SAK Principle for a class
of Grushin-type operators

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Abstract

We prove Fefferman's SAK Principle for a class of hypoelliptic operators on $\mathbb{R}^2$ whose nonnegative symbol vanishes anisotropically on the characteristic manifold.

1. Introduction

In his celebrated paper [5], Fefferman suggests a strategy to get a priori estimates (SAK principle). Suppose that one wants to know whether a given estimate of the form

$$(1.1) \qquad C\|q^w u\|_0 \leq \|L^w u\|_0 + \text{"small error terms"}, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

holds for given symbols $L, q \in S^2(\mathbb{R}^n)$. Here $S^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, is the set of "global symbols" $a(x,\xi)$ satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$ for all $(x,\xi) \in \mathbb{R}^{2n}$ and all $\alpha, \beta \in \mathbb{Z}_+^n$, and a^w denotes the Weyl quantization of a (see [10]). Fefferman's idea is to obtain (1.1) by a comparison of the symbols L and q. This is in analogy to the Fefferman-Phong inequality proved in [4]:

$$S^2(\mathbb{R}^n) \ni a \geq 0 \Longrightarrow (a^w u, \ u)_{L^2} \geq -C\|u\|_0^2, \qquad \forall \ u \in \mathcal{S}(\mathbb{R}^n).$$

Hérau proved (1.1) for $n = 1$ assuming that $L(x,\xi) \geq 0$ and $|q(x,\xi)| \leq L(x,\xi)$ for every $(x,\xi) \in \mathbb{R}^2$ (see [8]). Actually, Fefferman conjectured that (1.1) holds under a weaker assumption, namely

$$(1.2) \qquad \max_{\mathcal{B}_\mu} |q(x,\xi)| \leq c \, \max_{\mathcal{B}_\mu} L(x,\xi),$$

where $\{B_\mu\}_{\mu \in \mathbb{N}}$ is a suitable "partition" of the phase space $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ associated with $L(x,\xi)$.

In this paper, we prove this result in the case $n = 2$ and L^w being the Grushin operator with symbol

$$(1.3) \qquad L(x, \xi) = \xi_1^2 + x_1^{2h}\xi_2^2, \quad (x, \xi) = (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4, \ h \in \mathbb{Z}_+.$$

Similar arguments work in higher dimension for polynomial symbols having the same kind of anisotropic vanishing. More precisely, if $x = (x', x'') \in \mathbb{R}^\mu \times \mathbb{R}^{n-\mu}$ (accordingly $\xi = (\xi', \xi'')$), one can consider operators having real homogeneous polynomial symbols that near the characteristic manifold, $\{(x, \xi) = (x', x'', \xi', \xi'') \in \mathbb{R}^{2n} : x' = \xi' = 0\}$, behave like $|\xi|^2(|x'|^{2h} + |\xi'|^2/|\xi|^2)$. The required adjustments to treat these cases can be easily deduced and are here omitted. It is worthwhile to observe that the Grushin operator in $\mathbb{R}^2$ is "meaningful" in view of reduction theorems given in [5].

Before stating the precise result, let us observe that the weaker assumption (1.2) on the symbols has, in general, non trivial consequences. For example, considering real vector fields X_j with Weyl symbols p_j, a pointwise comparison of the symbols yields

$$\|X_j^2 u\|_0 \leq C\big(\|\sum_j X_j^2 u\|_0 + \|u\|_\epsilon\big).$$

In fact, we have $\sigma^w(X_k^2) = p_k^2 + i\{p_k, p_k\}/2 + S^0 = p_k^2 + S^0$, $p_k^2(x, \xi) \leq \sum_j p_j^2(x, \xi)$. The full SAK Principle would even imply that

$$\|X_j X_k u\|_0 \leq C(\|\sum_j X_j^2 u\|_0 + \|u\|_\epsilon)$$

since the symbol of $X_j X_k$ is $p_j p_k + i\{p_j, p_k\}/2 + S^0$ and it satisfies (1.2) for suitable rectangles B_μ. In fact, $|p_j p_k| \leq c \sum_j p_j^2$ but a pointwise comparison of $\{p_j, p_k\}$ by the symbol of $\sum_j X_j^2$ is in general not possible as one easily checks in our case $p_1 = \xi_1$ and $p_2 = x_1^h \xi_2$.

Some preliminary facts are needed in order to state the main Theorem of this paper. From (1.3), it is easily seen that $L(x, \xi) \geq 0$ with characteristic manifold given by

$$(1.4) \qquad \Sigma = \{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : x_1 = \xi_1 = 0, \ \xi_2 \neq 0\}.$$

Definition 1.1 *Let $R > 0$ be a suitable large fixed constant and consider, for $(x, \xi) \in \mathbb{R}^4$,*

$$D_\Sigma(x, \xi) = \Big(\frac{x_1^{2h}}{(1 + x_1^2)^h} + \frac{\xi_1^2}{R^2 + |\xi|^2} + \frac{R^{2h/(h+1)}}{(R^2 + |\xi|^2)^{h/(h+1)}}\Big)^{1/2}.$$

We then define, for every $(x, \xi), (y, \eta) \in \mathbb{R}^4$, the Hörmander metric

$$G_{(x,\xi)}(y, \eta) = \frac{|y|^2}{D_\Sigma(x, \xi)^{2/h}} + \frac{|\eta|^2}{(R^2 + |\xi|^2)D_\Sigma(x, \xi)^2}.$$

We want to point out that the metric G here considered is obtained by slightly modifying the metric used in [12] for the parametrix construction. As a consequence, the Weyl calculus defined by G is strictly related with Boutet de Monvel's calculus (see [3]). It is worth to observe that, near the characteristic manifold Σ, the metric G is equivalent to a metric defined only through the symbol L, namely

$$G^L_{(x,\xi)}(y,\eta) = \frac{|y|^2}{\langle\xi\rangle^{-2/h}(L(x,\xi) + \langle\xi\rangle^{2/(h+1)})^{1/h}} + \frac{|\eta|^2}{L(x,\xi) + \langle\xi\rangle^{2/(h+1)}},$$

for (x,ξ), $(y,\eta) \in \mathbb{R}^4$. In the following Theorem, we denote by c_{SV} the positive constant related to the slowly varying property of the metric G in Definition 1.1:

$$(1.5) \quad G_X(Y-X) \le c_{SV} \quad \Longrightarrow \quad c_{SV}G_Y(Z) \le G_X(Z) \le \frac{1}{c_{SV}}G_Y(Z),$$

$\forall X,Y,Z \in \mathbb{R}^2 \times \mathbb{R}^2$.

Theorem 1.2 *Let $q \in S^2(\mathbb{R}^2)$ and suppose that there exists a constant $0 < r \le c_{SV}$ and a covering $\{B^G_{(x_j,\xi_j)}(r)\}_{j\in\mathbb{N}}$ of $\mathbb{R}^2_x \times \mathbb{R}^2_\xi$ such that*

$$(1.6) \qquad\qquad \max_{B^G_{(x_j,\xi_j)}(r)} |q| \le c \max_{B^G_{(x_j,\xi_j)}(r)} L, \quad \forall j \in \mathbb{N},$$

where

$$B^G_{(x_j,\xi_j)}(r) = \{(y,\eta) \in \mathbb{R}^4 : G_{(x_j,\xi_j)}(y - x_j, \eta - \xi_j) \le r^2\}$$

and c is a constant **independent** *of $j \in \mathbb{N}$.*

Then, for every compact set $K \subset \mathbb{R}^2$, there exists a positive constant $C = C(K)$ such that

$$(1.7) \qquad\qquad \|L(x,D)u\|^2_0 + \|u\|^2_0 \ge C\|q^w u\|^2_0, \quad \forall u \in C^\infty_0(K).$$

Note that one cannot expect to have (1.7) uniformly for all u in $\mathcal{S}(\mathbb{R}^2)$, since L is not globally bounded in $\mathbb{R}^4$.

It is worth pointing out that the decomposition of the phase space $T^*\mathbb{R}^2$ by means of the balls $B^G_{(x_j,\xi_j)}(r)$ given above is different from the one proposed by Fefferman in [6]. This shows that there is no a-priori uniqueness in the choice of the "good" partition of $T^*\mathbb{R}^n$. A very difficult problem is to understand what a "natural" definition of ball associated with a second order pseudodifferential operator is. This problem is studied in [13], where a sharp definition of subunit ball for non-negative symbols of subelliptic operators is discussed.

The paper is organized as follows. In Section 2 we show that Theorem 1.2 can be "microlocalized" to Theorem 2.4 below and we develop some tools needed in Section 3 for the proof of Theorem 2.4. In the Appendix some technical results required throughout the paper are proved.

Acknowledgements. We thank A. Parmeggiani for introducing us to this argument and for the constant support.

2. First and second level microlocalizations

Definition 2.1 *Let $K \subset \mathbb{R}^2$ be an arbitrary fixed compact set and let χ be a function in $C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ on a neighborhood of K and $0 \leq \chi(x) \leq 1$. Define*

$$p(x, \xi) = \xi_1^2 + \left[\chi(x) x_1^{2h} + (1 - \chi(x)) \frac{x_1^{2h}}{(1 + x_1^2)^h} \right] \xi_2^2.$$

The symbol p is an extension of the Grushin operator outside the compact set K, whose properties are stated in the lemma below.

Lemma 2.2 *The following properties hold:*

1) *$p \in S^2(\mathbb{R}^2)$;*

2) *$p(x, \xi) \geq 0$, $\forall (x, \xi) \in \mathbb{R}_x^2 \times (\mathbb{R}_\xi^2 \setminus \{0\})$, and $p(x, \xi) = 0$ iff $(x, \xi) \in \Sigma$;*

3) *for every constant $C > 0$ and for every (closed) rectangle $Q \subseteq \mathbb{R}^4$ such that $\Pi_x(Q) \subseteq \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq C\}$ (here Π_x is the projection $(x, \xi) \in \mathbb{R}^4 \mapsto x \in \mathbb{R}^2$) one has*

$$\max_Q \; p(x, \xi) \sim \max_Q \; L(x, \xi) \quad \text{uniformly in } Q,$$

with L defined by (1.3);

4) *if $q \in S^2(\mathbb{R}^2)$ is such that*

$$(2.1) \qquad \|p^w u\|_0^2 + \|u\|_0^2 \geq c \|q^w u\|_0^2, \qquad \forall u \in \mathcal{S}(\mathbb{R}^2)$$

for some positive constant c, then estimate (1.7) holds.

The previous lemma reduces the proof of Theorem 1.2 to the proof, under the same assumptions, of (2.1). We further want to give a microlocal version of (2.1). In order to do this we need to introduce a partition of $\mathbb{R}^4$ in suitable rectangles having sides parallel to the coordinate axes.

For any $d > 0$ and for any $\bar{y} = (\bar{y}_1, \bar{y}_2) \in \mathbb{R}^2$, denote by $I_d(\bar{y})$ the rectangle $I_d(\bar{y}) = \{y \in \mathbb{R}^2 : |y_i - \bar{y}_i| \leq d/(2\sqrt{2}), \; i = 1, 2\}$ (d is referred to as the **diameter** of $I_d(\bar{y})$), and consider a partition $\{Q_\mu^1\}_{\mu \in \mathbb{N}}$ of $\mathbb{R}^4$ with $Q_\mu^1 = I_1(x_\mu) \times I_{M_\mu}(\xi_\mu)$, such that

- for every $(x,\xi) \in Q^1_\mu$ one has

  (2.2) $$2^{-4}(R + |\xi|) \le M_\mu \le 2^{-3}(R + |\xi|);$$

- $\mathring{Q}^1_\mu \cap \mathring{Q}^1_{\mu'} = \emptyset$ for each $\mu \ne \mu'$ ($\mathring{Q}^1_\mu$ is the interior of Q^1_μ) and $\bigcup_{\mu \in \mathbb{N}} Q^1_\mu = \mathbb{R}^4$.

