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Asymptotic behaviour of monomial ideals on regular sequences

Monireh Sedghi

Abstract

Let R be a commutative Noetherian ring, and let $\mathbf{x} = x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. An ideal I of R is said to be a monomial ideal with respect to $\mathbf{x}$ if it is generated by a set of monomials $x_1^{e_1} \dots x_d^{e_d}$. The monomial closure of I, denoted by $\widetilde{I}$, is defined to be the ideal generated by the set of all monomials m such that $m^n \in I^n$ for some $n \in \mathbb{N}$. It is shown that the sequences $\mathrm{Ass}_R R/\widetilde{I^n}$ and $\mathrm{Ass}_R \widetilde{I^n}/I^n$, $n = 1, 2, \dots$, of associated prime ideals are increasing and ultimately constant for large n. In addition, some results about the monomial ideals and their integral closures are included.

1. Introduction

Let R be a commutative Noetherian ring and I an ideal of R. In [9], L. J. Ratliff Jr., conjectured about the asymptotic behaviour of $\mathrm{Ass}_R R/I^n$ when R is a domain. Subsequently, M. Brodmann [1] showed that if R is Noetherian and M is a finitely generated R-module, then $\mathrm{Ass}_R M/I^nM$ is ultimately constant for large $n \in \mathbb{N}$. Furthermore, in [9], L. J. Ratliff Jr. showed that the sequence $\{\mathrm{Ass}_R R/(I^n)_a\}_{n\in\mathbb{N}}$ is increasing and ultimately constant, where $(I^n)_a$ denotes the classical integral closure of the ideal I^n. (We use $\mathbb{N}$ to denote the set of positive integers.)

The results of Brodmann and Ratliff have led to a large body of research. For example, McAdam and Ratliff (see [7, 3.9 and 11.16]) showed that, if the ideal I of R contains a non-zero divisor on R, then the sequences

$$\{\mathrm{Ass}_R R/(I^n)_a\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\mathrm{Ass}_R (I^n)_a/I^n\}_{n\in\mathbb{N}}$$

are increasing and eventually constant.

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There is considerable current interest in the concept of monomial ideal in a commutative Noetherian ring. Let R be a commutative Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence which is contained in the Jacobson radical of R. An ideal I of R is called a *monomial ideal* with respect to $\mathbf{x}$ if it can be generated by monomials $x_1^{e_1} \dots x_d^{e_d}$. For any monomial ideal I with respect to $\mathbf{x}$, the *monomial closure* of I, denoted by $\widetilde{I}$, is defined the ideal generated by all monomials m with $m^l \in I^l$ for some $l \in \mathbb{N}$.

The above-mentioned results of Brodmann, McAdam and Ratliff raise questions about asymptotic behaviour on monomial ideals on R-sequences. As the main result of the second section we shall prove the following:

Theorem 1.1. *Let I denote a monomial ideal of R with respect to a regular R-sequence contained in the Jacobson radical of R. Then the sequences*

$$\{\operatorname{Ass}_R R/\widetilde{I^n}\}_{n\in\mathbb{N}} \quad and \quad \{\operatorname{Ass}_R \widetilde{I^n}/I^n\}_{n\in\mathbb{N}}$$

of associated primes are increasing for large n and eventually constant.

For any regular R-sequence $\mathbf{x} := x_1, \dots, x_d$ contained in the Jacobson radical of R, let $(\mathbf{x})$ be a prime ideal of R. In third section we will obtain some results about the integral closure and monomial closure of a certain monomial ideal in R. In this section, among other things, we prove the following theorem.

Theorem 1.2. *Let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R, let $(\mathbf{x})$ be a prime ideal of R and let τ be a permutation of $\{1, \dots, d\}$. Let $I := (x_{\tau(1)}^{\alpha_1}, \dots, x_{\tau(s)}^{\alpha_s})$ be a monomial ideal of R where $1 \le s \le d$ and $\alpha_i \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, I^n and $\widetilde{I^n}$ are $(x_{\tau(1)}, \dots, x_{\tau(s)})$-primary ideals and $(I^n)_a = \widetilde{I^n}$.*

The proof of Theorem 1.2 is given in 3.4. For any unexplained notation and terminology we refer the reader to [6] and [2].

