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How smooth is almost every function in a Sobolev space?

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Abstract

We show that almost every function (in the sense of prevalence) in a Sobolev space is multifractal: Its regularity changes from point to point; the sets of points with a given Hölder regularity are fractal sets, and we determine their Hausdorff dimension.

1. Introduction

In order to answer the question raised in the title, one should first agree on what is meant by "almost every" in an infinite dimensional Banach space. Prevalence supplies a natural definition which is translation invariant and does not allow a specific measure to play a particular role. Since this notion is not widely used, we start by recalling its definition and basic properties.

1.1. Prevalence

In a finite dimensional space, "almost every" (without referring to a specific measure) means "for the Lebesgue measure" which enjoys a particular status since it is the only σ-finite translation invariant measure. No measure, in a metric infinite dimensional vector space, enjoys this property; but this does not mean that there exists no notion of "almost everywhere" which is translation invariant. A remarkable way to turn this problem and recover a canonical notion of almost everywhere was discovered by J. Christensen in 1972, see [3]. It is based on the following remark which allows to characterize Lebesgue measure-zero sets in $\mathbb{R}^d$ by a criterium which does not involve explicitly the Lebesgue measure.

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Lemma 1. *Let S be a Borel set in $\mathbb{R}^d$. If there exists a probability measure μ such that, for any $x \in \mathbb{R}^d$, $\mu(x+S)=0$, then the Lebesgue measure of S vanishes.*

This lemma can therefore be used as a definition in infinite-dimensional spaces, see [1, 3, 6]; "zero-measure sets" thus defined are called Haar-null (or shy).

Definition 1. *Let E be a complete metric vector space. A Borel set $A \subset E$ is Haar-null if there exists a compactly supported probability measure μ such that*

$$\forall x \in E, \quad \mu(x+A)=0. \tag{1.1}$$

If this property holds, the measure μ is said to be transverse to A.

A subset of E is called Haar-null if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a prevalent set.

The following results enumerate important properties of prevalence and show that these notions supply a natural generalization of "zero measure" and "almost every" in finite-dimensional spaces, see [1, 3, 6].

- If S is Haar-null, then $\forall x \in E$, $x+S$ is Haar-null.
- If $\dim(E) < \infty$, S is Haar-null if and only if $\text{meas}(S)=0$ (where meas denotes the Lebesgue measure).
- Prevalent sets are dense.
- The intersection of a countable collection of prevalent sets is prevalent.
- If $\dim(E)=\infty$, compact subsets of E are Haar-null.

Remarks. 1. In order to prove that a set is Haar-null, one can often use for transverse measure the Lebesgue measure on the unit ball of a finite dimensional subset V; (1.1) becomes

$$\forall x \in E, \quad (x+V)\cap A \quad \text{is of Lebesgue measure zero.}$$

In this case V is called a probe for the complement of A.

Note that in this case, the corresponding measure is supported in a countable union of compact sets so that the compacity assumption in Definition 1 is necessarily fulfilled.

2. If E is a function space, choosing a probability measure on E is equivalent to choosing a random process X_t whose sample paths are almost surely in E. Thus, the definition of a Haar-null set can be rewritten as follows: Let $\mathcal{P}$ be a property that can be satisfied by points of E and let

$$A = \{f \in E : \ \mathcal{P}(f) \quad \text{holds}\ \}.$$

The condition $\mu(f+A)=0$ means the event $\mathcal{P}(X_t - f)$ has probability zero. Therefore, a way to check that a property $\mathcal{P}$ holds only on a Haar-null set is to exhibit a random process X_t whose sample paths are in E and is such that

$$\forall f \in E, \ a.s. \quad X_t + f \ \text{ does not satisfy } \ \mathcal{P}$$

(provided that the set of sample paths almost surely belongs to a compact set.)

3. If E is separable, then every measure is tight, and therefore is supported by a countable union of compact sets, see [2]; the compactness assumption in the definition of prevalence is automatically fulfilled.

With a slight abuse of language, when a property holds on a prevalent set, we will say that it holds almost everywhere. Our goal in this paper is to investigate the regularity properties that hold almost everywhere in a given Sobolev or Besov space. The case of the Hölder spaces C^s_0 has previously been investigated by B. Hunt in [5]. In order to state his result, we need to recall the definition of pointwise Hölder regularity.

Definition 2. *Let $\alpha \geq 0$ and $C > 0$; a function $f : \mathbb{R}^d \to \mathbb{R}$ is (C,α) smooth at x_0 if there exists a polynomial P of degree less than $[\alpha]$ such that, if $|x - x_0| \leq 1$,*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^{\alpha}. \tag{1.2}$$

The function f is $C^{\alpha}(x_0)$ if there exists a $C > 0$ such that (1.2) holds. The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : f \in C^{\alpha}(x_0)\}.$$

Note that, if (1.2) holds for an $\alpha > 0$, f is bounded in a neighborhood of x_0; therefore Hölder exponents can only be defined for locally bounded functions. In [5], B. Hunt proved that, if $s > 0$, almost every function in $C^s_0(\mathbb{R}^d)$ satisfies

$$\forall x \in \mathbb{R}^d, \quad h_f(x) = s.$$

1.2. Statement of the main results

The example supplied by $C^s_0(\mathbb{R}^d)$, where almost all functions have everywhere the same regularity, is not typical: We will see that functions in a Sobolev or Besov space different from $C^s_0(\mathbb{R}^d)$ have (almost surely) a whole range of Hölder exponents; furthermore, one can determine the "size" of the sets of points with a given Hölder exponent. If H is a value taken by the Hölder exponent, let

$$E_H = \{x : \ h_f(x) = H\},$$

and

$$d_f(H) = \dim(E_H)$$

where dim denotes the Hausdorff dimension. The function $d_f(H)$ is called the *spectrum of singularities* of f, see [8, 4].

