

The existence of positive solution to some asymptotically linear elliptic equations in exterior domains

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Abstract

In this paper, we are concerned with the asymptotically linear elliptic problem $-\Delta u + \lambda_0 u = f(u)$, $u \in H_0^1(\Omega)$ in an exterior domain $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{O}}$ ($N \geq 3$) with \mathcal{O} a smooth bounded and star-shaped open set, and $\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = l, 0 < l < +\infty$. Using a precise deformation lemma and algebraic topology argument, we prove under our assumptions that the problem possesses at least one positive solution.

1. Introduction

In this paper, we are concerned with the existence of positive solution to some asymptotically linear elliptic equations

$$(1.1) \quad \begin{cases} -\Delta u + \lambda_0 u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda_0 > 0$ is a constant, $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{O}}$ ($N \geq 3$) is an exterior domain with \mathcal{O} a smooth bounded and star-shaped open set.

Problem (1.1) is a special case of the following more general elliptic problem

$$(1.2) \quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded or unbounded smooth domain in \mathbb{R}^N ($N \geq 2$).

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Assuming $f(x, 0) = 0$, the existence of nontrivial solutions or positive solutions to (1.2) has been studied extensively. Solutions to (1.2) can be viewed as critical points of the functional

$$(1.3) \quad I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx,$$

defined on $H_0^1(\Omega)$, where $F(x, u) = \int_0^u f(x, s) ds$.

When Ω is bounded, the Sobolev imbedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) (1 \leq q < \frac{2N}{N-2})$$

is compact. So if $f(x, u)$ is of “subcritical” at $u = +\infty$, i.e.,

$$(1.4) \quad |f(x, u)| \leq C(1 + |u|^q), \forall (x, u) \in \Omega \times \mathbb{R}^1, 2 \leq q < 2^*$$

and “superlinear” for u at 0 and $+\infty$, i.e.,

$$(1.5) \quad \lim_{t \rightarrow 0} \frac{f(x, t)}{t} = 0,$$

$$(1.6) \quad \lim_{t \rightarrow +\infty} \frac{f(x, t)}{t} = +\infty,$$

then $I(u)$ satisfies the so-called Palais-Smale compactness condition (see [1] for the definition) provided that we have the Ambrosetti-Rabinowitz condition: $\exists \theta \in (0, \frac{1}{2}), u_0 \geq 0$, such that

$$(1.7) \quad F(x, u) \leq \theta u f(x, u), \forall u \geq u_0, x \in \Omega.$$

And then, the critical point of (1.3) can be obtained by Mountain-Pass theorem of Ambrosetti and Rabinowitz (see e.g. [1]). (1.7) guarantees that every (PS) sequence of $I(u)$ is bounded, which is important in verifying the (PS) condition. In the mean time, (1.7) implies (1.6).

A extreme case is the “critical” case, i.e., $q = 2^*(N > 2)$ in (1.4). H. Brezis and L. Nirenberg [6] established a framework to deal with the problem.

Another extreme case is the so-called “asymptotically linear” case, i.e.,

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{t} = l, 0 < l < +\infty.$$

For this type of problems, a typical feature is that Ambrosetti-Rabinowitz condition (1.7) does not hold any more. Generally, it is not clear whether any (PS) sequence is bounded or not. For bounded Ω , (1.2) in asymptotically linear case has been studied by many authors, see e.g. [3, 7, 13, 16, 18, 19] and the references therein.

When Ω is unbounded, say $\Omega = \mathbb{R}^N$, problem is more difficult, due to the loss of compactness of Sobolev imbedding caused by the invariance of \mathbb{R}^N under the translations and rotations. In the “superlinear” and “subcritical” case, if $f(x, u) = f(u)$ or $f(x, u) = f(|x|, u)$, the existence of a nontrivial solution to (1.1) could be obtained by using radially symmetric Sobolev spaces $H_r^1(\mathbb{R}^N)$ together with constrained minimization (see e.g. [23], [5]). For general case, the Concentration-Compactness theory by P.L. Lions [21] makes it possible to find at least one nontrivial solution to (1.2) provided some extra conditions about $f(x, u)$ are imposed. A nontrivial solution could be obtained by either “natural constrained minimization” (see [21]) or Mountain-Pass theorem (see [26]). For “asymptotically linear” case when $\Omega = \mathbb{R}^N$, we mention that there are a series of works, see [10], [11], [14], [17], [20], [24], [27].

When $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{O}}$ where \mathcal{O} is an open bounded domain, problem (1.2) is much more complicated. Because there are no so-called ground state solutions with least energy even if

$$f(x, u) = |u|^{q-2}u - \lambda u \quad (2 < q < 2^*, \lambda > 0).$$

In fact, the infimum

$$\inf \left\{ \int_{\Omega} |\nabla u|^2 + \lambda u^2 : \int_{\Omega} |u|^q dx = 1 \right\}$$

is never achieved if $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{O}}$ (\mathcal{O} is a bounded open set). V. Benci and G. Cerami [4] studied the case for

$$f(x, u) = |u|^{q-2}u - \lambda u \quad (2 < q < 2^*, \lambda > 0),$$

and obtained the existence of a positive solution provided that \mathcal{O} is of small size. A. Bahri and P.L. Lions studied the case for general exterior domain and got the existence of a positive solution to (1.2) when

$$f(x, u) = b(x)u^p - \lambda_0 u, \quad \lambda_0 > 0, 1 < p < \frac{N+2}{N-2}, N > 2$$

by using mainly the topological method. Their works were known for the specialists although they were published much later [2]. Using the similar method as [2], G. Citti generalized the result of [2] to more general $f(x, u) = g(x, u) - \lambda u$, where $g(x, u)$ is “subcritical” and “superlinear” at 0 and $+\infty$ (see [9]).

A natural problem would be as follows: Could one get the existence of a positive solution for problem (1.2) if $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{O}}$, and $f(x, u) = g(x, u) - \lambda u$ where $g(x, u)$ is asymptotically linear at $+\infty$?

In this paper we try to answer this question for $g(x, u) = f(u)$, i.e., we want to get the existence of a solution to (1.1). We assume that the function $f(u)$ satisfies the following conditions:

- (H₁) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function of C^1 class and $f(t) = 0$ for all $t \leq 0$;
- (H₂) $\lim_{t \rightarrow 0^+} \frac{f(t)}{t} = 0, \lim_{t \rightarrow +\infty} \frac{f(t)}{t} = l$ ($0 < l < +\infty$);
- (H₃) $\frac{f(t)}{t}$ is strictly increasing in $t \in \mathbb{R}^+$, and $\frac{f(t)}{t^p}$ is decreasing in $t \in \mathbb{R}^+$ for some $p > 1$;
- (H₄) For any $M > 0$ large enough, there exists $\alpha \in (0, p - 1)$ such that $tf'(t) - (1 + \alpha)f(t) \geq 0$ for all $t \in [0, 2M]$.

Our main result is the following

Theorem 1.1 *Assume that (H₁)–(H₄) hold with $l > \lambda_0$ and the equation at infinity associated to (1.1)*

$$(1.8) \quad \begin{cases} -\Delta u + \lambda_0 u = f(u) & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

has a unique positive solution, then (1.1) has at least one solution.

Using the uniqueness theorem for positive solutions to (1.8) by Kwong and Zhang [15] and the method of [9], we have

Corollary 1.2 *Assume that (H₁)–(H₄) hold, and let $\theta > 0$ be such that $-\lambda_0 u + f(u) < 0$ in $(0, \theta)$ and $-\lambda_0 u + f(u) > 0$ in $(\theta, +\infty)$. If*

$$(1.9) \quad \frac{-\lambda_0 u + uf'(u)}{-\lambda_0 u + f(u)} \text{ is decreasing in } (\theta, +\infty),$$

then (1.1) has at least one solution.

A typical example of $f(u)$ satisfies (H₁)–(H₄) and the other assumptions in Corollary 1.2 is given by

$$f(t) = \frac{lt^2}{1+t}, t \geq 0, l > \lambda_0 = 1$$

with $p = 2$ in (H₃).

To prove our main theorem, we use the argument of contradiction as in [2] and [9]. First of all, we establish a representation theorem (see Proposition 2.1) for the Palais-Smale sequences related to the functional I . This

theorem analyzes the behavior of a (P-S) sequence and states that such a sequence either converges strongly to its weak limit or it differs from it by one or more sequences which after a suitable translation converges to a solution to (1.8). Hence the only obstructions to global compactness in the usual sense are the solutions to (1.8). Because our case is of “asymptotically linear”, the standard method, which can easily prove that (P-S) sequences are bounded, can not be applied here. However, we can adapt the method of [17] to prove that any (P-S) sequence is bounded. Then we construct a precise deformation lemma (see Lemma 2.4) as in [2, 9] under the assumption that (1.1) has no solution and equation (1.8) has a unique positive solution ω . This lemma shows that the energy level-set pair (W_m, W_{m-1}) of the functional I retracts to the pair $(W_{m-1}^\delta, W_{m-1})$ (see Section 2 for the definition of W_m and W_{m-1}^δ). Therefore, we can get the relationship between the homologies: $H_*(W_m, W_{m-1}) \cong H_*(W_{m-1}^\delta, W_{m-1})$. Furthermore, by the excision property of relative homology group, $H_*(W_m, W_{m-1}) = H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon))$ and the induced homomorphisms

$$i_* : H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \rightarrow H_*(W_m, W_{m-1})$$

are isomorphisms (for definition of $V(m, \varepsilon)$ see Section 2 below).