Remark 2.3 *For every $(x,\xi) \in 2Q^1_\mu$ one has[1]*

$$17^{-1}(R + |\xi|) \le M_\mu \le 7^{-1}(R + |\xi|).$$

Denoting by (x_μ, ξ_μ) the center of Q^1_μ one has in fact

$$(R + |\xi|) \le (R + |\xi_\mu|) + |\xi - \xi_\mu| \le (1 + 2^4)M_\mu$$

and
$$M_\mu \le 2^{-3}(R + |\xi|) + 2^{-3}|\xi - \xi_\mu| \le 2^{-3}(R + |\xi|) + 2^{-3}M_\mu.$$

As a consequence, one gets $M_{\mu'} \sim M_\mu$ whenever $2Q^1_\mu \cap 2Q^1_{\mu'} \ne \emptyset$.

Remark 2.3 allows us to apply Proposition A.2, so that the metric

(2.3) $$g^I_{(x,\xi)}(y,\eta) = |y|^2 + \frac{|\eta|^2}{M^2_\mu}, \qquad (x,\xi) \in \mathring{Q}_\mu,$$

can be extended to a Hörmander metric g^I. As a consequence of Theorem A.5 and of Lemma 2.2 we can finally reduce the proof of Theorem 1.2 to the proof of the following theorem which can be considered as its microlocal version.

Theorem 2.4 *With the same hypotheses of Theorem 1.2, let $\{\psi_\mu\}_{\mu \in \mathbb{N}}$, $\{\varphi_\mu\}_{\mu \in \mathbb{N}}$ be the partition of unity associated with the covering $\{Q^1_\mu\}_{\mu \in \mathbb{N}}$ in the sense of Proposition A.4. Then[2]*

(2.4) $\|(p\psi_\mu)^w \varphi^w_\mu u\|^2_0 \gtrsim \|(q\psi_\mu)^w \varphi^w_\mu u\|^2_0 - (r^w_\mu u,\ u)_{L^2}, \quad \forall \mu \in \mathbb{N},\ \forall u \in \mathcal{S}(\mathbb{R}^2),$

with p as in Definition 2.1 and $\sum_{\mu \in \mathbb{N}} r^w_\mu \in \mathcal{L}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$.

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[1]In what follows, given a rectangle Q in $\mathbb{R}^4$ and $k \in \mathbb{R}_+$, we denote by kQ the dilated of Q by the factor k keeping its center fixed

[2]From now on, for two non negative functions f, g on $\mathbb{R}^n$ we write $f \lesssim g$ (resp., $f \gtrsim g$) when there exists a constant $c > 0$ such that $f(x,\xi) \le c\, g(x,\xi)$ (resp., $f(x,\xi) \ge c\, g(x,\xi)$). We simply write $f \sim g$ when $f \lesssim g$ and $f \gtrsim g$. For function families $\{f_k\}_k$, $\{g_k\}_k$ and $\{h_k\}_k$ we write $f_k \lesssim g_k$ (resp., $f_k \gtrsim g_k$) whenever the constants in the inequalities don't depend on k; $f_k \gtrsim g_k - h_k$ whenever $f_k + h_k \gtrsim g_k$.

The proof of Theorem 2.4 requires a second microlocalization. Let $\{Q_\mu^1\}_{\mu\in\mathbb{N}}$ be the partition of $\mathbb{R}^4$ just described, we refer to the Q_μ^1 as "rectangles of first generation". Among them, we retain those that satisfy one of the following conditions:

either there exists $(x,\xi)\in Q_\mu^1$ such that $1\le 2[R/(R+|\xi|)]^{1/(h+1)}$
or $p(x,\xi)>0$ for every $(x,\xi)\in 4Q_\mu^1$.

If both these conditions fail we divide Q_μ^1 into $2^{2(h+1)}$ subrectangles $Q_{\mu'}^2$ (rectangles of second generation) by partitioning $I_1(x_\mu)$ into four congruent rectangles and $I_{M_\mu}(\xi_\mu)$ into 4^h congruent rectangles. We denote by $\mathrm{diam}_x Q_{\mu'}^2$, $\mathrm{diam}_\xi Q_{\mu'}^2$ respectively their x-diameter and ξ-diameter, and retain those satisfying one of the following conditions, for $j=2$,

$$(2.5)\qquad \exists (x,\xi)\in Q_{\mu'}^j \text{ such that } \mathrm{diam}_x Q_{\mu'}^j \le \frac{2R^{1/(h+1)}}{(R+|\xi|)^{1/(h+1)}};$$

$$(2.6)\qquad\qquad p(x,\xi)>0,\quad \forall (x,\xi)\in 4Q_{\mu'}^j.$$

We again divide the rectangles that were not retained into $2^{2(h+1)}$ subrectangles, $Q_{\mu'}^3$, as described above and retain those satisfying one of conditions (2.5), (2.6) for $j=3$. We continue this process and after a finite number of steps (because of the condition (2.5)) we obtain a partition of $\mathbb{R}^4$.

Lemma 2.5 *Let Q_μ^j be a rectangle, either retained or not, of the j-th generation, we then have*

$$(2.7)\quad 2^{-4}(R+|\xi|)\big(\mathrm{diam}_x Q_\mu^j\big)^h \le \mathrm{diam}_\xi Q_\mu^j \le 2^{-3}(R+|\xi|)\big(\mathrm{diam}_x Q_\mu^j\big)^h,$$

$\forall (x,\xi)\in Q_\mu^j$, *and*

$$(2.8)\quad 2^{-1}(R+|\xi|)\le R+|\eta|\le 2(R+|\xi|),\qquad (x,\xi),\ (y,\eta)\in 4Q_\mu^j.$$

Proof. When $j=1$, (2.7) is trivial. When $j\ge 2$, since

$$\mathrm{diam}_x Q_\mu^j = (1/2)\mathrm{diam}_x Q_\mu^{j-1} \quad\text{and}\quad \mathrm{diam}_\xi Q_\mu^j = (1/2^h)\mathrm{diam}_\xi Q_\mu^{j-1},$$

(2.7) follows by induction.

Consider now $(x,\xi)\in 4Q_\mu^j$, $j\ge 1$. From (2.7), taking into account that $\mathrm{diam}_x Q_\mu^j \le 1$, it follows, on denoting by $(\bar{x},\bar{\xi})$ the center of Q_μ^j, that $|\xi-\bar{\xi}|\le 2\,\mathrm{diam}_\xi Q_\mu^j \le 2^{-2}(R+|\bar{\xi}|)$. Thus

$$R+|\xi|\le R+|\bar{\xi}|+|\xi-\bar{\xi}|\le 5/4(R+|\bar{\xi}|)\quad\text{and}\quad R+|\xi|\ge 3/4(R+|\bar{\xi}|).$$

This concludes the proof. ∎

Definition 2.6 *Let $\{Q_{\mu\nu}\}$ ($\mu \in \mathbb{N}$, $\nu = 1, \ldots, N(\mu)$) be the partition of $\mathbb{R}^4$ obtained by means of conditions (2.5) and (2.6). The first index, μ, indicates the rectangle of the first generation Q_μ^1 whose decomposition gives $Q_{\mu\nu}$. We denote by $(x_{\mu\nu}, \xi_{\mu\nu})$ the center of $Q_{\mu\nu}$, by $\delta_{\mu\nu}$ and $\epsilon_{\mu\nu}$ respectively its x-diameter and the ξ-diameter and, finally, we write $h_{\mu\nu} = (\delta_{\mu\nu}\epsilon_{\mu\nu})^{-1}$.*

The following propositions describe the basic properties of the partition.

Proposition 2.7 *If $(x, \xi) \in Q_{\mu\nu}$, then*

$$(2.9) \qquad 2^{-4}(R + |\xi|)\delta_{\mu\nu}^h \leq \epsilon_{\mu\nu} \leq 2^{-3}(R + |\xi|)\delta_{\mu\nu}^h;$$

$$(2.10) \qquad \frac{R^{1/(h+1)}}{(R + |\xi|)^{1/(h+1)}} \leq \delta_{\mu\nu} \leq 1;$$

$$(2.11) \qquad 2^3(R + |\xi|)^{-1}\delta_{\mu\nu}^{-h-1} \leq h_{\mu\nu} \leq 2^4(R + |\xi|)^{-1}\delta_{\mu\nu}^{-h-1} < 16/R.$$

Moreover, if $Q_{\mu\nu}$ is retained by virtue of (2.5) one has

$$(2.12) \qquad 2^{2-h}R^{-1} \leq h_{\mu\nu};$$

$$(2.13) \qquad \delta_{\mu\nu} \leq 2^{(h+2)/(h+1)}\frac{R^{1/(h+1)}}{(R + |\xi|)^{1/(h+1)}}, \qquad \forall(x, \xi) \in 4Q_{\mu\nu}.$$

Proof. Inequality (2.9) is simply (2.7) written in this case. Inequality (2.10) is trivial if $\delta_{\mu\nu} = 1$. If $Q_{\mu\nu}$ is obtained by dividing some Q_μ^j this means, in particular, that Q_μ^j does not satisfy (2.5). Hence, taking into account that $2\delta_{\mu\nu} = \mathrm{diam}_x Q_\mu^j$ one proves (2.10) also in this case. Inequality (2.11) is obtained by multiplying (2.9) by $\delta_{\mu\nu}$ and by using (2.10).

Let us suppose now that $Q_{\mu\nu}$ is retained because of (2.5). From (2.11) we get (2.12). Finally, (2.13) is deduced from (2.5) and (2.8). ∎

Same arguments used in Proposition 1.4 of [7] prove the following Proposition.

Proposition 2.8 *If $2Q_{\mu\nu} \cap 2Q_{\mu'\nu'} \neq \emptyset$ then*

$$(2.14) \qquad 2^{-6}\delta_{\mu'\nu'} \leq \delta_{\mu\nu} \leq 2^6\delta_{\mu'\nu'} \qquad 2^{-(6h+3)}\epsilon_{\mu'\nu'} \leq \epsilon_{\mu\nu} \leq 2^{6h+3}\epsilon_{\mu'\nu'}.$$

In view of the results above we can apply Proposition A.2 and obtain that the metric

$$(2.15) \qquad g_{(x,\xi)}^{II}(y, \eta) = \delta_{\mu\nu}^{-2}|y|^2 + \epsilon_{\mu\nu}^{-2}|\eta|^2, \qquad (x, \xi) \in \overset{\circ}{Q}_{\mu\nu}$$

can be extended to a Hörmander metric g^{II} with Planck function h_{II} given by

$$h_{II}(x, \xi) = h_{\mu\nu}, \qquad (x, \xi) \in \overset{\circ}{Q}_{\mu\nu}.$$

Remark 2.9

1) If $Q_{\mu\nu}$ is obtained by partitioning a cube Q_μ^j of generation $j \geq 1$, then one has $4Q_\mu^j \cap \Sigma \neq \emptyset$ and it easily follows

$$|\xi_1| \leq 2^{2+h}\epsilon_{\mu\nu}, \quad |x_1| \leq 2^3\delta_{\mu\nu}, \qquad \forall(x_1, x_2, \xi_1, \xi_2) \in 2Q_{\mu\nu}.$$

2) If $Q_{\mu\nu}$ is retained by means of (2.6) then

$$either \quad |x_1| > \frac{\delta_{\mu\nu}}{\sqrt{2}} \quad or \quad |\xi_1| > \frac{\epsilon_{\mu\nu}}{\sqrt{2}}, \qquad \forall(x_1, x_2, \xi_1, \xi_2) \in 2Q_{\mu\nu}.$$

In fact, if by contradiction we suppose that there exists $(\bar{x}_1, \bar{x}_2, \bar{\xi}_1, \bar{\xi}_2) \in 2Q_{\mu\nu}$ such that $|\bar{x}_1| \leq \delta_{\mu\nu}/\sqrt{2}$ and $|\bar{\xi}_1| \leq \epsilon_{\mu\nu}/\sqrt{2}$, this implies $(0, \bar{x}_2, 0, \bar{\xi}_2) \in 4Q_{\mu\nu} \cap \Sigma$. In particular, if $Q_{\mu\nu}$ is retained by means of (2.6) then $p(x, \xi) + R^2 \gtrsim \epsilon_{\mu\nu}^2$ for every $(x, \xi) \in 2Q_{\mu\nu}$.