2. Finiteness of associated primes

In this section we show that if $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R, and I is a monomial ideal of R with respect to $\mathbf{x}$ then the sets $\operatorname{Ass}_R R/\widetilde{I^n}$ and $\operatorname{Ass}_R \widetilde{I^n}/I^n$ are constant for large n. The main results are Theorems 2.5 and 2.7. We begin with

Definition 2.1. Let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence of elements of R. Then:

(i) An element m of R is called a *monomial element w.r.t.* $\mathbf{x}$ if there exist non-negative integers $e_1, \dots, e_d$ such that $m = x_1^{e_1} \dots x_d^{e_d}$. In view of [6, Thm. 16.2] it is easy to see that $e_1, \dots, e_d$ are determined uniquely by m.

(ii) Suppose that $m = x_1^{e_1} \dots x_d^{e_d}$ is a monomial w.r.t. $\mathbf{x}$. The *support of* m, denoted by $\operatorname{supp}(m)$, is defined to be the set $\{j \mid j \in \{1, \dots, d\}$ and $e_j \neq 0\}$.

(iii) Let $\mathcal{M}$ denote the set of all monomials of R w.r.t. $\mathbf{x}$. An ideal I of R is called *monomial w.r.t.* $\mathbf{x}$ if it is generated by elements in $\mathcal{M}$. It follows from this that the zero ideal and R itself are monomial ideals.

(iv) Let I be a monomial ideal of R w.r.t. $\mathbf{x}$. The *monomial closure of* I, denoted by $\widetilde{I}$, is defined to be the ideal generated by all monomials m in $\mathcal{M}$ such that $m^n \in I^n$ for some $n \in \mathbb{N}$. Let $(I)_a$ denote the classical integral closure of I introduced by Northcott and Rees [8]. Then obviously $I \subseteq \widetilde{I} \subseteq (I)_a$. Also, if J is a second monomial ideal of R w.r.t. $\mathbf{x}$, then $\widetilde{\widetilde{I}\widetilde{J}} = \widetilde{IJ}$, and if $I \subseteq J$ then $\widetilde{I} \subseteq \widetilde{J}$. Recall that, by [5, Proposition 1], the set of all monomials of R is closed under finite products.

(v) Suppose that s is an integer such that $1 \leq s \leq d$, and let τ be a permutation of $\{1, \dots, d\}$ and let $e_1, \dots, e_s$ be positive integers. The monomial ideal generated by $x_{\tau(1)}^{e_1}, \dots, x_{\tau(s)}^{e_s}$ is called a *generalized-parameter ideal*.

Throughout this section, $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R w.r.t. $\mathbf{x}$, unless otherwise specified.

The following lemma plays a key role in the proof of the main results.

Lemma 2.2. *Let I be a non-zero monomial ideal of R. Then for all large n,*

$$(\widetilde{I^n} :_R \widetilde{I^k}) = (\widetilde{I^n} :_R I^k) = \widetilde{I^{n-k}},$$

for every integer k.

Proof. As $\widetilde{I^{n-k}} I^k \subseteq \widetilde{I^{n-k}} \widetilde{I^k} \subseteq \widetilde{I^n}$, it is enough to show that $(\widetilde{I^n} :_R I^k) \subseteq \widetilde{I^{n-k}}$. To this end, it will suffice to show that $(\widetilde{I^{n+1}} :_R I) \subseteq \widetilde{I^n}$ for all large n. Let $x \in R$ be such that $Ix \subseteq \widetilde{I^{n+1}}$ and let I be generated by s elements. Then there exists an integer $l \geq 1$ such that $x^l I^{sl} \subseteq I^{(n+s)l}$. Hence $x^l \in (I^{nl+sl} :_R I^{sl})$, and so in view of [7, Lemma 8.1], $x^l \in I^{nl}$. Therefore by definition $x \in \widetilde{I^n}$, as desired. ■