Next, by virtue of the properties of the domain Ω , similar to [2, 9] we define a map h from $S^m \times_{\sigma_m} \Delta_{m-1}$ to M^+ , where S is an $(N - 1)$ -dimensional sphere in Ω , Δ_{m-1} is an $(m - 1)$ -simplex, σ_m is a group of permutations of $\{1, 2, \dots, m\}$, $S^m \times_{\sigma_m} \Delta_{m-1}$ is the quotient of $S^m \times \Delta_{m-1}$ under the action σ_m , and $M^+ = \{u \in M : u \geq 0\}$ with

$$M = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx = \int_{\Omega} u f(u) dx \right\},$$

such that, there exists a $\mu \in \mathbb{N}$ with

- (1) $h : S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta \rightarrow W_{\mu-1}$,
- (2) $h(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta, \partial(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta)) \subset (V(m, \varepsilon_1), W_{m-1}) \subset (W_m, W_{m-1})$,

where $1 \leq m \leq \mu$, S_0^m and Δ_{m-1}^η are the appropriate subsets of S^m and Δ_{m-1} respectively. This is a key step for the whole argument. Following [9], it is clear that we need only to define a map $h : S^m \times \Delta_{m-1} \rightarrow M^+$. For the “superlinear case” as in [2, 9], it is trivial to show that $\forall u \in H_0^1(\Omega)$, there exists a unique $k \in \mathbb{R}^+$ such that $ku \in M$. Hence

$$k(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \in M^+$$

for any φ , m and λ , where φ is a cut-off function such that $0 \leq \varphi \leq 1$, and $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$, $\varphi = 1$ in the neighborhood of infinity. But for “asymptotically linear” case, the above statement is not true in general. So we need be more careful. We find that for fixed φ , $1 \leq m \leq \mu$ and $\lambda > 0$ large enough, there does exist a $k(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)) \in \mathbb{R}^+$ such that

$$k(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \in M^+$$

(see Lemma 3.2 below). We can then define

$$h(x_1, \dots, x_m, t_1, \dots, t_m) = k(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \in M^+,$$

for any $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$. Hence, using (H_1) – (H_4) , we can verify that the map h satisfies the properties (1) and (2) mentioned above.

Finally, using algebraic topological arguments, we obtain that the following diagram is commutative

$$\begin{CD} \mathbb{Z}_2 = H_{\mu N-1}(S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta, \partial(S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta)) @>\partial d_\alpha \cdots \partial d_\alpha>> H_{N-1}(S) = \mathbb{Z}_2 \\ @VV h_{\mu N-1} V @VV h_{N-1} V \\ H_{\mu N-1}(W_\mu, W_{\mu-1}) @>>> H_{N-1}(W_1) \\ @VV i_{\mu N-1}^{-1} V @VV i_{N-1}^{-1} V \\ H_{\mu N-1}(W_{\mu-1}^\delta \cap V(\mu, \varepsilon), W_{\mu-1} \cap V(\mu, \varepsilon)) @>\partial \bar{d}_\alpha \cdots \partial \bar{d}_\alpha>> H_{N-1}(W_0^\delta \cap V(1, \varepsilon)) \end{CD}$$

and that ∂d_α is an isomorphism, where i_*^{-1} are isomorphisms which we have mentioned above. On one hand, by (1) we know that $h_{\mu N-1} = 0$, then $h_{N-1} = 0$ by the commutative diagram. On the other hand, there is a function $s : V(1, \varepsilon) \rightarrow S$ such that s is a left inverse of h , hence h_{N-1} can not be zero. Thus we show that (1.1) has at least one solution.

Throughout this paper, we always use the standard notations, for instance, $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega (|\nabla u|^2 + \lambda_0 u^2) dx \right)^{\frac{1}{2}};$$

$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ is a Hilbert space with the norm

$$\|u\| = \|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_0 u^2) dx \right)^{\frac{1}{2}}.$$

We denote $\mathbb{R}^+ \equiv \{x \in \mathbb{R} : x > 0\}$.

2. Preliminaries

It is well known that the solution to (1.1) is a critical point of the functional

$$I(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx - \int_{\Omega} F(u) dx$$

constrained on the Finsler manifold

$$M = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx = \int_{\Omega} u f(u) dx \right\},$$

where $F(u) = \int_0^u f(t) dt$ (see [8] and [17]).

Similarly, the solution to the problem (1.8) at infinity is a critical point of the functional

$$I_{\infty}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_0 u^2) dx - \int_{\mathbb{R}^N} F(u) dx$$

constrained on the Finsler manifold

$$M_{\infty} = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_0 u^2) dx = \int_{\mathbb{R}^N} u f(u) dx \right\}.$$

It is easy to see that

$$(2.1) \quad \inf_M I(u) \geq \inf_{M_{\infty}} I_{\infty}(u).$$

By Theorem 1.1 in [17], $\inf_{M_{\infty}} I_{\infty}(u)$ is achieved by some positive $u_0 \in H^1(\mathbb{R}^N)$, which is a ground state solution to (1.8). From now on, we assume (1.8) has a unique positive solution ω . And by the result of [12], we see that $\omega \in C^{\infty}(\mathbb{R}^N)$ is spherically symmetric about some $x_0 \in \mathbb{R}^N$ and if we assume $x_0 = 0$, then $\omega(x) = \omega(r)$ satisfies

$$(2.2) \quad \begin{cases} \omega(r)r^{\frac{N-1}{2}} \exp(\sqrt{\lambda_0}r) \rightarrow c > 0 & \text{as } r \rightarrow +\infty, \\ \omega'(r)r^{\frac{N-1}{2}} \exp(\sqrt{\lambda_0}r) \rightarrow -c\sqrt{\lambda_0} & \text{as } r \rightarrow +\infty. \end{cases}$$

Since Palais-Smale condition may not be valid in general, we now give a precise compactness result which is similar to [2].

Proposition 2.1 *Let $\{u_n\}$ be a nonnegative sequence in M such that $I(u_n) \rightarrow c$ and $dI|_M(u_n) \rightarrow 0$ as $n \rightarrow \infty$, where $c \geq 0$. Then there exist a number $m \in \mathbb{N} \cup \{0\}$ and m sequences $\{y_n^j\}_{n \in \mathbb{N}}$ in \mathbb{R}^N ($j = 1, \dots, m$) such that $|y_n^j| \rightarrow +\infty$, for every $i \neq j$, $|y_n^i - y_n^j| \rightarrow +\infty$ as $n \rightarrow \infty$ and*

$$u_n(x) = u^0(x) + \sum_{i=1}^m \omega(x - y_n^i) + \psi_n(x)$$

where u^0 satisfies $dI(u^0) = 0$, $u^0 \in H_0^1(\Omega)$, $\psi_n \rightarrow 0$ in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$ and $I(u_n) \rightarrow I(u^0) + mI_{\infty}(\omega)$.

To prove Proposition 2.1, we need a “Vanishing” lemma. The following “Vanishing” lemma was proved by A. Bahri and P.L. Lions (see Lemma A.1 in Appendix of [2]).

Lemma 2.2 *Let*

$$Q_k(t) = \sup_{y \in \mathbb{R}^N} \int_{y+B_t} |\omega_k|^2 dx$$

for all $t \geq 0$, $\{\omega_n\}$ be bounded in $H^1(\mathbb{R}^N)$ and assume that for some $t_0 > 0$, $Q_k(t_0) \rightarrow 0$. Then $\omega_k \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for all $2 < q < \frac{2N}{N-2}$. If in addition ω_k satisfies $I'(\omega_k) \rightarrow 0$ in $H^{-1}(\mathbb{R}^N)$, then $\omega_k \rightarrow 0$ in $H^1(\mathbb{R}^N)$ as $k \rightarrow \infty$.

Proof of Proposition 2.1 To be concise, we denote subsequences of a sequence by the sequence itself.

First of all, we should prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Here we use the argument of [17]. If it were not true, we could assume $\|u_n\| \rightarrow +\infty$ (we always denote

$$\|u\| = \left(\int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx \right)^{\frac{1}{2}}$$

the equivalent norm of $u \in H_0^1(\Omega)$, and we denote

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_0 u^2) dx \right)^{\frac{1}{2}}$$

for $u \in H^1(\mathbb{R}^N)$ if there is no confusion).

There are two cases: (1) $c > 0$; (2) $c = 0$.

Case (1) $c > 0$.

Denote $v_n = \frac{2\sqrt{c}u_n}{\|u_n\|}$. So $\|v_n\| = 2\sqrt{c}$. Now we set $\rho_n = v_n^2$. If “Vanishing” occurs, i.e. for any $0 < R < +\infty$, as $n \rightarrow \infty$,

$$\sup_{\mathbb{R}^N} \int_{y+B_R} \rho_n(x) dx \rightarrow 0.$$

By virtue of Vanishing Lemma 2.2 (see also [21]),

$$(2.3) \quad v_n \rightarrow 0 \text{ in } L^\alpha(\mathbb{R}^N) \text{ as } n \rightarrow \infty, 2 < \alpha < \frac{2N}{N-2}.$$

Assumption (H_2) implies that $\forall \varepsilon > 0, \exists C_\varepsilon > 0$ such that

$$(2.4) \quad |f(t)| < \varepsilon|t| + C_\varepsilon|t|^\alpha, |F(t)| < \varepsilon|t|^2 + C_\varepsilon|t|^{\alpha+1}, 1 < \alpha < \frac{N+2}{N-2}.$$

Then (2.3), (2.4) imply that

$$\int_{\mathbb{R}^N} v_n f(v_n) dx \rightarrow 0, \int_{\mathbb{R}^N} F(v_n) dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So

$$(2.5) \quad I(v_n) = \frac{1}{2} \|v_n\|^2 - \int_{\mathbb{R}^N} F(v_n) dx = \frac{(2\sqrt{c})^2}{2} + o(1) = 2c + o(1).$$

On the other hand, for any $t > 0$,

$$\begin{aligned} I(tu_n) &= \frac{t^2}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} F(tu_n) dx, \\ \langle I'(u_n), u_n \rangle &= \|u_n\|^2 - \int_{\mathbb{R}^N} u_n f(u_n) dx. \end{aligned}$$

Then

$$I(tu_n) = \frac{t^2}{2} \int_{\mathbb{R}^N} u_n f(u_n) dx - \int_{\mathbb{R}^N} F(tu_n) dx.$$

Set $h(t) = \frac{t^2}{2} s f(s) - F(ts)$. It is easy to see that

$$h'(t) = tsf(s) - f(ts)s \begin{cases} \geq 0 & \text{if } 0 < t \leq 1, s > 0 \\ \leq 0 & \text{if } t \geq 1, s > 0 \\ = 0 & \text{if } t > 0, s \leq 0, \end{cases}$$

which means that $h(t) \leq h(1), \forall t > 0$. Then

$$I(tu_n) \leq \frac{1}{2} \int_{\mathbb{R}^N} u_n f(u_n) dx - \int_{\mathbb{R}^N} F(u_n) dx = I(u_n).$$

Therefore $I(v_n) = I(t_n u_n) \leq I(u_n) = c + o(1)$. This is impossible by (2.5). Hence, “Non-vanishing” occurs, i.e. there exists $\eta > 0, R > 0, \{y_n\} \subset \mathbb{R}^N$ such that

$$(2.6) \quad \lim_{n \rightarrow \infty} \int_{y_n + B_R} v_n^2 dx \geq \eta > 0.$$

We distinguish two cases : case (i) $y_n \rightarrow y_0 \in \mathbb{R}^N$ as $n \rightarrow \infty$; case (ii) $|y_n| \rightarrow +\infty$ as $n \rightarrow \infty$.