It is also convenient to compare the metric g^{II} with the metric G of Definition 1.1.

Lemma 2.10 *For every $\mu \in \mathbb{N}$ and $\nu = 1, \ldots, N(\mu)$ the following relations hold*

$$D_\Sigma(x, \xi) \sim \delta_{\mu\nu}^h, \qquad (R^2 + |\xi|^2)^{1/2}D_\Sigma(x, \xi) \sim \epsilon_{\mu\nu}, \qquad \forall(x, \xi) \in Q_{\mu\nu}.$$

Proof. By (2.9) it is enough to prove only one of the relations.

If $Q_{\mu\nu}$ is obtained by partition from a rectangle Q_μ^j of generation $j \geq 1$ then, by Remark 2.9, (2.9) and (2.10), $D_\Sigma(x, \xi) \lesssim \delta_{\mu\nu}^h$ on that rectangle. If it is a first generation rectangle the same holds because of $\delta_{\mu\nu} = 1$ and $D_\Sigma(x, \xi) \lesssim 1$.

The inequality $\delta_{\mu\nu}^h \lesssim D_\Sigma(x, \xi)$ follows from (2.13) when $Q_{\mu\nu}$ is retained by means of (2.5) and by Remark 2.9 and (2.9) in the other case. $\blacksquare$

As an immediate consequence of Lemma 2.10 and of Remark A.1 we get that G is equivalent to g^{II}. Moreover, since the radius r of the balls $B_{(x_j,\xi_j)}^G(r) = \{(y, \eta) \in \mathbb{R}^4 : G_{(x_j,\xi_j)}(y - x_j, \eta - \xi_j) \leq r^2\}$ satisfies the condition $r \leq c_{SV}$ (see (1.5)), Lemma 2.10 allows us to compare these balls with the rectangles $Q_{\mu\nu}$.

Corollary 2.11

1) *The metric G of Definition 1.1 is equivalent to the metric g^{II}.*

2) *There exists a positive constant k, independent of μ, ν, such that*

$$Q_{\mu\nu} \subseteq \bigcup_{j \in \mathcal{J}_{\mu\nu}} B_{(x_j,\xi_j)}^G(r) \subseteq kQ_{\mu\nu}, \quad \forall \mu \in \mathbb{N}, \nu = 1, \ldots, N(\mu),$$

where $\mathcal{J}_{\mu\nu} = \{j \in \mathbb{N} : B_{(x_j,\xi_j)}^G(r) \cap Q_{\mu\nu} \neq \emptyset\}$.

Definition 2.12 *For every $\mu \in \mathbb{N}$ we define*

$$(2.16) \qquad\qquad \rho_\mu = \left(\frac{R}{2^4 M_\mu}\right)^{1/(h+1)}.$$

Note that from (2.2) and (2.10) one has

$$(2.17) \quad \rho_\mu \le \frac{R^{1/(h+1)}}{(R+|\xi|)^{1/(h+1)}} \le \delta_{\mu\nu}, \ \ (x,\xi) \in Q_{\mu\nu}, \ \mu \in \mathbb{N}, \ \nu = 1,\ldots,N(\mu).$$

Proposition 2.13 *Let $p \in S^2(\mathbb{R}^2)$ be the symbol defined in Definition 2.1 and $\{\psi_\mu\}_\mu$, $\{\varphi_\mu\}_\mu$ be the partition of unity associated to the covering $\{Q_\mu^1\}$ in the sense of Proposition A.4 then $\rho_\mu^2 p \psi_\mu \in S(h_{II}^{-2}, g^{II})$ uniformly in μ.*

Proof. Let us recall that, by construction, $\operatorname{supp} \psi_\mu \subseteq 2Q_\mu^1$ and

$$|\partial_x^\alpha \partial_\xi^\beta \psi_\mu(x,\xi)| \lesssim M_\mu^{-|\beta|}, \qquad \forall \alpha, \beta \in \mathbb{Z}_+^2.$$

We have to evaluate the seminorms of $\rho_\mu^2 p \psi_\mu$ only on rectangles $Q_{\mu'\nu'}$ such that $Q_{\mu'\nu'} \cap 2Q_\mu^1 \ne \emptyset$. Hence, by Remark 2.3, $M_\mu \sim M_{\mu'}$ and $\rho_\mu \sim \rho_{\mu'}$. Thus

$$|\partial_x^\alpha \partial_\xi^\beta \psi_\mu(x,\xi)| \lesssim M_{\mu'}^{-|\beta|} \lesssim \epsilon_{\mu'\nu'}^{-|\beta|} \delta_{\mu'\nu'}^{-|\alpha|+(h|\beta|+|\alpha|)} \lesssim \epsilon_{\mu'\nu'}^{-|\beta|} \delta_{\mu'\nu'}^{-|\alpha|}, \quad (x,\xi) \in Q_{\mu',\nu'}.$$

We now have to show that

$$(2.18) \qquad |\partial_x^\alpha \partial_\xi^\beta (\rho_\mu^2 p)(x,\xi)| \lesssim \epsilon_{\mu'\nu'}^{2-|\beta|} \delta_{\mu'\nu'}^{2-|\alpha|}, \quad (x,\xi) \in Q_{\mu'\nu'}.$$

Note that, for $(x,\xi) \in Q_{\mu'\nu'}$ and $|\alpha| + |\beta| \ge 2h$,

$$|\partial_x^\alpha \partial_\xi^\beta (\rho_\mu^2 p)(x,\xi)| \lesssim \rho_{\mu'}^2 M_{\mu'}^{2-|\beta|} \lesssim (\epsilon_{\mu'\nu'} \delta_{\mu'\nu'}^{-h})^{2-|\beta|} \delta_{\mu'\nu'}^2 \lesssim \epsilon_{\mu'\nu'}^{2-|\beta|} \delta_{\mu'\nu'}^{2-|\alpha|}.$$

If $Q_{\mu'\nu'}$ is a retained rectangle of first generation, it must be $\delta_{\mu'\nu'} = 1$ and $\epsilon_{\mu'\nu'} = M_{\mu'}$, so that on $Q_{\mu'\nu'}$ one gets

$$\rho_\mu^2 p(x,\xi) \lesssim \rho_\mu^2 M_{\mu'}^2 \lesssim \delta_{\mu'\nu'}^2 \epsilon_{\mu'\nu'}^2.$$

If $Q_{\mu'\nu'}$ is obtained by partition from $Q_{\mu'}^j$, Remark 2.9 and (2.9) give, for every $(x,\xi) \in 2Q_{\mu'\nu'}$,

$$(2.19) \qquad |p(x,\xi)| \lesssim \epsilon_{\mu'\nu'}^2 + \delta_{\mu'\nu'}^{2h} (R+|\xi|)^2 \lesssim \epsilon_{\mu'\nu'}^2,$$

and this means that estimate (2.18) holds with $|\alpha| + |\beta| = 0$. The remaining cases in (2.18) are recovered by interpolation. Finally, by using the Leibniz Formula we complete the proof. ∎

Proposition 2.14 *If $q \in S^2(\mathbb{R}^2)$ satisfies hypotheses of Theorem 2.4 then, with the same notation of Proposition 2.13, $\rho_\mu^2 q \psi_\mu \in S(h_{II}^{-2}, g^{II})$ uniformly in μ.*

Proof. Since $q \in S^2(\mathbb{R}^2)$, the same arguments used in the proof of Proposition 2.13 show that, when $|\alpha| + |\beta|$ is large enough,

$$(2.20) \qquad |\partial_x^\alpha \partial_\xi^\beta (\rho_\mu^2 q \psi_\mu)(x,\xi)| \lesssim \epsilon_{\mu'\nu'}^{2-|\beta|} \delta_{\mu'\nu'}^{2-|\alpha|}, \quad (x,\xi) \in Q_{\mu'\nu'}.$$

Thus, it is enough to prove the estimate for $|\alpha| + |\beta| = 0$ to complete the proof as we did before. If $Q_{\mu'\nu'}$ is a retained rectangle of first generation, we have $\delta_{\mu'\nu'} = 1$ and $\epsilon_{\mu'\nu'} = M_{\mu'}$ so that, on $Q_{\mu'\nu'}$,

$$\rho_\mu^2 |(q\psi_\mu)(x,\xi)| \lesssim \rho_{\mu'}^2 M_{\mu'}^2 \lesssim \delta_{\mu'\nu'}^2 \epsilon_{\mu'\nu'}^2.$$

Assume now that $Q_{\mu'\nu'}$ is obtained by partitioning some $Q_{\mu'}^j$ and denote by B_j the ball

$$B_{(x_j,\xi_j)}^G(r) = \{(y,\eta) \in \mathbb{R}^4 : G_{(x_j,\xi_j)}(y - x_j, \eta - \xi_j) \leq r^2\}$$

with $j \in \mathbb{N}$. By Corollary 2.11 and by (1.6) we get

$$\max_{Q_{\mu'\nu'}} |q| \leq \sup_{j \in \mathcal{J}_{\mu'\nu'}} (\max_{B_j} |q|) \lesssim \sup_{j \in \mathcal{J}_{\mu'\nu'}} (\max_{B_j} L) \lesssim \max_{kQ_{\mu'\nu'}} L,$$

with $\mathcal{J}_{\mu'\nu'} = \{j \in \mathbb{N} : B_j \cap Q_{\mu'\nu'} \neq \emptyset\}$. Therefore the proof is complete if we show that

$$\max_{kQ_{\mu'\nu'}} L \lesssim \epsilon_{\mu'\nu'}^2.$$

To this purpose, denote by $(\bar{x}_1, \bar{x}_2, \bar{\xi}_1, \bar{\xi}_2) \in \mathbb{R}^4$ the center of $kQ_{\mu'\nu'}$. From 1) of Remark 2.9 it follows, for every $(x_1, x_2, \xi_1, \xi_2) \in kQ_{\mu'\nu'}$,

$$|x_1| \leq |\bar{x}_1| + |x_1 - \bar{x}_1| \leq (2^3 + k)\delta_{\mu'\nu'} \text{ and } |\xi_1| \leq |\bar{\xi}_1| + |\xi_1 - \bar{\xi}_1| \leq (2^{2+h} + k)\epsilon_{\mu'\nu'}.$$

Moreover, by virtue of (2.9),

$$\begin{aligned}
|\xi_2|^2 &\leq (R^2 + |\xi|^2) \leq 2|\xi - \bar{\xi}|^2 + 2(R^2 + |\bar{\xi}|^2) \\
&\leq 2k^2 \epsilon_{\mu'\nu'}^2 + 2^9 \frac{\epsilon_{\mu'\nu'}^2}{\delta_{\mu'\nu'}^{2h}} \leq (2k^2 + 2^9) \frac{\epsilon_{\mu'\nu'}^2}{\delta_{\mu'\nu'}^{2h}},
\end{aligned}$$

whence, for every $(x,\xi) \in kQ_{\mu'\nu'}$, one gets

$$L(x,\xi) = \xi_1^2 + x_1^{2h} \xi_2^2 \lesssim \epsilon_{\mu'\nu'}^2.$$

$\blacksquare$

3. Proof of Theorem 2.4

The proof of Theorem 2.4 requires some preliminary results whose aim is to microlocalize estimate (2.4) on the rectangles $Q_{\mu\nu}$ we constructed. The first step is a subelliptic estimate for p^w that we exploit in the sequel.