If $s \geq 0$, $L^{p,s}$ denotes the Sobolev space $\{f \in L^p, \ \ (-\Delta)^{s/2} f \in L^p\}$. We will prove the following result.

Theorem 1.
- *If $s - d/p \leq 0$, then almost every function in $L^{p,s}$ is nowhere locally bounded, and therefore its spectrum of singularities is not defined.*
- *If $s-d/p > 0$, then the Hölder exponent of almost every function f of $L^{p,s}$ takes values in $[s-d/p, s]$ and*

$$\forall H \in [s-d/p, s]\,, \qquad d_f(H) = Hp - sp + d; \tag{1.3}$$

  furthermore, for almost every x, $h_f(x) = s$.
- *If $s-d/p > 0$, let x_0 be an arbitrary given point in $\mathbb{R}^d$; then, for almost every function in $L^{p,s}$, $h_f(x_0) = s - d/p$.*

Remarks. 1. In the second case, when $p \neq \infty$, Theorem 1 states that almost every function f of $L^{p,s}$ is "multifractal". This means that the sets of points where f has a given Hölder exponent are fractal sets indexed by the Hölder exponent H which plays the role of a parameter continuously varying in $[s-d/p, s]$, see [4, 8]. Up to now it was commonly believed among mathematicians and physicists that multifractality was the signature of very peculiar properties of the function considered (such as self-similarity for instance). Therefore Theorem 1, which is the first "generic" result of multifractality in the setting provided by prevalence, reverts the common point of view in this field.

2. Similar results for Besov spaces will be stated in Theorem 2.

3. Special attention should be payed to the position of "almost every" in the second and third statements; indeed Fubini's theorem does not apply in prevalence. If one considers a "generic" function in $L^{p,s}$, its Hölder exponent is almost everywhere s, but when a point x_0 is fixed, the regularity at x_0 of almost every function f will be as bad as possible, i.e. $s-d/p$. Note that, in the previous case, this Hölder exponent was the one taken most exceptionally (on a set of dimension zero).

4. The second and third points will be sharpened in respectively Proposition 3 and Theorem 3 where exact moduli of continuity will be given.

5. The first and second points coincide with the Baire-type results of [11]. (However, usually, there is no implication between prevalent and Baire-type results.)

6. If $s-d/p>0$, the following upper bound of [8] holds for every function of $L^{p,s}$ or $B_p^{s,q}$

$$d_f(H) \leq Hp - sp + d.$$

Therefore, a generic function in $L^{p,s}$ (or in $B_p^{s,q}$) is as irregular as possible.

7. The proofs given below actually show that all the above properties also hold locally. In particular, in the second case, the spectrum of singularities of the restriction of f to any open ball is also given by (1.3). This implies that, on any open ball, the Hölder exponent takes all values in $[s-d/p, s]$ so that, for almost every function $f \in L^{p,s}$, the sets E_h are everywhere dense, and the Hölder exponent h_f is an everywhere discontinuous function.

1.3. Wavelet expansions in Sobolev and Besov spaces

The idea of the proof of Theorem 1 is to find appropriate probes in the corresponding Sobolev and Besov spaces. We will explicitly construct bases of these probes by defining their wavelet coefficients. We start by recalling some properties of wavelets expansions. We use 2^d-1 wavelets $\psi^{(i)}$, which belong to C^r, for $r \geq s+1$, and satisfy

$$\forall i, \ \forall \alpha \ \text{ such that } \ |\alpha| \leq r, \quad \partial^\alpha \psi^{(i)} \quad \text{has fast decay,}$$

and the set of functions

$$2^{dj/2}\psi^{(i)}(2^j x - k), \ \ j \in \mathbb{Z}^d, k \in \mathbb{Z}^d$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus any function $f \in L^2(\mathbb{R}^d)$ can be written

$$f = \sum c_{j,k}^{(i)} \psi^{(i)}(2^j x - k)$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int f(x)\psi^{(i)}(2^j x - k)dx.$$

(Note that we use an L^∞ normalization for the wavelets, which will simplify some formulas.) We introduce simpler notations; recall that a dyadic cube of scale j is a cube of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right),$$

where $k=(k_1, \dots k_d) \in \mathbb{Z}^d$. Instead of indexing the wavelets and wavelet coefficients with the three indices (i,j,k), we will use dyadic cubes.

Since i takes 2^d-1 values, we can assume that it takes values in $\{0,1\}^d-(0,\dots,0)$; we will use the following notations:

- $\lambda\ (=\lambda(i,j,k)) \ = \dfrac{k}{2^j}+\dfrac{i}{2^{j+1}}+\Big[0,\dfrac{1}{2^{j+1}}\Big)^d.$
- $c_\lambda = c^i_{j,k}$
- $\psi_\lambda(x)=\psi^{(i)}(2^jx-k)$,
- $\mu_\lambda = \dfrac{k}{2^j}.$

The wavelet ψ_λ is essentially localized near the cube λ; more precisely, when the wavelets are compactly supported

$$\exists C>0 \quad \text{such that} \quad \forall i,j,k, \qquad supp\,(\psi_\lambda)\subset C\cdot\lambda$$

(where $C\cdot\lambda$ denotes the cube of same center as λ and C times wider). Finally, Λ_j will denote the set of dyadic cubes λ which index a wavelet of scale j, i.e. wavelets of the form $\psi_\lambda(x)=\psi^{(i)}(2^jx-k)$ (note that Λ_j is a subset of the dyadic cubes of side $2^{-(j+1)}$).