In case (i), we know that there exists $R' > R$ such that

$$(2.7) \quad \int_{B_{R'}} v_n^2 dx \geq \frac{\eta}{2}.$$

Since $\|v_n\| = 2\sqrt{c}$, there exists a $v \in H_0^1(\Omega)$ such that

$$(2.8) \quad \begin{cases} v_n \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \\ v_n \rightarrow v \text{ strongly in } L_{loc}^\alpha(\Omega), 2 \leq \alpha < \frac{2N}{N-2}, \\ v_n \rightarrow v \geq 0 \text{ a.e. in } \Omega. \end{cases}$$

(2.7) implies $v \neq 0$. On the other hand, since $dI|_M(u_n) \rightarrow 0$, we have

$$\int_{\Omega} \nabla u_n \nabla \varphi + \lambda_0 u_n \varphi = \int_{\Omega} f(u_n) \varphi + o(\|\varphi\|).$$

Furthermore,

$$\int_{\Omega} \nabla v_n \nabla \varphi + \lambda_0 v_n \varphi = \int_{\Omega} \frac{f(2\sqrt{c}u_n)}{\|u_n\|} \varphi + o(\|\varphi\|), \forall \varphi \in C_0^\infty(\Omega).$$

Set

$$p_n(x) = \begin{cases} \frac{f(u_n)}{u_n}, & \text{if } u_n \neq 0, \\ 0, & \text{if } u_n = 0. \end{cases}$$

Then

$$(2.9) \quad \int_{\Omega} \nabla v_n \nabla \varphi + \lambda_0 v_n \varphi = \int_{\Omega} p_n(x) v_n \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Since $0 \leq p_n(x) \leq l$, there exists $h \in L_{loc}^\infty(\Omega)$ such that

$$p_n(x) \rightharpoonup h(x) \text{ weakly* in } L_{loc}^\infty(\Omega).$$

So $\forall \varphi \in C_0^\infty(\Omega)$, $\int_{\Omega} p_n(x) v_n(x) \varphi(x) dx \rightarrow \int_{\Omega} h(x) v \varphi dx$, which combined with (2.8) and (2.9) implies

$$(2.10) \quad \int_{\Omega} \nabla v \nabla \varphi + \lambda_0 v \varphi = \int_{\Omega} h(x) v \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Denote $A = \{x \in \Omega : v(x) > 0\}$. Then $p_n(x) \rightarrow l$ a.e. in A as $n \rightarrow \infty$. So $h(x) = l$ in A . Therefore, $\forall \varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} \nabla v \nabla \varphi + \lambda_0 v \varphi &= \int_A h(x) v \varphi + \int_{\Omega \setminus A} h(x) v \varphi \\ &= \int_A h(x) v \varphi = l \int_A v \varphi = l \int_{\Omega} h(x) v \varphi. \end{aligned}$$

Since $\lambda_0 < l$, we see that $-\Delta$ has an eigenvalue in $H_0^1(\Omega)$. But the Pohozaev type identity shows that $-\Delta$ has no eigenvalue in $H_0^1(\Omega)$ since $\Omega = \mathbb{R}^N \setminus \bar{\mathcal{O}}$, where \mathcal{O} is a smooth bounded and star-shaped open set (see Theorem 2.12 in Chapter 1 in [22]).

In case (ii), we set $\tilde{v}_n(x) = v_n(x + y_n)$. Then $\|\tilde{v}_n\| = \|v_n\|$. There exists some $\tilde{v} \in H_0^1(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{v}_n \rightharpoonup \tilde{v} \text{ weakly in } H^1(\mathbb{R}^N), \\ \tilde{v}_n \rightarrow \tilde{v} \text{ strongly in } L_{loc}^\alpha(\mathbb{R}^N), 2 \leq \alpha < \frac{2N}{N-2}, \\ \tilde{v}_n \rightarrow \tilde{v} \geq 0 \text{ a.e. in } \mathbb{R}^N. \end{cases}$$

(2.6) implies $\tilde{v} \neq 0$. Since $|y_n| \rightarrow +\infty$, we may assume that $\text{supp } \varphi(\cdot - y_n) \subset \Omega$, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$. So by $I'(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$ we have

$$\int_{\Omega} (\nabla u_n \nabla \varphi(x - y_n) + \lambda_0 u_n \varphi(x - y_n)) dx = \int_{\Omega} f(u_n) \varphi(x - y_n) dx + o(1).$$

Then $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla \tilde{u}_n \nabla \varphi + \lambda_0 \tilde{u}_n \varphi) dx = \int_{\mathbb{R}^N} f(\tilde{u}_n) \varphi dx + o(1),$$

where $\tilde{u}_n(x) = u_n(x + y_n)$. So

$$\int_{\mathbb{R}^N} (\nabla \tilde{v}_n \nabla \varphi + \lambda_0 \tilde{v}_n \varphi) dx = \int_{\mathbb{R}^N} \frac{f(\tilde{u}_n)}{\tilde{u}_n} \tilde{v}_n \varphi dx + o(1).$$

Therefore, we can deduce the contradiction similar to case (i). The only difference is that we need Pohozaev type identity on \mathbb{R}^N (see [21], [22]) in this case. Hence, $\{u_n\}$ is bounded in $H_0^1(\Omega)$ if $c > 0$.

Case (2) $c = 0$.

In this case, $I(u_n) \rightarrow 0, I'(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$. For any fixed $\alpha > 0$, let $t_n = \frac{\alpha}{\|u_n\|}, \omega_n = t_n u_n = \frac{\alpha u_n(x)}{\|u_n\|}$. Set $\rho_n(x) = \omega_n^2$. It is easy to rule out ‘‘Vanishing’’ case by a similar argument as in case (1) $c > 0$. So ‘‘Non-vanishing’’ occurs, i.e. there exist $\eta > 0, R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$(2.11) \quad \lim_{n \rightarrow \infty} \int_{y_n + B_R} \omega_n^2 dx \geq \eta > 0.$$

Set $\tilde{\omega}_n(x) = \omega_n(x + y_n)$, then $\|\tilde{\omega}_n\| = \|\omega_n\| = \alpha$. Then there exists some $\tilde{\omega} \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{\omega}_n \rightharpoonup \tilde{\omega} \text{ weakly in } H^1(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} \text{ strongly in } L_{loc}^2(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} \geq 0 \text{ a.e. in } \mathbb{R}^N. \end{cases}$$

By (2.11), we know $\tilde{\omega} \neq 0$.

For n large enough, we have

$$t_n = \frac{\alpha}{\|u_n\|} \in (0, 1), \quad \frac{f(t_n u_n)}{t_n u_n} \leq \frac{f(u_n)}{u_n}.$$

Hence,

$$\begin{aligned} \|\omega_n\|^2 - \int_{\mathbb{R}^N} f(\omega_n)\omega_n dx &= t_n^2 \left[\|u_n\|^2 - \int_{\mathbb{R}^N} \frac{f(t_n u_n)}{t_n u_n} u_n^2 dx \right] \\ &\geq t_n^2 \left[\|u_n\|^2 - \int_{\mathbb{R}^N} \frac{f(u_n)}{u_n} u_n^2 dx \right] = 0. \end{aligned}$$

Hence, by Fatou's Lemma, we have

$$\begin{aligned} I(\omega_n) &= \frac{1}{2}\|\omega_n\|^2 - \int_{\mathbb{R}^N} F(\omega_n) = \int_{\mathbb{R}^N} \left[\frac{1}{2}f(\omega_n)\omega_n - F(\omega_n) \right] \\ (2.12) \quad &= \int_{\mathbb{R}^N} \left[\frac{1}{2}f(\tilde{\omega}_n)\tilde{\omega}_n - F(\tilde{\omega}_n) \right] \geq \int_{\mathbb{R}^N} \left[\frac{1}{2}f(\tilde{\omega})\tilde{\omega} - F(\tilde{\omega}) \right] + o(1). \end{aligned}$$

But $I(\omega_n) = I(t_n u_n) \leq I(u_n) \rightarrow 0$. This and (2.12) imply

$$0 \leq \int_{\mathbb{R}^N} \left[\frac{1}{2}f(\tilde{\omega})\tilde{\omega} - F(\tilde{\omega}) \right] dx \leq 0.$$

Then $\tilde{\omega} = 0$. A contradiction.

Now, we have proved that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence there exists $u \in H_0^1(\Omega)$ such that

$$(2.13) \quad \begin{cases} u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_n \rightarrow u \text{ strongly in } L_{loc}^\alpha(\Omega), 2 \leq \alpha < \frac{2N}{N-2}, \\ u_n \rightarrow u \text{ a.e. in } \Omega \end{cases}$$

and $f(u_n) \rightarrow f(u)$ a.e. in Ω . Since $|f(t)| \leq l|t|$,

$$\begin{aligned} \left| \int_E f(u_n)\varphi dx \right| &\leq \left(\int_E |f(u_n)|^2 dx \right)^{\frac{1}{2}} \left(\int_E \varphi^2 dx \right)^{\frac{1}{2}} \\ &\leq l \left(\int_E u_n^2 dx \right)^{\frac{1}{2}} \left(\int_E \varphi^2 dx \right)^{\frac{1}{2}} \leq C \left(\int_E \varphi^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and any measurable set $E \subset \mathbb{R}^N$, Vitali's convergence theorem implies

$$(2.14) \quad \int_{\mathbb{R}^N} f(u_n)\varphi dx \rightarrow \int_{\mathbb{R}^N} f(u)\varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Therefore, by (2.13), (2.14), it is easy to check that u solves $-\Delta u + \lambda_0 u = f(u)$ in Ω and $u \geq 0$. By the assumptions on f , we can easily deduce that for any $\varphi \in C_0^\infty(\Omega)$,

$$(2.15) \quad \begin{cases} \int_{\mathbb{R}^N} [F(u_n) - F(u) - F(u_n - u)] dx \rightarrow 0, \\ \int_{\mathbb{R}^N} [f(u_n) - f(u) - f(u_n - u)] \varphi dx \rightarrow 0. \end{cases}$$

By (2.14), (2.15), we see that we may always assume that u_k converges weakly to 0 replacing if necessary $\{u_n\}$ by $\{u_n - u\}$.

