma_{N-1} \neq \varnothing$ then $R \geq 2d_{N-1} > d_{N-1}$ which means by the definition of k that $k = N-1$, $B(x,R) \supset \Omega_{N-1}$ and on the other hand $B(x,2R) \subset \Gamma_{N+1}$ which again gives (VD). ■

The elliptic Harnack inequality follows as in [8] from the fact that the Green functions are nearly radial. The linear resistance growth implies that

$$\rho(\{x\}, B^c(x,2R)) \simeq \rho(x,R,2R) \simeq R.$$

Let us also recall that from (VD), (H) and the linear resistance growth it follows that

$$\begin{aligned} cRV(x,2R) &\leq c\rho(x,R,2R)V(x,2R) \leq E(x,2R) \\ &\leq \rho(\{x\}, B^c(x,2R))V(x,2R) \leq CRV(x,2R). \end{aligned} \tag{5.3}$$

The conditions (TC) follows from (VD) and (5.3).

Let us remark that the mean value inequality is implied by the Harnack inequality and consequently the conditions of Theorem 1.1 and 3.1 are satisfied.

6. List of the main conditions

shortcut	*equation*	*name*
(p_0)	(1.9)	controlled weights condition
(VD)	(1.1)	volume doubling property
(TC)	(1.5)	time comparison principle
(TD)	(1.6)	time doubling property
(MV)	(1.10)	mean value inequality
(DUE)	(3.2)	diagonal upper estimate
(UE)	(1.12)	upper estimate
(E)	(1.15)	uniform mean exit time
(H)	(4.18)	elliptic Harnack inequality
(UE_F)	(1.18)	upper estimate w.r.t. F
(LE_F)	(1.19)	lower estimate w.r.t. F

References

[1] Barlow, M.T.: Diffusions on Fractals. In *Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-flour XXV-1995*, 1–121. Lecture Notes in Math. **1690**, Springer, 1998.

[2] Barlow, M.T., Bass, R.: Stability of the parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** (2004), no 4, 1501–1533.

[3] Barlow, M.T., Bass, R., Kumagai, T.: Stability of the parabolic Harnack inequalities on metric measure spaces. Preprint.

[4] Coulhon, T., Grigor'yan, A.: Random walks on graphs with regular volume growth. *Geom. Funct. Anal.* **8** (1998), 656–701.

[5] Delmotte, T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[6] Grigor'yan, A.: Heat kernel upper bounds on fractal spaces. Preprint.

[7] Grigor'yan, A., Telcs, A.: Sub-Gaussian estimates of heat kernels on infinite graphs. *Duke Math. J.* **109** (2001), no. 3, 452–510.

[8] Grigor'yan, A., Telcs, A.: Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Ann.* **324** (2002), 521–556.

[9] Gromov, M.: Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 57–73.

[10] Hambly, B., Kumagai, T.: Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, 233–259. Proc. Sympos. Pure Math. **72**, Part 2. Amer. Math. Soc., Providence, RI, 2004.

[11] Hebisch W., Saloff-Coste, L.: On the relation between elliptic and parabolic Harnack inequalities. *Ann. Inst. Fourier (Grenoble)* **51** (2001), no. 5, 1437–1481.

[12] Jones, O. D.: Transition probabilities for the simple random walk on the Sierpiński graph. *Stochastic Process. Appl.* **61** (1996), no. 1, 45–69.

[13] Kumagai, T., Sturm, K-T.: Construction of diffusion processes on fractals, d–sets, and general metric measure spaces. *J. Math. Kyoto Univ.* **45** (2005), no. 2, 307–327.

[14] Li, P., Wang, J.: Mean value inequalities. *Indiana Univ. Math. J.* **48** (1999), no. 4, 1257–1283.

[15] Telcs, A.: Random walks on graphs, electric networks and fractals. *Probab. Theory Related Fields* **82** (1989), 435–449.

[16] Telcs, A.: Volume and time doubling of graphs and random walks: the strongly recurrent case. *Comm. Pure Appl. Math.* **54** (2001), 975–1018.

[17] Telcs, A.: Some notes on the Einstein relation. To appear in *J. Stat. Phys.*

[18] Telcs, A.: Upper bound for transition probabilities on graphs and isoperimetric inequalities. To appear in *Markov Proc. and Rel. Fields*.

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