Sobolev spaces have the following characterization in terms of wavelet coefficients, see [14],

$$f\in L^{p,s} \Leftrightarrow \left(\sum_{\lambda\in\Lambda}|c_\lambda|^2(1+4^{js})\chi_\lambda(x)\right)^{1/2}\in L^p(\mathbb{R}^d), \tag{1.4}$$

where $\chi_\lambda(x)$ denotes the characteristic function of the cube λ and Λ is the set of all dyadics cubes. Besov spaces, which will also be considered, are characterized (for $p,q>0$) by

$$f\in B^{s,q}_p \iff \sum_j\left(\sum_{\lambda\in\Lambda_j}|c_\lambda|^p2^{(sp-d)j}\right)^{q/p}\le C \tag{1.5}$$

where Λ_j denotes the set of dyadics cubes at scale j, see [14]. Note that, if $p\in]0,1[$, Besov spaces are no more Banach spaces but nonetheless are separable complete metric vector spaces.

Theorem 2.
- *If either $s-d/p>0$ or if $s-d/p=0$ and $q\le\inf(p,1)$, then the conclusions of the second and third points of Theorem 1 hold for $B^{s,q}_p$.*
- *If $s-d/p=0$ and $0<p\le q\le 1$, then the Hölder exponent of almost every function in $B^{s,q}_p$ takes values in $[0,s]$ and*

  $$\forall H\in[0,d/q] \quad d_f(H)=Hq$$

  Furthermore, for almost every x, $h_f(x)=s$.
- *In the remaining cases almost every function in $B^{s,q}_p$ is nowhere locally bounded.*

Let us call "cone of influence above x_0 of width L" the set of couples (j,k) (or of cubes λ) such that

$$\left|\frac{k}{2^j} - x_0\right| \leq \frac{L}{2^j}$$

(we use the norm on $\mathbb{R}^d$: $|x| = \sup_{i=1,\dots,d} |x_i|$). The following lemma (see [7]) will be used as a simple irregularity criterium.

Lemma 2. *If f is (C,α) smooth at x_0, then there exists $C'(L)$ such that, in the cone of influence of width L above x_0,*

$$|c_{j,k}| \leq CC'(L)2^{-\alpha j} \tag{1.6}$$

where $C'(L)$ depends only on L and the wavelet, and can be bounded uniformly for $\alpha \in [0,A]$ (for any $A > 0$).

2. Proofs of Theorems 1 and 2

2.1. Spectra of singularities in the smooth cases

In this section, we prove the second point of Theorem 1 and, as regards Besov spaces, the corresponding cases $s - d/p > 0$ and $s - d/p = 0$ with $q \leq \inf(p,1)$.

Let us start with a few remarks. First, thanks to the Sobolev embeddings, if $s - d/p > 0$, any function in $L^{p,s}$ or in $B_p^{s,q}$ cannot have Hölder exponents less than $s - d/p$. Furthermore if $s - d/p = 0$ and $q \leq \inf(p,1)$, any function of $B_p^{s,q}$ is continuous, see [16]. Next, if $p \neq \infty$ and $q \neq \infty$, Besov spaces are separable spaces. On the other hand if $p = \infty$ or $q = \infty$, the measures we use are defined by a probe. So, in each case, the compacity assumption is fulfilled.

Let us now consider the Besov case (we will later mention modifications for the Sobolev case). We will prove a more precise result than the second point of Theorem 1: We will construct explicit sets E_α of dimension d/α and prove that almost every function of $B_p^{s,q}$ satisfies the following property: $\forall \alpha \in (1,\infty)$ at least on E_α the Hölder exponent of f is smaller than

$$H(\alpha) := s - \frac{d}{p} + \frac{d}{\alpha p}.$$

Let $l \in \mathbb{N}$ and $M = 2^{dl}$. We will construct an M-dimensional probe in $B_p^{s,q}$. The M generators of this space are defined by their wavelet coefficients. Let $j \geq 1$ and $k \in \{0,\dots,2^j - 1\}^d$; K and $J \leq j$ are defined by

$$\frac{k}{2^j} = \frac{K}{2^J} \qquad \text{where } K \in \mathbb{Z}^d - 2\mathbb{Z}^d.$$

Each dyadic cube λ is now split into M subcubes with side 2^{-j-l}. For each index $i \in \{1,\dots,M\}$ we choose a different subcube $i(\lambda)$. Let

$$a = \frac{1}{p} + \frac{2}{q}. \tag{2.1}$$

The probe is spanned by M functions g_i with the following wavelet coefficients

$$\begin{cases} \forall\lambda, \ d^i_{i(\lambda)} = j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J} \\ d^i_{\lambda'} = 0 \ \text{ if } \lambda' \text{ is not of the form } i(\lambda). \end{cases} \tag{2.2}$$

It follows directly from (1.5) that the functions $\{g_i\}_{i=\{1,\dots,M\}}$ belong to $B^{s,q}_p$.

Definition 3. *Let $\alpha \geq 1$; a point x_0 is said to be α-approximable (by dyadics) if there exists a sequence (J_n, K_n) such that*

$$\left| x_0 - \frac{K_n}{2^{J_n}} \right| \leq \frac{1}{2^{\alpha J_n}} \tag{2.3}$$

(clearly, we can assume that $K_n \in \mathbb{Z}^d - 2\,\mathbb{Z}^d$).

Lemma 3. *If x_0 is α-approximable, then there exists a sequence of distincts coefficients $(d^i_{j,k})$ in the cone of influence of width 2^l above x_0 such that*

$$\forall i \quad |d^i_{j,k}| \geq c(M) j^{-a} 2^{-Hj} \tag{2.4}$$

with $H (= H(\alpha)) = s - \frac{d}{p} + \frac{d}{\alpha p}$.