Next, in view of Lemma 2.2, either $u_k \rightarrow 0$ in $H^1(\mathbb{R}^N)$ and the proof is over or there exists an $\alpha > 0$ such that we have (up to a subsequence) $Q_k(t_0) \geq \alpha > 0$ and thus there exists $\{y_k\} \subset \mathbb{R}^N$ such that

$$Q_k(t_0) \triangleq \int_{y_k + B_{t_0}} |u_k|^2 dx \geq \alpha > 0.$$

Therefore, $\tilde{u}_k = u_k(\cdot + y_k)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $\tilde{u} \neq 0$ due to Rellich-Kondrakov theorem. Since $u_k \rightarrow 0$ in $H^1(\mathbb{R}^N)$, we deduce

$$(2.16) \quad |y_k| \rightarrow +\infty.$$

But by (2.14)-(2.16), we deduce that \tilde{u} solves $-\Delta \tilde{u} + \lambda_0 \tilde{u} = f(\tilde{u})$ in \mathbb{R}^N , $\tilde{u} \in H^1(\mathbb{R}^N)$ and $-\Delta v_k + \lambda_0 v_k = f(v_k) + o(1)$ in $H^{-1}(\mathbb{R}^N)$ where $v_k = \tilde{u}_k - \tilde{u}$. By the uniqueness of the solution to (1.8), we may assume $\tilde{u} = \omega$. Furthermore, we have

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_k|^2 dx - \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx - \int_{\mathbb{R}^N} |\nabla v_k|^2 dx = o(1), \\ \int_{\mathbb{R}^N} |u_k|^2 dx - \int_{\mathbb{R}^N} |\tilde{u}|^2 dx - \int_{\mathbb{R}^N} |v_k|^2 dx = o(1). \end{cases}$$

To conclude, we just iterate the above argument and this iteration procedure has to end in a finite number of steps since we have

$$m \int_{\mathbb{R}^N} \omega^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_k^2 dx.$$

Thus m can not go to ∞ because of $\omega \neq 0$. We can easily verify that all results of the proposition hold. That completes the proof. ■

Corollary 2.3 *If (1.1) has no solution, and $\{u_n\}$ satisfies the assumption as in Proposition 2.1, then there exist a number $m \in \mathbb{N} \cup \{0\}$ and m sequences $\{y_n^j\}_{n \in \mathbb{N}}$ in \mathbb{R}^N ($j = 1, \dots, m$) such that $|y_n^j| \rightarrow +\infty$, for every $i \neq j$, $|y_n^i - y_n^j| \rightarrow +\infty$ as $n \rightarrow \infty$ and*

$$\begin{cases} u_n = \sum_{i=1}^m \omega(\cdot - y_n^i) + o(1) & \text{as } n \rightarrow \infty \\ I(u_n) \rightarrow mI_\infty(\omega) & \text{as } n \rightarrow \infty. \end{cases}$$

Following [2], [9], we define

$$W_m = \{u \in M^+ : I(u) \leq (m + 1)I_\infty(\omega)\}$$

and

$$V(m, \varepsilon) = \left\{ u \in M^+ : \left\| u - \sum_{i=1}^m \omega(\cdot - y_i) \right\| < \varepsilon, |y_i| > \frac{1}{\varepsilon}, |y_i - y_j| > \frac{1}{\varepsilon}, \forall i \neq j \right\}$$

where $M^+ = \{u \in M : u \geq 0 \text{ a.e. in } \Omega\}$. From now on, we assume that (1.1) has no solution. So $I(u) > \inf_M I(u), \forall u \in M$. This and (2.1) imply $W_0 = \emptyset$.

For any $u_0 \in W_m$, we consider the Cauchy problem

$$\begin{cases} u'(t) = -\frac{I'_M(u)}{(1 + |I'_M(u)|^2)^{\frac{1}{2}}} \\ u(0) = u_0 \in W_m. \end{cases}$$

By the standard argument we know that this problem has a unique solution $u(t, u_0)$ for all $t \in \mathbb{R}$ and $u(t, u_0) \in M^+$. Denote

$$T_\delta(u_0) = \begin{cases} \inf\{s \geq 0 : I(u(s, u_0)) \leq mI_\infty(\omega) + \delta\} \\ +\infty, & \text{if the set is empty.} \end{cases}$$

Then $T_\delta(u_0)$ is continuous as a function of u_0 , for every $\delta \geq 0$ and if (1.1) has no solution, $T_\delta(u_0) < +\infty$ for all $\delta > 0$. Next we define $T(u_0) = \min\{T_\delta(u_0) + \sqrt{\delta}, T_0(u_0)\}$, $W_{m-1}^\delta = \{u(T(u_0), u_0) : u_0 \in W_m\}$ and $r : [0, 1] \times W_m \rightarrow W_m$ given by $r(t, u_0) = u(tT(u_0), u_0)$. From Corollary 2.3, the following lemma holds (See [2], [9]).

Lemma 2.4 *If (1.1) has no solution, then (W_m, W_{m-1}) retracts by deformation through r on $(W_{m-1}^\delta, W_{m-1})$, and for every $\varepsilon > 0, \varepsilon_1 \in (0, \varepsilon)$ such that $W_{m-1}^\delta \setminus W_{m-1} \subset V(m, \varepsilon)$ and $V(m, \varepsilon_1) \subset W_{m-1}^\delta \cap V(m, \varepsilon)$.*

3. Proof of Theorem 1.1

We will prove the theorem by contradiction as in [2], [9]. Assume that (1.1) has no solution, then the previous arguments are valid.

Let S be an $(N - 1)$ -dimensional sphere embedded in $\bar{\Omega}$ such that $\lambda S \subset \bar{\Omega}$ for $\lambda \geq 1$. Without loss of generality, we assume that S is a sphere of radius 1. Let σ_m be the group of permutations of $\{1, 2, \dots, m\}$,

$$D_m = \{(x_1, x_2, \dots, x_m) \in S^m : \exists i \neq j \text{ such that } x_i = x_j\},$$

\widehat{D}_m and $\widehat{\widehat{D}}_m$ be σ_m -invariant tubular neighborhoods of D_m , such that $\widehat{D}_m \subset \subset \widehat{\widehat{D}}_m$ and they retracts by deformation on D_m . Denote $S_0^m = \overline{S^m \setminus \widehat{D}_m}$. We denote

$$\Delta_{m-1} = \left\{ (t_1, \dots, t_m) \in \mathbb{R}^m : \sum_{i=1}^m t_i = 1, t_i \geq 0, \forall i = 1, \dots, m \right\}$$

and

$$\Delta_{m-1}^\eta = \left\{ (t_1, \dots, t_m) \in \Delta_{m-1} : \sup_{1 \leq i \leq m} |t_i - \frac{1}{m}| \leq \eta \right\}.$$

We will denote the quotient of $S^m \times \Delta_{m-1}$ and $S_0^m \times \Delta_{m-1}$ under the action σ_m respectively by $S^m_{\sigma_m} \times \Delta_{m-1}$ and $S_0^m_{\sigma_m} \times \Delta_{m-1}$.

Let φ be a cut-off function which is a fixed function of class C^∞ and identically zero in $\mathbb{R}^N \setminus \Omega$, 1 in a neighborhood of $+\infty$, $0 \leq \varphi \leq 1$ in \mathbb{R}^N .

By the definition of S_0^m for every $m \in \mathbb{N}$, there exist $\gamma_m > 0$ and $\tau > 0$ such that

$$(3.1) \quad \min_{i \neq j} |x_i - x_j| \geq \gamma_m, \quad \forall (x_1, \dots, x_m) \in S_0^m;$$

$$(3.2) \quad \min_{i \neq j} |x_i - x_j| \leq \frac{\tau}{2\sqrt{\lambda_0}}, \quad \forall (x_1, \dots, x_m) \in \partial S_0^m.$$

And we may assume $\tau < \sqrt{\lambda_0}$ as long as \widehat{D}_m is close enough to D_m . We claim that there exists $\mu \in \mathbb{N}$ such that

$$(3.3) \quad \min_{i \neq j} |x_i - x_j| \leq \frac{\tau}{2\sqrt{\lambda_0}}, \quad \forall (x_1, \dots, x_\mu) \in S_0^\mu.$$

Indeed, if by contradiction, $\forall \mu \in \mathbb{N}, \exists (x_1, \dots, x_\mu) \in S_0^\mu$ such that $|x_i - x_j| > \frac{\tau}{2\sqrt{\lambda_0}}, \forall i \neq j$, then

$$S \supseteq \bigcup_{i=1}^\mu \left(S \cap B\left(x_i, \frac{\tau}{4\sqrt{\lambda_0}}\right) \right).$$

It is easy to see that the $(N - 1)$ -measure of $S \cap B(x_i, \frac{\tau}{4\sqrt{\lambda_0}})$ is independent of x_i , we may denote it by $C(\tau)$. Then $|S|_{N-1} \geq \mu C(\tau)$ which is impossible for μ large enough.

The following lemma is important in proving our main result. It can be found in [2], [9].

Lemma 3.1 *Let $\varphi \in C(\mathbb{R}^N)$ be radially symmetric and assume that there exist $\alpha, \beta > 0, C \in \mathbb{R}$ such that*

$$\begin{aligned} \varphi(x) \exp(\alpha|x|)|x|^\beta &\rightarrow C \quad \text{as } |x| \rightarrow +\infty, \\ \int_{\mathbb{R}^N} |\psi(x)| \exp(\alpha|x|)(1 + |x|^\beta) dx &< +\infty. \end{aligned}$$

Then

$$\left(\int_{\mathbb{R}^N} \varphi(x+y)\psi(x) dx \right) \exp(\alpha|y|)|y|^\beta \rightarrow C \int_{\mathbb{R}^N} \psi(x) \exp(-\alpha\langle \gamma, x \rangle) dx$$

as $|y| \rightarrow +\infty$, for an arbitrary $\gamma \in \mathbb{R}^N, |\gamma| = 1$ (the right hand side is independent of γ since ψ is a radial function).

