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Riesz transforms for symmetric diffusion operators on complete Riemannian manifolds

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Dedicated to my mother for her sixtieth birthday

Abstract

Let (M, g) be a complete Riemannian manifold, $L = \Delta - \nabla\phi \cdot \nabla$ be a Markovian symmetric diffusion operator with an invariant measure $d\mu(x) = e^{-\phi(x)}d\nu(x)$, where $\phi \in C^2(M)$, ν is the Riemannian volume measure on (M, g). A fundamental question in harmonic analysis and potential theory asks whether or not the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$ and for certain $a \geq 0$. An affirmative answer to this problem has many important applications in elliptic or parabolic PDEs, potential theory, probability theory, the L^p-Hodge decomposition theory and in the study of Navier-Stokes equations and boundary value problems.

Using some new interplays between harmonic analysis, differential geometry and probability theory, we prove that the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $a > 0$ and $p \geq 2$ provided that L generates a ultracontractive Markovian semigroup $P_t = e^{tL}$ in the sense that $P_t 1 = 1$ for all $t \geq 0$, $\|P_t\|_{1,\infty} < Ct^{-n/2}$ for all $t \in (0, 1]$ for some constants $C > 0$ and $n > 1$, and satisfies

$$(K + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \mu)$$

for some constants $c \geq 0$ and $\epsilon > 0$, where $K(x)$ denotes the lowest eigenvalue of the Bakry-Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ on T_xM, i.e.,

$$K(x) = \inf\{Ric(L)(v, v) : v \in T_xM, \|v\| = 1\}, \quad \forall\ x \in M.$$

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Examples of diffusion operators on complete non-compact Riemannian manifolds with unbounded negative Ricci curvature or Bakry-Emery Ricci curvature are given for which the Riesz transform $R_a(L)$ is bounded in $L^p(\mu)$ for all $p \geq 2$ and for all $a > 0$ (or even for all $a \geq 0$).

1. Introduction

Let (M, g) be a complete non-compact Riemannian manifold, Δ be the non-positive Laplace-Beltrami operator, and $d\nu(x) = \sqrt{\det(g(x))}dx$ be the Riemanian volume measure on (M, g). Let $L = \Delta - \nabla\phi \cdot \nabla$ be a symmetric diffusion operator with an invariant measure $d\mu(x) = e^{-\phi(x)}d\nu(x)$, where $\phi \in C^2(M)$. For a non-negative constant a, let us define the Riesz transform $R_a(L)$ associated with the diffusion operator L by

$$R_a(L)f = \nabla(a - L)^{-1/2}f, \quad \forall\ f \in C_0^\infty(M).$$

where

$$C_0^\infty(M) = \begin{cases} C_c^\infty(M) & \text{if } a > 0 \text{ or } \mu(M) = +\infty, \\ \left\{f \in C_c^\infty(M) : \int_M f d\mu = 0\right\} & \text{otherwise.} \end{cases}$$

Integration by parts yields, for every $a \geq 0$ and $f \in C_0^\infty(M)$,

$$a\|f\|^2_{L^2(\mu)} + \|\nabla f\|^2_{L^2(\mu)} = \|\sqrt{a - L}f\|^2_{L^2(\mu)}.$$

This implies that for all $a \geq 0$ we have

$$\|R_a(L)f\|_{L^2(\mu)} \leq \|f\|_{L^2(\mu)}, \quad \forall f \in C_0^\infty(M).$$

It is very natural to pose the following fundamental

Problem 1.1 *What is the (weakest possible) condition on (M, g) and (L, ϕ) under which the Riesz transform $R_a(L)$ is bounded in $L^p(M, \mu)$ for all $1 < p < \infty$? That is, for any $1 < p < \infty$, there exists a constant $C = C_{p,a}$ such that*

$$\|R_a(L)f\|_{L^p(\mu)} \leq C_{p,a}\|f\|_{L^p(\mu)}, \qquad \forall\ f \in C_0^\infty(M).$$

In the special case where $L = \Delta$, $\phi = 0$ and $a = 0$, Problem 1.1 is exactly the famous Strichartz problem [65] concerning the L^p-boundedness of the Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ on a complete non-compact Riemannian manifold. For the historical background and for some partial affirmative answers to this problem, we refer the reader to [65, 48, 49, 3, 11, 40, 41, 17, 13, 14, 15, 39] and the references therein. See also [17, 13, 15, 39] for some counter-examples of complete or non-complete Riemannian manifolds on which the Riesz transform $R_0(\Delta)$ is not bounded in $L^p(\nu)$ for certain $p \in (1, \infty)$.

Note that, as indicated in Strichartz [65], the Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ is a zeroth order pseudo-differential operator. Well-known results in the theory of pseudo-differential operators imply that $R_0(\Delta)$ (and hence $R_a(\Delta) = \nabla(a-\Delta)^{-1/2}$ for all $a \geq 0$, see Theorem 3.2 below) is always bounded in $L^p(\nu)$ for all $1 < p < \infty$ if (M, g) is a compact Riemannian manifold.

In the general case where M is a complete Riemannian manifold and $\phi \in C^2(M)$, the study of Problem 1.1 has a great interest and very important applications in probability theory. Again, results in the theory of pseudo-differential operators imply that for all $a \geq 0$, $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(\mu)$ if (M, g) is a compact Riemannian manifold. While the situation when (M, g) is non-compact is very complicated. In [56], P.A. Meyer proved that if ∇ is the gradient operator and L is the Ornstein-Uhlenbeck diffusion operator on the Wiener space $W = \{x \in C([0,1], \mathbb{R}^d) : x(0) = 0\}$ equipped with the Wiener measure μ, then the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded from $L^p_0(\mu) = \{f \in L^p(\mu) : \int_W f d\mu = 0\}$ into $L^p(W, H, \mu)$ with respect to the Wiener measure for every $1 < p < \infty$, where $H = H^{1,2}([0,1], \mathbb{R}^d) \cap W$ is the Cameron-Martin space. This yields that, for all cylinder functions f on W, the Sobolev norm $\|f\|_p + \|\nabla f\|_p$ is equivalent to the Sobolev norm $\|(I - L)^{1/2} f\|_p$. This results, the so-called P.A. Meyer equivalence or the P.A. Meyer inequality, played a crucial rôle in the Malliavin calculus on the infinite dimensional Wiener space (see e.g. Malliavin [54]). For two different proofs of the P.A. Meyer inequality, see G. Pisier [57] and R. Gundy [31].

In a more general and geometric setting, using a probabilistic approach to the Littlewood-Paley theory, D. Bakry [5], see also [3, 6], proved the following remarkable

Theorem 1.1 *Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$, $L = \Delta - \nabla\phi \cdot \nabla$, $d\mu(x) = e^{-\phi(x)} d\nu(x)$. Suppose that there exists a non-negative constant a such that*

$$Ric(x) + \nabla^2\phi(x) \geq -a, \quad \forall\, x \in M. \tag{1.1}$$

Then for any $1 < p < \infty$, there exists a constant C_p independent of a and $n = dimM$ such that

$$\|R_a(L) f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)}, \quad \forall\, f \in C_0^\infty(M).$$

Here Ric denotes the Ricci curvature, $\nabla^2\phi$ denotes the Hessian of ϕ. Both of them are taken with respect to the Levi-Civita connection determined by the Riemannian metric g. In [6], Bakry introduced the symmetric 2-tensor

$$Ric(L) := Ric + \nabla^2\phi \tag{1.2}$$

and called it the Ricci curvature associated with the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$. In the literature, it is also called the Bakry-Emery Ricci curvature, see e.g. [50]. The inequality $Ric_x(L) \geq -a$ means that the lowest eigenvalue of the Bakry-Emery Ricci curvature tensor $Ric(L)$, considered as an endomorphism of the tangent space at $x \in M$, is at least $-a$. Note that, due to the free dependence of the constant C_p on $n = dimM$, Theorem 1.1 even applies to sub-elliptic diffusion operators L (formally given by $L = \Delta - \nabla\phi\cdot\nabla$) on some infinite dimensional manifolds provided that the Bakry-Emery Ricci curvature $Ric(L)$ defined by (1.2) is uniformly bounded from below. For example, if L is the Ornstein-Uhlenbeck operator on the Wiener space, then $Ric(L) = I_H$ (the identity operator on the Cameron-Martin space H). This enables us to recapture the P.A. Meyer inequality from Theorem 1.1 by taking $a = 0$ in (1.1).

Complete non-compact Riemannian manifolds are not necessarily to be those on which the Ricci curvature should be uniformly bounded from below. For example, one can easily construct a Cartan-Hadamard manifold and a rotational symmetric Riemannian manifold on which the Ricci curvature is not uniformly bounded from below. Moreover, starting from a complete non-compact Riemannian manifold (M, g) on which the Ricci curvature is uniformly bounded from below, we can use a conformal transformation to construct a new Riemannian metric $\widetilde{g}$ on M such that the Ricci curvature on $(M, \widetilde{g})$ is no longer uniformly bounded from below. Indeed, if $\widetilde{g}$ is a conformal change of g given by

$$\widetilde{g} = e^u g, \quad u \in C^2(M),$$

then, see e.g. Schoen and Yau [60], the Ricci curvature on $(M, \widetilde{g})$ is given by

$$(1.3)\quad Ric_{\widetilde{g}} = Ric_g - \frac{n-2}{2}\nabla^2 u + \frac{n-2}{4}\nabla u \otimes \nabla u - \frac{1}{2}\left(\Delta u + \frac{n-2}{2}|\nabla u|^2\right) g.$$

Hence, we can easily choose $u \in C^2(M) \setminus C_b^2(M)$ such that the Ricci curvature $Ric_{\widetilde{g}}$ corresponding to the new Riemannian metric $\widetilde{g}$ is not uniformly bounded from below eventhough Ric_g is uniformly bounded from below. In general, we can use a quasi-isometry to construct a new Riemannian metric $\widetilde{g}$ on M in the sense that there exist two constants c and C such that

$$cg \leq \widetilde{g} \leq Cg\,.$$

It this case, there is no explicit relationship between Ric_g and $Ric_{\widetilde{g}}$ but it is well-known that, even if the Ricci curvature Ric_g on (M, g) is uniformly bounded from below, the Ricci curvature $Ric_{\widetilde{g}}$ on $(M, \widetilde{g})$ is usually not uniformly bounded from below.

Similarly, there are many examples of diffusion operators L for which the Bakry-Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ is not uniformly bounded from below. As the most simple example, we consider the one-dimensional diffusion operator

$$L = \frac{d^2}{dx^2} - \phi'(x)\frac{d}{dx} \tag{1.4}$$

on the real line, where $\phi \in C^2(\mathbb{R})$. The Bakry-Emery Ricci curvature associated with L is

$$Ric_x(L) = \phi''(x), \quad \forall\, x \in \mathbb{R}\,.$$

Therefore, $Ric(L)$ is not uniformly bounded from below if we choose $\phi \in C^2(\mathbb{R})$ such that

$$\inf_{x\in\mathbb{R}} \phi''(x) = -\infty\,.$$

Moreover, starting from a symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ with Bakry-Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ uniformly bounded from below, we can use the conformal transformations to construct a new class of diffusion operators

$$\widetilde{L} = L + \nabla u \cdot \nabla, \quad u \in C^2(M)\,.$$

Then $\widetilde{\mu} = e^{-u}\mu$ is an invariant measure of $\widetilde{L}$ and the Bakry-Emery Ricci curvature of $\widetilde{L}$ is

$$Ric(\widetilde{L}) = Ric(L) + \nabla^2 u\,.$$

Hence, we can easily choose $u \in C^2(M)\backslash C_b^2(M)$ such that $Ric(\widetilde{L})$ is no longer uniformly bounded from below. In general, we can use the quasi-isometries to change either the Riemannian metric g or the density function $e^{-\phi}$ to obtain a new symmetric diffusion operator such that the Bakry-Emery Ricci curvature (of the new diffusion operator) is not uniformly bounded from below. Finally, let us mention that the Bakry-Emery Ricci curvatures associated with the Ornstein-Uhlenbeck diffusion operators on the path and the loop spaces (or the loop group) over a compact Riemannian manifold (or a compact non-Abelian Lie group) are not uniformly bounded from below, see Getzler [27] and Cruzeiro-Malliavin [18].

Due to the high importance of the Riesz transforms in potential theory will be illustrated in Section 3 below, it is very natural to ask whether or not the Riesz transforms associated with a general Markovian symmetric difusion operator $L = \Delta - \nabla\phi \cdot \nabla$ are bounded in L^p with respect to its invariant measure $\mu(dx) = e^{-\phi(x)}d\nu(x)$ even though the Bakry-Emery-Ricci curvature of L is not uniformly bounded from below. In [13], Coulhon and Duong have studied this problem for the case $1 < p \leq 2$ and proved that the

Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ (respectively, $R_0(L) = \nabla(-L)^{-1/2}$) is bounded in $L^p(M, \nu)$ (respectively, $L^p(M, \mu)$) for all $1 < p \leq 2$ provided that ν (respectively, μ) has the doubling volume property and the heat kernel of Δ (respectively, L) satisfies the inequality $p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}$ for all $x \in M$ and $t > 0$. Under a reasonable modification of these conditions, similar result holds for the L^p-boundedness of $R_a(L)$ for all $1 < p \leq 2$. Their result partially implies that the Riesz transform $R_0(\Delta)$ is bounded in $L^p(\nu)$ for $1 < p \leq 2$ if (M, g) is quasi-isometric to a complete Riemannian manifold with non-negative Ricci curvature. A complete non-compact Riemannian manifold (which is quasi-isometric to $\mathbb{R}^n$ but has a small piece of negative Ricci curvature) has been given in [13] which shows that the same conditions cannot imply the $L^p(\nu)$-boundedness of the Riesz transform $R_0(\Delta)$ for all $p > 2$. Indeed, it could be explicitly proved that it is not possible using duality argument to deduce from the results of Coulhon-Duong on the L^p-boundedness (for $p \in (1, 2]$) of Riesz transforms to the results for $p > 2$.

The purpose of this paper is to study the problem of the $L^p(\mu)$-boundedness of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ for $p \geq 2$ and $a \geq 0$ for a class of Markovian symmetric diffusion operators $L = \Delta - \nabla\phi \cdot \nabla$ on a complete Riemannian manifold (M, g) on which the Ricci curvature Ric or the Bakry-Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ is not necessarily uniformly bounded from below but satisfies some gaugeability or integrability conditions that we will explain in Section 2 below. Here we would like to point out that the (geodesically) completeness condition is only used to ensure that the Laplace-Beltrami operator Δ or the diffusion operator L is essentially self-adjoint in $L^2(\nu)$ or in $L^2(\mu)$, see e.g. [65, 5, 6].

In the case where M is not geodesically complete but the L-diffusion process does not explode (i.e., M is L-stochastically complete), such as the conic manifold (see e.g. [39]) or some infinite dimensional Riemaniann manifolds (see e.g. [55, 56, 62]), one can still study the L^p-boundedness of the Riesz transforms $R_a(L)$. While, the key semigroup domination inequalities (see Theorem 4.1) which play a crucial rôle in the whole paper cannot hold if one does not assume the stochastically completeness (i.e., the Markov property) of L (i.e. $P_t 1 = 1$ for all $t \geq 0$). Indeed, according to Page 254 in [23], the gaugeability in Theorem 2.1 or the $L^{\frac{n}{2}+\epsilon}$-integrability condition in Theorem 2.2 together with $dP_t f = e^{-t\Box_\phi} df$ for all $f \in C_c^\infty(M)$ (where $\Box_\phi$ is the Witten-Bismut operator on one-forms) will automatically imply that the L-diffusion process does not explode. By [51], it is well-known that the stochastically completeness of Brownian motion on complete non-compact Riemannian manifolds is not a stable property under general quasi-isometry. The above-mentioned counter-example due to Coulhon-Duong [13] shows that the L^p-boundedness of the Riesz transform

$R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ on complete non-compact Riemannian manifolds is not stable under general quasi-isometry at least for all $p \geq 2$ eventhough so is the Riesz transform $R_a(\Delta) = \nabla(a - \Delta)^{-1/2}$ for all $a > 0$ and all $p > 1$ on their manifold. We conjecture that the L^p-boundedness of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ (for *all* $a \geq 0$ or *all* $a > 0$) associated with a general symmetric diffusion operator L is not stable under general quasi-isometry on complete non-compact Riemannian manifolds. Our results (see Theorem 2.4, Theorem 2.5, Example 2.2 and Theorem 2.6 below) provide us with some examples of complete and stochastically complete Riemannian manifolds which are quasi-isometric to complete Riemannian manifolds with Ricci curvature bounded from below and with positive injectivity radius or to Cartan-Hadamard manifolds such that the Riesz transforms $R_a(L)$ are bounded in L^p for all $p \geq 2$ and all $a > 0$.

2. Main results

Throughtout this paper, let $L = \Delta - \nabla\phi\cdot\nabla$ be a Markovian symmetric diffusion operator with an invariant measure $d\mu(x) = e^{-\phi(x)}d\nu(x)$ on a complete Riemannian manifold (M, g).

Define the potential function K defined as follows: For all $x \in M$, let

$$K(x) := \inf\{< Ric(x)v, v > + < \nabla^2\phi(x)v, v >: \ v \in T_xM, \|v\| = 1\}, \quad (2.1)$$

Recall that L is a Markovian operator if it generates a Markovian semigroup $P_t = e^{tL}$ in the sense that $0 \leq P_tf \leq 1$ for all $f \in \mathcal{B}(M)$ with $0 \leq f \leq 1$ and $P_t1 = 1$ for all $t \geq 0$. Equivalently, the diffusion process $\{x_t\}$ generated by L on M has an infinite lifetime.

By Lemma A3 in [23], if there exists a constant $c \in \mathbb{R}^+$ such that the Ricci curvature on M satisfies $Ric(x) \geq -c[1+\rho^2(x)]$, and $d\rho(\nabla\phi(x)) \leq c[1+\rho(x)]$, where $\rho(x) = d(o, x)$ denotes the Riemannian distance function between x and a fixed point $o \in M$, then L is a Markovian symmetric diffusion operator. By Proposition A6 in [23], if we further suppose that

$$E_x\left[\sup_{s\leq t} \exp\left(-\int_0^s K(x_r)dr\right)\right] < +\infty, \ \forall\, x \in M, \ t > 0\,,$$

then $dP_tf(x) = e^{-t\Box_\phi}(df)(x)$ for all $x \in M, t > 0$ and for all $f \in C_c^\infty(M)$, where $\Box_\phi$ denotes the Witten-Bismut operator on one-forms with respect to μ (see Section 3 for details).

Using some new interplays between harmonic analysis, differential geometry and probability theory, we prove a criterion for the L^p-boundedness of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ for all $a > 0$ and $p \geq 2$.