Proof. Suppose that x_0 is α-approximable and let λ be the dyadic cube such that $\frac{k}{2^j} = \frac{K_n}{2^{J_n}}$ and $j = [\alpha J_n]$ (where J_n and K_n are given by (2.3)). The wavelet coefficient indexed by $i(\lambda)$ has size

$$d^i_{i(\lambda)} = j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J_n} \geq c(M) j^{-a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p\alpha}j}.$$

■

Let $\alpha \in (1,\infty)$ be fixed, and let $\gamma > H(\alpha)$. Let us first check that the set of functions in $B^{s,q}_p$, satisfying

$$|c_{j,k}| \leq CC'(L) 2^{-\gamma j} \tag{2.5}$$

at some α-approximable point x, is included in a Haar null Borel set. Indeed, for $i \in \mathbb{N}$ fixed, this set is included in the countable union over λ_j of the following sets F_{λ_j}: if

$$E_{\lambda_j} = \left\{ x : \ \left| x_0 - \frac{\lambda_j}{2^j} \right| \leq \frac{1}{2^{\alpha j}} \right\},$$

then,

$$F_{\lambda_j} = \left\{ f : \ \exists x \in E_{\lambda_j} \quad f \text{ satisfies (2.5) at } x \right\}.$$

Each F_{λ_j} is a closed set. Indeed, suppose that f_n converge to f in $B_p^{s,q}$. For each n, there exists $x_n \in E_{\lambda_j}$ such that f_n satisfies (1.6) at x_n. Since E_{λ_j} is compact, there exists $x \in E_{\lambda_j}$ an accumulation point of x_n, and a subsequence of x_n converging to a point x such that (2.5) still holds at x. Since, for each λ, the mapping $f \to c_\lambda$ is continuous on a given Besov or a Sobolev space, when n tends to infinity, f satisfies (1.6) at x. So f belongs to F_{λ_j}. So the set of functions in $B_p^{s,q}$ satisfying (2.5) with γ fixed is a Borel set. We will now prove that it is a Haar null set.

Let f be an arbitrary function in $B_p^{s,q}$ with wavelet coefficients c_λ. Consider the affine subspace of dimension M composed of the functions

$$f_\beta = f + \sum_{i=1}^{M} \beta^i g^i$$

where $\beta = (\beta^1, \dots, \beta^M)$. Let $x_0 \in \mathbb{R}^d$ and $\gamma > 0$. If f_β satisfies (1.6) at x_0 then, inside the cone of width 2^l above x_0,

$$\left| c_\lambda + \sum_{i=1}^{M} \beta^i d_\lambda^i \right| \le C\ c(M) 2^{-\gamma j}. \tag{2.6}$$

Denote by E_j^α the set of points x_0 such that

$$\exists k : \quad \left| x_0 - \frac{k}{2^j} \right| \le \frac{1}{2^{\alpha j}}.$$

(Note that x_0 is α-approximable if $x_0 \in E^\alpha := \limsup_{j\to\infty} E_j^\alpha$).

The set E_j^α is the union of 2^{dj} cubes of width $2 \cdot 2^{-\alpha j}$. Suppose that x and y are two points in the same cube and suppose furthermore that f_β satisfies (1.6) at x and $f_{\tilde\beta}$ satisfies (1.6) at y. Then $|x-y| \le 2 \cdot 2^{-\alpha j}$ and $\forall i = 1, \dots, M$

$$\left| c_{i(\lambda')} + \sum_{i=1}^{M} \beta^i d_{i(\lambda')}^i \right| \le Cc(M) 2^{-\gamma j'}, \quad \left| c_{i(\lambda')} + \sum_{i=1}^{M} \tilde\beta^i d_{i(\lambda')}^i \right| \le Cc(M) 2^{-\gamma j'} \tag{2.7}$$

for any dyadic cubes λ' at scale j' inside the cone of width 2 above x and y. But, since $|x-y| \le 2 \cdot 2^{-\alpha j}$, we can find such a λ' satisfying $j' = [\alpha j]$. Using Lemma 3 and (2.4) it follows from (2.7) that

$$\|\beta - \tilde\beta\| \le 2Cc(M) 2^{-(\gamma - H)j'} (j')^a (:= A(j'))$$

(where $\|\beta\| = \sup_{i=1,\dots,M} |\beta_i|$). Therefore the set of β satisfying

$$\exists x \in E_j^\alpha \quad \text{such that } f_\beta \text{ is } (C,\alpha) \text{ smooth at } x$$

is included in the union of 2^{dj} balls with radii $A(j')$.

It follows that the Lebesgue measure of the M-uples β satisfying

$$\exists x \in E^\alpha \quad \text{such that} \;\; f_\beta \text{ is } (C,\alpha) \text{ smooth at } x$$

is bounded by

$$\sum_{j=J}^{\infty} (Cc(M)[\alpha j]^a)^M 2^{-(\gamma-H)M[\alpha j]} 2^{dj} \tag{2.8}$$

(where J can be chosen arbitrary large); for $\gamma > H$, we can choose M large enough so that $d-(\gamma-H)M\alpha < 0$; thus, when J tends to ∞, (2.8) goes to 0, so that it vanishes. Therefore, the set of M-uples $\beta = (\beta_1, \dots, \beta_M)$ such that $f + \sum \beta^i g^i$ satisfies (1.6) at a point in E^α has measure zero. Since it is true for all $C > 0$, the set of β such that

$$\exists x \in E^\alpha : \quad f + \sum \beta^i g^i \;\; \text{is } C^\gamma$$

also has measure zero. Therefore

$$\forall \alpha > 1, \;\; \forall \gamma > H(\alpha), \quad \text{a.s. in } B_p^{s,q}, \quad \forall x \in E^\alpha \;\; h_f(x) \le \gamma.$$

Taking $\gamma_n \to H(\alpha)$ (with $\gamma_n > H(\alpha)$) it follows by countable intersection, that

$$\forall \alpha > 0, \quad \text{a.s. } \forall x \in E^\alpha \;\; h_f(x) \le H(\alpha). \tag{2.9}$$

Therefore, if α_n is a dense sequence in $]1,\infty[$, using the same argument, one obtains that

$$\text{a.s. in} \;\; B_p^{s,q}, \; \forall n, \; \forall x \in E^{\alpha_n}, \quad h_f(x) \le H(\alpha_n)\,. \tag{$\mathcal{P}$}$$