Now, we denote

$$Q(m, \lambda) = \left\{ \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) : (x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1} \right\}$$

for $m \leq \mu, m \in \mathbb{N}$ and $\lambda > 0$;

$$Q(\lambda) = \bigcup_{m=1}^{\mu} Q(m, \lambda), \quad \text{for } \lambda > 0.$$

It is obvious that $Q(m, \lambda) \subset Q(\lambda) \subset H_0^1(\Omega)$.

Lemma 3.2 *There exists $\tilde{\lambda} > 0$ such that, for all $\lambda \geq \tilde{\lambda}$ and $u \in Q(\lambda)$, there is a unique number $k(u) \in \mathbb{R}^+$ such that $k(u)u \in M$.*

Proof. We only need to show that for any $m \in \mathbb{N}$, there is $\lambda_m > 0$ such that for all $\lambda > \lambda_m$ and $u \in Q(m, \lambda)$ there is a unique number $k(u) \in \mathbb{R}^+$ such that $k(u)u \in M$.

Define $G : H_0^1(\Omega) \rightarrow \mathbb{R}$ and $H : S^m \times \Delta_{m-1} \rightarrow H_0^1(\Omega)$ by

$$G(u) = \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx - l \int_{\Omega} u^2 dx, \quad \forall u \in H_0^1(\Omega)$$

and $\forall (x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$,

$$H(x_1, \dots, x_m, t_1, \dots, t_m) = \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i).$$

Clearly, G and H are continuous.

Now, for any $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$,

$$\begin{aligned}
 G \circ H(x_1, \dots, x_m, t_1, \dots, t_m) &= \\
 &= \|\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)\|^2 - l \int_{\mathbb{R}^N} (\varphi \sum_{i=1}^m t_i \omega(x - \lambda x_i))^2 dx \\
 &= \sum_{i,j=1}^m t_i t_j \int_{\mathbb{R}^N} \nabla(\varphi \omega(x - \lambda x_i)) \nabla(\varphi \omega(x - \lambda x_j)) \\
 &\quad + (\lambda_0 - l) \varphi^2 \omega(x - \lambda x_i) \omega(x - \lambda x_j) dx \\
 &\leq \sum_{i,j=1}^m t_i t_j \int_{\mathbb{R}^N} \nabla \omega(x - \lambda x_i) \nabla \omega(x - \lambda x_j) + \lambda_0 \omega(x - \lambda x_i) \omega(x - \lambda x_j) \\
 &\quad - l \varphi^2 \omega(x - \lambda x_i) \omega(x - \lambda x_j) dx + \int_{\mathbb{R}^N} |\Delta \varphi| (\sum_{i=1}^m t_i \omega(x - \lambda x_i))^2 dx \\
 &\leq \sum_{i,j=1}^m t_i t_j \int_{\mathbb{R}^N} f(\omega(x - \lambda x_i)) \omega(x - \lambda x_j) - l \omega(x - \lambda x_i) \omega(x - \lambda x_j) dx \\
 &\quad + \int_{\mathbb{R}^N} (|\Delta \varphi| + l(1 - \varphi^2)) (\sum_{i=1}^m t_i \omega(x - \lambda x_i))^2 dx.
 \end{aligned}$$

Since $\varphi = 1$ in the neighborhood of infinity, we may assume that there exists a ball B such that $\varphi = 1$ in $\mathbb{R}^N \setminus B$. By (2.2), for λ large enough,

$$\begin{aligned}
 &\int_{\mathbb{R}^N} (|\Delta \varphi| + l(1 - \varphi^2)) (\sum_{i=1}^m t_i \omega(x - \lambda x_i))^2 dx \\
 &\leq C(\varphi, m) \int_B \sum_{i=1}^m \omega^2(x - \lambda x_i) dx \\
 &\leq C(\varphi, m) \int_B |x - \lambda x_i|^{-(N-1)} \exp\{-2\sqrt{\lambda_0}|x - \lambda x_i|\} dx \\
 &\leq C(\varphi, m) \int_B (\lambda - |x|)^{-(N-1)} \exp\{-2\sqrt{\lambda_0}(\lambda - |x|)\} dx \\
 &= o(1),
 \end{aligned}$$

where $\lim_{\lambda \rightarrow +\infty} o(1) = 0$ uniformly with respect to $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$ and $C(\varphi, m)$ is a positive constant which depends only on φ and m .

Therefore, for any $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$,

$$\begin{aligned} &G \circ H(x_1, \dots, x_m, t_1, \dots, t_m) \\ &= \sum_{i,j=1}^m t_i t_j \int_{\mathbb{R}^N} (f(\omega(x - \lambda x_i)) - l\omega(x - \lambda x_i))\omega(x - \lambda x_j) dx + o(1) \\ &\leq \sum_{i=1}^m t_i^2 \int_{\mathbb{R}^N} (f(\omega) - l\omega)\omega dx + o(1) \quad (\text{by } (H_2), (H_3)). \end{aligned}$$

It is easy to see that

$$\int_{\mathbb{R}^N} (f(\omega) - l\omega)\omega dx < 0 \quad \text{and} \quad \sum_{i=1}^m t_i^2 \geq \frac{1}{m}.$$

So, there exists $\lambda_m > 0$ such that for any $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times \Delta_{m-1}$ and $\lambda \geq \lambda_m$,

$$G \circ H(x_1, \dots, x_m, t_1, \dots, t_m) < 0,$$

i.e.,

$$(3.4) \quad G(u) < 0 \quad \text{for all } u \in Q(m, \lambda) \text{ and } \lambda \geq \lambda_m.$$

For any $u \in Q(m, \lambda)$, $\lambda \geq \lambda_m$, we define

$$g(t) = \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx - \int_{\Omega} \frac{f(tu)}{t} u dx, \quad t \geq 0.$$

It is obvious that $g : [0, +\infty) \rightarrow \mathbb{R}$ is continuous and

$$\begin{aligned} g(0) &= \lim_{t \rightarrow 0^+} g(t) = \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx > 0, \\ \lim_{t \rightarrow +\infty} g(t) &= \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx - l \int_{\Omega} u^2 dx = G(u) < 0 \quad \text{by (3.4)}. \end{aligned}$$

Therefore there exists a $k \in (0, +\infty)$ such that $g(k) = 0$, and it is not difficult to see that such a k is unique by the assumption (H_3) and that $k(u)$ is continuous in u . This means that for any $u \in Q(m, \lambda)$, $\lambda \geq \lambda_m$, there is a unique $k(u) \in \mathbb{R}^+$ such that $k(u)u \in M$. So the lemma is proved. ■

From now on, we always assume that $\lambda \geq \tilde{\lambda}, 0 \leq m \leq \mu$ and denote $\omega_i = \omega(\cdot - \lambda x_i)$ if there is no confusion.

Now, for any $(x_1, \dots, x_m, t_1, \dots, t_m) \in S^m \times_{\sigma_m} \Delta_{m-1}$, we define a continuous map $h : S^m \times_{\sigma_m} \Delta_{m-1} \rightarrow M^+$ by

$$h(x_1, \dots, x_m, t_1, \dots, t_m) = k(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i)) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i).$$

Denote $B_m = h(S^m \times_{\sigma_m} \Delta_{m-1})$.

It is well known that if x_1, \dots, x_m are distinct points in \mathbb{R}^N , $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ and $\varphi \sum_{j=1}^m \gamma_j \omega(\cdot - x_j) = 0$, then $\gamma_1 = \gamma_2 = \dots = \gamma_m = 0$ (the proof can be found in [2], [9]). Therefore we have

Lemma 3.3 (i) $B_{m-1} \subset B_m \subset M^+$, $\forall m \in \mathbb{N}$;
(ii) h defines an homeomorphism between

$$S^m \times_{\sigma_m} \Delta_{m-1} \setminus (D_m \times_{\sigma_m} \Delta_{m-1} \cup S^m \times_{\sigma_m} \partial \Delta_{m-1}) \quad \text{and} \quad B_m \setminus B_{m-1}.$$

For the sake of not to interrupt proving our main result, the following vital lemma will be proved in the next section.

Lemma 3.4 Let $\varepsilon, \varepsilon_1, \delta$ be the real numbers defined in Lemma 2.4. Then there exist $\eta \in [0, 1]$ and $\lambda > 0$ such that $\forall 1 \leq m \leq \mu$,

- (1) $h : S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta \rightarrow W_{\mu-1}$,
- (2) $h\left(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta, \partial(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta)\right) \subset (V(m, \varepsilon_1), W_{m-1}) \subset (W_m, W_{m-1})$.

Furthermore h sends any sufficiently small neighborhood of $\partial(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta)$ into W_{m-1} .