Theorem 2.1 *Let $p > 2$, $q = \frac{p}{p-1}$, and $L = \Delta - \nabla\phi \cdot \nabla$ be a Markovian symmetric diffusion operator on a complete Riemannian manifold (M, g) with an invariant measure $d\mu(x) = e^{-\phi(x)}d\nu(x)$. Suppose that K^- satisfies the following gaugeability condition: for some $\beta > p$,*

$$\sup_{0\le t\le 1, x\in M} E_x\left[\exp\left(\beta\int_0^t K^-(x_s)ds\right)\right] < +\infty. \tag{2.2}$$

Then, for all $a > 0$ and all $p \in [2, \beta)$, the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(\mu)$. That is, there exists a constant $C_{p,a}$ such that

$$\|R_a(L)f\|_{L^p(M,\mu)} \le C_{p,a}\|f\|_{L^p(M,\mu)}, \qquad \forall\ f \in C_0^\infty(M).$$

Suppose further that

$$\sup_{t>0, x\in M} E_x\left[\exp\left(-\beta\int_0^t K(x_s)ds\right)\right] dt < +\infty, \tag{2.3}$$

$$\sup_{x\in M}\int_0^\infty E_x\left[K^-(x_t)\exp\left(-q\int_0^t K(x_s)ds\right)\right] dt < +\infty\,. \tag{2.4}$$

Then the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $p \in [2, \beta)$.

Note that the gaugeability condition (2.2) is weaker than (1.1) in Theorem 1.1. Indeed, (1.1) implies

$$\sup_{0\le t\le 1, x\in M} E_x\left[\exp\left(\beta\int_0^t K^-(x_s)ds\right)\right] \le e^{\beta a}\,.$$

Moreover, if $Ric + \nabla^2\phi \ge 0$, then (2.3) and (2.4) hold. Therefore, Theorem 2.1 partially extends Theorem 1.1 to a class of Markovian symmetric diffusion operators with unbounded Bakry-Emery-Ricci curvature for all $p \ge 2$.

By Theorem 2.1, we will find a more effective sufficient condition for the L^p-boundedness of Riesz transforms associated with a class of ultracontractive Markovian symmetric diffusion operators. To state it, let us first recall the notion of ultracontractivity.

Following E.B. Davies [19] and Bakry [7], we call (L, μ) a ultracontractive diffusion operator with dimension $n = dim(L)$ if there exists a constant $C > 0$ such that the semigroup $P_t = e^{tL}$ generated by L satisfies

$$\|P_t f\|_\infty \le Ct^{-n/2}\|f\|_{L^1(\mu)},\ \forall t \in (0, 1]. \tag{2.5}$$

By Corollaire 4.6 in Bakry [7], this is the case if there exists an increasing and concave fonction $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Phi'(x)$ is equivalent to $\frac{n}{2x}$ at $x = \infty$ and the so-called Φ-energy-entropy inequality holds

$$\forall f \in C_0^\infty(M), \|f\|_{L^2(\mu)} = 1 \Longrightarrow Ent_\mu(f^2) \leq \Phi\left(\|\nabla f\|^2_{L^2(\mu)}\right), \tag{2.6}$$

where

$$Ent_\mu(f^2) := \int_M f^2 \log f^2 d\mu - \int_M f^2 d\mu \log\left(\int_M f^2 d\mu\right).$$

When $n > 2$, a well-known result due to Varopoulos [70] (see also Davies [19] and Bakry [7]) says that the ultracontractivity (2.5) is equivalent to the following L^2-Sobolev inequality for (L, μ) with two constants A and B:

$$\|f\|^2_{L^{\frac{2n}{n-2}}(\mu)} \leq A\|\nabla f\|^2_{L^2(\mu)} + B\|f\|^2_{L^2(\mu)}, \quad \forall\, f \in C_0^\infty(M). \tag{2.7}$$

When $n \in (1, 2]$, if for some $p \in [1, n]$ there exist A_p and B_p such that (L, μ) satisfies the L^p-Sobolev inequality

$$\|f\|_{L^{\frac{pn}{n-p}}(\mu)} \leq A_p\|\nabla f\|_{L^p(\mu)} + B_p\|f\|_{L^p(\mu)}, \quad \forall\, f \in C_0^\infty(M), \tag{2.8}$$

or when $n \geq 1$, if there exists certain constant C such that the Nash inequality holds

$$\|f\|^{2+4/n}_{L^2(\mu)} \leq C\|\nabla f\|^2_{L^2(\mu)}\|f\|^{4/n}_{L^1(\mu)}, \quad \forall f \in C_0^\infty(M), \tag{2.9}$$

then (L, μ) is ultracontractive with dimensional $dim(L) = n$. Indeed, for $n \in [1, 2]$, if (L, μ) satisfies (2.8) or (2.9), then the Φ-energy-entropy inequality holds for $\Phi(x) = \frac{n}{2}\log(a + bx)$ with some suitable constants a and b. Moreover, it is well-known that the L^2-Sobolev and the L^p-Sobolev inequalities and the Nash inequality as well as the Φ-energy-entropy inequality are stable under all bounded conformal transformations and all quasi-isometries. For instance, if (2.7) holds with two constants A and B on a complete Riemannian manifold (M, g), and if $\widetilde{g}$ is a new Riemannian metric $\widetilde{g}$ on M such that $\widetilde{g}$ is quasi-isometric to g with

$$cg \leq \widetilde{g} \leq Cg,$$

then we can verify that (2.7) holds on $(M, \widetilde{g})$ with two new constants $\widetilde{A}$ and $\widetilde{B}$ given by

$$\widetilde{A} = \frac{AC^{\frac{n}{2}}}{c^{\frac{n+4}{2}}}, \qquad \widetilde{B} = \frac{BC^{\frac{n-2}{2}}}{c^{\frac{n}{2}}}.$$

Now we are able to state the main result of this paper as follows.

Theorem 2.2 *Let $L = \Delta - \nabla\phi \cdot \nabla$ be a Markovian symmetric diffusion operator on a complete Riemannian manifold (M, g) with an invariant measure $d\mu(x) = e^{-\phi(x)}d\nu(x)$, $\phi \in C^2(M)$.*

Suppose that L generates a ultracontractive semigroup $P_t = e^{tL}$ in the sense that there exist some constant $C > 0$ and $n \geq 1$ such that

$$\|P_t f\|_\infty \leq Ct^{-n/2}\|f\|_{L^1(\mu)}, \ \forall t \in (0, 1] .$$

Suppose further that there exist two constants $c \geq 0$ and $\epsilon > 0$ such that

$$(K + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \mu) .$$

Then, for all $p \geq 2$ and all $a > 0$, the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(M, \mu)$.

Applying Theorem 2.2 to the one-dimensional diffusion operator (1.4) and using a result due to Kavian-Kerkyacharian-Roynette [36], we have the following result. For its proof and its high dimensional extension, see Section 9.2.

Theorem 2.3 *Let $\phi \in C^2(\mathbb{R}, \mathbb{R})$, L be the one-dimensional diffusion operator given by*

$$L = \frac{d^2}{dx^2} - \phi'(x)\frac{d}{dx} .$$

Suppose that there exist some constants $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}, c_3 \in \mathbb{R}^+$ and $\epsilon > 0$ such that

$$\phi(x) \leq c_1, \quad \forall x \in \mathbb{R} ,$$

$$\frac{\phi'^2(x)}{4} - \frac{\phi''(x)}{2} + c_2 \geq 0, \quad \forall\ x \in \mathbb{R} ,$$

and

$$\int_{\mathbb{R}} \left([\phi''(x) + c_3]^-\right)^{\frac{1}{2}+\epsilon} e^{-\phi(x)}dx < +\infty .$$

Then, for all $p \geq 2$ and all $a > 0$, the Riesz transform $R_a(L) = \frac{d}{dx}\left(a - L\right)^{-1/2}$ is bounded in $L^p(\mathbb{R}, e^{-\phi(x)}dx)$.

Remark 2.1 Let L_0 be the diffusion operator on the real line satisfying all the conditions in Theorem 2.3, and L_1 be the Ornstein-Uhlenbeck diffusion operator on the Wiener space. Then the Bakry-Emery Ricci curvature of the diffusion operator $L = L_0 + L_1$ on the product manifold $\mathbb{R} \times W$ is given by

$$Ric_x(L) = \begin{pmatrix} \phi''(x_0) & 0 \\ 0 & I_H \end{pmatrix}, \quad \forall x = (x_0, x_1) \in \mathbb{R} \times W,$$

where I_H is the identity operator on the Cameron-Martin space

$$H = H^{1,2}([0, 1], \mathbb{R}^d) \cap W .$$

By Theorem 2.1, one can verify that, under the same conditions as in Theorem 2.3, the Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mathbb{R}\times W, e^{-\phi(x_0)}dx_0\otimes\mu(dx_1))$ for all $p\geq 2$ and all $a>0$, where μ is the Wiener measure on the Wiener space.

Example 2.1 By [4], if $L=\Delta-\nabla\phi\cdot\nabla$ is a diffusion operator on a complete Riemannian manifold (M,g) with $Ric(L)=Ric+\nabla^2\phi$ uniformly bounded from below, then L is conservative. Suppose further that L satisfies the so-called curvature-dimension $CD(K,n)$ condition with $K=-(n-1)k^2$ where $k\geq 0$, that is,

$$\Gamma_2(u,u)\geq \frac{1}{n}(Lu)^2+K|\nabla u|^2,\ \forall\ u\in C^\infty(M)\,,$$

where $\Gamma_2(u,u):=\frac{1}{2}L|\nabla u|^2-<\nabla Lu,u>$. By the Bochner-Weitzenböck formula, we can show that $\Gamma_2(u,u)=|\nabla^2 u|^2+<Ric(L)\nabla u,\nabla u>$. Thus, the $CD(K,n)$ condition holds when L is the Laplace-Beltrami operator Δ on a complete Riemannian manifold with $Ric\geq K=-(n-1)k^2$. By a new preprint of Bakry-Qian [8], $CD(K,n)$ implies $L\rho(x)\leq(n-1)k\coth(k\rho(x))$ on $M\setminus cut(o)$, where $\rho(x)=d(x,o)$, $cut(o)$ denotes the cut-locus of a fixed point $o\in M$. By standard argument, see for example Bakry-Qian [8], Gong-Wang [29] and Lott [50], this differential inequality yields the Bishop-Gromov type volume comparison inequality for the weight measure μ, that is, $\mu(B(x,r))\leq Cr^n\mu(B(x,r_0))e^{kr\sqrt{(n-1)}}$ for all $x\in M$ and $r>r_0>0$. Using the generalized Karp-Li [35] or Grigory'an [30] conservativeness criterion of diffusion semigroup due to K. T. Sturm [68], it is easy to see that for all $u\in C_b(M)\cap C^1(M)$, the diffusion operator $\widetilde{L}=L+\nabla u\cdot\nabla$ is still conservative. Combining this with the stability of Sobolev inequalities or Nash inequality under bounded quasi-isometry and using Theorem 2.2, we can easily obtain the following result: if $L=\Delta-\nabla\phi\cdot\nabla$ is a ultracontractive diffusion operator on a complete Riemannian manifold (M,g) with $dim(L)=n$ and if L satisfies the $CD(K,n)$ condition with $K=-(n-1)k^2$ where $k\geq 0$, then for all $u\in C_b(M)\cap C^2(M)$, the Riesz transform associated with the diffusion operator $\widetilde{L}=L+\nabla u\cdot\nabla$, namely, $R_a(\widetilde{L})=\nabla(a-\widetilde{L})$, is bounded in $L^p(M,e^{-u}\mu)$ for all $a>0$ and $p\geq 2$ provided that for some $c\geq 0$ and $\epsilon>0$,

$$\int_M\left[\left(\lambda_{\min}(\nabla^2 u(x))+c\right)^-\right]^{\frac{n}{2}+\epsilon}d\mu(x)<+\infty\,,$$

where $\lambda_{\min}(\nabla u^2(x))$ denotes the lowest eigenvalue of $\nabla^2 u(x)$, $\forall x\in M$. See also Example 2.2 below for the particular case of conformal change of Riemannian metric on $\mathbb{R}^2$.

Below we focus on the Riesz transforms associated with the Laplace-Beltrami operator. Recall that, if the Ricci curvature on (M, g) satisfies $Ric(x) \geq -c[1 + \rho^2(x)]$ (where $\rho(x) = d(o, x)$ for a fixed $o \in M$), or if (M, g) is quasi-isometric to a complete Riemannian manifold with Ricci curvature bounded from below by a constant, then the well-known results due to Li [42], Karp-Li [35] and Grigor'yan [30] imply that (M, g) is stochastically complete. That is, the Brownian motion on M has an infinite lifetime. By Hoffman-Spruck [33], if (M, g) is (or is quasi-isometric to) a Cartan-Hadamard manifold (i.e., a complete, connected, simply connected Riemannian manifold with negative sectional curvature), then the Sobolev inequality holds for (Δ, ν) with $B = 0$. Moreover, see e.g. Hebey [32], if (M, g) is (or is quasi-isometric to) a complete Riemannian manifold with positive injectivity radius and Ricci curvature uniformly bounded from below, then the Sobolev inequality holds for (Δ, ν) with some constants A and B. Let

$$K_0(x) := \inf\{< Ric(x)v, v >: \ v \in T_xM, \|v\| = 1\}, \quad \forall x \in M\,.$$

Applying Theorem 2.2 to $L = \Delta$ and $\mu = \nu$, we obtain the following

Theorem 2.4 *Let (M, g) be a complete Riemannian manifold which is quasi-isometric to a complete Riemannian manifold with positive injectivity radius and Ricci curvature uniformly bounded from below. Suppose that for some constants $c \geq 0$ and $\epsilon > 0$,*

$$(K_0 + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \nu)\,.$$

Then $R_a(\Delta) = \nabla(a - \Delta)^{-1/2}$ is bounded in $L^p(M, \nu)$ for any $p \geq 2$ and $a > 0$.

Theorem 2.5 *Let (M, g) be a Cartan-Hadamard manifold with Ricci curvature $Ric(x) \geq -c[1 + \rho(x)^2]$, where $\rho(x) = d(o, x)$ for some fixed point $o \in M$. Suppose that for some constants $c \geq 0$ and $\epsilon > 0$,*

$$(K_0 + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \nu)\,.$$

Then $R_a(\Delta) = \nabla(a - \Delta)^{-1/2}$ is bounded in $L^p(M, \nu)$ for any $p \geq 2$ and $a > 0$.

Example 2.2 Consider $(M, g) = (\mathbb{R}^2, e^u g_0)$ endowed with a Riemannian metric $g = e^u g_0$ which is conformal to the standard Euclidean metric g_0. By Theorem 2.4 and (1.3), we can prove that, see Section 7.1 for details, if $u \in C^2(\mathbb{R}^2) \cap C_b(\mathbb{R}^2)$ satisfies

$$\int_{\mathbb{R}^2} \left[(c - \Delta_0 u(x))^-\right]^{1+\epsilon} dx < +\infty$$

for some constants $c \geq 0$ and $\epsilon > 0$, where Δ_0 denote the usual Laplace operator on $\mathbb{R}^2$, then for all $a > 0$ and all $p \geq 2$, the Riesz transform $R_a(\Delta) = \nabla(a-\Delta)^{-1/2}$ is bounded in $L^p(\mathbb{R}^2, e^{u(x)}dx)$, where ∇ and Δ denote the Riemannian gradient operator and the Laplace-Beltrami operator on $(\mathbb{R}^2, e^{u(x)}g_0)$. This can be regarded as a special case of Example 2.1.

By Theorem 2.4, and using an idea originally due to N. Lohoué [48] and re-formulated recently in Coulhon-Duong [13], see Theorem 3.3 below, we can prove the following

Theorem 2.6 *Let (M, g) be a Cartan-Hadamard manifold with sectional curvature* Sect $\leq -k < 0$ *and Ricci curvature $Ric(x) \geq -c[1+\rho^2(x)]$, where $\rho(x) = d(o, x)$ for a fixed point $o \in M$. Suppose that there exist two constants $c \geq 0$ and $\epsilon > 0$ such that*

$$(K_0 + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \nu)\,.$$

Then $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ is bounded in $L^p(M, \nu)$ for any $p \geq 2$ and $a > 0$.

Remark 2.2 In [48], N. Lohoué proved that: Let M be a Cartan-Hadamard manifold on which the (Riemannian) curvature tensor and its first and second order covariant derivatives are bounded. Suppose that there exists a constant $C > 0$ such that $\|f\|_{L^2(\nu)} \leq C\|\Delta f\|_{L^2(\nu)}$ (i.e., the Laplace-Beltrami operator is strictly negative in $L^2(M, \nu)$). Then for any vector fields X satisfying $\|X(x)\| = 1$, $\forall x \in M$, the Riesz transform $X(-\Delta)^{-1/2}$ is bounded in $L^p(M, \nu)$ for all $1 < p < \infty$. As pointed out in [48] (p. 164), the most important step to prove this result is to show that there exist two constants $C_1(p)$ and $C_2(p)$ such that

$$\|\nabla f\|_{L^p(\nu)} \leq C_1(p)\|\sqrt{-\Delta} f\|_{L^p(\nu)} + C_2(p)\|f\|_{L^p(\nu)}, \ \forall f \in C_0^\infty(M). \tag{2.10}$$

This is equivalent to say that there exists a certain $a > 0$ such that the Riesz transform $R_a(\Delta) = \nabla(a-\Delta)^{-1/2}$ is bounded in $L^p(M, \nu)$. Notice that the boundedness of the Riemannian curvature tensor implies that the Ricci curvature is bounded from below. Therefore, (2.10) can be obtained from Theorem 1.1.

Compare with N. Lohoué's theorem, Theorem 2.6 holds for all $p \geq 2$ on any Cartan-Hadamard manifold on which the Ricci curvature is not necessarily to be uniformly bounded from below but $Ric(x) \geq -c[1 + \rho^2(x)]$ and K_0 (the lowest eigenvalue of the Ricci curvature) satisfies the condition $(K_0+c)^- \in L^{\frac{n}{2}+\epsilon}$ for certain $c \geq 0$ and $\epsilon > 0$. In other words, we do not need to assume that the Riemannian curvature tensor and its first two covariant derivatives are bounded.

By the second part of Theorem 2.1 and combining two arguments due to Rosenberg-Yang [59] and Davies-Simon [20] for the positivity of the principal eigenvalue and the Lyapunov exponent of the Schrödinger operators, we can prove the following result.

Theorem 2.7 *Let $L = \Delta - \nabla\phi \cdot \nabla$ be a Markovian symmetric diffusion operator on a complete Riemannian manifold (M, g) on which there exist two constants A and B such that*

$$\|f\|^2_{L^{\frac{2n}{n-2}}(M,\mu)} \leq A\|\nabla f\|^2_{L^2(M,\mu)} + B\|f\|^2_{L^2(M,\mu)}, \quad \forall\, f \in C_0^\infty(M)\,.$$

Suppose that $n > 4$ and there exist some constants $\beta > 2$, $c \geq 0$, $\alpha > 0$ and $\epsilon > 0$ such that

$$(K + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \mu)$$

and

$$\|(K - \alpha)^-\|_{L^{\frac{n}{2}}(\mu)} < \min\{(\beta A)^{-1}, \alpha B^{-1}\}\,.$$

Then, for all $p \in [2, \beta)$, the Riesz transform

$$R_0(L) = \nabla(-L)^{-1/2}$$

is bounded in $L^p(M, \mu)$.