Theorem 3.1 *Let $p \in S^2(\mathbb{R}^2)$ be the symbol introduced in Definition 2.1 and let $\{\varphi_\mu\}_{\mu\in\mathbb{N}}$, $\{\psi_\mu\}_{\mu\in\mathbb{N}}$ the partition of unity associated to the covering $\{Q_\mu^1\}$ in the sense of Proposition A.4. Let us consider a bounded family $\{\phi_\mu\}_{\mu\in\mathbb{N}}$ in $S(1, g^I)$ with $\phi_\mu = 1$ on $(9/8)Q_\mu^1$ and supp $\phi_\mu \subseteq (5/4)Q_\mu^1$. Then the following estimates hold*

$$(3.1) \quad \|(p\psi_\mu)^w \varphi_\mu^w u\|_0^2 \gtrsim M_\mu^{4/(h+1)} \|\varphi_\mu^w u\|_0^2 - \|(1-\psi_\mu)^w (M_\mu^2 \varphi_\mu^w) u\|_0^2$$
$$- \|(1-\phi_\mu)^w (M_\mu^2 \varphi_\mu^w) u\|_0^2 - \|\varphi_\mu^w u\|_0^2, \quad u \in \mathcal{S}(\mathbb{R}^2).$$

Proof. Let us denote by U_1 the set of $\mu \in \mathbb{N}$ such that the center

$$(x_{1,\mu}, x_{2,\mu}, \xi_{1,\mu}, \xi_{2,\mu}) = (x_\mu, \xi_\mu)$$

of Q_μ^1 satisfies the following condition: either $|x_{1,\mu}| \geq 2$ or $|\xi_{1,\mu}| \geq 2|\xi_{2,\mu}|$. By U_2 we denote the set $\mathbb{N} \setminus U_1$. The following relations hold for every $(x, \xi) \in 2Q_\mu^1$

$$(3.2) \qquad \exists \gamma_1 > 0 \text{ such that } p(x, \xi) + R^2 \geq \gamma_1 M_\mu^2, \quad \forall \mu \in U_1;$$

$$(3.3) \quad \exists \gamma_2 > 0 \text{ such that } p(x, \xi) + R^2 \geq \gamma_2(\xi_1^2 + x_1^{2h} M_\mu^2), \quad \forall \mu \in U_2.$$

The ellipticity property (3.2) gives

$$(3.4) \qquad \|(p\psi_\mu)^w \varphi_\mu^w u\|_0^2 \gtrsim M_\mu^4 \|\varphi_\mu^w u\|_0^2 - \|(1-\psi_\mu)^w M_\mu^2 \varphi_\mu^w u\|_0^2 - \|\varphi_\mu^w u\|_0^2,$$

$u \in \mathcal{S}(\mathbb{R}^2)$, $\forall \mu \in U_1$. We now prove

$$(3.5) \quad \|(p\psi_\mu)^w \varphi_\mu^w u\|_0^2 \gtrsim M_\mu^{4/(h+1)} \|\varphi_\mu^w u\|_0^2 - \|(1-\phi_\mu)^w M_\mu^2 \varphi_\mu^w u\|_0^2 - \|\varphi_\mu^w u\|_0^2,$$

$u \in \mathcal{S}(\mathbb{R}^2)$, $\forall \mu \in U_2$. Note that in view of (3.3), for $\mu \in U_2$ one has

$$p\psi_\mu + R^2 = (p + R^2)\psi_\mu + (1-\psi_\mu)R^2 \geq \gamma_2(\xi_1^2 + x_1^{2h} M_\mu^2)\psi_\mu,$$

and that $p\psi_\mu + R^2 - \gamma_2(\xi_1^2 + x_1^{2h} M_\mu^2)\psi_\mu \in S(h_{g^I}^{-2}, g^I)$. This allows an application of the Fefferman-Phong inequality (see [10, Lemma 18.6.10]). Taking into account that the lowest eigenvalue of the anharmonic oscillator $D_{x_1}^2 + M_\mu^2 x_1^{2h}$ is $\lambda \sim M_\mu^{2/(h+1)}$, we get for every $u \in \mathcal{S}(\mathbb{R}^2)$

$$(3.6) \qquad \|(p\psi_\mu)^w \phi_\mu^w u\|_0^2 \gtrsim M_\mu^{4/(h+1)} \|\phi_\mu^w u\|_0^2$$
$$- \|\left((\xi_1^2 + M_\mu^2 x_1^{2h})(1-\psi_\mu)\right)^w \phi_\mu^w u\|_0^2 - \|\phi_\mu^w u\|_0^2.$$

In order to estimate the term $\|\big((\xi_1^2 + M_\mu^2 x_1^{2h})(1 - \psi_\mu)\big)^w \phi_\mu^w u\|_0^2$, let us now consider the Hörmander metric G_μ defined by

$$G_{\mu,(x,\xi)}(y,\eta) = \frac{|y|^2}{\langle x - x_\mu \rangle^2} + \frac{|\eta|^2}{M_\mu^2 + |\xi - \xi_\mu|^2}, \qquad (x,\xi) \in \mathbb{R}^4.$$

Note that $G_\mu \sim g^I$ on $2Q_\mu^1$. Moreover $\xi_1^2 + x_1^{2h} M_\mu^2 \in S(m, G_\mu)$, with (admissible) weight $m(x,\xi) = \langle x - x_\mu \rangle^{2h}(M_\mu^2 + |\xi - \xi_\mu|^2)$, and $1 - \psi_\mu$, $\phi_\mu \in S(1, G_\mu)$ with seminorms independent of μ. Thus $(\xi_1^2 + M_\mu^2 x_1^{2h})(1 - \psi_\mu) \sharp \phi_\mu \in S(1, G_\mu)$ (if $a \in S(m_1, g)$ and $b \in S(m_2, g)$, $a \sharp b \in S(m_1 m_2, g)$ denotes the symbol of the composition $a^w b^w$) with seminorms not depending on μ. Hence, from (3.6), we get

$$\|(p\psi_\mu)^w \phi_\mu^w \varphi_\mu^w u\|_0^2 \gtrsim \|\phi_\mu^w \varphi_\mu^w M_\mu^{2/(h+1)} u\|_0^2 - \|\varphi_\mu^w u\|_0^2$$
$$\gtrsim M_\mu^{4/(h+1)} \|\varphi_\mu^w u\|_0^2 - M_\mu^{4/(h+1)}\|(1 - \phi_\mu)^w \varphi_\mu^w u\|_0^2 - \|\varphi_\mu^w u\|_0^2,$$

for $u \in \mathcal{S}(\mathbb{R}^2)$. From this the conclusion follows once we observe that

$$\|(p\psi_\mu)^w \phi_\mu^w \varphi_\mu^w u\|_0^2 \lesssim \|(p\psi_\mu)^w \varphi_\mu^w u\|_0^2 + \left\|\Big(\frac{p\psi_\mu}{M_\mu^2}\Big)^w (1 - \phi_\mu)^w M_\mu^2 \varphi_\mu^w u\right\|_0^2, \quad u \in \mathcal{S}(\mathbb{R}^2)$$

with $p\psi_\mu / M_\mu^2 \in S(1, g^{II})$ uniformly in μ. The proof of the theorem readily follows from (3.4) and (3.5). ∎

Theorem 3.1 is used to prove the following Proposition.

Proposition 3.2 *Let p and q be as in Theorem 2.4. Then estimate (2.4) holds provided one has*

(3.7) $$\|(\rho_\mu^2 p\psi_\mu)^w u\|_0^2 \gtrsim \|(\rho_\mu^2 q\psi_\mu)^w u\|_0^2 - \|u\|_0^2 - (r_{\mu,1}^w u, \ u)_{L^2},$$

$\forall u \in \mathcal{S}(\mathbb{R}^2), \ \forall \mu \in \mathbb{N}, \ where \ \displaystyle\sum_{\mu \in \mathbb{N}} \rho_\mu^{-4} \varphi_\mu^w r_{\mu,1}^w \varphi_\mu^w \in \mathcal{L}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2)).$

Proof. We divide all terms in (3.7) by ρ_μ^4 and write $\varphi_\mu^w u$ instead of u. From (2.16) we then have

$$\|(p\psi_\mu)^w \varphi_\mu^w u\|_0^2 \gtrsim \|(q\psi_\mu)^w \varphi_\mu^w u\|_0^2 - M_\mu^{4/(h+1)}\|\varphi_\mu^w u\|_0^2 - (\rho_\mu^{-4} \varphi_\mu^w r_{\mu,1}^w \varphi_\mu^w u, \ u)_{L^2},$$

$\forall u \in \mathcal{S}(\mathbb{R}^2)$. The conclusion follows by applying estimate (3.1) to control the term $M_\mu^{4/(h+1)}\|\varphi_\mu^w u\|_0^2$ and by using (A.13) and Proposition A.9 to control the remainder terms. ∎

Until now we reduced the proof of Theorem 2.4 to the proof of (3.7). We now want to reduce the proof of (3.7) to the proof of suitable inequalities on the rectangles of the partition $\{Q_{\mu\nu}\}$ constructed in Section 2.

Definition 3.3 *Let $\bar{\mu}$ be a fixed arbitrary index in $\mathbb{N}$. If $\{Q_{\mu\nu}\}_{\mu\in\mathbb{N},\nu=1,\dots,N(\mu)}$ is the partition constructed in Section 2, we define*

1) $I_{\bar{\mu}} = \{(\mu,\nu) : \dfrac{13}{8}Q_{\mu\nu} \cap 2Q_{\bar{\mu}} \neq \emptyset\}$;

2) $I_{\bar{\mu}}^e = \{(\mu,\nu) \in I_{\bar{\mu}} \text{ satisfying } (2.6)\}$;

3) $I_{\bar{\mu}}^s = I_{\bar{\mu}} \setminus I_{\bar{\mu}}^e = \{(\mu,\nu) \in I_{\bar{\mu}} \text{ satisfying only} (2.5)\}$.

Note that $I_{\bar{\mu}}^e \cap I_{\bar{\mu}}^s = \emptyset$ and that $I_{\bar{\mu}}^e \cup I_{\bar{\mu}}^s = I_{\bar{\mu}}$.

Remark 3.4 *One has $\rho_{\bar{\mu}} \sim \rho_{\mu} \lesssim \delta_{\mu\nu}$ for every $\mu\nu$ such that $(\mu,\nu) \in I_{\bar{\mu}}$. In fact, from $(13/8)Q_{\mu\nu} \cap 2Q_{\bar{\mu}} \neq \emptyset$ it follows $2Q_{\mu} \cap 2Q_{\bar{\mu}} \neq \emptyset$ which in turn, by Remark 2.3, gives $M_{\mu} \sim M_{\bar{\mu}}$. The conclusion then follows by (2.16) and (2.17).*

The next proposition clarifies the meaning of Definition 3.3. We, in fact, see that $I_{\bar{\mu}}$ describes the set of the indices that "count" when one microlocalizes (3.7) with $\mu = \bar{\mu}$ on the rectangles of the partition $\{Q_{\mu\nu}\}$.