Before bringing the next result we fix some notations. Let U be an arbitrary subset of $\{1, \dots, d\}$. We use $\mathfrak{q}_U$ to denote the ideal of R generated by the set $\{x_i : i \in U\}$ and $\mathcal{P}_U$ to denote the set $\operatorname{Ass}_R R/\mathfrak{q}_U$. In particular, if $U = \{1, \dots, d\}$ then we write,

$$\mathfrak{q} := \mathfrak{q}_U \text{ and } \mathcal{P} := \operatorname{Ass}_R R/\mathfrak{q}.$$

The following lemma is originally shown by K. Kiyek and J. Stückrad in [5, Lemma 1], we give a direct proof by using the important notion of the generalized-parameter monomial ideals introduced by W. Heinzer *et al.* [3].

Lemma 2.3. *Let I be a monomial ideal of R generated by monomials $m_1, \ldots, m_r$. Then*

$$\operatorname{Ass}_R R/I \subseteq \bigcup \{\mathcal{P}_U : U \subseteq \operatorname{supp}(m_1) \cup \cdots \cup \operatorname{supp}(m_r)\}.$$

Proof. Since $\operatorname{Ass}_R R = \mathcal{P}_\emptyset$, it is enough to consider $I \neq 0$. Then by [5, Lemma 3] and [3, Theorem 4.9] I is an irredundant finite intersection of generalized-parameter monomial ideals which is unique up to the order of the factors. Hence without loss of generality we may assume that I is a generalized-parameter monomial ideal w.r.t. $\mathbf{x}$. Because $\mathbf{x}$ is contained in the Jacobson radical of R, we can assume $I = (x_1^{e_1}, \ldots, x_s^{e_s})$ where $1 \leq s \leq d$ and $e_i \in \mathbb{N}$ for all $i = 1, \ldots, s$. If $I = (x_1, \ldots, x_s)$ then the assertion follows. Now, let $e_i \geq 2$ for some $i = 1, \ldots, s$. As $\mathbf{x}$ is contained in the Jacobson radical of R, we may assume that $i = 1$. Consider $J = (x_1^{e_1 - 1}, x_2^{e_2}, \ldots, x_s^{e_s})$ and let $\mu : R/J \longrightarrow R/I$ denote the multiplication homomorphism by x_1. One can easily see that μ is a monomorphism, and then from the exact sequence,

$$0 \longrightarrow R/J \longrightarrow R/I \longrightarrow R/(x_1, x_2^{e_2}, \ldots, x_s^{e_s}) \longrightarrow 0$$

we have

$$\operatorname{Ass}_R R/I \subseteq \operatorname{Ass}_R R/J \cup \operatorname{Ass}_R R/(x_1, x_2^{e_2}, \ldots, x_s^{e_s}).$$

Hence the assertion follows by induction. ■

Lemma 2.4. *Let I be a non-zero monomial ideal of R. Then for all large n,*

$$\operatorname{Ass}_R R/\widetilde{I^n} = \operatorname{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}.$$

Proof. For large n, let $\mathfrak{p} \in \operatorname{Ass}_R R/\widetilde{I^n}$. Then there exists $x \in R \setminus \widetilde{I^n}$ such that $\mathfrak{p} = (\widetilde{I^n} :_R x)$. Since $I \subseteq \mathfrak{p}$, it follows that $x \in (\widetilde{I^n} :_R I)$. Hence by Lemma 2.2, $x \in \widetilde{I^{n-1}}$ and so $\mathfrak{p} \in \operatorname{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}$. This completes the proof. ■

We can now state and prove our the main results, which concern the asymptotic stability of $\operatorname{Ass}_R R/\widetilde{I^n}$ and $\operatorname{Ass}_R \widetilde{I^n}/I^n$.