Remark 2.3 Taking $\phi \equiv 0$, Theorem 2.7 provides us with another non-trivial example of complete and stochastically Riemannian manifolds on which the Ricci curvature is not uniformly bounded below while the Riesz transform $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ is bounded in $L^p(\nu)$ for all $p \in [2, \beta)$. For details, see Theorem 9.1 in Section 9.1.

The rest part of this paper is organised as follows: In Section 3 we give some applications of the L^p-continuity of the Riesz transforms. In Section 4 we prove some semigroup domination inequalities and the Littlewood-Paly inequalities on functions and on one-forms. In Section 5 we deal with the potential theory of Schrödinger operator. In Section 6 we prove Theorem 2.1 and a useful criterion for the $L^p(\mu)$-boundedness of $R_0(L) = \nabla(-L)^{-1/2}$ for all $p \geq 2$. In Section 7 we prove Theorem 2.2. In Section 7 we prove Theorem 2.7. In Section 9, we prove Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6 and describe how to construct a one-dimensional diffusion operator

$$L = \frac{d^2}{dx^2} - \phi'(x)\frac{d}{dx}$$

with unbounded negative Bakry-Emery Ricci curvature and satisfying the conditions required in Theorem 2.3.

3. Applications

In this section we give some basic properties of the Riesz transforms and some important applications of the L^p-continuity of the Riesz transforms. Here we state the results under the assumption that $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$.

If $R_a(L)$ is bounded in $L^p(\mu)$ for a fixed $p \in (1,\infty)$, the Marcinkiewicz interpolation theorem implies that $R_a(L)$ is bounded in $L^q(\mu)$ for all $q \in [\min\{2,p\}, \max\{2,p\}]$. In this case, the reader can easily reformulate these results in a similar way by himself.

The following result is well-known to experts and is the starting point to develop a nice L^p-potential theory as we will explain later.

Theorem 3.1 *The following statements are equivalent:*

(1) *The Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$. That is, for every $1 < p < \infty$, there exists a constant $\|R_a(L)\|_{p,p} \in (0,\infty)$ such that*

$$\|R_a(L)f\|_{L^p(\mu)} \le \|R_a(L)\|_{p,p}\|f\|_{L^p(\mu)}, \quad \forall f \in C_0^\infty(M)\,.$$

(2) *The Sobolev norms $\|\cdot\|_{1,p}$ and $|||\cdot|||_{1,p}$ are equivalent, where*

$$\begin{aligned} \|f\|_{1,p} &= \sqrt{a}\|f\|_{L^p(\mu)} + \|\nabla f\|_{L^p(\mu)}\,, \\ |||f|||_{1,p} &= \|(a-L)^{1/2}f\|_{L^p(\mu)}\,. \end{aligned}$$

(3) *The domain of the Cauchy operator $\sqrt{a-L}$ in $L^p(M,\mu)$ coincides with the domain of the closure of the gradient operator ∇ in $L^p(M,\mu)$. That is, the Sobolev space $H^{1,p}(M,\mu)$ coincides with the Sobolev space $W^{1,p}(M,\mu)$, where*

$$\begin{aligned} H^{1,p}(M,\mu) &= \{u \in L^p(M,\mu) : |\nabla u| \in L^p(\mu)\}\,, \\ W^{1,p}(M,\mu) &= (a-L)^{-1/2}(L^p(\mu))\,. \end{aligned}$$

(4) *The $(1,p)$-capacities $c_{1,p}$ and $C_{1,p}$ are equivalent, where*

$$\begin{aligned} c_{1,p}(O) &= \inf\{\|f\|_{1,p}^p : f \in H^{1,p}(M,\mu),\ f \ge 0,\ f \ge 1\ \mu - a.s. \text{ on } O\}\,, \\ C_{1,p}(O) &= \inf\{|||f|||_{1,p}^p : f \in W^{1,p}(M,\mu), f \ge 0, f \ge 1\ \mu - a.s. \text{ on } O\}\,, \end{aligned}$$

for all open sets $O \subset M$, and for all $A \subset M$,

$$\begin{aligned} c_{1,p}(A) &= \inf\{c_{1,p}(O) : A \subset O \subset M, O \text{ is open}\}\,, \\ C_{1,p}(A) &= \inf\{C_{1,p}(O) : A \subset O \subset M, O \text{ is open}\}\,. \end{aligned}$$

Proof. The proof of the equivalence between (2), (3) and (4) is easy and is omitted. Here we only prove (1) $\Longleftrightarrow$ (2). First, we prove (2) $\Longrightarrow$ (1). Indeed, if (2) is true, then for some constant $A_{p,a}$,

$$\|\nabla f\|_p \le \|f\|_{1,p} \le A_{p,a} |||f|||_{1,p} = A_{p,a} \|\sqrt{a-L} f\|_p.$$

By density argument, we can replace f by $\sqrt{a-L}f$ in the above inequality. Thus $\|R_a(L)f\|_p \le A_{p,a}\|f\|_p$ and (1) is proved.

Next, we prove (1) $\Longrightarrow$ (2). Suppose that $R_a(L)$ is bounded in $L^p(\mu)$ for all $p > 1$. Then

$$\|\nabla f\|_p \le \|R_a(L)\|_{p,p} \|\sqrt{a-L} f\|_p\,.$$

By Lemma 4.2 in Bakry [5], there exist two constants c_p and C_p such that

$$c_p\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right) \le \|\sqrt{a-L}f\|_p \le C_p\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right). \tag{3.1}$$

Hence $\|\nabla f\|_p \le C_p \|R_a(L)\|_{p,p}\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right)$. This yields

$$\begin{aligned} \|f\|_{1,p} &\le (1 + C_p\|R_a(L)\|_{p,p})\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right) \\ &\le c_p^{-1}(1 + C_p\|R_a(L)\|_{p,p})|||f|||_{1,p}\,. \end{aligned}$$

This is to say $\|f\|_{1,p} \le B_{p,a}|||f|||_{1,p}$, where $B_{p,a} := c_p^{-1}(1 + C_p\|R_a(L)\|_{p,p})$.

On the other hand, using the duality argument as used in the proof of Corollaire 4.3 in [5], for $q = \frac{p}{p-1}$,

$$\begin{aligned} \|\sqrt{a-L}f\|_q &\le \sup\{\|f\|_{1,q}\|g\|_{1,p} :\ g \in C_0^\infty(M),\ \|\sqrt{a-L}g\|_p \le 1\} \\ &\le B_{p,a} \sup\{\|f\|_{1,q}|||g|||_{1,p} :\ g \in C_0^\infty(M),\ \|\sqrt{a-L}g\|_p \le 1\} \\ &= B_{p,a}\|f\|_{1,q}\,. \end{aligned}$$

Hence $|||f|||_{1,q} \le B_{p,a}\|f\|_{1,q}$. Combining this with the inverse inequality, we have

$$B_{q,a}^{-1}|||f|||_{1,p} \le \|f\|_{1,p} \le B_{p,a}|||f|||_{1,p}, \quad \forall f \in C_0^\infty(M)\,.$$

The proof of Theorem 3.1 is completed. ■

The following result is very useful. It seems that one cannot find it in the literature.

Theorem 3.2 *Let* $p \in (1,\infty)$. *Suppose that for some* $a \ge 0$, $R_a(L) = \nabla(a-L)^{-1/2}$ *is bounded in* $L^p(\mu)$. *Then* $R_b(L) = \nabla(b-L)^{-1/2}$ *is bounded in* $L^p(\mu)$ *for all* $b > \min\{a, 0\}$. *Moreover, there exists a constant* A_p *such that*

$$\|R_b(L)\|_{p,p} \le A_p \max\{\sqrt{ab^{-1}}, 1\}\|R_a(L)\|_{p,p}\,.$$

Proof. Indeed, saying that $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$ is equivalent to saying that

$$\|\nabla f\|_p \leq \|R_a(L)\|_{p,p}\|(a-L)^{1/2}f\|_p \quad \text{with} \quad \|R_a(L)\|_{p,p} < +\infty\,.$$

By (3.1), if $R_a(L)$ is bounded in $L^p(\mu)$, then for all $b \geq a$,

$$\begin{aligned}
\|\nabla f\|_p &\leq \|R_a(L)\|_{p,p}C_p\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right) \\
&\leq \|R_a(L)\|_{p,p}C_p\left(\sqrt{b}\|f\|_p + \|\sqrt{-L}f\|_p\right) \\
&\leq \|R_a(L)\|_{p,p}c_p^{-1}C_p\|(b-L)^{1/2}f\|_p\,.
\end{aligned}$$

This yields, for all $b \geq a$,

$$\|R_b(L)\|_{p,p} \leq c_p^{-1}C_p\|R_a(L)\|_{p,p}\,.$$

On the other hand, for $a > b > 0$,

$$\begin{aligned}
\|\nabla f\|_p &\leq \|R_a(L)\|_{p,p}C_p\left(\sqrt{a}\|f\|_p + \|\sqrt{-L}f\|_p\right) \\
&\leq \|R_a(L)\|_{p,p}C_p\sqrt{\frac{a}{b}}\left(\sqrt{b}\|f\|_p + \sqrt{\frac{b}{a}}\|\sqrt{-L}f\|_p\right) \\
&\leq \|R_a(L)\|_{p,p}C_p\sqrt{\frac{a}{b}}\left(\sqrt{b}\|f\|_p + \|\sqrt{-L}f\|_p\right) \\
&\leq \|R_a(L)\|_{p,p}c_p^{-1}C_p\sqrt{\frac{a}{b}}\|(b-L)^{1/2}f\|_p.
\end{aligned}$$

Hence

$$\|R_b(L)\|_{p,p} \leq c_p^{-1}C_p\sqrt{ab^{-1}}\|R_a(L)\|_{p,p}\,.$$

Taking $A_p = c_p^{-1}C_p$, the proof of Theorem 3.2 is completed. ■

The following result is due to N. Lohoué [48], see also Coulhon-Duong [13, p. 1154].

Theorem 3.3 *Let $p > 1$ be fixed. Suppose that the Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$ for some $a > 0$, and the bottom of spectrum of $-L$ in $L^2(\mu)$ is strictly positive, i.e.,*

$$\lambda_2(-L) := \inf_{f\in L^2(\mu)\setminus\{0\}} \frac{\int_M |\nabla f(x)|^2 d\mu(x)}{\|f\|_2^2} > 0\,.$$

Then the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded in $L^p(\mu)$.

Now we give an important application of the L^p-continuity of the Riesz transforms in the regularity theory of parabolic PDEs (the heat semigroup and the Poisson semigroup).

Theorem 3.4 *Suppose that the Riesz transform $R_a(L)$ is bounded in $L^p(\mu)$. Then*[1]

(1) *there exist two constants A_p and C_p such that*

$$\|\nabla e^{tL} f\|_p \leq C_p \|R_a(L)\|_{p,p} \left(\sqrt{a} + \frac{A_p}{\sqrt{t}}\right) \|f\|_p, \quad \forall\, t > 0\,.$$

(2) *there exists a constant $A_p > 0$ such that, for any $\epsilon \in (0,1)$,*

$$\|\nabla e^{tL} f\|_p \leq \frac{A_p \|R_a(L)\|_{p,p}}{\sqrt{\epsilon}} \frac{e^{a\epsilon t}}{\sqrt{t}} \|f\|_p, \quad \forall\, t > 0\,.$$

(3) *if the heat semigroup e^{tL} is hypercontractive, i.e., $\|e^{tL}\|_{p,q(t)} = m_{p,q}(t)$, then*

$$\|\nabla e^{tL}\|_p \leq \frac{C_p \|R_a(L)\|_{p,p}}{\sqrt{\epsilon}} \frac{e^{a\epsilon t} m_{p,q}((1-\epsilon)t)}{\sqrt{t}} \|f\|_q\,.$$

(4) *there exists a constant $A_p > 0$ such that, for any $\epsilon \in (0,1)$, and any $t > 0$,*

$$\|\nabla e^{-t\sqrt{a-L}} f\|_p \leq \frac{A_p \|R_a(L)\|_{p,p}}{\epsilon} \frac{e^{-\sqrt{a}\epsilon t}}{t} \|f\|_p\,.$$

Proof. From the proof of Theorem 3.1, for the same constant C_p as in (3.1), we have

$$\|\nabla f\|_p \leq C_p \|R_a(L)\|_{p,p} \left(\|\sqrt{-L} f\|_p + \sqrt{a}\|f\|_p\right)\,.$$

Replacing f by $e^{tL} f$ we obtain

$$\|\nabla e^{tL} f\|_p \leq C_p \|R_a(L)\|_{p,p} \left(\|\sqrt{-L} e^{tL} f\|_p + \sqrt{a}\|e^{tL} f\|_p\right).$$

Since L is a sub-Markovian operator, the semigroup e^{tL} is analytic. Hence there exists a constant A_p such that

$$\|\sqrt{-L} e^{tL} f\|_p = \|\sqrt{-L} e^{\frac{t}{2}L} e^{\frac{t}{2}L} f\|_p \leq A_p t^{-1/2} \|e^{\frac{t}{2}L} f\|_p.$$

[1] In a recent paper [1], for the case where $L = \Delta$ and $\mu = \nu$, P. Auscher, T. Coulhon, X.T. Duong and S. Hoffman have shown that the L^p-regularity property of e^{tL} together with the exponential growth condition of the geodesic ball is indeed a sufficient condition for the L^p-boundedness of $R_a(\Delta) = \nabla(a - \Delta)^{-1/2}$.

Moreover, the semigroup e^{tL} is contractive in L^p for all $1<p<\infty$. Hence

$$\begin{aligned}\|\nabla e^{tL}f\|_p &\le C_p\|R_a(L)\|_{p,p}\left(\frac{A_p}{\sqrt{t}}\|e^{\frac{t}{2}L}f\|_p+\sqrt{a}\|e^{tL}f\|_p\right)\\ &\le C_p\|R_a(L)\|_{p,p}\left(\frac{A_p}{\sqrt{t}}+\sqrt{a}\right)\|f\|_p\,.\end{aligned}$$

Similarly, since the semigroup $e^{-t(a-L)}$ is analytic and since e^{tL} is L^p-contractive, for $\epsilon\in(0,1)$, there exists a constant A_p such that, for every $\epsilon\in(0,1)$,

$$\begin{aligned}\|\nabla e^{-t(a-L)}f\|_p &\le \|R_a(L)\|_{p,p}\|\sqrt{a-L}e^{-t(a-L)}f\|_p\\ &= \|R_a(L)\|_{p,p}\|\sqrt{a-L}e^{-\epsilon t(a-L)}e^{-t(1-\epsilon)(a-L)}f\|_p\\ &\le \|R_a(L)\|_{p,p}\frac{A_p}{\sqrt{\epsilon t}}\|e^{-t(1-\epsilon)(a-L)}f\|_p\\ &\le \frac{A_p\|R_a(L)\|_{p,p}}{\sqrt{\epsilon t}}e^{-a(1-\epsilon)t}\|e^{(1-\epsilon)tL}f\|_p\\ &\le \frac{A_p\|R_a(L)\|_{p,p}}{\sqrt{\epsilon t}}e^{-a(1-\epsilon)t}\|f\|_p\,.\end{aligned}$$

Hence

$$\|\nabla e^{tL}f\|_p\le\frac{A_p\|R_a(L)\|_{p,p}}{\sqrt{\epsilon}}\frac{e^{a\epsilon t}}{\sqrt{t}}\|f\|_p\,.$$

Moreover, if $\|e^{tL}\|_{p,q}=m_{p,q}(t)$, then

$$\|\nabla e^{tL}f\|_p\le\frac{A_p\|R_a(L)\|_{p,p}}{\sqrt{\epsilon}}\frac{e^{a\epsilon t}m_{p,q}((1-\epsilon)t)}{\sqrt{t}}\|f\|_q\,.$$

On the other hand, the analyticity of $e^{-t\sqrt{a-L}}$ implies, for some $A_p>0$ and for all $\epsilon\in(0,1)$,

$$\begin{aligned}\left\|\nabla e^{-t\sqrt{a-L}}f\right\|_p &\le \|R_a(L)\|_{p,p}\left\|\sqrt{a-L}e^{-t\sqrt{a-L}}f\right\|_p\\ &\le \|R_a(L)\|_{p,p}\frac{A_p}{(1-\epsilon)t}\|e^{-t\epsilon\sqrt{a-L}}f\|_p\\ &\le \frac{A_p\|R_a(L)\|_{p,p}}{1-\epsilon}\frac{e^{-t\epsilon\sqrt{a}}}{t}\|f\|_p,\end{aligned}$$

where in the last step we have used the following estimate

$$\begin{aligned}\left\|e^{-t\sqrt{a-L}}f\right\|_p &= \left\|\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}(a-L)}f(\cdot)e^{-u}u^{-1/2}du\right\|_p\\ &\le \frac{1}{\sqrt{\pi}}\int_0^\infty \left\|e^{-\frac{t^2}{4u}(a-L)}f\right\|_p e^{-u}u^{-1/2}du\\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{at^2}{4u}}\left\|e^{\frac{t^2}{4u}L}f\right\|_p e^{-u}u^{-1/2}du\\ &\le \frac{1}{\sqrt{\pi}}\left(\int_0^\infty e^{-\frac{at^2}{4u}}e^{-u}u^{-1/2}du\right)\|f\|_p\\ &= e^{-t\sqrt{a}}\|f\|_p\,.\end{aligned}$$

The proof of Theorem 3.4 is completed. ■

In the case $M=\mathbb{R}^n$, the L^p-boundedness of the Riesz transforms and the Sobolev inequalities yield the regularity of the solution to Poisson equation $\Delta u=f$, see e.g. E.M. Stein [63] and Gilbarg-Trudinger [28]. Similarly, we have the following result for a general ultracontractive diffusion operator on a complete Riemannian manifold.