Proposition 3.5 *Let p and q be as in Theorem 2.4. Then, for $\bar{\mu} \in \mathbb{N}$, the estimate*

$$(3.8) \quad \|(\rho_{\bar{\mu}}^2 p\psi_{\bar{\mu}})^w u\|_0^2 \gtrsim \|(\rho_{\bar{\mu}}^2 q\psi_{\bar{\mu}})^w u\|_0^2 - \|u\|_0^2 - (r_{\bar{\mu},2}^w u,\ u)_{L^2}, \quad \forall u \in \mathcal{S}(\mathbb{R}^2)$$

with $\sum_{\bar{\mu}\in\mathbb{N}} \rho_{\bar{\mu}}^{-4} \varphi_{\bar{\mu}}^w r_{\bar{\mu},2}^w \varphi_{\bar{\mu}}^w \in \mathcal{L}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$ holds if both the following estimates hold:

- *for any $\mu\nu$ such that $(\mu,\nu) \in I_{\bar{\mu}}^e$ and for every $u \in \mathcal{S}(\mathbb{R}^2)$*

$$(3.9) \quad \|(\rho_{\bar{\mu}}^2 p\psi_{\bar{\mu}}\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 \gtrsim \|(\rho_{\bar{\mu}}^2 q\psi_{\bar{\mu}}\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 - \|\varphi_{\mu\nu}^w u\|_0^2$$
$$- \|\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2((1-\psi_{\bar{\mu}})\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 - \|\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2(1-\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2.$$

- *for any $\mu\nu$ such that $(\mu,\nu) \in I_{\bar{\mu}}^s$ and for every $u \in \mathcal{S}(\mathbb{R}^2)$*

$$(3.10) \quad \|(\rho_{\bar{\mu}}^2 p\psi_{\bar{\mu}}\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 \gtrsim \|(\rho_{\bar{\mu}}^2 q\psi_{\bar{\mu}}\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 - \|\varphi_{\mu\nu}^w u\|_0^2.$$

Proof. By adding (3.9) to (3.10) and by using Theorem A.5, we see that it suffices to prove the following relations:

$$(3.11) \quad \sum_{(\mu,\nu)\in I_{\bar{\mu}}} \|(\rho_{\bar{\mu}}^2 q\psi_{\bar{\mu}}\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 \gtrsim \|(\rho_{\bar{\mu}}^2 q\psi_{\bar{\mu}})^w u\|_0^2 - \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^2);$$

$$(3.12) \quad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2(1-\psi_{\mu\nu})^w \varphi_{\mu\nu}^w u\|_0^2 \lesssim \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^2);$$

and

$$(3.13) \quad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\,\epsilon_{\mu\nu}^2(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w\varphi_{\mu\nu}^w u\|_0^2 \lesssim (RM_{\bar{\mu}}^h)^{4/(h+1)}\|(1-\psi_{\bar{\mu}})^w u\|_0^2$$

$$+ \left\|\rho_{\bar{\mu}}^2\big((p+R^2)(1-\psi_{\bar{\mu}})\big)^w u\right\|_0^2 + \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^2).$$

Relations (3.11) and (3.12) are an immediate consequence of (A.14) and Proposition A.9. We proceed to prove (3.13). We first observe that

$$(3.14)$$

$$\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w\varphi_{\mu\nu}^w u\|_0^2 \lesssim \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\varphi_{\mu\nu}^w(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w u\|_0^2$$

$$+ \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \left\|\left[\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w,\ \varphi_{\mu\nu}^w\right]u\right\|_0^2.$$

We now want to estimate the terms in the right hand side of (3.14). As concerns the first one, we can apply Proposition A.9 (in view of Remark 3.4) and get

$$\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\varphi_{\mu\nu}^w(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w u\|_0^2 \lesssim \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\varphi_{\mu\nu}^w(1-\psi_{\bar{\mu}})^w u\|_0^2 + \|u\|_0^2.$$

Moreover, $\rho_{\bar{\mu}}^4\epsilon_{\mu\nu}^4 \lesssim (R/M_{\bar{\mu}})^{4/(h+1)}M_\mu^4\delta_{\mu\nu}^{4h} \lesssim (RM_{\bar{\mu}}^h)^{4/(h+1)}$ then, applying (A.13),

$$\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\varphi_{\mu\nu}^w(1-\psi_{\bar{\mu}})^w u\|_0^2 \lesssim (RM_{\bar{\mu}}^h)^{4/(h+1)}\|(1-\psi_{\bar{\mu}})^w u\|_0^2.$$

Thus

$$(3.15) \quad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\varphi_{\mu\nu}^w(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w u\|_0^2 \lesssim (RM_{\bar{\mu}}^h)^{4/(h+1)}\|(1-\psi_{\bar{\mu}})^w u\|_0^2 + \|u\|_0^2.$$

As for the second term in the right hand side of (3.14), we show that

$$(3.16) \quad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \left\|\left[\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2(\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w,\ \varphi_{\mu\nu}^w\right]u\right\|_0^2 \lesssim$$

$$\lesssim \left(\rho_{\bar{\mu}}^2\big((p+R^2)(1-\psi_{\bar{\mu}})\big)^w u, u\right)_{L^2} + \|u\|_0^2.$$

Note that the family $\{\varphi_{\mu\nu}\}$ is bounded in $S(1,g^{II})$ and that the family of positive functions $\{\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\psi_{\mu\nu}(1-\psi_{\bar{\mu}})\}$ is bounded in $S(h_{II}^{-2},g^{II})$ with supp $\varphi_{\mu\nu} \subseteq (9/8)Q_{\mu\nu}$ and supp $\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\psi_{\mu\nu}(1-\psi_{\bar{\mu}}) \subseteq (13/8)Q_{\mu\nu} \subseteq (7/4)Q_{\mu\nu}.$

Thus (A.25) applies yielding

$$(3.17) \qquad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \left\| \left[\varphi_{\mu\nu}^w, \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (\psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w \right] u \right\|_0^2 \lesssim$$

$$\lesssim \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \left(\psi_{\mu\nu}^w (\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 \psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w \psi_{\mu\nu}^w u, u \right)_{L^2} + \|u\|_0^2.$$

Moreover, from (A.24) it follows

$$(3.18) \qquad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^w (\rho_{\bar{\mu}}^2 \,\epsilon_{\mu\nu}^2 \psi_{\mu\nu}(1-\psi_{\bar{\mu}}))^w \psi_{\mu\nu}^w =$$

$$= \left(\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}}) \right)^w + r_{\bar{\mu},3}^w$$

with $r_{\bar{\mu},3}^w \in \mathcal{L}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$ having norm bounded uniformly in $\bar{\mu}$. Let us now define $J_{\bar{\mu}} = \{\mu' \in \mathbb{N} : \exists (\mu,\nu) \in I_{\bar{\mu}}, \ (13/8)Q_{\mu\nu} \cap (9/8)Q_{\mu'} \neq \emptyset\}$ and observe that, since $\mathrm{supp}\,\psi_{\mu\nu}^3 \subseteq (13/8)Q_{\mu\nu}$, we have

$$(3.19) \qquad \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}}) = \sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}})(\sum_{\mu'\in J_{\bar{\mu}}} \varphi_{\mu'}^2).$$

From Remark 2.9 one has, for $(\mu,\nu) \in I_{\bar{\mu}}^e$,

$$\psi_{\mu,\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}}) \lesssim \rho_{\bar{\mu}}^2 (p + R^2)(1-\psi_{\bar{\mu}}).$$

Hence by the finite overlapping properties and (3.19)

$$\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}}) \lesssim \sum_{\mu'\in J_{\bar{\mu}}} \rho_{\bar{\mu}}^2 (p + R^2)(1-\psi_{\bar{\mu}})\varphi_{\mu'}^2.$$

We can now apply the Fefferman-Phong inequality in the class $S(h_{II}^{-2}, g^{II})$ and get

$$(3.20) \quad \left((\sum_{(\mu,\nu)\in I_{\bar{\mu}}^e} \psi_{\mu\nu}^3 \rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 (1-\psi_{\bar{\mu}}))^w u, u \right)_{L^2} \lesssim$$

$$\lesssim \left((\sum_{\mu'\in J_{\bar{\mu}}} \varphi_{\mu'}^2 \rho_{\bar{\mu}}^2 (p + R^2)(1-\psi_{\bar{\mu}}))^w u, u \right)_{L^2} + \|u\|_0^2.$$

Finally, since (again by finite overlapping)

$$\sum_{\mu'\in J_{\bar{\mu}}} \varphi_{\mu'}^2 \rho_{\bar{\mu}}^2 (p + R^2)(1-\psi_{\bar{\mu}}) \lesssim \rho_{\bar{\mu}}^2 (p + R^2)(1-\psi_{\bar{\mu}}),$$

we can again apply the Fefferman-Phong inequality in the class $S(h_I^{-2}, g^I)$ and obtain the inequality (3.16) by (3.17), (3.18) and (3.20). An application of (3.15) and (3.16) to (3.14) completes the proof. ∎

Proof of Theorem 2.4. We use Propositions 3.2 and 3.5 and reduce the proof of Theorem 2.4 to the proof of estimates (3.9) and (3.10).

In order to prove (3.9), let us first observe that, for every $(\mu, \nu) \in I_{\bar{\mu}}^e$, the following inequality holds for $u \in \mathcal{S}(\mathbb{R}^2)$

$$(3.21) \quad \left(\left(\rho_{\bar{\mu}}^2 (p + R^2)\psi_{\bar{\mu}}\psi_{\mu\nu} + (1 - \psi_{\mu\nu})\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 \right)^w u, u \right)_{L^2} \geq$$
$$\geq - \left(\rho_{\bar{\mu}}^2 \epsilon_{\mu\nu}^2 \left(\psi_{\mu\nu}(1 - \psi_{\bar{\mu}}) \right)^w u, u \right)_{L^2} + \epsilon_{\mu\nu}^2 \rho_{\bar{\mu}}^2 \|u\|_0^2 - C\|u\|_0^2.$$

If, in fact, we consider the family of constant metrics

$$g_{\mu\nu}(y, \eta) = \frac{|y|^2}{\delta_{\mu\nu}^2} + \frac{|\eta|^2}{\epsilon_{\mu\nu}^2},$$

by (2.17), Proposition 2.13 and Remark 3.4 one has, for every $(\mu, \nu) \in I_{\bar{\mu}}^e$, that

$$\rho_{\bar{\mu}}^2(p+R^2)\psi_{\bar{\mu}}\psi_{\mu\nu}+(1-\psi_{\mu\nu})\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2, \ \ \rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2 \ \ \text{and} \ \ \rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\psi_{\mu\nu}(1-\psi_{\bar{\mu}}) \in S(h_{g_{\mu\nu}}^{-2}, g_{\mu\nu})$$

with seminorms independent of $\mu, \nu, \bar{\mu}$.

Moreover, from Remark 2.9 it follows that

$$\rho_{\bar{\mu}}^2(p + R^2)\psi_{\bar{\mu}}\psi_{\mu\nu} + (1 - \psi_{\mu\nu})\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2 \geq \rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2 - \rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2\psi_{\mu\nu}(1 - \psi_{\bar{\mu}}).$$

Thus (3.21) follows by the Fefferman-Phong inequality.

If we observe that $R^2\rho_{\bar{\mu}}^2\psi_{\bar{\mu}}\psi_{\mu\nu}$ is in $S(1, g_{\mu\nu})$ with seminorms independent of $\bar{\mu}, \mu, \nu$, we then get

$$(3.22) \quad \|\left(\rho_{\bar{\mu}}^2 p\psi_{\mu}\psi_{\mu\nu} \right)^w u\|_0^2 \geq \epsilon_{\mu\nu}^4\rho_{\bar{\mu}}^4\|u\|_0^2 - C'\|u\|_0^2 - \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2((1 - \psi_{\bar{\mu}})\psi_{\mu\nu})^w u\|_0^2$$
$$- \|\rho_{\bar{\mu}}^2\epsilon_{\mu\nu}^2(1 - \psi_{\mu\nu})^w u\|_0^2, \ \forall(\mu, \nu) \in I_{\bar{\mu}}^e, \ \forall u \in \mathcal{S}(\mathbb{R}^2).$$

Proceeding as in the proof of Proposition 2.14 we also have

$$\delta_{\mu\nu}^2 q\psi_{\bar{\mu}}\psi_{\mu\nu} \in S(h_{II}^{-2}, g^{II})$$

uniformly in $\bar{\mu}$, μ, ν and this yields

$$(3.23) \quad \|\rho_{\bar{\mu}}^2(q\psi_{\bar{\mu}}\psi_{\mu\nu})^w u\|_0^2 \lesssim \epsilon_{\mu\nu}^4\rho_{\bar{\mu}}^4\|u\|_0^2, \qquad \forall(\mu, \nu) \in I_{\bar{\mu}}^e, \ \forall u \in \mathcal{S}(\mathbb{R}^2).$$

Finally, from (3.22) and (3.23), (3.9) readily follows.