Let f be a function such that ($\mathcal{P}$) holds. Let α be fixed and $\alpha_{\varphi(n)}$ a subsequence of α_n such that $\alpha_{\varphi(n)}$ is non decreasing and tends to α. The subsets $E^{\alpha_{\varphi(n)}}$ are decreasing and their intersection ($:= \tilde{E}^\alpha$) contains E^α. Therefore $\forall x \in \tilde{E}^\alpha$, $h_f(x) \le H(\alpha)$ and thus

$$\forall x \in E^\alpha, \;\; h_f(x) \le H(\alpha). \tag{2.10}$$

But (see [9]) there exists a measure m_α supported on E^α such that any set E of dimension less than d/α satisfies $m_\alpha(E) = 0$, and $m_\alpha(E^\alpha) > 0$. Moreover, if $G_H = \{x : \; h_f(x) \le H\}$, then

$$\forall f \in B_p^{s,q} \quad \dim_H(G_H) \le Hp - sp + d \tag{2.11}$$

(see [8] if $s - d/p > 0$ or [13] if $s - d/p = 0$ and $q \le \inf(p,1)$).

In particular, if F_α denotes the set of points where $h_f(x) < H(\alpha)$, F_α can be written as a countable union of sets with dimension less than d/α (because $H(\alpha) = s - \frac{d}{p} + \frac{d}{\alpha p}$, so that $\frac{d}{\alpha} = pH(\alpha) - sp + d$). It follows that $m_\alpha(F_\alpha) = 0$ and $m_\alpha(E^\alpha - F^\alpha) > 0$; but $E^\alpha - F^\alpha$ is a set of point where the Hölder exponent is exactly $H(\alpha)$. Thus

$$\forall H \in \left[s - \frac{d}{p}, s\right] \quad d_f(H) = Hp - sp + d,$$

and (1.3) holds on a prevalent set.

Moreover $E^1 = [0,1]^d$, so that we can take m_1 equal to the Lebesgue measure, and (2.10) yields, if $\alpha = 1$,

$$\text{a.s.} \quad \forall x \in [0,1]^d, \; h_f(x) \leq s. \tag{2.12}$$

Furthermore, as before, almost every function f of $B_p^{s,q}$ satisfies

$$\text{meas}\left(\{x : \quad h_f(x) < s\}\right) = 0,$$

so that

$$\text{a.s. for almost every } x \text{ in } [0,1]^d \quad h_f(x) = s. \tag{2.13}$$

Results (2.12) and (2.13) are not specific to the unit cube, but they also hold for any cube. By countable intersection, it follows that, almost surely, $\forall x \in \mathbb{R}^d$, $h_f(x) \leq s$ and, almost surely, a.e., $h_f(x) = s$. Therefore the second point of Theorem 1 holds in the case of $B_p^{s,q}$. The proof for the Sobolev case is similar: The functions g_i are defined by picking $q = 1$ in (2.1). Since $B_p^{s,1} \hookrightarrow L^{p,s}$, the g_i belong to $L^{p,s}$ and the remaining of the proof is unchanged.

2.2. Regularity at a fixed point

The last point of Theorem 1, and the corresponding result for $B_p^{s,\infty}$, will be the consequence of a more precise result concerning the pointwise modulus of continuity at a given point for a prevalent set of functions. Let us recall some definitions (see [12]).

Definition 4. *A function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity if*

- $\theta(0) = 0$
- θ *is increasing*
- $\exists c > 0 \quad \theta(2x) \leq c\theta(x)$.

θ is a modulus of continuity of f at x_0 if there exists a polynomial P and a constant c such that

$$|f(x) - P(x - x_0)| \leq c\,\theta(|x - x_0|); \tag{2.14}$$

θ is a uniform modulus of continuity if (2.14) holds for all x_0, and if the constant c does not depend on x_0.

Let $D_j(f) = \sup_k |c_{j,k}|$. The characterization of moduli of continuity using decay conditions on the $D_j(f)$, see [12], immediately implies the following result.

Proposition 1. *Let $f \in L^{p,s}$ (or $f \in B_p^{s,q}$ with $q \neq \infty$):*

- *If $s - d/p \in \mathbb{R}^+ - \mathbb{N}$ then, for all x_0, there exists a polynomial P with degree less than $s - d/p$ such that*

  $$|f(x) - P(x - x_0)| = o\left(|x - x_0|^{s-d/p}\right) \tag{2.15}$$

  (if $f \in B_p^{s,\infty}$, the o has to be replaced by a O).

- *If $s - d/p \in \mathbb{N}^+$ (and $q \neq \infty$ in the Besov case)*

  $$|f(x) - P(x - x_0)| = o\left(|x - x_0|^{s-d/p} \log|x - x_0|\right) \tag{2.16}$$

  (and if $f \in B_p^{s,\infty}$ the o has to be replaced by a O).

- *If $s - d/p = 0$, and $q \leq 1$ (Besov case), f is continuous at x_0.*

We will prove the following theorem which implies the last point of Theorem 1 and the corresponding point of Theorem 2.

Theorem 3. *Let $x_0 \in \mathbb{R}^d$ be an arbitrary fixed point and let $s > d/p$; if θ is a modulus of continuity which is a $o(x^{s-d/p})$, then the set of functions f in $L^{p,s}$ or in $B_p^{s,q}$ such that*

$$|f(x) - P(x - x_0)| = O(\theta(|x - x_0|))$$

is a Haar-null set.

Furthermore, if $s - d/p = 0$, the same result holds for every given modulus of continuity θ.

Remark. Consider, for instance, the case $s - d/p \in \mathbb{R}^+ - \mathbb{N}$ and $q \neq \infty$. Every function of $L^{p,s}$ satisfies (2.15), but Theorem 3 states that there exists no modulus of continuity which is a $o(|x - x_0|^{s-d/p})$ and which will work for a prevalent set of functions. Thus Theorem 3 together with Proposition 1 show that, at a given point the worst possible irregularity behavior is almost sure.