Theorem 3.5 *Suppose that there exists a constant $C_1>0$ such that*

$$\|e^{tL}\|_{1,\infty}\le C_1t^{-n/2},\quad \forall\ t>0\,.$$

For a fixed $p<n$, suppose that $R_0(L)$ is bounded in $L^q(\mu)$ for $q=\frac{pn}{n-p}$. Let $f\in L^p(M)$ satisfying $\int_M fd\mu=0$, u be a solution to the Poisson equation

$$Lu=f\,.$$

Then there exists a constant $C=C_{p,n,C_1}$ such that

$$\|\nabla u\|_{\frac{pn}{n-p}}\le C\|f\|_p\,.$$

Proof. Write

$$\nabla u=\nabla L^{-1}f=\nabla(-L)^{-1/2}(-L)^{-1/2}f=R_0(L)(-L)^{-1/2}f.$$

By [71],

$$\|e^{tL}\|_{1,\infty}\le C_1t^{-n/2}\qquad\text{for all } t>0$$

implies $\|(-L)^{-1/2}\|_{q,p}<+\infty$. Hence

$$\|\nabla u\|_q\le\|R_0(L)\|_{q,q}\|(-L)^{-1/2}\|_{p,q}\|f\|_p\,.$$

This finishes the proof of Theorem 3.5. ■

Below we describe some important applications of the L^p-continuity of the Riesz transforms in probability theory. Since the gradient operator ∇ is a local derivative operator while the square root operator $(a-L)^{1/2}$ is a global integral operator, it is easy to deal with the Sobolev norm $\|f\|_{1,p}$ and the $(1,p)$-capacity $c_{1,p}$ while usually it is very hard to deal with the Sobolev norm $|||f|||_{1,p}$ and the $(1,p)$-capacity $C_{1,p}$. For example, under some suitable conditions we have two effective approaches (see [58] [44]) to prove that $c_{1,p}$ is tight, that is, for any $\epsilon > 0$, there exists a compact subset $K \subset M$ such that $c_{1,p}(M \setminus K) \leq \epsilon$. Moreover, this is true if for any $\epsilon > 0$ there exists $f \in C_0^\infty(M, [0,1])$ such that $\mu(M \setminus \mathrm{supp} f) < \epsilon/2$ and $\|\nabla f\|_p \leq \epsilon/2$. However, usually it is very hard to prove the tightness of $C_{1,p}$ except for $p = 2$. If one can prove the tightness of $c_{1,p}$ by the above method and the approaches in [58] [44] and if one can further prove that the Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, then Theorem 3.4 implies that the $(1,p)$-capacity $C_{1,p}$ is also tight. In this case, we have the following nice properties:

(1) By Sugita [69] and Kazumi-Shigekawa [37], for every positivity-preserving distribution $\Phi \in W^{-1,q}(M,\mu)$, there exists a unique finite tight measure ν_Φ such that

$$\Phi(f) = \int_M f(x) d\nu_\Phi(x), \quad \forall\ f \in W^{1,p}(M,\mu)\,.$$

Here $W^{-1,q}(M,\mu)$ is the dual of $W^{1,p}(M,\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, for all open set $O \subset M$,

$$\nu_\Phi(O) \leq \|\Phi\|_{-1,q} \left(C_{1,p}(O)\right)^{1/p}. \tag{3.2}$$

In particular, $\nu_\Phi(A) = 0$ for any Borel set $A \in \mathcal{B}(M)$ with $C_{1,p}(A) = 0$.

(2) By Fukushima [26] and the references therein, the Hunt process associated with the Dirichlet form $\mathcal{E}(f,g) = \int_M |\nabla f|^2 d\mu$ can be constructed uniquely up to $C_{1,p}$-equivalence on M via the Dirichlet form theory if $C_{1,p}$ is tight.

As the last application of the Riesz transforms, we would like to mention that the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ (and its generalisations on k-forms) is naturally involved in the Hodge decomposition theorem with respect to the weight measure

$$d\mu(x) = e^{-\phi(x)} dv(x).$$

Let

$$\Box_{\phi,k} = dd_\phi^* + d_\phi^* d$$

be the Witten-de Rham-Hodge operator on k-forms, where d is the exterior differential operator, d_ϕ^* denotes its $L^2(\mu)$-adjoint operator. Suppose that the Green operators L^{-1} and $\Box_{\phi,k}^{-1}$ exist, $k = 1, 2$. Similarly to the well-known integral representation formulae in the standard Hodge decomposition theorem, see e.g. [21], we can prove that every 1-forms ω in $L^2(\mu)$ can be decomposed into

$$\omega = \omega_0 + \omega_1 + \omega_2\,,$$

where ω_1 is $\Box_{\phi,1}$-harmonic, i.e., $\Box_{\phi,1}\omega_1 = 0$, and

$$\begin{aligned}\omega_0 &= d(-L)^{-1}\delta_\phi\omega = d(-L)^{-1/2}(d(-L)^{-1/2})^*\omega = R_0(L)R_0^*(L)\omega,\\ \omega_2 &= \delta\Box_{\phi,2}^{-1}d\omega = \delta\Box_{\phi,2}^{-1/2}(\delta\Box_{\phi,2}^{-1/2})^*\omega = R_0(\Box_{\phi,2})R_0^*(\Box_{\phi,2})\omega\,.\end{aligned}$$

Here $R_0(L) = d(-L)^{-1/2}$ (we identify it with $\nabla(-L)^{-1/2}$), $R_0(\Box_{\phi,2}) = \delta\,\Box_{\phi,2}^{-1/2}$, $R_0^*(L)$ (respectively, $R_0^*(\Box_{\phi,2})$) denotes the $L^2(\mu)$-adjoint of $R_0(L)$ (respectively, $R_0(\Box_{\phi,2})$). Thus, we have the following result related to the L^p-Hodge decomposition theory.

Theorem 3.6 *Let $p > 1$, $q = \frac{p}{p-1}$. Suppose that the Riesz transforms*

$$R_0(L) = \nabla(-L)^{-1/2} \quad \text{and} \quad R_0(\Box_{\phi,2}) = \delta\Box_\phi^{-1/2}$$

are bounded in $L^p(\mu)$ and in $L^q(\mu)$. Then the harmonic projection

$$\omega \to \omega_1 := (I - R_0(L)R_0^*(L) - R_0(\Box_{\phi,2})R_0^*(\Box_{\phi,2}))\,\omega$$

is bounded in $L^p(\mu)$. That is, there exists a constant C_p such that

$$\|\omega_1\|_{L^p(\mu)} \leq C_p\|\omega\|_{L^p(\mu)}\,.$$

To obtain the L^p-boundedness of $R_0(\Box_{\phi,2})$, we need the Ricci curvature and the Weitzenböck curvature on 2-forms. Here we will not discuss this issue and refer the reader to Bakry [5] in which it was proved that $R_0(\Box_{\phi,2})$ is bounded in $L^p(\mu)$ provided that the Ricci curvature and the Weitzenböck curvature are non-negative. To end this section, we would like to mention that the L^p-(Helmotz)-Hodge decomposition theory plays an important rôle (via the Leray projection) in the study of the Navier-Stokes equations and boundary value problems, see e.g. [47, 61]. We also refer the reader to [65, 66, 34, 1] for the discussion of the L^p-Hodge decomposition theory related to the de Rham-Hodge operator on complete Riemaniann manifolds.

4. Littlewood-Paley inequalities

4.1. Semigroup domination inequalities

Let d be the exterior differential operator on $\Lambda(T^*M)$, d^* be the $L^2(v)$-adjoint of d, d^*_ϕ be the $L^2(\mu)$-adjoint of d. Then $d^*_\phi = d^* - i_{\nabla\phi}$, where $i_{\nabla\phi}$ is the interior multiplication by the vector field $X = \nabla\phi$. The Hodge-de Rham operator $\Box$ and the Witten-Bismut operator $\Box_\phi$ on one-forms are defined by

$$\begin{aligned}\Box &= dd^* + d^*d,\\ \Box_\phi &= dd^*_\phi + d^*_\phi d.\end{aligned}$$

Moreover, see e.g. Formula (2.5) in [25], we have $\Box_\phi = \Box - L_{\nabla\phi}$, where $L_{\nabla\phi}$ is the Lie derivative along the direction of $\nabla\phi$.

In the proof of Theorem 1.1, Bakry [5] used the following two semigroup domination inequalities: if $Ric(x) + \nabla^2\phi(x) \geq -a$, $\forall\ x \in M$, then for any $f \in C_0^\infty(M)$ and $\omega \in C_0^\infty(\Lambda^1(T^*(M)))$,

$$\left|e^{-t\Box_\phi}\omega(x)\right| \leq e^{at}e^{tL}\left|\omega\right|(x), \quad \forall\ x \in M, \tag{4.1}$$

$$\left|\nabla e^{tL}f(x)\right| \leq e^{at}e^{tL}|\nabla f|(x), \quad \forall\ x \in M. \tag{4.2}$$

These type semigroup domination inequalities cannot be expected if we do not assume that the Bakry-Emery Ricci curvature $Ric+\nabla^2\phi$ is uniformly bounded from below[2]. For this reason the study of the L^p-boundedness of the Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ for a diffusion operator L with unbounded negative Bakry-Emery Ricci curvature is very complicated. One of the most important ingredients of this paper is that we can replace (4.1) and (4.2) in an appropriate way such that our new semigroup domination inequalities (see Theorem 4.1 below) hold for any Markovian symmetric diffusion operator L on a complete Riemannian manifold on which the Bakry-Emery Ricci curvature is not necessarily to be uniformly bounded from below. Moreover, these new semigroup domination inequalities enable us to prove the L^p-boundedness of the Littlewood-Paly function $g_{a-L}(f)$ for all $p \in (1,\infty)$ and the L^p-boundedness of the Littlewood-Paly function $g_{a+\Box_\phi}$ for all $p \in (1, 2)$, which imply the L^p-boundedness of the Riesz transforms $R_a(L) = \nabla(a-L)^{-1/2}$ for all $p \geq 2$. In this subsection we will first recall the probabilistic representations of the heat semigroup $e^{-t\Box_\phi}$. Then we prove our new semigroup domination inequalities which will play a crucial rôle in the proof of Theorem 2.1.

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[2]By Bakry-Emery [2], for diffusion operator $L = \Delta-\nabla\phi\cdot\nabla$ on a complete Riemannian manifold M, (4.2) holds if and only if $Ric(x) - \nabla^2\phi(x) \geq -a$ for all $x \in M$.

Let $O(M)$ be the orthonormal frame bundle over M, $\pi : O(M) \to M$ be the projection. Let $A_1, \dots, A_n$ be the canonical horizontal vector fields on $O(M)$, X be the unique horizontal vector field on $O(M)$ such that

$$\pi_{*u}(X) = \nabla\phi(x), \quad \forall\, u \in \pi^{-1}(x)\,.$$

Then, for any $u \in O(M)$ with $\pi(x) = x$, there exists a unique non-explosive Markov process $\{u_t, t \in \mathbb{R}^+\}$ on $O(M)$ such that

$$\begin{aligned} du_t &= \sum_{i=1}^n A_i(u_t) \circ dw_t^i - X(u_t)dt \\ u_0 &= u. \end{aligned}$$

Moreover, $x_t = \pi(u_t)$ is a non-explosive Markovian diffusion process on M starting at x with infinitesimal generator $L = \Delta - \nabla\phi \cdot \nabla$. We call $\{u_t, t \in \mathbb{R}^+\}$ the stochastic horizontal lift of $\{x_t, t \in \mathbb{R}^+\}$ with respect to the Levi-Civita connection on (M, g).

Let E_x be the expectation with respect to the measure μ_x, the law of $\{x_t, t \in \mathbb{R}^+\}$ on the path space $P_x(M) = \{\gamma \in C(\mathbb{R}^+, M) : \gamma(0) = x\}$. Let $P_t = e^{tL}$, $P_t^1 = e^{-t\Box_\phi}$. Then for any $\omega \in \Lambda^1(T^*M)$ and $v \in T_xM$, see e.g. [53, 54, 22, 24, 5], it is well-known that

$$< P_t^1\omega, v > (x) = E_x\left[< \omega(x_s), v_t >\right], \tag{4.3}$$

where v_t is an $T_{x_t}M$-valued process defined by the following covariant differential equation:

$$\frac{D}{\partial t} v_t = -Ric_{x_t}(L)v_t, \quad v_0 = v. \tag{4.4}$$

Here

$$\frac{D}{\partial t} v_t = u_t \frac{d}{dt}(u_t^{-1} v_t)$$

denotes the Itô stochastic covariant derivative along the diffusion process $\{x_t\}$ on M. By [53, 22, 5, 24], we have

$$\|v_t\| \;\leq\; \|v\| \exp\left(-\int_0^t K(x_s)ds\right), \tag{4.5}$$

$$\left|P_t^1\omega(x)\right| \;\leq\; E_x\left[|\omega(x_t)| \exp\left(-\int_0^t K(x_s)ds\right)\right], \tag{4.6}$$

$$|\nabla P_t f(x)| \;\leq\; E_x\left[|\nabla f(x_t)| \exp\left(-\int_0^t K(x_s)ds\right)\right]. \tag{4.7}$$

For a reasonable function $V \in \mathcal{B}(M, \mathbb{R})$, the Schrödinger semigroup $P_t^V := e^{-t(-L+V)}$ and its associated Poisson semigroup $Q_t^V := e^{-t\sqrt{-L+V}}$ are defined as follows: for any $f \in \mathcal{B}(M, \mathbb{R}^+)$,

$$\begin{aligned} P_t^V f(x) &= E_x\left[f(x_s)\exp\left(-\int_0^t V(x_s)ds\right)\right], \ \forall x \in M, \\ Q_t^V f(x) &= \int_0^\infty m(t,s)P_s^V f(x)ds, \ \forall x \in M, \end{aligned}$$

where

$$m(t,s) = \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-\frac{t^2}{4s}}, \quad s > 0.$$

Let

$$Q_t := e^{-t\sqrt{-L}}, \qquad Q_t^1 := e^{-t\sqrt{\Box_\phi}}$$

be the Poisson semigroups associated with L and $\Box_\phi$.

Theorem 4.1 *For all $x \in M$, $\omega \in C_0^\infty(\Lambda^1(T^*M))$ and $f \in C_0^\infty(M)$, we have*

$$\left|P_t^1\omega(x)\right|^2 \leq P_t^{2K}1(x) \cdot P_t|\omega|^2(x), \tag{4.8}$$

$$|\nabla P_t f(x)|^2 \leq P_t^{2K}1(x) \cdot P_t|\nabla f|^2(x), \tag{4.9}$$

$$|\nabla Q_t f(x)|^2 \leq Q_t^{2K}1(x) \cdot Q_t(|\nabla f|^2)(x), \tag{4.10}$$

$$|\nabla Q_t f(x)|^2 \leq Q_{\frac{t}{2}}^{2K}1(x) Q_{\frac{t}{2}}\left(|\nabla Q_{\frac{t}{2}} f(\cdot)|^2\right)(x), \tag{4.11}$$

$$|\partial_t Q_t f(x)|^2 \leq 4Q_{\frac{t}{2}}\left(|\partial_t Q_{\frac{t}{2}} f(\cdot)|\right)^2(x). \tag{4.12}$$

Proof. By the Cauchy-Schwarz inequality, (4.6) (resp., (4.7)) implies (4.8) (resp., (4.9)). Note that $\int_0^\infty m(t,s)ds = 1$. By (4.9) and using the Cauchy-Schwarz inequality,

$$\begin{aligned} |\nabla Q_t f(x)|^2 &\leq \left[\int_0^\infty m(t,s)\sqrt{P_s^{2K}1(x) \cdot P_s|\nabla f|^2(x)}ds\right]^2 \\ &\leq \left(\int_0^\infty m(t,s)P_s^{2K}1(x)ds\right)\int_0^\infty m(t,s)P_s|\nabla f|^2(x)ds. \end{aligned}$$

This proves (4.10). Replacing (t, f) in (4.10) by $(t/2, Q_{t/2}f)$, we get (4.11). To prove prove (4.12), notice that

$$\frac{\partial}{\partial t}Q_t f(x) = \sqrt{-L}Q_t f(x) = Q_{t/2}\sqrt{-L}Q_{t/2}f(x).$$

The Poisson kernel of L is given by $k_t(x,y) = \int_0^\infty m(t,s)p_s(x,y)ds$, where $p_s(x,y)$ is the heat kernel of L. By Fubini's theorem and since

$\int_M p_s(x,y)dy = 1$, $\forall\ s > 0, x \in M$, we have $\int_M k_t(x,y)dy = 1$, $\forall\ t > 0$, $x \in M$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}\left|\frac{\partial}{\partial t}Q_t f(x)\right|^2 &= \left[\int_M \sqrt{-L}Q_{t/2}f(y)k_{t/2}(x,y)dy\right]^2 \\ &\le \int_M \left|\sqrt{-L}Q_{t/2}f(y)\right|^2 k_{t/2}(x,y)dy = 4Q_{t/2}\left[\left|\frac{\partial}{\partial t}Q_{t/2}f(x)\right|^2\right].\end{aligned}$$

The proof of Theorem 4.1 is completed. ■

4.2. The Littlewood-Paley inequality for g_{a-L}

Recall the definition of the Littlewood-Paley function g_{a-L}: for any $f \in C_0^\infty(M)$ and any constant $a \ge 0$,

$$g_{a-L}(\omega)(x) = \left[\int_0^\infty t\left(|\frac{\partial}{\partial t}e^{-t\sqrt{a-L}}f(x)|^2 + |\nabla e^{-t\sqrt{a-L}}f(x)|^2\right)dt\right]^{1/2}. \tag{4.13}$$

Here the Poisson semigroup $e^{-t\sqrt{a-L}}$ is given by

$$e^{-t\sqrt{a-L}}f(x) = \int_0^\infty m(t,s)e^{-s(a-L)}f(x)ds.$$

In this subsection we prove the following Littlewood-Paley inequality.

Theorem 4.2 *Let $L = \Delta - \nabla\phi\cdot\nabla$ be a Markovian symmetric diffusion operator on a complete Riemannian manifold (M,g) with an invariant measure $d\mu(x) = e^{-\phi(x)}dv(x)$, where $\phi \in C^2(M)$.*

(1) *For any $a \ge 0$ and $1 < p \le 2$, there exists a constant C_p such that*

$$\|g_{a-L}(f)\|_{L^p(\mu)} \le C_p\|f\|_{L^p(\mu)}.$$

(2) *Suppose that*

$$C_K = \sup_{x\in M, t>0} Q_t^{2K}1(x) < \infty. \tag{4.14}$$

Then for all $p > 2$, there exists a constant C_p such that

$$\|g_{a-L}(f)\|_{L^p(\mu)} \le C_p\|f\|_{L^p(\mu)}. \tag{4.15}$$

(3) *Under the same condition of* (2)*, for any $1 < p < \infty$, there exists a constant B_p such that for any $f \in L^p(\mu)$,*

$$\|f - E_0 f\|_{L^p(\mu)} \le B_p\|g_{a-L}(f)\|_{L^p(\mu)}, \tag{4.16}$$

where E_0 denotes the orthogonal projection from $L^2(\mu)$ onto

$$Ker(L) = \{u \in L^2(\mu) : Lu = 0\}.$$

Proof. By Stein [64], the horizontal Littlewood-Paley function

$$g_{1,a-L}(f)(x) := \left[\int_0^\infty t\left|\frac{\partial}{\partial t}e^{-t\sqrt{a-L}}f(x)\right|^2 dt\right]^{1/2}, \quad \forall\, x\in M,$$

is bounded in $L^p(\mu)$ for all $1<p<\infty$. Hence we need only to show the vertical Littlewood-Paley function

$$g_{2,a-L}(f)(x) := \left[\int_0^\infty t\left|\nabla e^{-t\sqrt{a-L}}f(x)\right|^2 dt\right]^{1/2}, \quad \forall\, x\in M,$$

is bounded in $L^p(\mu)$ for all $1<p\leq 2$ on any complete Riemannian manifold, and for $p>2$ on any complete Riemannian manifold satisfying the condition in (2).

To prove (1), we use Stein's argument as one has used in the proof of Theorem 1.4 in [16].