For what concerns estimate (3.10), note that by (2.11) and (2.12) we have $\delta_{\mu\nu}\epsilon_{\mu\nu} \sim 1$ when $(\mu, \nu) \in I_{\bar{\mu}}^s$. Hence (3.10) follows by L^2-continuity and this concludes the proof. $\blacksquare$

A. Appendix

Let us fix a sequence $\{(x_k, \xi_k)\}_{k \in \mathbb{N}}$ of points in the phase space $T^* \mathbb{R}^n \simeq \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and two sequences $\{\delta_k\}_{k \in \mathbb{N}}, \{\epsilon_k\}_{k \in \mathbb{N}}$ of real positive numbers. We denote by B_k the rectangles defined by

(A.1)
$$B_k = \{(x, \xi) \in T^* \mathbb{R}^n \mid |x_j - x_{k,j}| \le \frac{\delta_k}{2\sqrt{n}}, \ |\xi_j - \xi_{k,j}| \le \frac{\epsilon_k}{2\sqrt{n}}, \ j = 1, \dots, n\}$$

and assume that $\{B_k\}_{k \in \mathbb{N}}$ is a partition of $T^* \mathbb{R}^n$, i.e.

$$\bigcup_{k \in \mathbb{N}} B_k = T^* \mathbb{R}^n \quad \text{and} \quad \mathring{B}_k \cap \mathring{B}_{k'} = \emptyset \quad \text{for every } k, k' \in \mathbb{N}, \ k \ne k'.$$

Consider the functions $\delta, \epsilon : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \longrightarrow \mathbb{R}_+$ defined by

$$\delta(x, \xi) = \delta_\gamma, \quad \epsilon(x, \xi) = \epsilon_\gamma \qquad \text{where } \gamma = \min \{k \in \mathbb{N} : (x, \xi) \in B_k\}$$

and notice that $\delta(x, \xi) = \delta_k, \epsilon(x, \xi) = \epsilon_k$ when $(x, \xi) \in \mathring{B}_k$.

Remark A.1 *If the partition is such that $\delta_k \sim \delta_{k'}$ and $\epsilon_k \sim \epsilon_{k'}$ whenever $2B_k \cap 2B_{k'} \ne \emptyset$ then*

$$(x, \xi) \in B_k \Rightarrow \delta(x, \xi) \sim \delta_k, \quad \epsilon(x, \xi) \sim \epsilon_k.$$

For any $(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, we consider a positive definite quadratic form $g_{(x,\xi)}$ of $(t, \tau) \in \mathbb{R}^{2n}$ as follows

(A.2)
$$g_{(x,\xi)}(t, \tau) = \frac{|t|^2}{\delta(x, \xi)^2} + \frac{|\tau|^2}{\epsilon(x, \xi)^2}.$$

It is important to point out that the Riemannian metric $g : (x, \xi) \longrightarrow g_{(x,\xi)}$ is not, in general, continuous on the boundary of each B_k, but that causes no problem as shown in [14].

In this setting, for any $(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ the dual metric g^σ of g (with respect to the canonical symplectic form σ on $T^* \mathbb{R}^n$) is given by

$$g^\sigma_{(x,\xi)}(t, \tau) = \epsilon(x, \xi)^2 |t|^2 + \delta(x, \xi)^2 |\tau|^2, \quad \forall (t, \tau) \in \mathbb{R}^{2n},$$

while the Planck function $h_g : T^* \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is given by

$$h_g(x, \xi) = \delta(x, \xi)^{-1} \epsilon(x, \xi)^{-1}.$$

The following proposition establishes sufficient conditions under which g is a Weyl-Hörmander metric.

Proposition A.2 *Fix a positive integer h and a real positive number $R \geq 1$. Suppose that, for the partition $\{B_k\}_{k \in \mathbb{N}}$ defined in $(A.1)$, the following conditions hold:*

$$(A.3) \qquad\qquad \epsilon_k \sim (R + |\xi|)\delta_k^h, \quad \forall (x, \xi) \in B_k;$$

$$(A.4) \qquad R^{1/(h+1)}(R + |\xi|)^{-1/(h+1)} \leq \delta_k \leq 1, \quad \forall (x, \xi) \in B_k;$$

$$(A.5) \qquad\quad If \ 2B_k \cap 2B_{k'} \neq \emptyset \Longrightarrow \delta_k \sim \delta_{k'} \ and \ \epsilon_k \sim \epsilon_{k'}.$$

Then g defined by $(A.2)$ is a Weyl-Hörmander metric provided R is chosen large enough, i.e. g is slowly varying and σ-temperate (see [9]). Moreover, $(A.5)$ determines an a priori bounded number $K = K(n)$ of overlappings among the rectangles $2B_k$. Namely, one has $|J_k| \leq K$ for every $k \in \mathbb{N}$, where

$$J_k = \{k' \in \mathbb{N} : 2B_{k'} \cap 2B_k \neq \emptyset\}$$

and $|J_k|$ denotes its cardinality.

Proof. We start off by proving that g is a slowly varying metric. Suppose that $g_{(y,\eta)}(x - y, \xi - \eta) \leq 1/(4n)$ with $(x, \xi) \in B_k$. Then $(x, \xi) \in 2B_\gamma$ with $\gamma = \min\{\omega \in \mathbb{N} : (y, \eta) \in B_\omega\}$. This gives, in view of Remark A.1 and $(A.5)$, the existence of positive constants c_1, c_2 such that

$$c_1 g_{(x,\xi)}(t, \tau) \leq g_{(y,\eta)}(t, \tau) \leq c_2 g_{(x,\xi)}(t, \tau), \quad \forall (t, \tau) \in \mathbb{R}^{2n},$$

for such (x, ξ). Hence, g is slowly varying.

From $(A.3)$ and $(A.4)$ it follows, if R is chosen large enough, $h_g(x, \xi) < 1$ for any $(x, \xi) \in \mathbb{R}^{2n}$, whence $g_{(x,\xi)}(t, \tau) \leq g_{(x,\xi)}^\sigma(t, \tau)$. It remains to show that there exist a positive constant c and a positive integer Q for which

$$(A.6) \qquad g_{(x,\xi)}(t, \tau) \leq c\, g_{(y,\eta)}(t, \tau)\big(1 + g_{(x,\xi)}^\sigma(y - x, \eta - \xi)\big)^Q,$$

whenever $(x, \xi), (y, \eta) \in \mathbb{R}^{2n}$. Actually we prove the following equivalent inequality

$$(A.7) \qquad g_{(y,\eta)}^\sigma(t, \tau) \leq c\, g_{(x,\xi)}^\sigma(t, \tau)\big(1 + g_{(x,\xi)}^\sigma(y - x, \eta - \xi)\big)^{h+1}.$$

We first observe that, from Remark A.1 it follows

$$(A.8) \qquad \sup_{(t,\tau) \in \mathbb{R}^{2n}} \frac{g_{(y,\eta)}^\sigma(t, \tau)}{g_{(x,\xi)}^\sigma(t, \tau)} \sim \max\left\{\frac{\epsilon_{k'}^2}{\epsilon_k^2}, \frac{\delta_{k'}^2}{\delta_k^2}\right\}, \quad \forall (x, \xi) \in B_k, \ \forall (y, \eta) \in B_{k'};$$

therefore, if $g_{(y,\eta)}(x - y, \xi - \eta) \leq 1/(4n)$, inequality $(A.7)$ easily follows from the slowly varying property just because the right hand side of $(A.8)$

is bounded. When $g_{(y,\eta)}(x-y,\xi-\eta) > 1/(4n)$, again by using Remark A.1, we have two cases:

$$\text{either } |\xi - \eta|^2 \gtrsim \epsilon_{k'}^2 \quad \text{or} \quad |x - y|^2 \gtrsim \delta_{k'}^2.$$

Assuming $|\xi - \eta|^2 \gtrsim \epsilon_{k'}^2$, we get

(A.9) $$\left(1 + g_{(x,\xi)}^\sigma(y-x, \eta-\xi)\right)\epsilon_k^2 \gtrsim \epsilon_k^2 \delta_k^2 |\xi - \eta|^2 \gtrsim \epsilon_{k'}^2.$$

If $|\xi - \eta| \geq |\xi|/2$, from (A.4) it follows

$$\begin{aligned}
\left(1 + g_{(x,\xi)}^\sigma(y-x, \eta-\xi)\right)\delta_k^2 &\gtrsim \left(1 + \delta_k^2|\xi|^2\right)\delta_k^2 \geq \frac{1}{R^2}\delta_k^4\left(R + |\xi|\right)^2 \\
&\geq \frac{1}{R^2}\frac{R^{4/(h+1)}}{(R+|\xi|)^{4/(h+1)}}\left(R + |\xi|\right)^2 \gtrsim 1 \gtrsim \delta_{k'}^2.
\end{aligned}$$

Suppose now that $|\xi - \eta| \leq |\xi|/2$, in view of (A.3) and (A.9) we have

$$\begin{aligned}
\left(1 + g_{(x,\xi)}^\sigma(y-x, \eta-\xi)\right)\delta_k^{2h} &\gtrsim \left(\left(1 + g_{(x,\xi)}^\sigma(y-x, \eta-\xi)\right)\epsilon_k^2\right)\left(R + |\xi|\right)^{-2} \\
&\gtrsim \epsilon_{k'}^2\left(R + |\eta|\right)^{-2} \gtrsim \delta_{k'}^{2h}.
\end{aligned}$$

Inequality (A.7) follows in this case from (A.8). The case $|x - y|^2 \gtrsim \delta_{k'}^2$ is achieved similarly.

It remains to show the existence of an a priori bounded number of overlappings for the covering $\{B_k\}_{k\in\mathbb{N}}$. Suppose that $2B_{k'} \cap 2B_k \neq \emptyset$. Then there exists $(x, \xi) \in 2B_{k'} \cap 2B_k$, and from (A.5) it follows, for a constant $c > 0$ independent of k, k', $\delta_k/c \leq \delta_{k'} \leq c\delta_k$, $\epsilon_k/c \leq \epsilon_{k'} \leq c\epsilon_k$. Upon denoting by (x_k, ξ_k) the center of the rectangle B_k, one has, for every $(y, \eta) \in 2B_{k'}$,

$$|y_i - x_{k,i}| \leq |y_i - x_i| + |x_i - x_{k,i}| \leq 2\delta_{k'} + \delta_k \leq (1 + 2c)\delta_k, \quad i = 1, \ldots, n,$$

and, in a similar way, $|\eta_i - \xi_{k,i}| \leq (1 + 2c)\epsilon_k$, $i = 1, \ldots, n$. Therefore one gets

$$\bigcup_{k' \in J_k} B_{k'} \subseteq (1 + 2c)\, 2\sqrt{n}\, B_k,$$

whence

(A.10) $$\sum_{k' \in J_k} m(B_{k'}) = m\Big(\bigcup_{k' \in J_k} B_{k'}\Big) \leq m\big((1 + 2c)\, 2\sqrt{n}\, B_k\big),$$

where $m(A)$ denotes the Lebesgue measure of a set A, and from this the conclusion follows since $m(B_{k'}) \sim \epsilon_k^n \delta_k^n$ for each $k' \in J_k$ and $m(B_k) \sim \epsilon_k^n \delta_k^n$.   ∎

In the sequel we assume that the hypotheses of Proposition A.2 hold for the partition defined by (A.1). As a consequence, g is slowly varying and σ−temperate, and an easy check shows that the Planck function h_g is an admissible weight for g.

We now define the class $S(h_g^s, g)$ $(s \in \mathbb{R})$ as follows.

Definition A.3 *Let j be a non negative integer and (x, ξ) be a point of $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$; define, for any smooth function $b \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$,*

$$|b|_j^g(x, \xi) = \max_{|\alpha+\beta| \leq j} \delta(x, \xi)^{|\alpha|} \epsilon(x, \xi)^{|\beta|} |D_x^\alpha D_\xi^\beta b(x, \xi)|.$$

We denote by $S(h_g^s, g)$ $(s \in \mathbb{R})$ the class of the functions $b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for any integer $l \geq 0$,

$$(A.11) \qquad \|b\|_{(l, h_g^s)} = \sup_{(x, \xi) \in \mathbb{R}^{2n}} h_g(x, \xi)^{-s} |b|_l^g(x, \xi) < +\infty.$$

It turns out that $S(h_g^s, g)$ is a Fréchet space with the topology induced by the seminorms defined in $(A.11)$.