Now we continue our proof of theorem 1.1. Firstly, we mention that all homologies below are with \mathbb{Z}_2 -coefficients. Let $\widehat{\Delta}_{m-1}$ and $\widehat{\widehat{\Delta}}_{m-1}$ with $\widehat{\Delta}_{m-1} \subset \subset \widehat{\widehat{\Delta}}_{m-1}$ be neighborhoods of $\partial \Delta_{m-1}$, which retract by deformation of $\partial \Delta_{m-1}$. Then $\widehat{B}_{m-1} \triangleq h(\widehat{D}_m \times_{\sigma_m} \Delta_{m-1} \cup S^m \times_{\sigma_m} \widehat{\Delta}_{m-1})$ and $\widehat{\widehat{B}}_{m-1} \triangleq h(\widehat{\widehat{D}}_m \times_{\sigma_m} \Delta_{m-1} \cup S^m \times_{\sigma_m} \widehat{\widehat{\Delta}}_{m-1})$ are neighborhoods of B_{m-1} which retract by deformation on B_{m-1} . If we denote t^η the dilatation,

$$t^\eta : (\Delta_{m-1}, \partial \Delta_{m-1}) \rightarrow (\Delta_{m-1}^\eta, \partial \Delta_{m-1}^\eta),$$

and define $r^\eta : S^m \times \Delta_{m-1} \rightarrow S^m \times \Delta_{m-1}^\eta$ by

$$r^\eta(x, y) = (x, t^\eta(y)), \forall (x, y) \in S^m \times \Delta_{m-1}.$$

From Lemma 3.3 and Lemma 3.4, it follows that

$$h \circ r^\eta \circ h^{-1}(B_m \setminus \widehat{B}_{m-1}, \widehat{\widehat{B}}_{m-1} \setminus \widehat{B}_{m-1}) \subset (W_m, W_{m-1}).$$

Therefore we can consider the following commutative diagram:

$$\begin{array}{ccc}
 H_{*+1}(B_m \setminus \widehat{B}_{m-1}, \widehat{\widehat{B}}_{m-1} \setminus \widehat{B}_{m-1}) & \xrightarrow{(hr^\eta h^{-1})_{*+1}} & H_{*+1}(W_m, W_{m-1}) \\
 \downarrow \partial_* & & \downarrow \partial_* \\
 H_*(\widehat{\widehat{B}}_{m-1} \setminus \widehat{B}_{m-1}) & \xrightarrow{(hr^\eta h^{-1})_*} & H_*(W_{m-1}) \\
 \downarrow j_* & & \parallel \\
 H_*(\widehat{\widehat{B}}_{m-1} \setminus \widehat{B}_{m-2}) & \xrightarrow{(hr^\eta h^{-1})_*} & H_*(W_{m-1}) \\
 \downarrow r_*^B & & \parallel \\
 H_*(B_{m-1} \setminus \widehat{B}_{m-2}) & \xrightarrow{(hr^\eta h^{-1})_*} & H_*(W_{m-1}) \\
 \downarrow p_* & & \downarrow q_* \\
 H_*(B_{m-1} \setminus \widehat{B}_{m-2}, \widehat{\widehat{B}}_{m-2} \setminus \widehat{B}_{m-2}) & \xrightarrow{(hr^\eta h^{-1})_*} & H_*(W_{m-1}, W_{m-2})
 \end{array}$$

where ∂_* are the connecting homomorphisms, j_* is induced by the natural embedding, r_*^B is induced by the retraction of $\widehat{\widehat{B}}_{m-1}$ on B_{m-1} , p_* and q_* are the projections on the quotients.

Since $h \circ (r^\eta)^{-1}$ is an homeomorphism,

$$\begin{aligned}
 & H_*((B_m \setminus \widehat{B}_{m-1}, \widehat{\widehat{B}}_{m-1} \setminus \widehat{B}_{m-1})) \\
 &= H_*\left(S^m_{\sigma_m} \times \Delta_{m-1}^\eta \setminus (\widehat{D}_m \times_{\sigma_m} \Delta_{m-1}^\eta \cup S^m_{\sigma_m} \times \widehat{\Delta}_{m-1}^\eta), \widehat{D}_m \times_{\sigma_m} \Delta_{m-1}^\eta \right. \\
 &\quad \left. \cup S^m_{\sigma_m} \times \widehat{\Delta}_{m-1}^\eta \setminus (\widehat{D}_m \times_{\sigma_m} \Delta_{m-1}^\eta \cup S^m_{\sigma_m} \times \widehat{\Delta}_{m-1}^\eta)\right) \\
 &= H_*\left(S^m_{\sigma_m} \times \Delta_{m-1}^\eta, \partial(S^m_{\sigma_m} \times \Delta_{m-1}^\eta)\right).
 \end{aligned}$$

Hence the function $p_* \cdot r_*^B \cdot j_* \cdot \partial_*$ naturally induces an homomorphism, which we call again ∂_* ,

$$\begin{aligned}
 \partial_* : H_{*+1}\left(S^m_{\sigma_m} \times \Delta_{m-1}^\eta, \partial(S^m_{\sigma_m} \times \Delta_{m-1}^\eta)\right) \\
 \longrightarrow H_*\left(S^{m-1}_{\sigma_{m-1}} \times \Delta_{m-2}^\eta, \partial(S^{m-1}_{\sigma_{m-1}} \times \Delta_{m-2}^\eta)\right).
 \end{aligned}$$

So the following diagram is commutative

$$\begin{array}{ccc}
 H_*(E_m, \partial E_m) & \xrightarrow{\partial_*} & H_{*-1}(E_{m-1}, \partial E_{m-1}) \\
 \downarrow h_* & & \downarrow h_{*-1} \\
 H_*(W_m, W_{m-1}) & \xrightarrow{\partial_*} & H_{*-1}(W_{m-1}, W_{m-2})
 \end{array}$$

Now, let us denote

$$E_m = S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta, \quad F_m = W_{m-1}^\delta \cap V(m, \varepsilon), \quad G_m = W_{m-1} \cap V(m, \varepsilon).$$

From Lemma 2.4 and the excise property of homology groups,

$$H_*(W_m, W_{m-1}) = H_*(W_{m-1}^\delta, W_{m-1}) = H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)).$$

The natural embedding

$$i_* : H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \rightarrow H_*(W_m, W_{m-1})$$

is an isomorphism, and it induces an homomorphism

$$\begin{aligned}
 \partial_* : H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon)) \\
 \rightarrow H_{*-1}(W_{m-2}^\delta \cap V(m-1, \varepsilon), W_{m-2} \cap V(m-1, \varepsilon))
 \end{aligned}$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
 H_*(E_m, \partial E_m) & \xrightarrow{\partial_*} & H_{*-1}(E_{m-1}, \partial E_{m-1}) \\
 \downarrow h_* & & \downarrow h_{*-1} \\
 H_*(W_m, W_{m-1}) & \xrightarrow{\partial_*} & H_{*-1}(W_{m-1}, W_{m-2}) \\
 \downarrow i_*^{-1} & & \downarrow i_{*-1}^{-1} \\
 H_*(F_m, G_m) & \xrightarrow{\partial_*} & H_{*-1}(F_{m-1}, G_{m-1}).
 \end{array}$$

However, by Lemma 3.4 (2), h_* sends $H_*(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta, \partial(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta))$ to $H_*(W_{m-1}^\delta \cap V(m, \varepsilon), W_{m-1} \cap V(m, \varepsilon))$, so the above diagram simply reduces to

$$\begin{array}{ccc}
 H_*(E_m, \partial E_m) & \xrightarrow{\partial_*} & H_{*-1}(E_{m-1}, \partial E_{m-1}) \\
 \downarrow h_* & & \downarrow h_{*-1} \\
 H_*(F_m, G_m) & \xrightarrow{\partial_*} & H_{*-1}(F_{m-1}, G_{m-1}).
 \end{array}$$

Lemma 3.5 *There exists a continuous map*

$$s : V(m, \varepsilon) \longrightarrow S^m / \sigma_m$$

such that the following diagram is commutative

$$\begin{array}{ccc} S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta & \xrightarrow{h} & W_{m-1}^\delta \cap V(m, \varepsilon) \\ \downarrow t & & \parallel \\ S^m / \sigma_m & \xleftarrow{s} & W_{m-1}^\delta \cap V(m, \varepsilon) \end{array}$$

where t is the projection on the first coordinate.

The proof of this lemma is obvious due to Proposition III.1 in [2]. We omit the details.

Let us denote

$$\begin{aligned} s^{N-1} &: H^{N-1}(S^m / \sigma_m) \rightarrow H^{N-1}(V(m, \varepsilon)), \\ \text{and } t^{N-1} &: H^{N-1}(S^m / \sigma_m) \rightarrow H^{N-1}(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta) \end{aligned}$$

the homomorphisms induced on the cohomology groups. Then for any $\alpha \in H^{N-1}(S^m / \sigma_m)$, we can define

$$\bar{d}_\alpha : H_{mN-1}(F_m, G_m) \rightarrow H_{mN-N}(F_m, G_m)$$

by $\bar{d}_\alpha(\xi) = s^*(\alpha) \cap \xi$ and

$$d_\alpha : H_{mN-1}(E_m, \partial E_m) \rightarrow H_{mN-N}(E_m, \partial E_m)$$

by $d_\alpha(\xi) = t^*(\alpha) \cap \xi$, where \cap is the cap product, and we have the following commutative diagram:

$$\begin{array}{ccc} H_{mN-1}(E_m, \partial E_m) & \xrightarrow{d_\alpha} & H_{mN-N}(E_m, \partial E_m) \\ \downarrow h_{mN-1} & & \downarrow h_{mN-N} \\ H_{mN-1}(F_m, G_m) & \xrightarrow{\bar{d}_\alpha} & H_{mN-N}(F_m, G_m). \end{array}$$

Furthermore, we have $H_{mN-1}(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta, \partial(S_0^m \times_{\sigma_m} \Delta_{m-1}^\eta)) = \mathbb{Z}_2$ and there exists $\alpha \in H^{N-1}(S^m / \sigma_m)$ such that

$$\partial \cdot d_\alpha : H_{mN-1}(E_m, \partial E_m) \rightarrow H_{(m-1)N-1}(E_{m-1}, \partial E_{m-1})$$

sends the generator to the generator (see [2] for details).

Therefore we have the commutative diagram:

$$\begin{CD} H_{\mu N-1}(E_\mu, \partial E_\mu) @>\partial d_\alpha \cdots \partial d_\alpha>> H_{N-1}(E_1, \partial E_1) \\ @VV h_{\mu N-1} V @VV h_{N-1} V \\ H_{\mu N-1}(F_\mu, G_\mu) @>\partial \bar{d}_\alpha \cdots \partial \bar{d}_\alpha>> H_{N-1}(F_1, G_1). \end{CD}$$

Since $W_0 = \emptyset$ and $S_0 = S$, the above diagram becomes

$$\begin{CD} H_{\mu N-1}\left(S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta, \partial(S_0^\mu \times_{\sigma_\mu} \Delta_{\mu-1}^\eta)\right) @>\partial d_\alpha \cdots \partial d_\alpha>> H_{N-1}(S) \\ @VV h_{\mu N-1} V @VV h_{N-1} V \\ H_{\mu N-1}\left(W_{\mu-1}^\delta \cap V(\mu, \varepsilon), W_{\mu-1} \cap V(\mu, \varepsilon)\right) @>\partial \bar{d}_\alpha \cdots \partial \bar{d}_\alpha>> H_{N-1}(W_0^\delta \cap V(1, \varepsilon)). \end{CD}$$

By Lemma 3.4, $h_{\mu N-1} = 0$. By the commutativity of the above diagram and the fact that ∂d_α is an isomorphism, we know $h_{N-1} = 0$. But this is impossible. Indeed, if $m = 1$, then the commutative diagram in Lemma 3.5 implies that $s \circ h = id$, which implies $s_* \circ h_* = id$. Then h_{N-1} is one to one. However, $H_{N-1}(S) = \mathbb{Z}_2$. Then $h_{N-1} \neq 0$. A contradiction. We thus complete the proof of Theorem 1.1.