Theorem 2.5. *Let I be a monomial ideal of R. Then the sequence*

$$\{\operatorname{Ass}_R R/\widetilde{I^n}\}_{n \geq 1}$$

of associated prime ideals, is increasing for large n and eventually constant.

Proof. For large n, let $\mathfrak{p} \in \mathrm{Ass}_R R/\widetilde{I^n}$. Then there exists $x \in R \setminus \widetilde{I^n}$ such that $\mathfrak{p} = (\widetilde{I^n} :_R x)$. Hence in view of Lemma 2.2, $\mathfrak{p} = (\widetilde{I^{n+1}} :_R Ix)$. Therefore $\mathfrak{p} \in \mathrm{Ass}_R R/\widetilde{I^{n+1}}$. It follows that the sequence $\{\mathrm{Ass}_R R/\widetilde{I^n}\}_{n\geq 1}$ is increasing for all large n. On the other hand by Lemma 2.3, the set $\bigcup_{n\geq 1} \mathrm{Ass}_R R/\widetilde{I^n}$ is finite. Consequently the assertion now follows. ■

An immediate consequence of Theorem 2.5 is the following.

Corollary 2.6. *Let I be a monomial ideal of R. Then the sequences*

$$\{\mathrm{Ass}_R (I^n)_a/\widetilde{I^n}\}_{n\geq 1} \text{ and } \{\mathrm{Ass}_R \widetilde{I^{n-1}}/\widetilde{I^n}\}_{n\geq 1}$$

of associated primes are increasing for large n and eventually constant.

Theorem 2.7. *Let I be a monomial ideal of R. Then the sequence*

$$\{\mathrm{Ass}_R \widetilde{I^n}/I^n\}_{n\geq 1}$$

is increasing for large n and ultimately constant.

Proof. In view of Brodmann's result [1], it is enough to show that the sequence $\{\mathrm{Ass}_R \widetilde{I^n}/I^n\}_{n\geq 1}$ is increasing for large n. To do this, by [7, Lemma 8.1] for all large n we have $(I^{n+1} :_R I) = I^n$. Now for large n, let $\mathfrak{p} \in \mathrm{Ass}_R \widetilde{I^n}/I^n$ and we write $\mathfrak{p} = (I^n :_R x)$ for some $x \in \widetilde{I^n} \setminus I^n$. Then we have

$$\mathfrak{p} = ((I^{n+1} :_R I) :_R x) = (I^{n+1} :_R Ix).$$

Since $Ix \subseteq \widetilde{I^{n+1}}$, it follows that there exists $y \in \widetilde{I^{n+1}}$ such that $\mathfrak{p} = (I^{n+1} :_R y)$. This shows that $\mathfrak{p} \in \mathrm{Ass}_R \widetilde{I^{n+1}}/I^{n+1}$ for all large n, and so the assertion follows. ■

3. Some results about monomial ideals

Recall that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R.

Remark 3.1. Let $\mathfrak{q} := (\mathbf{x})$ be a prime ideal of R. Then it follows from [2, Theorem 1.1.8] that the associated graded ring $\oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$ is a domain. As $\mathfrak{q}$ is contained in the Jacobson radical of R, we have $\cap_{n\geq 1}\mathfrak{q}^n = 0$, and hence R is also a domain. Indeed if x and y are non-zero elements of R with $xy = 0$, we may choose integers m, n such that $x \in \mathfrak{q}^m \setminus \mathfrak{q}^{m+1}$ and $y \in \mathfrak{q}^n \setminus \mathfrak{q}^{n+1}$. Then $(x + \mathfrak{q}^{m+1})(y + \mathfrak{q}^{n+1}) = 0$ and this is a contradiction. Moreover, since $\mathfrak{q}$ is a prime ideal of R generated by *height* $\mathfrak{q}$ elements it follows that $R_\mathfrak{q}$ is a regular local ring. So that, if $U \subseteq \{1, \ldots, d\}$ and $\mathfrak{q}_U$ is an ideal of R generated by $\{x_i : i \in U\}$, then $\mathfrak{q}_U R_\mathfrak{q}$ is a prime ideal of $R_\mathfrak{q}$.