As a consequence of Remark A.1, we have

$$\|b\|_{(l, h_g^s)} \sim \sup_{k \in \mathbb{N}} \max_{\substack{(x, \xi) \in B_k \\ |\alpha + \beta| \leq l}} \delta_k^{s+|\alpha|} \epsilon_k^{s+|\beta|} |D_x^\alpha D_\xi^\beta b(x, \xi)|.$$

By proceeding as in [2] we get a partition of unity related to the covering $\{B_k\}_{k \in \mathbb{N}}$.

Proposition A.4 *There exist two families of smooth functions $\{\varphi_k\}_{k \in \mathbb{N}}$, $\{\psi_k\}_{k \in \mathbb{N}}$ defined in $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$, uniformly bounded in the space $S(1, g)$, with the following properties:*

1) *for every $j = 0, 1, \ldots$, there exists a positive constant C_j independent of R, such that*

$$(A.12) \qquad \|\varphi_k\|_{(j, 1)} + \|\psi_k\|_{(j, 1)} \leq C_j, \qquad \forall k \in \mathbb{N};$$

2) *for any $k \in \mathbb{N}$,*

$$0 \leq \varphi_k(x, \xi) \leq 1, \ \ 0 \leq \psi_k(x, \xi) \leq 1, \qquad \forall (x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n$$

$$\text{supp } \varphi_k \subseteq \frac{9}{8} B_k, \quad \text{supp } \psi_k \subseteq \frac{13}{8} B_k, \quad \psi_k \equiv 1 \ \ on \ \frac{3}{2} B_k;$$

3) *for every $(x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n$, one has $\sum_{k \in \mathbb{N}} \varphi_k(x, \xi)^2 \equiv 1$.*

In what follows, we refer to the pair $\{\varphi_k\}_{k \in \mathbb{N}}$, $\{\psi_k\}_{k \in \mathbb{N}}$ as a partition of unity associated with the covering $\{B_k\}_{k \in \mathbb{N}}$.

The following theorem is the main result we use to reduce the proof of inequality (1.7) to microlocal estimates on suitable rectangles of the cotangent space.

Theorem A.5 *Let $\{\varphi_k\}_{k\in\mathbb{N}}$, $\{\psi_k\}_{k\in\mathbb{N}}$ be a partition of unity associated with $\{B_k\}_{k\in\mathbb{N}}$. Then*

1) *there exists a constant $c_1 > 0$ independent of R such that*

$$(A.13) \qquad \sum_{k\in\mathbb{N}} \|\varphi_k^w u\|_0^2 \le c_1 \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n);$$

2) *for any $p \in S(h_g^{-2}, g)$, there exist positive constants c_2, c_2' such that*

$$(A.14) \qquad \|p^w u\|_0^2 \le c_2 \sum_{k\in\mathbb{N}} \|(p\psi_k)^w \varphi_k^w u\|_0^2 + c_2' \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

with c_2 independent of R.

Furthermore, if $p \in S(h_g^{-2}, g)$ is a real non-negative symbol, then, for suitable positive constants c_3, c_3', we have

$$(A.15) \qquad c_3 \|p^w u\|_0^2 + c_3' \|u\|_0^2 \ge \sum_{k\in\mathbb{N}} \|(p\psi_k)^w \varphi_k^w u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Moreover, if $\delta_k^{-2+|\alpha|} \epsilon_k^{-2+|\beta|} |\partial_x^\alpha \partial_\xi^\beta p_{|B_k}|$ are uniformly bounded in R, for any rectangle B_k having center, (x_k, ξ_k), satisfying $|\xi_k| \ge 2R$, then c_3 can be chosen not depending on R.

The proof of this proposition requires some preliminary results we give below.

Lemma A.6 *Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of symbols in $S(h_g^s, g)$ for any $s \in \mathbb{R}$ and assume that, for any $j = 0, 1, \ldots$, there exists a constant $C_j > 0$ such that*

$$(A.16) \qquad \sum_{k\in\mathbb{N}} |a_k|_j^g(x, \xi) \le C_j h_g(x, \xi)^s, \qquad \forall (x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n.$$

Then one has $\sum_{k\in\mathbb{N}} a_k \in S(h_g^s, g)$ and, for every $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\sum_{k\in\mathbb{N}} a_k^w u = \Big(\sum_{k\in\mathbb{N}} a_k \Big)^w u,$$

where $\sum_{k\in\mathbb{N}} a_k^w u$ denotes the $\mathcal{S}(\mathbb{R}^n)-$limit of $\{\sum_{k\le N} a_k^w u\}_{N\in\mathbb{N}}$ as N tends to $+\infty$. Furthermore, if $s = 0$ then $\Big(\sum_{k\in\mathbb{N}} a_k \Big)^{\overline{w}}$ gives rise to a linear continuous operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof. As a consequence of (A.16), $\{\sum_{k\leq N} a_k\}_{N\in\mathbb{N}}$ is a bounded sequence in $C^\infty(\mathbb{R}^{2n})$. In fact, for any compact set K of $\mathbb{R}^{2n}$, $K \subset \bigcup_{\mu=1}^{N_K} B_\mu$ and this gives

$$\delta(x,\xi) \sim c_K, \ \epsilon(x,\xi) \sim C_K \qquad \text{for any } (x,\xi) \in K.$$

From standard arguments we then get that $\{\sum_{k\leq N} a_k\}_{N\in\mathbb{N}}$ is a convergent sequence in C^∞ if it converges pointwise. To this purpose, note that, by Theorem 14.4 of [17], it has a subsequence converging to a smooth function $a(x,\xi)$. By (A.16), choosing $j = 0$, we then have that $\{\sum_{k\leq N} a_k\}_{N\in\mathbb{N}}$ converges pointwise to a.

As a consequence, a is in $S(h_g^s, g)$ and $\{\sum_{k\leq N} a_k^w u\}$ is a bounded sequence in $\mathcal{S}(\mathbb{R}^n)$. Again, if we prove that it converges to $a^w u$ pointwise we get convergence in $\mathcal{S}(\mathbb{R}^n)$. Since the pointwise convergence is an immediate consequence of the Lebesgue Theorem the proof is thus complete. ∎

The following lemma is proved in [9, Theorem 4.2, pages 390-391].

Lemma A.7 *Suppose that $q_k \in C_0^\infty(\frac{7}{4}B_k)$ and $q_{k'} \in C_0^\infty(\frac{7}{4}B_\gamma)$ with $k, k' \in \mathbb{N}$. For $N = 0, 1, \ldots,$ set*

$$T_N(q_k, q_{k'})(x,\xi) = \sum_{j<N} \frac{1}{j!} \left(\frac{1}{2i}\sigma(D_x, D_\xi; D_y, D_\eta) \right)^j q_k(x,\xi) q_{k'}(y,\eta) \Big|_{\substack{y = x \\ \eta = \xi}}$$

where σ is the standard symplectic 2-form, i.e.

$$\sigma\big((x,\xi); (y,\eta)\big) = \langle \xi, y \rangle - \langle x, \eta \rangle \qquad \text{for every } (x,\xi), (y,\eta) \in \mathbb{R}^{2n}.$$

Then, for any $j, N = 0, 1, \ldots,$ for any $s \geq 0$ and any r, t, there exist positive constants c_j and l independent of R such that

$$|q_k \sharp q_{k'} - T_N(q_k, q_{k'})|_j^g(x,\xi) \leq$$
$$\leq c_j h_g(x,\xi)^{r+t+N} \big(2 + d_k(x,\xi) + d_{k'}(x,\xi) \big)^{-s} \|q_k\|_{(l,h_g^r)} \|q_{k'}\|_{(l,h_g^t)},$$

where $a \sharp b$ is the Weyl symbol of the composition $a^w b^w$ (i.e. $(a \sharp b)^w = a^w b^w$) and d_k is defined by

$$d_k(x,\xi) = \inf_{(y,\eta) \in \frac{15}{8} B_k} g^\sigma_{(x,\xi)}(x - y, \xi - \eta).$$

The next lemma, due to Fujiwara (see [7, Proposition A2.4] or [9, Thm. 5.3]), is used in the proof of Proposition A.9 which is the main tool we need to prove Theorem A.5.

Lemma A.8 *Let $s \in \mathbb{R}$ be such that $s > (Q+1)n$ (where Q is the constant appearing in $(A.6)$) and let A be a positive number. Define*

$$d_{\nu k} = \inf_{(x,\xi) \in \frac{15}{8} B_\nu} d_k(x,\xi).$$

Then there exists a constant $C > 0$ independent of R such that, for any $(x,\xi) \in \mathbb{R}^{2n}$,

(A.17)
$$\sum_{k \in \mathbb{N}} \left(2 + A + d_k(x,\xi)\right)^{-s} \leq C(2+A)^{(Q+1)n-s},$$

(A.18)
$$\sum_{k \in \mathbb{N}} \left(2 + A + d_{\nu k}\right)^{-s} \leq C(2+A)^{(Q+1)n-s}.$$

Proposition A.9 *Let $\{f_k\}_{k \in \mathbb{N}}, \{g_k\}_{k \in \mathbb{N}}$ be two bounded sequences in $S(h_g^r, g)$ and in $S(h_g^t, g)$, respectively, for some $r,t \in \mathbb{R}$. Assume that*

1) $\operatorname{supp} f_k \cap \operatorname{supp} g_k = \emptyset$ *for every $k \in \mathbb{N}$;*

2) *one of the following conditions holds: either $\operatorname{supp} f_k \subseteq \frac{7}{4} B_k$ for every $k \in \mathbb{N}$ or $\operatorname{supp} g_k \subseteq \frac{7}{4} B_k$ for every $k \in \mathbb{N}$.*

Setting $l_k = f_k \sharp g_k$, one has

$$\sum_{k \in \mathbb{N}} \bar{l}_k \sharp l_k \in S(1,g) \quad and \quad \sum_{k \in \mathbb{N}} l_k^{w*} l_k^w u = \Big(\sum_{k \in \mathbb{N}} \bar{l}_k \sharp l_k\Big)^w u \ \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Furthermore, $\sum_{k \in \mathbb{N}} l_k^{w*} l_k^w$ can be extended to a linear continuous operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof. We can assume that $\operatorname{supp} f_k \subseteq (7/4)B_k$ for every $k \in \mathbb{N}$ because the arguments below work analogously in the other case. In view of Lemma A.6, it is enough to show that

(A.19)
$$\sum_{k \in \mathbb{N}} |\bar{l}_k \sharp l_k|_j^g(x,\xi) \leq C_j, \qquad \forall (x,\xi) \in \mathbb{R}^{2n}.$$

We start by proving that, for any real constant $s > (Q+1)n$ and any integer j one has

$$|l_k|_j^g(x,\xi) \leq c\big(2 + d_\varrho(x,\xi)\big)^{(Q+1)n-s}, \quad \forall (x,\xi) \in \mathbb{R}^{2n},$$

where Q is the constant appearing in (A.6). This estimate follows from Lemma A.7, but it is not straightforward since the symbols $f_k(x,\xi)$ and

$g_k(x,\xi)$ are not both supported in the rectangle $(7/4)B_k$. In view of that, we use the partition of unity given in Proposition A.4. Set

$$g_{k\nu}(x,\xi) = \varphi_\nu(x,\xi)^2 g_k(x,\xi) \quad \text{for any } k,\nu \in \mathbb{N}.$$

From Leibniz formula, we have, for any $j \in \mathbb{N}$,

$$(A.20) \qquad h_g(x,\xi)^{-t}|g_{k\nu}|_j^g(x,\xi) \le C_j, \qquad \forall \nu,k \in \mathbb{N}, \; \forall(x,\xi) \in \mathbb{R}^{2n}.$$