Proof. Consider first the case of $B_p^{s,q}$ with $s - d/p \leq 0$. Let θ be a modulus of continuity which is a $o(x^{s-d/p})$; therefore

$$\theta(2^{-j}) = a_j 2^{-(s-d/p)j}$$

with $a_j \to 0$ when $j \to \infty$. We will use the following lemma.

Lemma 4. *There exists $(b_j) \in l^q$ and an increasing sequence j_n such that*

$$a_{j_n} = o(b_{j_n}).$$

Proof. One can take a subsequence j_n such that $|a_{j_n}| \leq 2^{-n}$ and pick $b_{j_n} = 2^{-n/2}$ and $b_j = 0$ if j is not one of the j_n. ■

Using wavelet decomposition, the set of function with a smaller modulus of continuity than the one provided by Proposition 1 is included in a Haar null Borel set. To show this, we use the following condition on wavelet coefficients, given in [12].

Proposition 2. *If θ is a modulus of continuity of f at x_0 then:*

$$\exists\, c > 0 \qquad \forall j,k \qquad |c_{j,k}| \leq c\left(\theta(2^{-j}) + \theta\left(\left|x_0 - \frac{k}{2^j}\right|\right)\right). \tag{2.17}$$

Since x_0 and θ are fixed, the set of function whose wavelet coefficients satisfy (2.17) is clearly a countable union of closed sets.

We define a function g by its wavelet coefficients as follows

- $d_{j,k} = 0$ if $j \neq j_n$.
- if $j = j_n$ only one $d_{j,k}$ does not vanish, and the corresponding k_n is such that $|2^{j_n} x_0 - k_n| \leq 2$, in which case

$$d_{j_n,k_n} = 2^{-(s-d/p)j_n} b_{j_n}.$$

The sequence b_j belongs to l^q so that $g \in B_p^{s,q}$. The probe used is the one-dimensional subspace spanned by g. Let f be an arbitrary function in $B_p^{s,q}$. Let us assume that there exist λ_1 and λ_2 such that $f + \lambda_1 g$ and $f + \lambda_2 g$ share θ as modulus of continuity at x_0. Using Proposition 1.1 of [12], the wavelet coefficients of $f + \lambda_1 g$ and of $f + \lambda_2 g$ are a $O(\theta(2^{-j}))$ inside the cone $|2^j x_0 - k| \leq 2$. Therefore, there exist constants c, c' such that, inside this cone,

$$|c_{j,k} + \lambda_1 d_{j,k}| \leq c\theta(2^{-j}) \quad \text{and} \quad |c_{j,k} + \lambda_2 d_{j,k}| \leq c'\theta(2^{-j}).$$

In particular

$$|\lambda_1 - \lambda_2||d_{j_n,k_n}| \leq (c + c')\theta(2^{-j_n}) \leq (c + c')a_{j_n} 2^{-(s-\frac{d}{p})j_n}.$$

It follows that $\forall n \geq 0$, $|\lambda_1 - \lambda_2| \leq 2^{-\frac{n}{2}}$ so that $\lambda_1 = \lambda_2$. Thus, each line $f + \lambda g$ has at most one λ such that the modulus of continuity of $f + \lambda g$ at x_0 is a $O(\theta)$.

The case $L^{p,s}$ can be proved the same way using the function g build for $B^{s,1}_p$. ■

2.3. The irregular case

We will now consider the Besov spaces $B^{s,q}_p$ with $s - \frac{d}{p} < 0$ or $s - \frac{d}{p} = 0$ and $q > 1$ and the Sobolev spaces $L^{p,s}$ with $s - \frac{d}{p} \leq 0$. Those spaces share the following property: In each of them, there exists a function g which is nowhere locally bounded, see [15] or [16], i.e.

$$\forall x \in \mathbb{R}^d,\ \forall r > 0 \quad g.\mathbf{1}_{B(x,r)} \notin L^\infty.$$

In order to prove prevalent results of nowhere boundedness, we need to construct such functions defined by explicit values of their wavelet coefficients.

We assume that $s = d/p$ and $q > 1$ (this is the hardest case; if $s < d/p$, the cases of $B^{s,q}_p$ or $L^{p,s}$ are easier to handle). We use in the following compactly supported smooth wavelets issued from a multiresolution analysis such that the associated function φ is also compactly supported, see [14]. Let $P_j(f)$ denote the orthogonal projection on the space V_j. We have

$$P_j(f)(x) = \sum_{k \in \mathbb{Z}^d} 2^{dj} \int f(t)\varphi(2^j t - k)dt\varphi(2^j x - k).$$

Recall that f is bounded at x if

$$\exists r > 0 \quad f.\mathbf{1}_{B(x,r)} \in L^\infty.$$

The following lemma follows immediately from the hypothesis of compact support for φ.

Lemma 5. *If $f \in L^1$ and if f is bounded at x_0, then there exists $r > 0$ such that the sequence $P_j(f)$ is uniformly bounded on $B(x_0, r)$.*

We now start the construction of nowhere bounded functions. Since the $2^d - 1$ wavelets are continuous we can pick one of them (which we denote by ψ) which (after perhaps an integer translation) satisfies the following condition: There exist μ_0 dyadic subcube of $[0,1]^d$, and a constant $C > 0$ such that

$$\forall x \in \mu_0, \quad \psi(x) \geq C.$$

Furthermore, we can assume that the support of $\psi(2^l x)$ is supported in a dyadic cube of width less than 1/2. We first construct a function f which is not bounded at a point $x_0 \in [0,1]^d$.