Let $f\in C_0^\infty(M)$, and set $u_a(x,t)=e^{-t(a-L)}f(x)$. One can assume that f is non-negative and non identically zero. Then by standard estimates u is smooth and positive everywhere. Let

$$H_a(f)(x) = \left[\int_0^{+\infty} |\nabla u_a(x,t)|^2 dt\right]^{1/2}.$$

Using the Bochner subordination formula

$$e^{-\sqrt{a-L}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}(a-L)}e^{-u}u^{-1/2}du,$$

one can prove (for details, see [16, p. 40])

$$g_{2,a-L}(f)(x)\leq CH_a(f)(x), \quad \forall\, x\in M.$$

Now, for any $1<p\leq 2$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-L\right)u_a^p(x,t) &= pu_a(x,t)^{p-1}\left(\frac{\partial}{\partial t}-L\right)u_a(x,t)-p(p-1)u(x,t)^{p-2}|\nabla u_a(x,t)|^2\\ &= -\,apu_a(x,t)^p - p(p-1)u(x,t)^{p-2}|\nabla u_a(x,t)|^2,\end{aligned}$$

which yields

$$|\nabla u_a(x,t)|^2 = -\frac{1}{p(p-1)}u_a(x,t)^{2-p}\left(\frac{\partial}{\partial t}-L+ap\right)u_a^p(x,t). \tag{4.17}$$

Hence

$$\begin{aligned}H_a^2(f)(x): &= \int_0^{+\infty} |\nabla u_a(x,t)|^2\, dt\\ &= -\,C_p\int_0^{+\infty} u_a(x,t)^{2-p}\left(\frac{\partial}{\partial t}-L+ap\right)u_a^p(x,t)\,dt\\ &\le C_p\sup_{t>0} u_a(x,t)^{2-p}J_a(x),\end{aligned}$$

where

$$J_a(x)=-\int_0^{+\infty}(\frac{\partial}{\partial t}-L+ap)u_a^p(x,t)dt\,.$$

Using the Hölder inequality with exponent $\frac{2}{2-p}$ and $\frac{2}{p}$,

$$\begin{aligned}\int_M H_a^p(f)(x)\,d\mu(x) &\le C_p\int_M \sup_{t>0} u_a(x,t)^{\frac{(2-p)p}{2}}J_a(x)^{\frac{p}{2}}\,d\mu(x)\\ &\le C_p\left[\int_M\sup_{t>0}u_a(x,t)^p\,d\mu(x)\right]^{(2-p)/2}\left[\int_M J_a(x)\,d\mu(x)\right]^{p/2}.\end{aligned}$$

By the maximal inequality for the symmetric L^p-contractive semigroups (see [64]), there exists a constant A_p such that

$$\|\sup_{t>0}u_a(\cdot,t)\|_p\le A_p\|f\|_p\,.$$

By (4.17), $\left(L-\frac{\partial}{\partial t}-ap\right)u_a^p(x,t)\ge 0$, $\forall (x,t)\in M\times\mathbb{R}^+$. Applying first the Fubini theorem, then integrating by parts and using $L1=0$, we obtain

$$\begin{aligned}\int_M J_a(x)\,d\mu(x) &= \int_M\int_0^\infty\left(L-\frac{\partial}{\partial t}-ap\right)u_a^p(x,t)dt\,d\mu(x)\\ &\le -\int_M u_a^p(x,t)|_0^\infty\,d\mu(x)+\int_0^\infty\int_M Lu_a^p(x,t)\,d\mu(x)dt\\ &= \int_M u_a(x,0)^p\,d\mu(x)=||f||_p^p.\end{aligned}$$

Combining this with the previous inequalities, we have

$$||H_a(f)||_p\le C_p'||f||_p^{\frac{2-p}{2}}||f||_p^{\frac{p}{2}}\le C_p'||f||_p.$$

Therefore, we have proved $\|g_{2,a-L}(f)\|_p\le C_p\|f\|_p$ for all $1<p\le 2$.

Next we prove (4.15) for the case $p>2$. Without loss of the generality, we only consider the case where $a=0$.

Let $u(x,t)=Q_tf(x)$. Combining (4.11) with (4.12) in Theorem 4.1, we have

$$|\partial_t u(x,t)|^2+|\nabla u(x,t)|^2\le(4+C_K)\,Q_{t/2}\left(|\partial_t u(x,t/2)|^2+|\nabla u(x,t/2)|^2\right).$$

Therefore

$$\begin{aligned} g(f)^2 &= \int_0^\infty \left[|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right] tdt \\ &\leq (4+C_K)\int_0^\infty e^{-\frac{t}{2}\sqrt{-L}}\left(|\nabla u(x,t/2)|^2 + |\partial_t u(x,t/2)|^2\right) tdt. \end{aligned}$$

Changing the variable $t \to 2t$, we get

$$g(f)^2 \leq 4\,(4+C_K)\int_0^\infty e^{-t\sqrt{-L}}\left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right) tdt.$$

whence

$$g(f) \leq 2\sqrt{4+C_K}H(f), \tag{4.18}$$

where

$$H(f) := \left[\int_0^\infty e^{-t\sqrt{-L}}\left(|\nabla e^{-t\sqrt{-L}}(x)|^2 + |\frac{\partial}{\partial t}e^{-t\sqrt{-L}}f(x)|^2\right) tdt\right]^{1/2},$$

By P.A. Meyer [55], Bakry [5], see also Shigekawa-Yoshida [62], for any $p \geq 2$, there exists a constant C_p such that

$$\|H(f)\|_p \leq C_p\|f\|_p. \tag{4.19}$$

Hence $\|g(f)\|_p \leq 2C_p\sqrt{4+C_K}\|f\|_p$. In fact, we can prove (2) by the same argument as used in Stein [64] (p. 51-55). Due to the limit of the paper, we leave this to the reader.

Finally, let $L = -\int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of L in $L^2(\mu)$. Then, for all $f \in L^2(\mu)$,

$$\begin{aligned} \|g_{1,a-L}(f)\|_2^2 &= \int_M\int_0^\infty \left|\frac{\partial}{\partial t}e^{-t\sqrt{a-L}}f(x)\right|^2 tdtd\mu(x) \\ &= \int_0^\infty tdt\int_M |\sqrt{a-L}e^{-t\sqrt{a-L}}f(x)\|_2^2 d\mu(x) \\ &= \int_0^\infty tdt\int_0^\infty |\sqrt{a+\lambda}e^{-t\sqrt{a+\lambda}}|^2 d\|E_\lambda f\|_2^2 \\ &= \int_0^\infty (a+\lambda)\left(\int_0^\infty e^{-2t\sqrt{a+\lambda}}tdt\right) d\|E_\lambda f\|_2^2 \\ &= \frac{1}{4}\|f-E_0 f\|_2^2. \end{aligned}$$

Hence

$$\|f-E_0 f\|_2^2 = 4\|g_{1,a-L}(f)\|_2^2.$$

By polarization, for any $f, h \in L^2(\mu)$, we have

$$< f - E_0 f, h - E_0 h >_{L^2(\mu)} = 4\int_M \int_0^\infty t \frac{\partial}{\partial t} e^{-t\sqrt{a-L}} f(x) \frac{\partial}{\partial t} e^{-t\sqrt{a-L}} h(x) dt d\mu(x).$$

By duality argument, we have

$$\|f - E_0 f\|_p \le B_p \|g_{1,a-L}(f)\|_p \le B_p \|g_{a-L}(f)\|_p \le C_p \|f\|_p$$

for all $1 < p < \infty$. The proof of Theorem 4.2 is completed. ■

4.3. The Littlewood-Paley inequality for $g_{a+\Box_\phi}$

Recall the definition of the Littlewood-Paley function $g_{a+\Box_\phi}$: for any $\omega \in C_0^\infty(\Lambda^1(T^*M))$ and any constant $a \ge 0$,

(4.20)
$$g_{a+\Box_\phi}(\omega)(x) = \left[\int_0^\infty t \left(|\frac{\partial}{\partial t} e^{-t\sqrt{a+\Box_\phi}} \omega(x)|^2 + |\nabla e^{-t\sqrt{a+\Box_\phi}} \omega(x)|^2 \right) dt \right]^{1/2},$$

where ∇ denotes the Levi-Civita covariant derivative on M, and

$$e^{-t\sqrt{a+\Box_\phi}} \omega(x) = \int_0^\infty m(t,s) e^{-s(a+\Box_\phi)} \omega(x) ds. \tag{4.21}$$

In this subsection we prove the following Littlewood-Paley inequality.

Theorem 4.3 *Let $1 < p \le 2$, $L = \Delta - \nabla\phi \cdot \nabla$ be a Markovian symmetric diffusion operator on a complete Riemannian manifold (M, g) with an invariant measure $d\mu(x) = e^{-\phi(x)} dv(x)$, where $\phi \in C^2(M)$. Suppose that for some $\beta > q = \frac{p}{p-1}$ we have*

$$C_{\beta,K} = \sup_{t>0, x\in M} P_t^{\beta(a+K)} 1(x) < \infty, \tag{4.22}$$

and

$$\sup_{x\in M} G_{p(a+K)} (a+K)^-(x) < \infty, \tag{4.23}$$

where $G_{p(a+K)}$ is the Green potential operator of the Schrödinger operator $-L + p(a+K)$. Then there exists a constant A_p such that

$$\|g_{a+\Box_\phi}(\omega)\|_p \le A_p \|\omega\|_p.$$

To prove Theorem 4.3, we need two lemmas.

Lemma 4.4 *Suppose that* (4.22) *holds on* (M, g). *Then for any* $1 < p \leq 2$, *there exists a constant* C_p *such that*

$$\int_M \sup_{t>0} \left| e^{-t\sqrt{a+\Box_\phi}}\omega(x)\right|^p d\mu(x) \leq C_p \|\omega\|^p_{L^p(M,\mu)}. \tag{4.24}$$

Proof. By (4.3), (4.5) and (4.21), we have

$$\begin{aligned}
\left|e^{-t\sqrt{a+\Box_\phi}}\omega(x)\right| &= \sup_{\|v\|=1} \left| \int_0^\infty m(t,s) e^{-as} E_x[< \omega(x_s), v_s >] ds \right| \\
&\leq \sup_{\|v\|=1} \int_0^\infty m(t,s) e^{-as} E_x \left[|\omega(x_s)| \|v_s\|\right] ds \\
&\leq \int_0^\infty m(t,s) e^{-as} E_x \left[|\omega|(x_s) e^{-\int_0^s K(x_u)du}\right] ds \\
&= \int_0^\infty m(t,s) E_x \left[|\omega|(x_s) e^{-\int_0^s (a+K)(x_u)du}\right] ds.
\end{aligned}$$

By the Hölder inequality, for any $\alpha = \frac{\beta}{\beta-1} \in (1,p)$ and $\beta > q = \frac{p}{p-1}$, we have

$$\begin{aligned}
\sup_{t>0} \left|e^{-t\sqrt{a+\Box_\phi}}\omega(x)\right| &\leq \sup_{s>0} E_x \left[|\omega|(x_s) e^{-\int_0^s (a+K)(x_u)du}\right] \\
&\leq \sup_{s>0} \left(\{E_x [|\omega|^\alpha (x_s)]\}^{1/\alpha} \left\{ E_x \left[e^{-\beta \int_0^s (a+K)(x_u)du}\right]\right\}^{1/\beta}\right) \\
&\leq \sup_{s>0, x\in M} \left\{ E_x \left[e^{-\beta \int_0^s (a+K)(x_u)du}\right]\right\}^{1/\beta} \sup_{s>0} [P_s|\omega|^\alpha(x)]^{1/\alpha}.
\end{aligned}$$

Note that

$$E_x \left[e^{-\beta \int_0^t (a+K)(x_u)du}\right] = P_t^{\beta(a+K)} 1(x).$$

Thus under the assumption (4.22), we get

$$\sup_{t>0} \left|e^{-t\sqrt{a+\Box_\phi}}\omega(x)\right| \leq C^{1/\beta}_{\beta,K} \sup_{s>0} [P_s|\omega|^\alpha(x)]^{1/\alpha},$$

from which and using the maximal inequality for symmetric L^p-contractive semigroups (see [64]), we get

$$\begin{aligned}
\int_M \sup_{s>0} \left|e^{-t\sqrt{a+\Box_\phi}}\omega(x)\right|^p d\mu(x) &\leq C^{p/\beta}_{\beta,K} \int_M \left| \sup_{s>0} P_s|\omega|^\alpha(x)\right|^{p/\alpha} d\mu(x) \\
&\leq C^{p/\beta}_{\beta,K} A^{p/\alpha}_{p/\alpha} \int_M |\omega|^{\alpha\cdot p/\alpha}(x) d\mu(x) \\
&= C^{p/\beta}_{\beta,K} A^{p/\alpha}_{p/\alpha} \|\omega\|^p_p.
\end{aligned}$$

The proof of the lemma is finished. ■

To state the second lemma, for any $\omega \in C_0^\infty(\Lambda^1(T^*M))$ and any $\epsilon > 0$, let us define

$$\begin{aligned}\omega(x,t) &= e^{-t\sqrt{a+\Box_\phi}}\omega(x),\\ \omega_\epsilon(x,t) &= \sqrt{|\omega(x,t)|^2+\epsilon^2}.\end{aligned}$$

Lemma 4.5 *Under the above notation, for any $1 < p \le 2$, we have*

$$\begin{aligned}|\overline{\nabla}\omega(x,t)|^2 &\le \frac{1}{p(p-1)}|\omega(x,t)|^{2-p}\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p\\ &\quad+\frac{1}{p-1}(a+K)^-(x)|\omega(x,t)|^2,\end{aligned}$$

where $\overline{\nabla} = (\frac{\partial}{\partial t}, \nabla)$, ∇ is the Levi-Civita covariant derivative on M.

Proof. The modified Bochner-Weitzenböck formula reads (cf. Bakry [5] Formula (0.3))

$$L|\omega|^2 = -2 < \omega, \Box_\phi\omega > +2|\nabla\omega|^2 + 2 < Ric(L)\omega, \omega >,$$

which implies that

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega(x,t)|^2 &= 2|\overline{\nabla}\omega(x,t)|^2 + 2 < (\frac{\partial^2}{\partial t^2}-\Box_\phi)\omega(x,t), \omega(x,t) >\\ &\quad +2 < Ric_x(L)\omega(x,t), \omega(x,t) > .\end{aligned}$$

Notice that

$$\Big(\frac{\partial^2}{\partial t^2}-a-\Box_\phi\Big)\omega(x,t) = 0$$

and

$$< Ric_x(L)\omega(x,t), \omega(x,t) > \ge K(x)|\omega(x,t)|^2 .$$

Hence

$$\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega(x,t)|^2 \ge 2|\overline{\nabla}\omega(x,t)|^2 + 2\left(a+K(x)\right)|\omega(x,t)|^2. \tag{4.25}$$

On the other hand, we have

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}+L\Big)|\omega_\epsilon(x,t)|^p &= \Big(\frac{\partial^2}{\partial t^2}+L\Big)(|\omega_\epsilon(x,t)|^2)^{\frac{p}{2}}\\ &= \frac{p}{2}(|\omega_\epsilon(x,t)|^2)^{\frac{p}{2}-1}\Big(\frac{\partial^2}{\partial t^2}+L\Big)|\omega_\epsilon(x,t)|^2\\ &\qquad+\frac{p}{2}(\frac{p}{2}-1)(|\omega_\epsilon(x,t)|^2)^{\frac{p}{2}-2}|\overline{\nabla}|\omega_\epsilon(x,t)|^2|^2\\ &= \frac{p}{2}|\omega_\epsilon(x,t)|^{p-2}\Big(\frac{\partial^2}{\partial t^2}+L\Big)|\omega_\epsilon(x,t)|^2+\frac{p}{2}(\frac{p}{2}-1)|\omega_\epsilon(x,t)|^{p-4}|\overline{\nabla}|\omega_\epsilon(x,t)|^2|^2.\end{aligned}$$

Hence

$$\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p = \frac{p}{2}|\omega_\epsilon(x,t)|^{p-2}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega(x,t)|^2 + \frac{p}{2}(\frac{p}{2}-1)|\omega_\epsilon(x,t)|^{p-4}|\overline{\nabla}|\omega(x,t)|^2|^2. \tag{4.26}$$

Moreover

$$\begin{aligned}|\overline{\nabla}|\omega(x,t)|^2|^2 &= |\nabla|\omega(x,t)|^2|^2+\left|\partial_t|\omega(x,t)|^2\right|^2\\ &= 4|<\nabla\omega(x,t),\omega(x,t)>|^2+4|<\omega(x,t),\partial_t\omega(x,t)>|^2\\ &\le 4|\omega(x,t)|^2|\overline{\nabla}\omega(x,t)|^2.\end{aligned}$$

Hence

$$|\overline{\nabla}|\omega(x,t)|^2|^2\le 4|\omega_\epsilon(x,t)|^2|\overline{\nabla}\omega(x,t)|^2. \tag{4.27}$$

From (4.25), (4.26) and (4.27), for any $1<p<2$, we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p &\ge \frac{p}{2}|\omega_\epsilon(x,t)|^{p-2}\left(2|\overline{\nabla}\omega(x,t)|^2+2\left(a+K(x)\right)|\omega(x,t)|^2\right)\\ &\quad +\frac{p}{2}(\frac{p}{2}-1)|\omega_\epsilon(x,t)|^{p-4}\cdot 4|\omega_\epsilon(x,t)|^2|\overline{\nabla}|\omega(x,t)|^2|^2\\ &= p(p-1)|\omega_\epsilon(x,t)|^{p-2}|\overline{\nabla}\omega(x,t)|^2\\ &\quad +p\left(a+K(x)\right)|\omega_\epsilon(x,t)|^{p-2}|\omega(x,t)|^2.\end{aligned}$$

Hence

$$\begin{aligned}|\overline{\nabla}\omega(x,t)|^2 &\le \frac{1}{p(p-1)}|\omega_\epsilon(x,t)|^{2-p}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p\\ &\quad +\frac{1}{p-1}(a+K)^-(x)|\omega(x,t)|^2.\end{aligned}$$

Taking $\epsilon\to 0$, the proof of Lemma 4.5 is completed. ■

Proof of Theorem 4.3. By (4.20) and Lemma 4.5, we have

$$g_{a+\Box_\phi}(\omega)\le I_1(\omega)+I_2(\omega), \tag{4.28}$$

where

$$\begin{aligned}I_1^2(\omega) &= \frac{1}{p(p-1)}\int_0^\infty t|\omega(x,t)|^{2-p}\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt,\\ I_2^2(\omega) &= \frac{1}{p-1}\int_0^\infty t(a+K)^-(x)|\omega(x,t)|^2dt.\end{aligned}$$

Note that

$$\begin{aligned} I_1^2(\omega) &\le \frac{1}{p(p-1)}\sup_{t>0}|\omega(x,t)|^{2-p}\int_0^\infty t\lim_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt \\ I_2^2(\omega) &\le \frac{1}{p-1}(a+K)^-(x)\sup_{t>0}|\omega(x,t)|^{2-p}\int_0^\infty t|\omega(x,t)|^p dt. \end{aligned}$$