Since f_k and $g_{k\nu}$ have disjoint supports, one gets $T_N(f_k,g_{k\nu}) = 0$ for every N. Hence Lemma A.7 yields, on choosing $N \in \mathbb{N}$, $N+r+t \ge 0$,

$$|f_k \sharp g_{k\nu}|_k^g(x,\xi) \le c_1 h_g(x,\xi)^{r+t+N}\big(2 + d_k(x,\xi) + d_\nu(x,\xi)\big)^{-s}\|f_k\|_{(l,h_g^r)}\|g_{k\nu}\|_{(l,h_g^t)}$$
$$(A.21) \qquad \le c_2\big(2 + d_k(x,\xi) + d_\nu(x,\xi)\big)^{-s}, \qquad \forall(x,\xi) \in \mathbb{R}^{2n}.$$

In view of the finite overlapping property of $\{B_\nu\}_{\nu\in\mathbb{N}}$ (see Proposition A.2) and of (A.20), the sequence $\{\sum_{\nu\le N} g_{k\nu}\}_{N\in\mathbb{N}}$ is bounded in $S(h_g^t,g)$ and converges in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ to g_k. Therefore, from the weak continuity of the map $\sharp$, it follows that the sequence $\{\sum_{\nu\le N} f_k \sharp g_{k\nu}\}_{N\in\mathbb{N}}$ converges in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ to $f_k \sharp g_k$ as N tends to $+\infty$. Hence by (A.21) one has

$$|f_k \sharp g_k|_k^g(x,\xi) \le \sum_{\nu\in\mathbb{N}} |f_k \sharp g_{k\nu}|_k^g(x,\xi) \le c_2 \sum_{\nu\in\mathbb{N}} \big(2 + d_k(x,\xi) + d_\nu(x,\xi)\big)^{-s}.$$

By (A.17) we finally get, for any $s > (Q+1)n$,

$$(A.22) \quad |l_k|_j^g(x,\xi) = |f_k \sharp g_k|_j^g(x,\xi) \le c_3\big(2 + d_k(x,\xi)\big)^{(Q+1)n-s}, \quad \forall(x,\xi) \in \mathbb{R}^{2n}.$$

In order to prove (A.19), we want to apply Lemma A.7 to the symbols $\bar{l}_k, l_k$, thus we have to localize them in the rectangles $\{\frac{7}{4}B_\nu\}_{\nu\in\mathbb{N}}$. From Leibniz formula and from (A.22), it follows, for any $j \in \mathbb{N}$ and any $s > (Q+1)n$,

$$|\varphi_\nu^2 l_k|_j^g(x,\xi) \le c_4\big(2 + d_{\nu k}\big)^{(Q+1)n-s}, \qquad |\varphi_\mu^2 \bar{l}_k|_j^g(x,\xi) \le c_5\big(2 + d_{\mu k}\big)^{(Q+1)n-s}.$$

An application of Lemma A.7 yields, for any $s > (Q+1)n$,

$$(A.23) \quad |\big(\varphi_\nu^2 \bar{l}_k\big)\sharp\big(\varphi_\mu^2 l_k\big)|_j^g(x,\xi) \le$$
$$\le c_6\big(2 + d_\nu(x,\xi) + d_\mu(x,\xi)\big)^{-s}\big(2 + d_{\nu k}\big)^{(Q+1)n-s}\big(2 + d_{\mu k}\big)^{(Q+1)n-s}.$$

Furthermore, an easy check shows that the sequences $\{\sum_{\nu\le N} \varphi_\nu^2 l_k\}_{N\in\mathbb{N}}$, $\{\sum_{\mu\le N} \varphi_\mu^2 \bar{l}_k\}_{N\in\mathbb{N}}$ are bounded in $S(1,g)$ and that

$$\sum_{\nu\le N} \varphi_\nu^2 l_k \xrightarrow[N\to+\infty]{C^\infty(\mathbb{R}^{2n})} l_k, \qquad \sum_{\mu\le N} \varphi_\mu^2 \bar{l}_k \xrightarrow[N\to+\infty]{C^\infty(\mathbb{R}^{2n})} \bar{l}_k,$$

whence the weak continuity of $\sharp$ gives

$$\sum_{\nu,\mu \leq N} (\varphi_\nu^2 \bar{l}_k) \sharp (\varphi_\mu^2 l_k) \xrightarrow[N \to +\infty]{C^\infty(\mathbb{R}^{2n})} \bar{l}_k \sharp l_k.$$

This result and (A.23) finally yield

$$|\bar{l}_k \sharp l_k|_j^g(x,\xi) \leq \sum_{\nu,\mu \in \mathbb{N}} |(\varphi_\nu^2 \bar{l}_k) \sharp (\varphi_\mu^2 l_k)|_j^g(x,\xi)$$

$$\leq c_6 \sum_{\nu,\mu \in \mathbb{N}} (2 + d_\nu(x,\xi) + d_\mu(x,\xi))^{-s} (2 + d_{\nu k})^{(Q+1)n-s} (2 + d_{\mu k})^{(Q+1)n-s},$$

and a repeated application of Lemma A.8 gives then, choosing $s > 2n(Q+1)$,

$$\sum_{k \in \mathbb{N}} |\bar{l}_k \sharp l_k|_j^g(x,\xi) \leq$$

$$\leq c_6 \sum_{\nu,\mu \in \mathbb{N}} (2 + d_\nu(x,\xi) + d_\mu(x,\xi))^{-s} \left[\sum_{k \in \mathbb{N}} (2 + d_{\nu k})^{(Q+1)n-s} (2 + d_{\mu k})^{(Q+1)n-s} \right]$$

$$\leq c_7 \sum_{\nu \in \mathbb{N}} \sum_{\mu \in \mathbb{N}} (2 + d_\nu(x,\xi) + d_\mu(x,\xi))^{-s} \leq C.$$

This proves (A.19). ∎

The following Lemma is essentially due to D.Fujiwara [7] and to F.Hérau (see [8, Lemma 1.11]).

Lemma A.10 *Let $\{f_k\}_{k \in \mathbb{N}}$, $\{g_k\}_{k \in \mathbb{N}}$ be bounded sequences of real functions in $S(h_g^{-2}, g)$ and in $S(1,g)$, respectively. Let J be an arbitrary (finite or infinite) subset of $\mathbb{N}$. The following holds:*

1) *if* supp $f_k \subseteq (7/4)B_k$, supp $g_k \subseteq (7/4)B_k$ *for every* $k \in \mathbb{N}$. *Then* $\sum_{k \in J} g_k^2 f_k \in S(h_g^{-2}, g)$ *and*

   $$(A.24) \qquad \sum_{k \in J} g_k^w f_k^w g_k^w u = \left(\sum_{k \in J} g_k^2 f_k \right)^w u + R_J u, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

   where R_J is a linear continuous operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with norm independent of the subset J.

2) *if $f_k \geq 0$ with* supp $f_k \subseteq (7/4)B_k$ *and* supp $g_k \subseteq (9/8)B_k$ *for every $k \in \mathbb{N}$, there exist positive constants C, C' independent of J such that*

   $$(A.25) \quad \sum_{k \in J} \|[g_k^w, f_k^w]u\|_0^2 \leq C \sum_{k \in J} (f_k^w \psi_k^w u, \psi_k^w u) + C' \|u\|_0^2, \ \ \forall u \in \mathcal{S}(\mathbb{R}^n),$$

   where $\{\psi_k\}_{k \in \mathbb{N}}$ is the family of functions defined in Lemma A.4. Moreover, if the seminorms of the functions f_k and g_k don't depend on R, the constant C can be chosen not depending on R.

We complete this appendix by proving Theorem A.5.

Proof of Theorem A.5. Inequality (A.13) is an immediate consequence of Lemma A.6 and Lemma A.7.

As for (A.14), set $p_k(x, \xi) = (p\psi_k)(x, \xi)$ for any $k \in \mathbb{N}$. Thus by (A.24) there exists an operator $R \in \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ such that

$$\sum_{k \in \mathbb{N}} \varphi_k^w p_k^w \varphi_k^w u = \Big(\sum_{k \in \mathbb{N}} \varphi_k^2 p \Big)^w u + Ru = p^w u + Ru, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

whence

$$\|p^w u\|_0^2 = \sum_{k \in \mathbb{N}} \big(p_k^w \varphi_k^w u, \varphi_k^w p^w u \big)_{L^2} - \big(Ru, p^w u \big)_{L^2}, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

By the Cauchy-Schwarz inequality and (A.13) one gets, for an arbitrary constant $\epsilon > 0$,

$$\|p^w u\|_0^2 \leq \frac{1}{\epsilon^2} \sum_{k \in \mathbb{N}} \|p_k^w \varphi_k^w u\|_0^2 + \big(1 + c_1 \big) \epsilon^2 \|p^w u\|_0^2 + \frac{c}{\epsilon^2} \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

By choosing $\epsilon^2 < (1 + c_1)^{-1}$, (A.14) readily follows.

It remains to prove (A.15). We start by observing that

$$\|\varphi_k^w p_k^w u\|_0^2 \leq 2\|\varphi_k^w p^w u\|_0^2 + 2\|\varphi_k^w \big((1 - \psi_k)p \big)^w u\|_0^2.$$

An application of Proposition A.9 with $f_k = \varphi_k$ and $g_k = (1 - \psi_k)p$ yields

$$\sum_{k \in \mathbb{N}} \|\varphi_k^w p_k^w u\|_0^2 \leq 2 \sum_{k \in \mathbb{N}} \|\varphi_k^w p^w u\|_0^2 + C_1 \|u\|_0^2, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Hence, from (A.13) it follows

$$\sum_{k \in \mathbb{N}} \|\varphi_k^w p_k^w u\|_0^2 \leq C_2 \|p^w u\|_0^2 + C_2' \|u\|_0^2, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

with C_2 independent of R.

Since $p_k^w \varphi_k^w = \varphi_k^w p_k^w + [p_k^w, \varphi_k^w]$ one has, for every $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\sum_{k \in \mathbb{N}} \|\varphi_k^w p_k^w u\|_0^2 \geq \frac{1}{2} \sum_{k \in \mathbb{N}} \|p_k^w \varphi_k^w u\|_0^2 - \sum_{k \in \mathbb{N}} \|[p_k^w, \varphi_k^w]u\|_0^2,$$

so that (A.15) is proved if we show that

$$(A.26) \qquad \sum_{k \in \mathbb{N}} \|[p_k^w, \varphi_k^w]u\|_0^2 \leq C_3 \|p^w u\|_0^2 + C_3' \|u\|_0^2, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Let us now set $\bar{N} := \{k \in \mathbb{N} : |\xi_k| \geq R\}$ and note that $p_k \in S(1, g)$, uniformly in k, when $k \notin \bar{N}$. Hence we can apply (A.25) with $f_k = p_k$ and $g_k = \varphi_k$ (here we also use (A.12)) and get

$$\sum_{k \in \mathbb{N}} \|[p_k^w, \varphi_k^w]u\|_0^2 \leq C_4 \sum_{k \in \bar{N}} \left(p_k^w \psi_k^w u, \psi_k^w u\right)_{L^2} + C_4' \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Observe that C_4 is independent of R when $\delta_k^{-2+|\alpha|} \epsilon_k^{-2+|\beta|} |\partial_x^\alpha \partial_\xi^\beta p_{|B_k}|$ are uniformly bounded in R. Therefore, by (A.24), one has

$$(\text{A.27}) \quad \sum_{k \in \mathbb{N}} \|[p_k^w, \varphi_k^w]u\|_0^2 \leq C_5 \left(\left(\sum_{k \in \bar{N}} \psi_k^3 p\right)^w u, u\right)_{L^2} + C_5' \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

An application of the Fefferman-Phong inequality gives

$$\left(\left(\sum_{k \in \bar{N}} \psi_k^3 p\right)^w u, u\right)_{L^2} \leq C_6 \left(p^w u, u\right) + C_6' \|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

This estimate together with (A.27) yields (A.26). The proof of (A.15) is complete. ∎

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