4. Proof of Lemma 3.4

First of all, we prove some lemmas.

Lemma 4.1 *From the hypotheses we have made on f , it follows that*

$$(4.1) \quad f(t) \leq t f'(t) \leq p f(t), \quad \forall t \geq 0;$$

$$(4.2) \quad f\left(\sum_{i=1}^m a_i\right) \geq \sum_{i=1}^m f(a_i), \quad \forall a_i \geq 0, i = 1, \dots, m;$$

$$(4.3) \quad F\left(\sum_{i=1}^m a_i\right) \geq \sum_{i=1}^m F(a_i) + \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \sum_{\substack{i,j=1 \\ i \neq j}}^m f(a_i) a_j, \quad \forall a_i \in [0, M]$$

where M, α are given by (H_4) .

Proof (4.1) is a direct result of (H_3) .

Since f is convex, for every $j \in \{1, \dots, m\}$, we have

$$f\left(a_j + \sum_{\substack{i=1 \\ i \neq j}}^m a_i\right) \geq f(a_j) + f'(a_j) \sum_{\substack{i=1 \\ i \neq j}}^m a_i.$$

So

$$\begin{aligned} f\left(\sum_{i=1}^m a_i\right)a_j &\geq f(a_j)a_j + f'(a_j)a_j \sum_{\substack{i=1 \\ i \neq j}}^m a_i \\ &\geq f(a_j)a_j + f(a_j) \sum_{\substack{i=1 \\ i \neq j}}^m a_i = f(a_j) \sum_{i=1}^m a_i. \end{aligned}$$

Then

$$f\left(\sum_{i=1}^m a_i\right) \sum_{j=1}^m a_j \geq \sum_{j=1}^m f(a_j) \sum_{i=1}^m a_i, \text{ i.e. } f\left(\sum_{i=1}^m a_i\right) \geq \sum_{j=1}^m f(a_j).$$

Hence (4.2) is proved.

We only need to show (4.3) when $m = 2$, i.e. we will show for any $a, b \in [0, M]$, $F(a + b) \geq F(a) + F(b) + \frac{2^{2+\alpha}-2}{2^{2+\alpha}+\alpha}(f(a)b + f(b)a)$.

Indeed, we may assume $a \geq b$.

$$\begin{aligned} &F(a + b) - F(a) - F(b) \\ &= \int_0^1 f(a + \theta b)b d\theta - \int_0^1 f(\theta b)b d\theta = \int_0^1 \int_0^1 f'(ta + \theta b)ab dtd\theta \\ &= \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \int_0^1 \int_0^1 f'(ta + \theta b)ab dtd\theta + \frac{\alpha + 2}{2^{2+\alpha} + \alpha} \int_0^1 \int_0^1 f'(ta + \theta b)ab dtd\theta \\ &\geq \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \int_0^1 \int_0^1 f'(ta)ab dtd\theta + \frac{\alpha + 2}{2^{2+\alpha} + \alpha} \int_0^1 \int_0^1 f'((t + \theta)b)ab dtd\theta \\ &= \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} f(a)b + \frac{\alpha + 2}{2^{2+\alpha} + \alpha} \int_0^1 \int_t^{1+t} f'(sb)ab ds dt \\ &= \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} f(a)b + \frac{\alpha + 2}{2^{2+\alpha} + \alpha} \int_0^1 (f((1 + t)b) - f(tb))a dt \\ &\geq \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} f(a)b + \frac{\alpha + 2}{2^{2+\alpha} + \alpha} \int_0^1 ((1 + t)^{1+\alpha} - t^{1+\alpha})f(b)a dt \quad (\text{by } (H_4)) \\ &= \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} (f(a)b + f(b)a) \quad \text{for all } a, b \in [0, M]. \end{aligned}$$

Therefore (4.3) is proved. ■

Lemma 4.2 $\forall u \in Q(\lambda)$ and $m > 0$, we have

$$\min \left\{ \left(\frac{k(u)}{m} \right)^{p-1}, 1 \right\} \leq \frac{m^2 \int_{\Omega} (|\nabla u|^2 + \lambda_0 u^2) dx}{\int_{\Omega} f(mu) mudx} \leq \max \left\{ \left(\frac{k(u)}{m} \right)^{p-1}, 1 \right\}.$$

Proof. The proof is easy due to Lemma 3.2 and (H_3) . We omit the details. ■

Now, let us proof Lemma 3.4. By (3.1)-(3.3), as in [2], we need only to prove

Lemma 4.3 *Suppose that the assumptions in Theorem 1.1 hold, then*

$$(4.4) \quad \exists \bar{\lambda}, \bar{\eta} > 0, \text{ such that } \forall \lambda \geq \bar{\lambda}, \forall \eta \leq \bar{\eta},$$

if $\min |x_i - x_j| \geq \gamma_m$ and $(t_1, \dots, t_m) \in \Delta_{m-1}^\eta$, then

$$k \left(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \right) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \in V(m, \varepsilon_1);$$

$$(4.5) \quad \forall m \geq 2, \exists \bar{\lambda}, \bar{\eta} > 0 \text{ such that } \forall \lambda \geq \bar{\lambda}, \forall \eta \leq \bar{\eta},$$

if $\gamma_m \leq \min |x_i - x_j| \leq \frac{\tau}{2\sqrt{\lambda_0}}$ and $(t_1, \dots, t_m) \in \Delta_{m-1}^\eta$, then

$$I \left(k \left(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \right) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \right) < m I_\infty(\omega);$$

$$(4.6) \quad \forall m \geq 2, \exists \bar{\eta} > 0 \text{ such that } \forall \eta \leq \bar{\eta}, \exists \bar{\lambda} > 0, \text{ such that } \forall \lambda \geq \bar{\lambda},$$

if $\gamma_m \leq \min |x_i - x_j|$ and $(t_1, \dots, t_m) \in \partial \Delta_{m-1}^\eta$, then

$$I \left(k \left(\varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \right) \varphi \sum_{i=1}^m t_i \omega(\cdot - \lambda x_i) \right) < m I_\infty(\omega).$$

Proof. We will write k instead of $k \left(\varphi \sum_{i=1}^m t_i \omega_i \right)$. Since

$$k^2 \int_{\Omega} \left(\left| \nabla \left(\varphi \sum_{i=1}^m t_i \omega_i \right) \right|^2 + \lambda_0 \left(\varphi \sum_{i=1}^m t_i \omega_i \right)^2 \right) dx = \int_{\Omega} k \varphi \sum_{i=1}^m t_i \omega_i f \left(k \varphi \sum_{i=1}^m t_i \omega_i \right) dx,$$

we have

$$(4.7)$$

$$\int_{\Omega} \left(\left| \nabla \left(\varphi \sum_{i=1}^m t_i \omega_i \right) \right|^2 + \lambda_0 \left(\varphi \sum_{i=1}^m t_i \omega_i \right)^2 \right) dx = \int_{\Omega} \frac{\varphi \sum_{i=1}^m t_i \omega_i f \left(k \varphi \sum_{i=1}^m t_i \omega_i \right)}{k} dx.$$

If $\min |x_i - x_j| \geq \gamma_m$, then it is easy to show that

$$(4.8) \quad \int_{\Omega} \left(\left| \nabla \left(\varphi \sum_{i=1}^m t_i \omega_i \right) \right|^2 + \lambda_0 \left(\varphi \sum_{i=1}^m t_i \omega_i \right)^2 \right) dx \rightarrow \frac{\|\omega\|^2}{m} = \frac{1}{m} \int_{\mathbb{R}^N} \omega f(\omega) dx$$

as $\lambda \rightarrow +\infty$ and $t_i \rightarrow \frac{1}{m}$.

Now, denote

$$\liminf_{\substack{\lambda \rightarrow +\infty \\ t_i \rightarrow 1/m}} k = k_1, \quad \limsup_{\substack{\lambda \rightarrow +\infty \\ t_i \rightarrow 1/m}} k = k_2.$$

Then by (4.7), (4.8), we have

$$\begin{aligned} \frac{1}{m} \int_{\mathbb{R}^N} \omega f(\omega) dx &= \sum_{i=1}^m \int_{\mathbb{R}^N} \frac{\frac{1}{m} \omega f(k_1 \frac{1}{m} \omega)}{k_1} dx = \int_{\mathbb{R}^N} \frac{\omega f(\frac{k_1}{m} \omega)}{k_1} dx \\ \frac{1}{m} \int_{\mathbb{R}^N} \omega f(\omega) dx &= \sum_{i=1}^m \int_{\mathbb{R}^N} \frac{\frac{1}{m} \omega f(k_2 \frac{1}{m} \omega)}{k_2} dx = \int_{\mathbb{R}^N} \frac{\omega f(\frac{k_2}{m} \omega)}{k_2} dx \end{aligned}$$

by Lebesgue’s Dominated Convergence Theorem and translation. These two equalities become

$$\int_{\mathbb{R}^N} \omega f(\omega) dx = \int_{\mathbb{R}^N} \frac{\omega f(\frac{k_1}{m} \omega)}{\frac{k_1}{m}} dx = \int_{\mathbb{R}^N} \frac{\omega f(\frac{k_2}{m} \omega)}{\frac{k_2}{m}} dx.$$

But by the assumption (H_3) , $\frac{k_1}{m} = \frac{k_2}{m} = 1$. Therefore,

$$\lim_{\substack{\lambda \rightarrow +\infty \\ t_i \rightarrow 1/m}} k = m$$

and this shows (4.4).