Denote by l the scale of μ_0 ($|\mu_0| = 2^{-l}$); f has all its wavelet coefficients vanishing except for the scales which are nonnegative multiples of l, in which case, f will have only one nonvanishing wavelet coefficient. If $j = 0$ it corresponds to $k = 0$ and has value 1. It follows that $P_0(f) \geq C$ on μ_0. At scale l, the nonvanishing wavelet coefficient corresponds to the wavelet indexed by μ_0. The corresponding wavelet coefficient has size 1/2. There exists a subcube μ_1 of scale $2l$ where ψ_{μ_0} is larger than C. We continue this procedure, thus constructing a sequence of nested dyadic cubes μ_n such that the associated wavelet coefficient has size $1/k$ so that

$$\forall x \in \mu_n, \quad P_{nl}(f)(x) \geq C \sum_{k=1}^{n} \frac{1}{k}.$$

Therefore f is not bounded at x_0, but one immediately checks that $f \in B_p^{d/p,q}$.

We now deduce from f another function g which is nowhere locally bounded on $[0,1]^d$. The function $f - P_j(f)$ is supported inside a dyadic cube of width 1/2 included in $[0,1]^d$. We denote by $\tilde{f}_i$ its 2^d translates by translations of vectors in $(1/2)\mathbb{Z}^d$ which are also supported inside $[0,1]^d$. The supports of these functions do not intersect.

Let J_1 be such that $P_{J_1} f(x) \geq 1$ on μ_{J_1}. Up to the scale 1, g has the same wavelet coefficients as f. We consider now $P_{J_1}(f) + 4^{-d} \sum \tilde{f}_i$. There exists an index J_2 such that this function is larger than 2 on 2^d dyadic subcubes of each of the subcubes of width 1/2. The function g has the same wavelet coefficients as this function up to the index J_2. This construction is continued and clearly yields the required example. For every $x \in (0,1)^d$ and for every $r > 0$, $B(x,r)$ intersect one of dyadic cubes μ_j where $P_j(g)$ is greater than j.

Now, using Lemma 5, we show that the set M of function f such that there exists $x \in \mathbb{R}^d$, $r > 0$ and $(P_j(f))_j$ is uniformly bounded on $B(x,r)$ is Haar null. Since $P_j(f)$ is a continuous function, M is included in a countable union over a dense sequence $(x_n)_n$ and countable $r_n > 0$ of functions f such that the sequence $(P_j(f))$ is uniformly bounded on $B(x,r)$. Since P_j is a continuous linear operator, this set is Borelian, as a countable union of the continuous inverse of closed sets.

We take for probe the subspace spanned by g. Let f be an arbitrary given function in $L^{p,s}$ or in $B_p^{s,q}$, and let

$$A = \{\lambda : \ \exists x \in \mathbb{R}^d, \ \exists r > 0 \ \ (P_j(f + \lambda g))_j \ \text{ is uniformly bounded on } B(x,r)\}.$$

If A is not a countable set there exist $\lambda_1 \neq \lambda_2$ such that the two corresponding balls $B(x_1, r_1)$ and $B(x_2, r_2)$ have an intersection (denoted by I) with a non-empty interior. But $P_j(f+\lambda_1 g)$ and $P_j(f+\lambda_2 g)$ are uniformly bounded on I, and therefore $(\lambda_1-\lambda_2)P_j(g)$, is also bounded in I which contradicts the fact that $P_j(g)$ is nowhere locally bounded. Hence the first point of Theorem 1 holds, as well as the corresponding result in the Besov case.

Remark. The last part of the argument would work using any nowhere locally bounded function; but the first part requires the explicit wavelet construction.

2.4. The critical Besov case

The case of the Besov spaces with critical exponents $s = d/p$ and $p \le q \le 1$ still has to be considered. Functions in these spaces are bounded (and even continuous). Let $G_H = \{x,\ h_f(x) \le H\}$; then

$$\text{if} \quad p \le q \le 1, \quad d(H) \le qH,$$

see [13]. The proof of the second point of Theorem 2 will essentially follow the proof of the third point (where $s - d/p > 0$). As in this proof, the set of point with a given Hölder exponent is included in a Haar null Borel set. Namely, we will prove that the set of function satisfying (1.6) at a point x (with an given Hölder exponent γ which will be fixed later), is a Haar null Borel set. We have already proved it is a Borel set and the use of a probe implies that the measure is implicitly compactly supported.

We construct a number M large enough of functions g_i defined by their wavelet coefficients as follows: The g_i are lacunaries wavelet series; they have at most one coefficient different from zero at each scale, of size

$$a(j) = [j(\log j)^2]^{-1/q}; \tag{2.18}$$

we define

$$g_i(x) = \sum_{j=1}^{\infty} \varepsilon_j a(j)\psi(2^j x - m_{i,j}) \tag{2.19}$$

with $\varepsilon_j = 0$ or 1 (these values will be fixed later), and $m_{i,j}$ satisfy the following conditions: Let (r_n) be the sequence defined by $r_1 = 1$ and $r_{n+1} = dr_n + 1$. When j is such that

$$2^{r_n} < j \le 2^{r_n} + 2^{dr_n},$$

for each i, $\frac{m_{i,j}}{2^j}$ takes all 2^{dr_n} values $\frac{k}{2^{r_n}} \in [0,1]^d$. We choose $m_{i,j}$ such that

$$\forall j \ \ i \ne i' \ \Rightarrow \ m_{i,j} \ne m_{i',j} \quad \text{but} \quad |m_{i,j} - m_{i',j}| \le M. \tag{2.20}$$

Note that our choice for the r_n implies that the intervals $[2^{r_n}+1, 2^{r_n}+2^{dr_n}]$ do not intersect. In (2.19) we take $\varepsilon_j = 1$ if there exists n such that $j \in [2^{r_n}+1, 2^{r_n}+2^{dr_n}]$ and $\varepsilon_j = 0$ else. The functions g_i clearly belong to $B_p^{s,q}$.