Using the Hölder inequality with exponent $\frac{2}{2-p}$ and $\frac{2}{p}$, we have

$$\begin{aligned} \|I_1(\omega)\|_p^p &\le C_p\int_M \sup_{t>0}|\omega(x,t)|^{\frac{p(2-p)}{2}}\left(\int_0^\infty t\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt\right)^{p/2} d\mu(x) \\ &\le C_p\|\sup_{t>0}|\omega(x,t)|\|_p^{\frac{p(2-p)}{2}}\left(\int_M\int_0^\infty t\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p d\mu(x)dt\right)^{p/2}. \end{aligned}$$

Using Lemma 4.4, we have

$$\begin{aligned} &\|I_1(\omega)\|_p^p \\ &\le C_p\|\omega\|_p^{\frac{p(2-p)}{2}}\left(\int_M\int_0^\infty t\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p d\mu(x)dt\right)^{p/2}. \end{aligned} \tag{4.29}$$

Let B_t be a one-dimensional Brownian motion with the generator $\frac{d^2}{dt^2}$ starting at $T\in\mathbb{R}^+$. Let $\tau=\inf\{t\ge 0: B_t=0\}$. Then (x_t, B_t) is a diffusion process on $M\times\mathbb{R}$ with infinitesimal generator $L+\frac{\partial^2}{\partial t^2}$. Let

$$Z_t = |\omega_\epsilon(x_{t\wedge\tau}, B_{t\wedge\tau})|^p\,.$$

Then $Z_t - Z_0$ is a non-negative continuous submartingale with the Doob-Meyer decomposition $Z_t - Z_0 = M_t + A_t$, where M_t is a continuous martingale and A_t is a continuous increasing process given by

$$A_t = \int_0^{t\wedge\tau}\left(\frac{\partial^2}{\partial s^2}+L\right)|\omega_\epsilon(x_s,B_s)|^p ds\,.$$

Denote $P_{x,T}$ the law of the diffusion process (x_t, B_t). For any $f\in\mathcal{B}(M,\mathbb{R})$, let $E_T[f]=\int_M E_{x,T}(f)dx$. By P. A. Meyer [55] and Bakry [5], see also Shigekawa-Yoshida [62],

$$\int_M\int_0^\infty (T\wedge t)\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt = E_T\left[\int_0^\tau\left(\frac{\partial^2}{\partial s^2}+L\right)|\omega_\epsilon(x_s,B_s)|^p ds\right].$$

On the other hand, using the Lenglart-Lépingle-Pratelli inequality [38], we have

$$E[A_\infty]\le 2E[Z_\infty - Z_0]\,.$$

Hence

$$\begin{aligned} E_T\left[\int_0^\tau \left(\frac{\partial^2}{\partial s^2}+L\right)|\omega_\epsilon(x_s,B_s)|^p ds\right] &\le 2E_T\left[\left(|\omega(x_\tau,0)|^2+\epsilon^2\right)^{\frac{p}{2}}-\epsilon^p\right] \\ &\le 2E_T\left[|\omega(x_\tau,0)|^p\right] \\ &\le 2\int_M |\omega(x,0)|_p d\mu(x) = 2\|\omega\|_p^p. \end{aligned}$$

where in the second step we have used the elementary inequality: for any $p\in(1,2)$ and for any $x,y\in\mathbb{R}^+$, it holds that

$$(x+y)^{\frac{p}{2}}-y^{\frac{p}{2}}\le x^{\frac{p}{2}}\,.$$

By Fatou's lemma

$$\begin{aligned} &\int_M\int_0^\infty t\liminf_{\epsilon\to 0}\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt d\mu(x)\le \\ &\qquad \le \liminf_{\epsilon\to 0}\lim_{T\to\infty}\int_M\int_0^\infty (T\wedge t)\left(\frac{\partial^2}{\partial t^2}+L\right)|\omega_\epsilon(x,t)|^p dt d\mu(x) \\ &\qquad \le 2\|\omega\|_p^p. \end{aligned}$$

Combining this with (4.29), we have

$$\|I_1(\omega)\|_p\le C\|\omega\|_p. \tag{4.30}$$

Next we estimate $\|I_2(\omega)\|_p$.

Using the Hölder inequality with exponent $\frac{2}{2-p}$ and $\frac{2}{p}$, we have

$$\begin{aligned} \|I_2(\omega)\|_p^p &= A_p\int_M \sup_{t>0}|\omega(x,t)|^{\frac{p(2-p)}{2}}\left((a+K)^-(x)\int_0^\infty t|\omega(x,t)|^p dt\right)^{p/2} d\mu(x) \\ &\le A_p\|\sup_{t>0}|\omega(x,t)|\|_p^{\frac{p(2-p)}{2}}\left[\int_M\int_0^\infty t(a+K)^-(x)|\omega(x,t)|^p dt d\mu(x)\right]^{p/2}. \end{aligned}$$

By Lemma 4.4 we have

$$\|I_2(\omega)\|_p\le A_p\|\omega\|_p^{\frac{(2-p)}{2}}\left[\int_{M\times\mathbb{R}^+} t(a+K)^-(x)|\omega(x,t)|^p d\mu(x)dt\right]^{1/2}. \tag{4.31}$$

Therefore it is enough to estimate

$$J:=\int_{M\times\mathbb{R}^+} t(a+K)^-(x)|\omega(x,t)|^p d\mu(x)dt. \tag{4.32}$$

To do so, combining (4.3), (4.5) with (4.21), we obtain

$$\begin{aligned} J &= \int_{M\times\mathbb{R}^+} t(a+K)^-(x)\left|\int_0^\infty m(t,s)e^{-s(a+\square_\phi)}\omega(x)ds\right|^p dtd\mu(x)\\ &\le \int_{M\times\mathbb{R}^+} t(a+K)^-(x)\int_0^\infty m(t,s)\left|e^{-s(a+\square_\phi)}\omega(x)\right|^p dsd\mu(x)dt. \end{aligned}$$

The Fubini formula yields

$$J \le \int\limits_{M\times\mathbb{R}^+}\left(\int_0^\infty tm(t,s)dt\right)(a+K)^-(x)\left(E_x\left[|\omega(x_s)|e^{-\int_0^s(a+K(x_r))dr}\right]\right)^p d\mu(x)ds.$$

Notice that for any $s>0$, by changing variable $t=\sqrt{s}v$, we have

$$\int_0^\infty tm(t,s)dt = \int_0^\infty \frac{t^2}{2\sqrt{\pi}s^{3/2}}e^{\frac{-t^2}{4s}}dt = \int_0^\infty \frac{v^2}{2\sqrt{\pi}}e^{\frac{-v^2}{4}}dv = \frac{1}{2}.$$

Hence

$$J \le C\int_{M\times\mathbb{R}^+}(a+K)^-(x)E_x\left[|\omega(x_s)|^p e^{-p\int_0^s(a+K(x_r))dr}\right]dx. \tag{4.33}$$

Notice that $q_s(x,y)$ is the transition density of x_s with respect to the reference measure μ. Taking conditional expectation and then setting $\widehat{x}_r := x_{s-r}$, $r\in[0,s]$, we have

$$\begin{aligned} J &\le C\int\limits_{M\times\mathbb{R}^+}(a+K)^-(x)\int\limits_M|\omega(y)|^pE_x\left[e^{-p\int_0^s(a+K(x_r))dr}|x_s=y\right]q_s(x,y)d\mu(y)dsd\mu(x)\\ &= C\int\limits_M|\omega(y)|^p\int\limits_{M\times\mathbb{R}^+}(a+K)^-(x)\widehat{E}_y\left[e^{-p\int_0^s(a+K(\widehat{x}_r))dr}|\widehat{x}_s=x\right]q_s(x,y)d\mu(x)dsd\mu(y). \end{aligned}$$

By the reversibility of the L-diffusion process, using the Fubini theorem and since $q_s(x,y)=q_s(y,x)$, we obtain

$$\begin{aligned} J &\le C\int\limits_M|\omega(y)|^p\int\limits_{M\times\mathbb{R}^+}(a+K)^-(x)E_y\left[e^{-p\int_0^s(a+K(x_r^y))dr}|x_s=x\right]q_s(y,x)d\mu(x)dsd\mu(y)\\ &= C\int\limits_M|\omega(y)|^p\int_0^\infty E_y\left[(a+K)^-(x_s^y)e^{-p\int_0^s(a+K(x_r^y))dr}\right]dsd\mu(y)\\ &= C\int\limits_M|\omega(y)|^pE_y\left[\int_0^\infty(a+K)^-(x_s^y)e^{-p\int_0^s(a+K(x_r^y))dr}ds\right]d\mu(y), \end{aligned}$$

where $\{x_s^y, s\in[0,\infty)\}$ is a sample of L-diffusion process on M starting at $y\in M$.

Moreover, by the definition of the Green potential operator of $-L+p(a+K)$,

$$G_{p(a+K)}(a+K)^-(y) = E_y\left[\int_0^\infty (a+K)^-(x_s^y)e^{-p\int_0^s (a+K(x_r^y))dr}ds\right].$$

Thus, under the assumptions of Theorem 4.3,

$$J \leq C_p\|\omega\|_p^p. \tag{4.34}$$

From (4.31), (4.32) and (4.34), we have $\|I_2(\omega)\|_p \leq C\|\omega\|_p$. Combining this with (4.30) and (4.28), the proof of Theorem 4.3 is completed. ■

Remark 4.1 The heat kernel of the Schrödinger operator $-L+pK$ can be given by

$$q_t^{pK}(x,y) = p_t(x,y)E_x\left[\exp\left(-p\int_0^t K(x_s)ds\right)\Big| x_t = y\right].$$

In view of this, one can reformulate the above proof and the proof of Lemma 4.4 in an analytic way by avoiding the use of the probabilistic representation of the Schrödinger semigroup.

5. Schrödinger semigroups on Riemannian manifold

Let $\{x_t, t \in \mathbb{R}^+\}$ be the diffusion process generated by the Markovian symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ on a complete Riemannian manifold (M,g). Throughout this section, we let V denote a potential on M, i.e., a Borel measurable real valued function on M. For $V \in \mathcal{B}(M,\mathbb{R})$ such that $\int_0^t |V(x_s)|ds < +\infty$, $\forall t > 0, P_\mu = P_x \otimes \mu(dx) - a.s.$, the Schrödinger semigroup generated by $-L+V$ is defined by the Feynman-Kac formula

$$P_t^V f(x) = E_x\left[f(x_t)\exp\left(-\int_0^t V(x_s)ds\right)\right], \quad \forall\, f \in \mathcal{B}(M,\mathbb{R}^+).$$

Definition 5.1 ([9], [12]) *The Kato class $\mathcal{K}(M,L)$ associated with the L-diffusion operator L on a complete Riemannian manifold M is defined as the collection of Borel measurable real valued functions V on M such that*

$$\lim_{t\downarrow 0}\sup_{x\in M} E_x\left[\int_0^t |V|(x_s)ds\right] = 0.$$

Definition 5.2 ([72]) *The weak Kato class $\mathcal{K}_w(M,L)$ is defined as the collection of Borel measurable real valued functions V on M such that*

$$\sup_{x\in M} E_x\left[\exp\left(\int_0^t |V(x_s)|ds\right)\right] < +\infty, \quad \forall t \geq 0.$$

Proposition 5.3 ([12, 67]) *$\mathcal{K}(M,L)\subset\mathcal{K}_w(M,L)$. Indeed, if $V\in\mathcal{K}(M,L)$, then there exist two constants $C_1, C_2(V)$ such that*

$$\sup_{x\in M} E_x\left[\exp\left(\int_0^t |V(x_s)|ds\right)\right]\le C_1\exp(C_2(V)t),\quad \forall t>0,$$

where C_1 can be chosen independent of V.

Definition 5.4 *Let $V\in\mathcal{K}_w(M,L)$. For any $p\in[2,\infty]$, the L^p-bottom of spectrum $\lambda_p(-L+V)$ of $-L+V$ is defined as follows:*

$$\lambda_p(-L+V)=-\lim_{t\to\infty}\frac{1}{t}\log\|P_t^V\|_{L^p(\mu),L^p(\mu)}.$$

Proposition 5.5 *Suppose that $\sup_{0\le t\le 1}\|P_t^V\|_{\infty,\infty}<+\infty$. Then*

$$-\log\gamma(-L+V)\le\lambda_\infty(-L+V)\,,$$

where

$$\gamma(-L+V):=\sup_{0\le t\le 1}\|P_t^V\|_{\infty,\infty}\,.$$

Moreover, if $\sup_{0\le t\le 1}\|P_t^{-|V|}\|_{\infty,\infty}<+\infty$, then $V\in\mathcal{K}_w(M,L)$.

Proof. Indeed, for any $t>0$, let $t=[t]+\{t\}$, where $[t]$ is the integer part of t. Then, the semigroup (or the strong Markov) property implies

$$\|P_t^V\|_{\infty,\infty}\le\|P_{\{t\}}^V\|_{\infty,\infty}\|P_1^V\|_{\infty,\infty}^{[t]}\le\sup_{0\le s\le 1}\|P_s^V\|_{\infty,\infty}^{[t]+1}. \tag{5.1}$$

Hence, if $\sup_{0\le s\le 1}\|P_s^V\|_{\infty,\infty}<+\infty$, then

$$-\log\gamma(-L+V)\le\lambda_\infty(-L+V)\,.$$

Similarly

$$\|P_t^{-|V|}\|_{\infty,\infty}\le\sup_{0\le s\le 1}\|P_s^{-|V|}\|_{\infty,\infty}^{[t]+1},\quad\forall t>0. \tag{5.2}$$

This implies $V\in\mathcal{K}_w(M,L)$. ■

Corollary 5.6 *Let $V^-\in\mathcal{K}(M,L)$. Then $\lambda_\infty(-L+V)>-\infty$.*

Proof. This is an easy consequence of Proposition 5.3 and Proposition 5.5. ■

Proposition 5.7 *Let $V \in \mathcal{K}_w(M, L)$. Then*

(1) *For any $a \in \mathbb{R}$ and $p \in [2, \infty]$, $\lambda_p(-L + a + V) = a + \lambda_p(-L + V)$. Moreover*

$$\inf\{V(x), x \in M\} \leq \lambda_p(-L + V), \quad \forall p \in [2, \infty].$$

(2) *The function $p \to \lambda_p(-L + V)$ is decreasing on $[2, \infty]$.*

(3) *The function $r \to \lambda_\infty(-L + rV)$ is concave on $[0, \infty)$.*

(4) *The function $r \to \lambda_\infty(-L + rV)/r$ is decreasing on $(0, \infty)$.*

Proof. The proof of (1) is trivial. For (2), (3) and (4), we use the Hölder inequality as in the case where $V \in \mathcal{K}(M, L)$, see e.g. [67]. ∎

Proposition 5.8 *Suppose that*

$$\sup_{0 \leq s \leq 1} \|P_s^V\|_{\infty,\infty} < +\infty \qquad \text{and} \qquad \lambda_\infty(-L + V) \geq 0.$$

Then

$$\sup_{t>0} \sup_{x \in M} P_t^V 1(x) \quad < \quad +\infty, \tag{5.3}$$

$$\sup_{t>0} \sup_{x \in M} Q_t^V 1(x) \quad < \quad +\infty. \tag{5.4}$$

Proof. By $\lambda_\infty(-L + V) \geq 0$, there exists $t_0 > 0$ such that

$$\sup_{t \geq t_0} \|P_t^V\|_{\infty,\infty} \leq 1.$$

Combining this with (5.1) we finish the proof. ∎

Following Chung-Zhao [12], let us introduce the class $\mathcal{F}(M, V^-)$ of functions $f \in \mathcal{B}(M, \mathbb{R})$ satisfying

$$|f(x)| \leq C_1 + C_2 V^-(x),$$

for some constants $C_1 > 0$ and $C_2 > 0$ and for all $x \in M$.

Proposition 5.9 *Suppose that*

$$\sup_{0 \leq s \leq 1} \|P_s^{-V^-}\|_{\infty,\infty} < +\infty, \quad f \in \mathcal{F}(M, V^-).$$

Then

$$\left\| \int_0^t P_s^V |f| ds \right\|_\infty \leq (C_1 t + C_2) \sup_{0 \leq s \leq 1} \|P_s^{-V^-}\|_{\infty,\infty}^{[t]+1}, \ \forall t > 0.$$

Proof. We modify the proof of Prop. 3.6 in [12, pp. 80-81]). Notice that

$$\begin{aligned}
\int_0^t P_s^V|f|ds &\leq C_1\int_0^t P_s^V 1ds + C_2\int_0^t P_s^V V^- ds\\
&\leq C_1 t\sup_{0\leq s\leq t}\|P_s^V 1\|_\infty + C_2\sup_{x\in M}E_x\left[\int_0^t \exp\left(\int_0^s V^-(x_u)du\right)V^-(x_s)ds\right]\\
&= C_1 t\sup_{0\leq s\leq t}\|P_s^V 1\|_\infty + C_2\sup_{x\in M}E_x\left[\int_0^t d\exp\left(\int_0^s V^-(x_u)du\right)\right]\\
&= C_1 t\sup_{0\leq s\leq t}\|P_s^V 1\|_\infty + C_2\sup_{x\in M}E_x\left[\exp\left(\int_0^t V^-(x_u)|du\right)-1\right]\\
&\leq C_1 t\sup_{0\leq s\leq 1}\|P_s^V\|_{\infty,\infty}^{[t]+1} + C_2\left[\sup_{0\leq s\leq 1}\|P_s^{-V^-}\|_{\infty,\infty}^{[t]+1}-1\right];
\end{aligned}$$

in the last step we have used (5.1) and (5.2). This finishes the proof. ■

Definition 5.10 *The Green operator of the Schördinger operator $-L+V$ is defined as follows: for any $f\in\mathcal{B}(M,\mathbb{R}^+)$,*

$$G_V f(x) = \int_0^\infty P_t^V f(x)dt.$$

Proposition 5.11 *Suppose that*

$$\sup_{0\leq s\leq 1}\|P_s^{-V^-}\|_{\infty,\infty} < +\infty \quad \text{and} \quad \sup_{x\in M} G_V 1(x) < +\infty\,.$$

Then for any $f\in\mathcal{F}(M,V^-)$, we have

$$G_V|f|\in L^\infty(M)\,.$$

In particular,

$$\sup_{x\in M} G_V V^-(x) < \infty\,.$$

Proof. For any $x\in M$ and $t>0$, write

$$G_V|f|(x) = \int_0^t P_s^V|f|(x)ds + G_V(P_t^V|f|)(x). \tag{5.5}$$

By Proposition 5.9, there exist $a>0$ and $A>0$ such that for every $t\in(0,a]$,

$$\left\|\int_0^t P_s^V|f|ds\right\|_\infty \leq A\,.$$

Integrating with respect to t from 0 to a on both sides of (5.5) and then dividing by a, we get

$$G_V|f|(x) \;\leq\; A + G_V\left(\frac{1}{a}\int_0^a P_t^V|f|dt\right)(x) \leq A\left(1+a^{-1}\|G_V 1\|_\infty\right).$$

Hence $G_V|f|\in L^\infty(M)$. ■

Proposition 5.12 *Suppose that*

$$\sup_{0\leq s\leq 1}\|P_s^{-V^-}\|_{\infty,\infty}<+\infty \quad \textit{and} \quad \lambda_\infty(-L+V)>0\,.$$

Then

$$\sup_{x\in M} G_V V^-(x)<\infty. \tag{5.6}$$

Proof. Choose T big enough such that $\|P_t^V 1\|_\infty \leq e^{-\lambda_\infty(-L+V)t/2}$, $\forall t\geq T$. Then $\left\|\int_T^\infty P_t^V 1dt\right\|_\infty<+\infty$. Note that

$$\left\|\int_0^T P_t^V 1dt\right\|_\infty \leq T\|P_T^{-V^-}1\|_\infty \leq T\sup_{0\leq s\leq 1}\|P_s^{-V^-}\|_{\infty,\infty}^{[T]+1}\,.$$

Hence $\|G_V 1\|_\infty = \left\|\int_0^\infty P_t^V 1dt\right\|_\infty < +\infty$. Applying Proposition 5.11, we obtain (5.6). ■

6. General case: Proof of Theorem 2.1

To prove Theorem 2.1, we need the following result which first appeared in Bakry [5] (p. 161) and has been used by many authors (see e.g. [3, 5, 6, 11, 14, 15, 44, 45, 62]).