To prove (4.5), we denote $\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda_0 uv) dx$.

$$\begin{aligned} &\int_{\Omega} \left(|\nabla (\varphi \sum_{i=1}^m kt_i \omega_i)|^2 + \lambda_0 (\varphi \sum_{i=1}^m kt_i \omega_i)^2 \right) dx \\ &\leq \left\| \sum_{i=1}^m kt_i \omega_i \right\|^2 + \int_{\Omega} |\Delta \varphi| \left(\sum_{i=1}^m kt_i \omega_i \right)^2 dx \\ &\leq \sum_{i=1}^m (kt_i)^2 \|\omega\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^m t_i t_j k^2 \langle \omega_i, \omega_j \rangle + C \sum_{i=1}^m \int_{\Omega} |\Delta \varphi| \omega_i^2 dx. \end{aligned}$$

Since ω is a solution to (1.8), we have

$$\langle \omega_i, \omega_j \rangle = \int_{\mathbb{R}^N} f(\omega_i) \omega_j dx \quad \text{and} \quad \|\omega\|^2 = \int_{\mathbb{R}^N} f(\omega) \omega dx.$$

From Lemma 3.1, we have

$$\sum_{i=1}^m \int_{\Omega} |\Delta \varphi| \omega_i^2 dx \leq C \exp(-\lambda \tau) \lambda^{-\frac{N-1}{2}}.$$

Hence, we obtain

$$\begin{aligned} & \int_{\Omega} \left(|\nabla(\varphi \sum_{i=1}^m kt_i \omega_i)|^2 + \lambda_0(\varphi \sum_{i=1}^m kt_i \omega_i)^2 \right) dx \\ & \leq \sum_{i=1}^m (kt_i)^2 \int_{\mathbb{R}^N} f(\omega) \omega dx + \sum_{\substack{i,j=1 \\ i \neq j}}^m t_i t_j k^2 \int_{\mathbb{R}^N} f(\omega_i) \omega_j dx + C \exp(-\lambda\tau) \lambda^{-\frac{N-1}{2}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\Omega} F\left(\varphi \sum_{i=1}^m kt_i \omega_i\right) & \geq \int_{\mathbb{R}^N} F\left(\sum_{i=1}^m kt_i \omega_i\right) - \int_{\mathbb{R}^N} \left(F\left(\sum_{i=1}^m kt_i \omega_i\right) - F\left(\varphi \sum_{i=1}^m kt_i \omega_i\right) \right) \\ & \geq \int_{\mathbb{R}^N} F\left(\sum_{i=1}^m kt_i \omega_i\right) + C \int_{\mathbb{R}^N} (\varphi^p - 1) F\left(\sum_{i=1}^m kt_i \omega_i\right) \\ & \geq \int_{\mathbb{R}^N} F\left(\sum_{i=1}^m kt_i \omega_i\right) + C \int_{\mathbb{R}^N} (\varphi^p - 1) F(\omega_i) \\ & \geq \int_{\mathbb{R}^N} F\left(\sum_{i=1}^m kt_i \omega_i\right) - C \exp(-\lambda\tau) \lambda^{-\frac{N-1}{2}}. \end{aligned}$$

For λ large enough and t_i close enough to $\frac{1}{m}$, we have $kt_i \leq 2$ by (4.4). Therefore using the assumption (H_4) and (4.3) for $M = 2 \sup_{x \in \mathbb{R}^N} \omega(x)$, we have

$$\begin{aligned} & \int_{\Omega} F\left(\varphi \sum_{i=1}^m kt_i \omega_i\right) \\ & \geq \sum_{i=1}^m \int_{\mathbb{R}^N} F(kt_i \omega_i) + \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_{\mathbb{R}^N} f(kt_i \omega_i) (kt_j \omega_j) - C \exp(-\lambda\tau) \lambda^{-\frac{N-1}{2}} \\ & \geq \sum_{i=1}^m \int_{\mathbb{R}^N} F(kt_i \omega_i) + \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \sum_{\substack{i,j=1 \\ i \neq j}}^m \min(1, (kt_i)^{p-1}) kt_j \int_{\mathbb{R}^N} f(\omega_i) \omega_j \\ & \quad - C \exp(-\lambda\tau) \lambda^{-\frac{N-1}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} I(\varphi \sum_{i=1}^m kt_i \omega_i) & \leq \frac{1}{2} \sum_{i=1}^m (kt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(kt_i \omega) \\ & \quad + \sum_{\substack{i,j=1 \\ i \neq j}}^m \left(\frac{1}{2} t_i k - \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \min(1, (kt_i)^{p-1}) \right) kt_j \int_{\mathbb{R}^N} f(\omega_i) \omega_j + C \exp(-\tau\lambda) \lambda^{-\frac{N-1}{2}}. \end{aligned}$$

Since (4.4) holds, for all $i = 1, \dots, m$, $kt_i \rightarrow 1$ as $\lambda \rightarrow +\infty$ and $\eta \rightarrow 0$,

$$\frac{1}{2}t_i k - \frac{2^{2+\alpha} - 2}{2^{2+\alpha} + \alpha} \min(1, (kt_i)^{p-1}) \rightarrow \frac{\alpha + 4 - 2^{2+\alpha}}{2(2^{2+\alpha} + \alpha)} < 0.$$

On the other hand, by Lemma 3.1 and $\min |x_i - x_j| \leq \frac{\tau}{2\sqrt{\lambda_0}}$, it follows that there exist $i, j \in \{1, \dots, m\}$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} f(\omega_i)\omega_j &\geq C \exp\left(-\lambda\sqrt{\lambda_0}|x_i - x_j|\right) \left(\lambda\sqrt{\lambda_0}|x_i - x_j|\right)^{-\frac{N-1}{2}} \\ &\geq C \exp\left(-\frac{\tau\lambda}{2}\right) \left(\frac{\tau\lambda}{2}\right)^{-\frac{N-1}{2}}. \end{aligned}$$

Therefore, for λ large enough we have

$$\begin{aligned} I\left(\varphi \sum_{i=1}^m kt_i\omega_i\right) &< \frac{1}{2} \sum_{i=1}^m (kt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(kt_i\omega) \\ &\leq \frac{1}{2} \sum_{i=1}^m \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(\omega) \\ &= m\left(\frac{1}{2}\|\omega\|^2 - \int_{\mathbb{R}^N} F(\omega)\right) = mI_\infty(\omega), \end{aligned}$$

i.e. (4.5) is proved.

Finally we prove (4.6). We have

$$\begin{aligned} \int_{\Omega} f\left(\varphi \sum_{i=1}^m mt_i\omega_i\right) \varphi \sum_{i=1}^m mt_i\omega_i &= \int_{\mathbb{R}^N} f\left(\sum_{i=1}^m mt_i\omega_i\right) \sum_{i=1}^m mt_i\omega_i \\ &\quad - \int_{\mathbb{R}^N} \left(f\left(\sum_{i=1}^m mt_i\omega_i\right) - f\left(\varphi \sum_{i=1}^m mt_i\omega_i\right)\varphi\right) \sum_{i=1}^m mt_i\omega_i \\ &\geq \sum_{i=1}^m \int_{\mathbb{R}^N} f(mt_i\omega_i)mt_i\omega_i + \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_{\mathbb{R}^N} f(mt_i\omega_i)mt_j\omega_j \\ &\quad + \int_{\mathbb{R}^N} (\varphi^p - 1)f\left(\sum_{i=1}^m mt_i\omega_i\right) \sum_{i=1}^m mt_i\omega_i \\ &\geq \sum_{i=1}^m \int_{\mathbb{R}^N} f(mt_i\omega_i)mt_i\omega_i - C \exp(-\tau\lambda)\lambda^{-\frac{N-1}{2}} \end{aligned}$$

and by Lemma 4.2 and the first part of the proof of (4.5),

$$\min \left(1, \left(\frac{k}{m} \right)^{p-1} \right) \leq \frac{\sum_{i=1}^m (mt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) + o(\lambda)}{\sum_{i=1}^m \int_{\mathbb{R}^N} mt_i \omega f(mt_i \omega) + o(\lambda)},$$

where $\lim_{\lambda \rightarrow +\infty} \frac{o(\lambda)}{\lambda} = 0$. It is not difficult to prove that the function

$$(t_1, \dots, t_m) \rightarrow \sum_{i=1}^m (mt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} mt_i \omega f(mt_i \omega)$$

takes a strict maximum on the manifold

$$\sum_{i=1}^m t_i = 1, t_i \geq 0, i = 1, \dots, m,$$

at the point $(\frac{1}{m}, \dots, \frac{1}{m})$. Therefore, $\exists \bar{\eta} > 0$, such that $\forall \eta < \bar{\eta}, \exists \theta(\eta), \lambda(\eta)$ such that $\forall (t_1, \dots, t_m) \in \partial \Delta_{m-1}^\eta, \lambda > \lambda(\eta)$,

$$\min \left(1, \left(\frac{k}{m} \right)^{p-1} \right) \leq \frac{\sum_{i=1}^m (mt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) + o(\lambda)}{\sum_{i=1}^m \int_{\mathbb{R}^N} mt_i \omega f(mt_i \omega) + o(\lambda)} < 1 - \frac{\theta}{2}.$$

Consequently, $\forall \eta \leq \bar{\eta}$, there exists $\lambda(\eta)$ and $\bar{\theta}(\eta)$ such that $k < m - \bar{\theta}(\eta)$.

But

$$I \left(\varphi \sum_{i=1}^m kt_i \omega_i \right) \leq \frac{1}{2} \sum_{i=1}^m (kt_i)^2 \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(kt_i \omega) + o(\lambda).$$

The function

$$(t_1, \dots, t_m) \rightarrow \frac{1}{2} \sum_{i=1}^m t_i^2 \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(t_i \omega)$$

takes a strict maximum at the point $(1, \dots, 1)$. Since

$$\min |x_i - x_j| \geq \gamma_m$$

by (4.4), we can assume that kt_i belongs to an arbitrary small neighborhood of 1 for all $i = 1, \dots, m$.

However,

$$\sum_{i=1}^m |kt_i - 1| \geq m - \sum_{i=1}^m kt_i = m - k > \bar{\theta}(\eta).$$

Hence, $\exists \bar{\theta} > 0$ such that

$$\begin{aligned} I\left(\varphi \sum_{i=1}^m kt_i \omega_i\right) &\leq \frac{1}{2} \sum_{i=1}^m \int_{\mathbb{R}^N} \omega f(\omega) - \sum_{i=1}^m \int_{\mathbb{R}^N} F(\omega) - \bar{\theta}(\eta) + o(\lambda) \\ &< mI_\infty(\omega) \end{aligned}$$

if λ is large enough. Hence (4.6) is proved and we have completed the proof of the lemma.

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