A point x_0 is said to be α-approximable by dyadics *at the scales* r_n if there exists an infinite number of values of n and a sequence (k_n) such that, for these values of n,

$$\left| x_0 - \frac{k_n}{2^{r_n}} \right| \leq \frac{c}{2^{\alpha r_n}} \tag{2.21}$$

If such is the case, by definition of $m_{i,j}$,

$$\forall i \;\; \exists j \in]2^{r_n}+1, 2^{r_n}+2^{dr_n}], \;\; \exists m_{i,j} : \;\; \frac{k_n}{2^{r_n}} = \frac{m_{i,j}}{2^j} \tag{2.22}$$

and the corresponding wavelet coefficient is such that

$$d^i_{j,m_{i,j}} = a(j) = \frac{1}{(j(\log j)^2)^{\frac{1}{q}}} \geq \frac{c}{2^{dr_n/q}(dr_n)^{2/q}} \tag{2.23}$$

(since $j \leq 2^{dr_n} + 2^{r_n}$).

As usual, let

$$f_\beta = f + \sum_{i=1}^{M} \beta^i g^i,$$

where f is an arbitrary function in $B^{s,q}_p$.

Recall that $E^\alpha_{r_n}$ is the set of points x_0 such that

$$\exists k : \;\; \left| x_0 - \frac{k}{2^{r_n}} \right| \leq \frac{1}{2^{\alpha r_n}}. \tag{2.24}$$

Assume that x and y are in the same cube defined by (2.24) and that, simultaneously, f_β satisfies (1.6) at x and $f_{\tilde{\beta}}$ satisfies (1.6) at y, with $\gamma > d/q\alpha$. We will use the following smoothness criterium from [7].

Lemma 6. *If f is (C,γ) smooth at x_0, then*

$$\forall j,k \quad |c_{j,k}| \leq CC' \left(2^{-j} + \left| x_0 - \frac{k}{2^j} \right| \right)^\gamma$$

where C' depends only the wavelet chosen.

Therefore, at the scale j defined in (2.22)

$$\forall k, \quad \left| c_{j,k} + \sum_{i=1}^{M} \beta^i d^i_{j,k} \right| \leq C \left(2^{-j} + \left| x - \frac{k}{2^j} \right| \right)^\gamma .$$

and

$$\forall k, \quad \left| c_{j,k} + \sum_{i=1}^{M} \tilde{\beta}^i d^i_{j,k} \right| \leq C \left(2^{-j} + \left| y - \frac{k}{2^j} \right| \right)^{\gamma}.$$

Taking $k = m_{i,j}$, the right hand sides are bounded by

$$c(2^{-j} + 2^{-\alpha r_n})^{\gamma} \leq c' 2^{-\alpha\gamma r_n}$$

(as $j \geq 2^{r_n}$). Following the argument already developed above, it follows that

$$\|\beta\| a(j) \leq c' 2^{-\alpha\gamma r_n}$$

so that, using (2.23), and since $\gamma > d/q\alpha$,

$$\|\beta\| \leq c 2^{-\delta r_n} \text{ for a } \delta > 0.$$

We conclude as usual that, for M large enough, the Lebesgue measure of the set of β considered is zero. The end of the proof follows, using the fact that points α-approximable at scale r_n have a d/α-dimensional Hausdorff measure which is strictly positive, see [9].

2.5. Almost everywhere modulus of continuity

Proposition 3. *Let $s > d/p$ and let θ be a modulus of continuity such that*

$$\theta(2^{-j}) = 2^{-sj}\omega_j \qquad \text{with } \omega_j \in l^q; \tag{2.25}$$

(which is well defined if $q < \infty$); then, almost every function f in $B^{s,q}_p$, or $L^{p,s}$, satisfies the following property (in the Sobolev case, $q = 1$ must be picked in (2.25)):

For a.e. x_0, the modulus of continuity of f at x_0 is not a $o(\theta)$.

Proof. First note that the set of functions which is considered clearly is a closed set so that it is a Borelian set. If $q \neq \infty$, the measure defined by the following stochastic process is clearly compactly supported. Indeed, it can be seen as the continuous image of a compact set. On $[0,1]^d$ we consider the stochastic process

$$X_x = \sum_{j=0}^{\infty} \sum_{\lambda \in [0,1]^d} \varepsilon_{j,k} 2^{-sj} \omega_j \psi(2^j x - k)$$

where $\{\varepsilon_{j,k}\}_{j,k}$ is a Rademacher sequence, that is the $\varepsilon_{j,k}$ are I.I.D. random variables such that

$$\mathbb{P}(\varepsilon_{j,k} = 1) = \mathbb{P}(\varepsilon_{j,k} = -1) = \frac{1}{2}.$$

One checks immediately that $X_x \in B^{s,q}_p$ and $X_x \in L^{p,s}$ if $q = 1$.

Let f be an arbitrary function in $B_p^{s,q}$ (or $L^{p,s}$). Following the second remark after Definition 1, we will prove that, almost surely the following result holds:

For almost every x_0 fixed, $f(x)+X_x$ has not at x_0 a modulus of continuity which is a $o(\theta)$. Using Fubini's theorem, it is sufficient to prove that, if f and x_0 are arbitrary and fixed, with probability 1, $f(x)+X_x$ has not a modulus of continuity which is a $o(\theta)$. Indeed, if such was the case, using Proposition 1.1 of [12], it would follow that

$$c_{j,k} + \varepsilon_{j,k} 2^{-sj} \omega_j = o(\theta(2^{-j}))$$

inside the cone $|2^j x - k| \leq 2$. Since θ is a modulus of continuity, $\omega_j \neq 0$ so that

$$\varepsilon_{j,k} = \frac{-c_{j,k} 2^{-sj}}{\omega_j} + o(1).$$

Since the $c_{j,k}$ are fixed (deterministic), this result implies that for j large enough all $\varepsilon_{j,k}$ inside the cone have a fixed, deterministic value, which is of probability zero. The case of $[0,1]^d$ or $\mathbb{R}^d$ follows using countable intersection. ∎

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