Lemma 6.1 *For any* $f\in C_0^\infty(M)$ *and* $\omega\in C_0^\infty(\Lambda^1(T^*M))$,

$$\|< R_a(L)f,\omega >\|_{L^1(\mu)} \leq 4\int_M g_{2,a-L}(f)(x)g_{1,a+\square_\phi}(\omega)d\mu(x),$$

where

$$\begin{aligned} g_{2,a-L}(f)(x) &= \left(\int_0^\infty |\nabla e^{-t\sqrt{a-L}}f(x)|^2 tdt\right)^{1/2}, \\ g_{1,a+\square_\phi}(\omega)(x) &= \left(\int_0^\infty \left|\frac{\partial}{\partial t}e^{-t\sqrt{a+\square_\phi}}\omega(x)\right|^2 tdt\right)^{1/2}. \end{aligned}$$

Proof. (See [5]. See also [14] for the special case where $L=\Delta$ and $a=0$.) For the completeness of the paper and for the convenience of the reader, we would like to give the proof here for general diffusion operator L and $a\geq 0$.

Similarly to the proof of (3) in Theorem 4.2, using spectral decomposition and polarization, we can prove that for all $\omega,\eta\in L^2(\Lambda^1(T^*M),\mu)$

with $E_0(\eta) = E_0(\omega) = 0$, where E_0 denote the orthogonal projection from $L^2(\Lambda^1(T^*M), \mu)$ onto $\mathrm{Ker}(\Box) = \{\omega \in L^2(\Lambda^1(T^*M), \mu), \Box_\phi \omega = 0\}$, we have

$$< \eta, \omega >_{L^2(\mu)} = 4 \int_0^\infty < \frac{\partial}{\partial t} e^{-t\sqrt{a+\Box_\phi}} \eta, \frac{\partial}{\partial t} e^{-t\sqrt{a+\Box_\phi}} \omega >_{L^2(\mu)} tdt.$$

Applying the above formula to $\eta = d(a-L)^{-1/2} f$, and using the fact

$$de^{-t\sqrt{a-L}}(f) = \frac{\partial}{\partial t} e^{-t\sqrt{a+\Box_\phi}} (d(a-L)^{-1/2}) f) \,,$$

we obtain

$$\left| < R_a(L) f, \omega >_{L^2(\mu)} \right| = 4 \left| \int_0^\infty < de^{-t\sqrt{a-L}} f, \frac{\partial}{\partial t} e^{-t\sqrt{a+\Box_\phi}} \omega >_{L^2(\mu)} tdt \right|$$
$$\leq 4 \int_{M \times \mathbb{R}^+} \left| \nabla e^{-t\sqrt{a-L}} f(x) \right| \left| \frac{\partial}{\partial t} e^{-t\Box a + \Box_\phi} \omega(x) \right| tdtd\mu(x).$$

Using the Cauchy-Schwarz inequality, we complete the proof of Lemma 6.1. ■

Combining Theorem 4.2, Theorem 4.3 and Lemma 6.1, we have the following

Theorem 6.2 *Let $p \geq 2$, $q = \frac{p}{p-1}$. Let M be a complete Riemannian manifold, $L = \Delta - \nabla\phi \cdot \nabla$ be a symmetric Markovian diffusion operator. Suppose that*

$$\sup_{t>0, x \in M} E_x \left[e^{-\beta \int_0^t (a+K)(x_s) ds} \right] < \infty \quad \text{for some } \beta > p,$$

and

$$\sup_{x \in M} G_{q(a+K)}(a+K)^-(x) < \infty.$$

Then the Riesz transform $R_a(L) = \nabla(a-L)^{-1/2}$ is bounded in $L^p(\mu)$.

Now we are able to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\gamma_\beta := \sup_{0 \leq s \leq 1} \|P_s^{\beta K}\|_{\infty,\infty}$. Then

$$\gamma_\beta \leq \sup_{0 \leq s \leq 1} \|P_s^{-\beta K^-}\|_{\infty,\infty} < +\infty.$$

Proposition 5.5 yields $\lambda_\infty(-L + \beta K) \geq -\log\gamma_\beta > -\infty$. If we assume that $a > \max\{-\lambda_\infty(-L+\beta K)/\beta, 0\}$, then $\lambda_\infty(-L+\beta(a+K)) > 0$. This implies $\lambda_\infty(-L + 2(a+K)) > 0$.

By Proposition 5.8 we get $\sup_{t>0,x\in M} Q_t^{2(a+K)}1(x) < \infty$. By Theorem 4.2, for any $p > 1$, there exists $A_p > 0$ such that

$$\|g_{a-L}(f)\|_p \leq A_p\|f\|_p, \quad \forall f \in C_0^\infty(M). \tag{6.1}$$

By Proposition 5.7, for all $q \in (1, 2]$, we have

$$\frac{\lambda_\infty(-L + q(a + K))}{q} \geq \frac{\lambda_\infty(-L + \beta(a + K))}{\beta} > 0.$$

By Proposition 5.8 and Proposition 5.12, (4.22) and (4.23) hold when replacing p there by $q = \frac{p}{p-1}$ here. Applying Theorem 4.3, for all $p \in [2, \beta)$ and all $a > \max\{-\lambda_\infty(-L + \beta K)/\beta, 0\}$, there exists $B_q > 0$ such that

$$\|g_{a+\Box_\phi}\omega)\|_q \leq B_q\|\omega\|_q, \quad \forall\omega \in C_0^\infty(\Lambda^1(T^*M)). \tag{6.2}$$

By Lemma 6.1 and the Hölder inequality,

$$\|< R_a(L)f, \omega >\|_{L^1(\mu)} \leq 4\|g_{a-L}(f)\|_p\|g_{a+\Box_\phi}(\omega)\|_q.$$

Combining this with (6.2) and (6.1),

$$\|< R_a(L)f, \omega >\|_{L^1(\mu)} \leq 4A_pB_q\|f\|_p\|\omega\|_q$$

for all $p \in [2, \beta)$ and $a > \max\{-\lambda_\infty(-L + \beta K)/\beta, 0\}$. Taking the supremum over all $\omega \in C_0^\infty(\Lambda^1(T^*(M))$ with $\|\omega\|_q = 1$, we obtain

$$\|R_a(L)f\|_p \leq 4A_pB_q\|f\|_p$$

for all $p \in [2, \beta)$ and all $a > \max\{-\lambda_\infty(-L + \beta K)/\beta, 0\}$. By Theorem 3.2, $R_a(L)$ is bounded in $L^p(\mu)$ for all $p \in [2, \beta)$ and all $a > 0$.

Finally, if (2.3) and (2.4) hold, then

$$\lambda_\infty(-L + \beta K) \geq 0 \qquad \text{and} \qquad \sup_{x\in M} G_{qK}K^-(x) < +\infty.$$

Hence $\|g_{2,-L}(f)\|_p \leq A_p\|f\|_p$ for all $p \geq 2$ and $\|g_{1,\Box_\phi}(\omega)\|_q \leq B_q\|\omega\|_q$ for $q = \frac{p}{p-1}$. By Lemma 6.1, we have $\|R_0(L)f\|_p \leq 4A_pB_p\|f\|_p$ for all $p \in [2, \beta)$. ■

The following result gives a more effective criterion for the L^p-boundedness of $R_0(L) = \nabla(-L)^{-1/2}$. The proof is very similar to the above one and is omitted here.

Theorem 6.3 *Suppose that for some $\beta > 2$ we have*

$$\sup_{0\leq t\leq 1, x\in M} e^{t(L+\beta K^-)}1(x) < +\infty,$$

and $\lambda_\infty(-L + \beta K) > 0$, or $\lambda_\infty(-L + \beta K) \geq 0$ and $\lambda_\infty(-L + K) > 0$. Then the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $a \geq 0$ and $p \geq [2, \beta)$.

7. Case of ultracontractive diffusion operator

From now on we suppose that (L, μ) is a ultracontractive diffusion operator on a complete Riemannian manifold (M, g) with dimension $dim(L) = n$. That is, $P_t = e^{tL}$ satisfies

$$\|e^{tL} f\|_\infty \le C t^{-\frac{n}{2}} \|f\|_{L^1(\mu)}, \quad \forall\, t \in (0, 1], f \in C_0^\infty(M), \tag{7.1}$$

where C is a positive constant. Note that (7.1) holds if and only if the heat kernel $q_t(x, y)$ of the diffusion operator L with respect to its invariant measure μ satisfies

$$\sup_{x,y\in M} q_t(x, y) = \|e^{tL}\|_{1,\infty} \le C t^{-\frac{n}{2}}, \ \forall\, t \in (0, 1].$$

The following result is well-known to experts.

Proposition 7.1 *Let (L, μ) be a ultracontractive diffusion operator with $dim(L) = n$. Then $V \in \mathcal{K}(M, L)$ provided that*

$$V \in L^{\frac{n}{2}+\epsilon}(\mu) \ \text{ for some } \epsilon > 0\,.$$

Proof. For the convenience of the reader, we give a proof here. By the Riesz-Thorin interpolation, (7.1) implies that, for all $p \ge 1$, there exists a constant C_p such that

$$\|e^{tL} f\|_{p,\infty} \le C_p t^{-\frac{n}{2p}}, \quad \forall\, t \in (0, 1].$$

Hence, for $0 < t \le 1$, we have

$$\begin{aligned} \sup_{x\in M} E_x\left[\int_0^t |V(x_s)| ds\right] &\le \int_0^t \|E_x[|V(x_s)|]\|_\infty ds = \int_0^t \|e^{sL}|V|\|_\infty ds \\ &\le C_p \int_0^t s^{-\frac{n}{2p}} \|V\|_p ds = \frac{C_p}{1-\frac{n}{2p}} t^{1-\frac{n}{2p}} \|V\|_p, \end{aligned}$$

provided that $V \in L^p(\mu)$ with $p > \frac{n}{2}$. Taking $t \to 0$, Proposition 7.1 follows. ■

Proof of Theorem 2.2. By Proposition 7.1, $(K + c)^- \in L^{\frac{n}{2}+\epsilon}$ implies $(K + c)^- \in \mathcal{K}(M, L)$. Hence, for all $\beta > 2$, $\beta(K + c)^- \in \mathcal{K}(M, L)$. Proposition 5.3 yields

$$\sup_{0\le s\le 1} \|P_s^{-\beta(K+c)^-}\|_{\infty,\infty} < +\infty.$$

Note that $K^- \le (K + c)^- + c^+$. Hence,

$$\sup_{0\le s\le 1} \|P_s^{-\beta K^-}\|_{\infty,\infty} < +\infty\,.$$

Theorem 2.2 follows immediately from Theorem 2.1. ■

8. From principal eigenvalue to Lyapunov exponent

From Theorem 6.3, the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded in $L^p(\mu)$ if the Lyapunov exponent $\lambda_\infty(-L+\beta K) > 0$, or if $\lambda_\infty(-L+\beta K) \geq 0$ an $\lambda_\infty(-L+K) > 0$. While it is not easy to estimate the Lyapunov exponent if we do not assume that K (the lowest eigenvalue of the Bakry-Emery Ricci curvature) is uniformly bounded from below. We now give an effective approach to estimate the Lyapunov exponent $\lambda_\infty(-L+V)$ from the principal eigenvalue $\lambda_2(-L+V)$. To state it, let us define $L^\nu_w(M,\mu)$ (for $\nu \geq 1$) be the collection of Borel measurable functions $f : M \to \mathbb{R}$ such that

$$\|f\|_{\nu,w} := \left[\sup_{\xi>0} \mu\left\{x \in M : |f(x)| \geq \xi\right\} \xi^\nu\right]^{1/\nu} < +\infty\,.$$

Note that for all $\nu \geq 1$, $L^p(M,\mu) \subset L^\nu_w(M,\mu)$. However, $\|\cdot\|_{\nu,w}$ is not a norm.

Theorem 8.1 *Let (L,μ) be a ultracontractive diffusion operator with $\dim(L) = n$. Suppose that $V \in \mathcal{B}(M,\mathbb{R})$, $V^- \in \mathcal{K}(M,L)$ and there exists $\nu > 2$ such that*

$$(V - \lambda_2^+(V))^- \in L^\nu_w(M,\mu)\,,$$

where $\lambda_2^+(V) := \max\{0, \lambda_2(-L+V)\}$. Then

$$\|e^{-t(-L+V)}\|_{\infty,\infty} \leq C(1+t)^{\nu/2} e^{-\lambda_2(-L+V)t}, \quad \forall\ t > 0\,,$$

and

$$\lambda_\infty(-L+V) = \lambda_2(-L+V)\,.$$

Proof. We modify the argument used in Davies-Simon [20]. Let $H = -L+V$, $H^+ = -L+V^+$. Then

$$e^{-tH} = e^{-tH^+} + \int_0^t e^{-sH} V^- e^{-(t-s)H^+} ds\,.$$

Therefore

$$0 \leq e^{-tH}1 = e^{-tH^+}1 + \int_0^t e^{-sH} V^- e^{-(t-s)H^+} 1 ds \leq 1 + \int_0^t e^{-sH} V^- ds$$

and

$$0 \leq e^{-(t+1)H}1 \leq e^{-H}1 + \int_0^t e^{-(s+1)H} V^- ds\,.$$

Recall an interpolation inequality [3] used in Davies-Simon [20] for positivity-preserving linear operators T mapping $L^2 + L^\infty$ into L^∞:

$$\|Tf\|_\infty \le \frac{\nu}{\nu-2}\|T\|_{\infty,\infty}^{1-2/\nu}\|T\|_{\infty,2}^{2/\nu}\|f\|_{\nu,w}\,,$$

where $f \in L^\nu_w(M,\mu)$, $\nu > 2$. By this interpolation inequality and using Proposition 5.3, we have

$$\|e^{-(t+1)H}\|_{\infty,\infty} \le c_1 + c_2\int_0^t \|e^{-(s+1)H}\|_{2,\infty}^{2/\nu}\|e^{-(s+1)H}\|_{\infty,\infty}^{1-2/\nu}\|V^-\|_{\nu,w}ds. \tag{8.1}$$

Let $g(s) = \|e^{-sH}\|_{2,2}$, $n(t) = \sup\{\|e^{-(s+1)H}\|_{\infty,\infty} : 0 \le s \le t\}$. Then

$$\|e^{-(s+1)H}\|_{2,\infty} \le \|e^{-H}\|_{2,\infty}\|e^{-sH}\|_{2,2} = \|e^{-H}\|_{2,\infty}g(s). \tag{8.2}$$

By the Feynman-Kac formula and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|e^{-H}\|_{2,\infty} &= \sup_{\|f\|_2\le 1}\left\|e^{-(-L+V)}f\right\|_\infty\\
&\le \sup_{\|f\|_2\le 1}\sup_{x\in M} E_x\left[|f(x_1)|e^{-\int_0^1 V(x_s)ds}\right]\\
&\le \sup_{\|f\|_2\le 1}\sup_{x\in M}\left\{E_x\left[|f(x_1)|^2\right]\right\}^{1/2}\left\{E_x\left[e^{-2\int_0^1 V(x_s)ds}\right]\right\}^{1/2}\\
&\le \left\|e^{-(-L+2V)}\right\|_{\infty,\infty}^{1/2}\sup_{\|f\|_2\le 1}\sup_{x\in M}\left\{e^L|f|^2(x)\right\}^{1/2}\\
&\le \left\|e^{-(-L+2V)}\right\|_{\infty,\infty}^{1/2}\sup_{\|f\|_2\le 1}\left(\left\|e^L\right\|_{1,\infty}^{1/2}\|f\|_2\right)\\
&\le \left\|e^{L+2V^-}\right\|_{\infty,\infty}^{1/2}\left\|e^L\right\|_{1,\infty}^{1/2}.
\end{aligned}$$

Combining this with (7.1) and using Proposition 5.3, we have $\|e^{-H}\|_{2,\infty} < \infty$. Hence

$$n(t) \le c_1 + c_3\int_0^t g(s)^{2/\nu}n(s)^{1-2/\nu}ds \tag{8.3}$$

$$\le c_1 + c_3 n(t)^{1-2/\nu}\int_0^t g(s)^{2/\nu}ds. \tag{8.4}$$

To simplify the notation, let $\lambda_p(V) = \lambda_p(-L+V)$. By Proposition 5.7, it is always true that $\lambda_\infty(V) \le \lambda_2(V)$. We now prove the converse inequality in the following three cases.

[3]For its proof, see for example the MathSciNet Review of [20] given by J. A. Van Casteren.

Case 1. Suppose that $\lambda_2(V)=0$. Then $g(s)\le C$ and $n(t)\le c_1+c_4n(t)^{1-2/\nu}t$. Hence

$$n(t)\le Ctn(t)^{1-2/\nu},\quad \forall\, t\ge 1\,.$$

Thus

$$\left\|e^{-tH}\right\|_{\infty,\infty}\le n(t)\le C(1+t)^{\nu/2}\,.$$

This yields $\lambda_\infty(V)\ge 0$ and hence $\lambda_\infty(V)=0$.

Case 2. Suppose that $\lambda_2(V)=a^2$. Let $H_a:=H-a^2=-L+V_a$, where $V_a=V-a^2$. Then $\lambda_2(V_a)=0$, $V_a\in\mathcal{K}(M,L)$ and $V_a^-\in L^\nu_w(M)$. Using the result in the *Case* 1, we have

$$\|e^{-tH_a}\|_{\infty,\infty}\le C(1+t)^{\nu/2}\,.$$

Hence

$$\|e^{-tH}\|_{\infty,\infty}\le C(1+t)^{\nu/2}e^{-a^2t}\,.$$

This yields $\lambda_\infty(V)\ge a^2$ and hence $\lambda_\infty(V)=a^2$.

Case 3. Suppose that $\lambda_2(V)=-a^2$. Then $g(s)=\|e^{-sH}\|_{2,2}\le e^{a^2s}$. Hence

$$n(t)\le c_1+c_3n(t)^{1-2/\nu}\int_0^t e^{2a^2s/\nu}ds\le c_1+c_4n(t)^{1-2/\nu}e^{\frac{2a^2t}{\nu}}\,.$$

Let $y(t)=n(t)e^{-a^2t}$. Then

$$y(t)\le c_1e^{-a^2t}+c_4y(t)^{1-2/\nu}\le c_5+c_4y(t)^{1-2/\nu}.$$

This yields that $y(t)\le C$ for some constant $C>0$. Hence

$$\left\|e^{-tH}\right\|_{\infty,\infty}\le n(t)\le Ce^{a^2t}\,.$$

This yields $\lambda_\infty(V)\ge -a^2$ and hence $\lambda_\infty(V)=-a^2$. ∎

Remark 8.1 Suppose that $\lambda_2(V)=a^2>0$ and $V^-\in L^\nu_w(M,\mu)$ for some $\nu>2$. Then

$$\|e^{-t(-L+V)}\|_{\infty,\infty}\le C,\quad\text{and}\quad \lambda_\infty(-L+V)\ge 0\,.$$

Indeed, in the proof of the Case 1, $g(s)=\|e^{-sH}\|_{2,2}\le e^{-a^2s}$. Hence

$$n(t)\le c_1+c_3\int_0^t e^{-2a^2s/\nu}n(s)^{1-2/\nu}ds\le c_1+c_4n(t)^{1-2/\nu}.$$

The function $f(x)=x-c_4x^{1-2/\nu}-c_1$ is increasing on $[c_4^{2/\nu}(1-2/\nu)^{2/\nu},\infty)$ and $f(+\infty)=+\infty$. Let $C=\max\{x>0:f(x)=0\}$. Then $n(t)\le C$, $\forall\, t>0$. Thus

$$\left\|e^{-tH}\right\|_{\infty,\infty}\le C,\quad \forall\, t>0,\ \text{and}\ \lambda_\infty(V)\ge 0\,.$$

Similarly to the proof of Case 2 in Theorem 8.1, we have the following

Proposition 8.2 *Let (L, μ) be a ultracontractive Markovian diffusion operator with dim$(L) = n$. Suppose that $V \in \mathcal{B}(M, \mathbb{R})$, $V^- \in \mathcal{K}(M, L)$. If*

$$\lambda_2(-L+V) \geq a^2 \qquad \text{and} \qquad (V-a^2)^- \in L^\nu_w(M, \mu)\,,$$

then

$$\|e^{-t(-L+V)}\|_{\infty,\infty} \leq Ce^{-a^2 t}, \quad \text{and} \quad \lambda_\infty(-L+V) \geq a^2\,.$$

Proof of Theorem 2.7. By the same argument as used in [59], we can prove

$$\lambda_2(-L+\beta K) \geq a^2 := \beta\alpha - \beta\|(K-\alpha)^-\|_{n/2} B > 0.$$

Indeed, integration by parts and the Sobolev inequality yield

$$\begin{aligned}\int_M (-Lf+\beta Kf, f) d\mu &\geq \|\nabla f\|_2^2 + \beta\alpha \int_M f^2 d\mu - \int_M \beta(K-\alpha)^- f^2 d\mu \\ &\geq \|\nabla f\|_2^2 + \beta\alpha\|f\|_2^2 - \beta\|(K-\alpha)^-\|^2_{n/2}\|f\|^2_{2n/n-2} \\ &\geq \|\nabla f\|_2^2(1-\beta\|(K-\alpha)^-\|_{n/2}A) + \beta\|f\|_2^2(\alpha - \|(K-\alpha)^-\|_{n/2}B) \\ &\geq \beta(\alpha - \|(K-\alpha)^-\|_{n/2}B)\|f\|_2^2.\end{aligned}$$

Note that

$$(\beta K - a^2)^- = (\beta K + \beta B\|(K-\alpha)^-\|_{n/2} - \beta\alpha)^- \leq \beta(K-\alpha)^-.$$

Therefore

$$\|(\beta K - a^2)^-\|_{L^{\frac{n}{2}}(\mu)} \leq \beta\|(K-\alpha)^-\|_{L^{\frac{n}{2}}(\mu)} \leq \min\{A^{-1}, \alpha\beta B^{-1}\}.$$

Hence

$$(\beta K - a^2)^- \in L^{\frac{n}{2}}(M, \mu) \subset L_w^{\frac{n}{2}}(M, \mu).$$

Since $\frac{n}{2} > 2$, Proposition 8.2 applies and yields

$$\lambda_\infty(-L+\beta K) \geq a^2 > 0.$$

By Theorem 2.1, $R_0(L) = \nabla(-L)^{-1/2}$ is bounded in $L^p(\mu)$ for all $p \in [2, \beta)$. ∎

9. Examples

9.1. Riesz transforms for the Laplace-Beltrami operator

Suppose that (M, g) is a n-dimensional complete and stochastically complete Riemannian manifold on which the following Sobolev inequality holds

$$\|f\|^2_{L^{\frac{2n}{n-2}}(\nu)} \leq A\|\nabla f\|^2_{L^2(\nu)} + B\|f\|^2_{L^2(\nu)}, \quad n \geq 3. \tag{9.1}$$

By Proposition VIII.3.3 in Chavel [10], see also Proposition 3.6 in Hebey [32], if (M, g) is a complete Riemannian manifold with positive injectivity radius and Ricci curvature bounded from below, then

$$\|e^{t\Delta}\|_{1,\infty} \leq Ct^{-n/2}, \quad \forall t \in (0, 1]$$

and in particular, when $n > 2$, the Sobolev inequality (9.1) holds with some constants A and B. On the other hand, by Hoffman-Spruck [33], see also Theorem 8.3 in Hebey [32], if (M, g) is a Cartan-Hadamard manifold, then the Sobolev inequality holds with some constant A (which depends only on $n = \dim M$) and $B = 0$. In both cases, the Laplace-Beltrami operator (Δ, ν) is a ultracontractive diffusion operator with $dim(\Delta) = n = \dim M$. Note that, see Section 1 above, the family of Sobolev inequalities is stable under the quasi-isometry (and in particular the bounded conformal transformation).

Proof of Theorem 2.4 and Theorem 2.5. They follow immediately from Theorem 2.2, the Sobolev inequality on the Cartan-Hadamard manifolds and on the complete Riemannian manifolds with positive injectivity radius and with Ricci curvature bounded from below, and the stability of the Sobolev inequalities under quasi-isometries. ■

Proof of Theorem 2.6. By Theorem 2.4, $R_a(\Delta)$ is bounded in $L^p(\nu)$ for all $p \geq 2$ and all $a > 0$. By H.P. McKean's theorem (see [52]), if M is a n-dimensional Cartan-Hadamard manifold with sectinal curvature $\mathrm{Sect} \leq -k < 0$, then

$$\|f\|^2_{L^2(\nu)} \leq C_{k,n}\|\Delta f\|_{L^2(\nu)}$$

holds for all $f \in C_0^\infty(M)$ with $C_{k,n} = \frac{(n-1)^2 k}{4}$. Hence $-\Delta$ is strictly positive in $L^2(\nu)$. As in Lohoué [48] and Coulhon-Duong [13], Theorem 3.3 yields that $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ is bounded in $L^p(\nu)$ for all $p \geq 2$. ■

Applying Theorem 2.7 to the case $L = \Delta$, we obtain immediately the following

Theorem 9.1 *Let (M, g) be an n-dimensional complete and stochastically complete Riemannian manifold satisfying the Sobolev inequality* (9.1), $n \geq 5$. *Let*

$$K_0(x) = \inf\{< Ric(x)v, v >: \ v \in T_xM, \ \|v\| = 1\}.$$

Suppose that there exist some constants $\beta > 2$, $c \geq 0$, $\epsilon > 0$ and $\alpha > 0$ such that

$$(K_0 + c)^- \in L^{\frac{n}{2}+\epsilon}(M, \nu),$$

and

$$\|(K_0 - \alpha)^-\|_{L^{\frac{n}{2}}(\nu)} < \min\{(\beta A)^{-1}, \alpha B^{-1}\}.$$

Then, $R_0(\Delta) = \nabla(-\Delta)^{-1/2}$ is bounded in $L^p(\nu)$ for all $p \in [2, \beta)$.

Remark 9.1 By [4, 46], if the Ricci curvature on M is bounded from below, then $\lambda_\infty(-\Delta + K_0) > 0$ implies that M has finite volume and finite universal covering. Thus, the Ricci curvature on a complete non-compact Riemannian manifold with infinite volume and satisfying the conditions in the second part of Theorem 9.1 must to be unbounded from below.

9.2. Case of $(\mathbb{R}^n, e^{u(x)}g_0)$

In particular, let us consider the complete Riemannian manifold

$$(M, g) = (\mathbb{R}^n, e^u g_0)$$

on which the Riemannian metric g is conformal to the standard Euclidean metric g_0 on $\mathbb{R}^n$. Let $u \in (C^2(\mathbb{R}^n) \setminus C_b^2(\mathbb{R}^2)) \cap C_b(\mathbb{R}^n)$. Then $(\mathbb{R}^n, e^u g_0)$ is stochastically complete. As pointed out in Hebey [32] (p. 62), the Sobolev imbedding $H^{1,p}(M, \nu) \subset L^{p^*}(M, \nu)$ holds with $B_p = 0$ for all $p \in [1, n]$, $n \geq 2$. Indeed, this is a consequence of the stability of Sobolev inequalities under the quasi-isometries. Hence, (Δ, ν) is a ultracontractive operator on $(\mathbb{R}^n, e^u g_0)$ with $dim(\Delta) = n = dim\mathbb{R}^n$, $\forall n \geq 2$.

By (1.3), the Ricci curvature on $(\mathbb{R}^n, e^u g_0)$ is given by

$$Ric = -\frac{n-2}{2}\nabla_0^2 u + \frac{n-2}{4}\nabla_0 u \otimes \nabla_0 u - \frac{1}{2}\left(\Delta_0 u + \frac{n-2}{2}|\nabla_0 u|^2\right) g_0,$$

where ∇_0 and Δ_0 denote the standard gradient and Laplace operators on $(\mathbb{R}^n, g_0)$. Suppose $u \in C_b^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ but $u \notin C_b^2(\mathbb{R}^n)$. Then $Ric \geq K(x)g$. Here

$$K(x) = c_0 - \frac{1}{2}e^{-u(x)}\Delta_0 u(x) - \frac{n-2}{2}e^{-u(x)} \sup\{< \nabla_0^2 u(x)v, v >: \|v\| = 1\},$$

where $\| \ \|$ and $< \ >$ denote the standard Euclidean norm and inner-product on $\mathbb{R}^n$, and

$$c_0 := \inf \left\{ \frac{n-2}{4} e^{-u(x)} \nabla_0 u \otimes \nabla_0 u - \frac{n-2}{4} e^{-u(x)} |\nabla_0 u(x)|^2 : x \in \mathbb{R}^n \right\}.$$

To simplify the computation, below we consider the *case* $n = 2$. Then $Ric = -\frac{1}{2}\left(e^{-u}\Delta_0 u\right) g$. Hence $K(x) = -\frac{1}{2} e^{-u(x)} \Delta_0 u(x)$. The volume element on $(\mathbb{R}^2, e^{u(x)} g_0)$ is $d\nu(x) = e^{u(x)} dx$. The condition $(K + c)^- \in L^{\frac{2}{2}+\epsilon}(\nu)$ for some $c \in \mathbb{R}^+$ writes

$$\int_{\mathbb{R}^2} \left[\left(2c - e^{-u(x)} \Delta_0 u(x)\right)^- \right]^{1+\epsilon} e^{u(x)} dx < +\infty,$$

which is true provided that

$$\int_{\mathbb{R}^2} \left[\left(2c e^{\min u} - \Delta_0 u(x)\right)^- \right]^{1+\epsilon} dx < +\infty. \tag{9.2}$$

Hence, if $u \in C^2(\mathbb{R}^2) \cap C_b(\mathbb{R}^2)$ satisfies (9.2) for some constant $c \in \mathbb{R}^+$, then for all $a > 0$ and $p \geq 2$, the Riesz transform $R_a(\Delta) = \nabla(a - \Delta)^{-1/2}$ is bounded in $L^p(\mathbb{R}^2, e^{u(x)} dx)$, where ∇ and Δ denote the Riemannian gradient operator and the Laplace-Beltrami operator on $(\mathbb{R}^2, e^{u(x)} g_0)$. This proves Example 2.2.

9.3. Riesz transforms for diffusion operators on $\mathbb{R}^1$ and $\mathbb{R}^d$

Let $\phi \in C^2(\mathbb{R}, \mathbb{R})$, L be the one-dimensional diffusion operator on the real line given by

$$Lf(x) = f''(x) - \phi'(x) f'(x), \quad \forall\, f \in C_0^\infty(\mathbb{R}), \forall\, x \in \mathbb{R}.$$

Then $\mu(dx) = e^{-\phi(x)} dx$ is an invariant measure of L and the Bakry-Emery Ricci curvature of L is $Ric_x(L) = \phi''(x), \ \forall\, x \in \mathbb{R}$. Hence

$$K(x) = \phi''(x), \quad \forall\, x \in \mathbb{R}.$$

By Remarque 2.2 or 5.2 in [36], $P_t = e^{tL}$ is a ultracontractive Markovian diffusion semigroup provided that $u(x) := e^{-\phi(x)}$ is bounded from below by a strictly positive constant and $\frac{\Delta u^{1/2}}{u^{1/2}} + c_2 \geq 0$ holds for some constant $c_2 \in \mathbb{R}$. Equivalently, $P_t = e^{tL}$ is ultracontractive and Markovian if there exist two constants $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ such that

$$\phi(x) \leq c_1, \quad \forall\, x \in \mathbb{R},$$

and

$$\frac{\phi'^2(x)}{4} - \frac{\phi''(x)}{2} + c_2 \geq 0, \quad \forall\, x \in \mathbb{R},$$

More precisely, under the above conditions,

$$\|P_t f\|_\infty \le C e^{c_2 t} t^{-1/4} \|f\|_{L^2(\mu)}, \quad \forall\, t > 0\,.$$

By Lemma 2.1.2 in Davies [19], $\|P_t\|_{1,\infty} = \|P_{\frac{t}{2}}\|^2_{2,\infty}$. Hence, for a suitable constant C,

$$\|P_t\|_{1,\infty} \le C t^{-1/2}, \quad \forall\, t \in (0,1]\,.$$

That is, (L,μ) is a ultracontractive diffusion operator with $dim(L) = 1$.

Applying Theorem 2.2 to this special case of (L,μ,K), we obtain immediately Theorem 2.3. In general, using the same argument as above and by Remarque 2.2 or 5.2 in [36] for general case of $\mathbb{R}^d$, we can prove the following

Theorem 9.2 *Let $\phi \in C^2(\mathbb{R}^d,\mathbb{R})$, $L = \Delta - \nabla\phi\cdot\nabla$. Suppose that there exist some constants $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}, c_3 \in \mathbb{R}^+$ and $\epsilon > 0$ such that*

$$\phi(x) \le c_1, \quad \forall x \in \mathbb{R}^d\,,$$

$$\frac{|\nabla\phi(x)|^2}{4} - \frac{\Delta\phi(x)}{2} + c_2 \ge 0, \quad \forall\, x \in \mathbb{R}^d,$$

and

$$\int_{\mathbb{R}^d} \left([K(x)+c_3]^-\right)^{\frac{d}{2}+\epsilon} e^{-\phi(x)}dx < +\infty\,,$$

where $K(x)$ is the lowest eigenvalue of

$$\nabla^2\phi(x) = \Big(\frac{\partial^2}{\partial x_i \partial x_j}\phi(x)\Big)_{1\le i,j\le d}, \quad \forall x \in \mathbb{R}^d\,.$$

Then, for all $p \ge 2$ and all $a > 0$, the Riesz transform $R_a(L) = \nabla\,(a-L)^{-1/2}$ is bounded in $L^p(\mathbb{R}^d, e^{-\phi(x)}dx)$.

9.4. Realisation of a one dimensional model

To end this paper, let us describe how to construct a diffusion operator $L = \frac{d^2}{dx^2} - \phi'(x)\frac{d}{dx}$ such that the Bakry-Emery Ricci curvature $Ric(L) = \phi''$ is not uniformly bounded from below but the Riesz transform $R_a(L) = \frac{d}{dx}(a-L)^{-1/2}$ is bounded in $L^p(\mathbb{R}, e^{-\phi(x)}dx)$ for all $p \ge 2$ and all $a > 0$. To this end, we need only to construct a C^2-smooth function ϕ such that ϕ satsifies all the conditions required in Theorem 2.3 and $\inf_{x\in\mathbb{R}} \phi''(x) = -\infty$.

Let c be a fixed positive constant, $\{a_k, k \in \mathbb{Z}\}$ and $\{b_k, k \in \mathbb{Z}\}$ be two sequences such that $a_k < b_k < a_{k+1} < b_{k+1}$, $\lim_{k\to-\infty} a_k = -\infty$ and $\lim_{k\to+\infty} b_k = +\infty$. Let

$$\phi(x) = \sum_{k=-\infty}^{+\infty} c1_{[a_k,b_k]}(x) + \phi_k(x) 1_{(b_k,a_{k+1})}(x)\,.$$

where $\phi_k \in C^2((b_k, a_{k+1}), \mathbb{R})$ is a "V"-sharp function which is only concave in a very narrow well $I_k \subset (b_k, a_{k+1})$.

Suitably control $|I_k|$ (the length of I_k) and choose $0 \leq \phi_k \leq c$ so that

$$\inf\{\phi_k''(x) : x \in I_k\} \to -\infty, \quad \text{when } |k| \to \infty\,,$$

and

$$\sum_{k=-\infty}^{\infty} \int_{I_k} \left[(\phi''(x))^-\right]^{\frac{1}{2}+\epsilon} dx < +\infty\,.$$

Then $Ric(L) = \phi''$ is not uniformly bounded from below and ϕ satisfies the conditions required in Theorem 2.3. This provides us with a possible way to construct explicitly a one-dimensional diffusion operator

$$L = \frac{d^2}{dx^2} - \phi'(x)\frac{d}{dx}$$

with unbounded negative part of Bakry-Emery Ricci curvature $Ric(L) = \phi''$ and for which the Riesz transform $R_a(L) = \frac{d}{dx}(a-L)^{-1/2}$ is bounded in $L^p(\mathbb{R}, \mu)$ for all $p \geq 2$ and all $a > 0$. Modifying this example, we can construct a complete non-compact rotational symmetric Riemannian manifold with unbounded negative part of Ricci curvature and for which the Riesz transform $R_a(\Delta) = \nabla(a-\Delta)^{-1/2}$ is bounded in $L^p(\nu)$ for all $p \geq 2$ and all $a > 0$.

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