Thus $\mathfrak{q}_U$ will be a prime ideal of R. Now, let τ be a permutation on $\{1,\ldots,d\}$ and $I := (x_{\tau(1)}^{e_1},\ldots,x_{\tau(s)}^{e_s})$ a generalized-parameter monomial ideal. Then $\mathrm{Ass}_R\, R/I = \{(x_{\tau(1)},\ldots,x_{\tau(s)})\}$ by Lemma 2.3. As $(x_{\tau(1)},\ldots,x_{\tau(s)})$ is a prime ideal it follows that I is a $(x_{\tau(1)},\ldots,x_{\tau(s)})$-primary monomial ideal of R. Furthermore, since I is generated by a regular R-sequence it follows from [4, Thm. 125 and Ex. 13] that $\mathrm{Ass}_R\, R/I^n = \mathrm{Ass}_R\, R/I$ for any $n \in \mathbb{N}$, and so I^n is also a $(x_{\tau(1)},\ldots,x_{\tau(s)})$-primary monomial ideal of R.

Proposition 3.2. *Let $\mathfrak{q} = (\mathbf{x})$ be as in Remark* 3.1 *and let I be a non-zero monomial ideal of R. Then I has a primary decomposition each primary component of which is monomial ideal.*

Proof. In view of [3, Theorem 4.9] I has a unique generalized-parametric monomial decomposition. Now since $\mathfrak{q} = (\mathbf{x})$ is a prime ideal it follows that from Remark 3.1 that the mentioned decomposition is the desired primary decomposition of I. ■

Lemma 3.3. *Let $\mathfrak{q} = (\mathbf{x})$ and let $I := (x_{\tau(1)}^{e_1},\ldots,x_{\tau(s)}^{e_s})$ be as in Remark* 3.1. *Then $\widetilde{I}$ is a primary ideal of R.*

Proof. Let $\mathfrak{p} := (x_{\tau(1)},\ldots,x_{\tau(s)})$. Then by Remark 3.1 we have $\mathrm{Ass}_R\, R/I = \{\mathfrak{p}\}$. Let $\widetilde{I} := (m_1,\ldots,m_r)$ where $m_i \in \mathcal{M}$ for all $i = 1,\ldots,r$. In view of Lemma 2.3 and Remark 3.1, it is enough to show that $\mathrm{supp}(m_i) \subseteq \{\tau(1),\ldots,\tau(s)\}$ for each $1 \leq i \leq r$. Assume the contrary. Then there exists $e \in \mathbb{N}, 1 \leq j \leq r$ and $1 \leq t \leq d$ such that $m_j = x_t^e m'$ where $m' \in \mathcal{M}$, $\mathrm{supp}(m') \subseteq \{\tau(1),\ldots,\tau(s)\}$, $m' \notin \widetilde{I}$ and $t \notin \{\tau(1),\ldots,\tau(s)\}$. Then there is an integer k such that $m_j^k \in I^k$. Hence $x_t^{ek} \in (I^k :_R m'^k)$. As $m'^k \notin I^k$, it follows that $x_t^{ek} \in \mathfrak{p}$. So that $x_t \in \mathfrak{p}$, which is contradiction. ■

The following result, which was shown by K. Kiyek and J. Stuckrad in [5, Proposition 7], extends the original argument that the integral closure of a monomial ideal in a polynomial ring over a field in a finite number of indeterminates is a monomial ideal (see [11, section 6.6, Ex. 6.6.1]).

Theorem 3.4. *Let $\mathfrak{q} := (\mathbf{x})$ be a prime ideal of R and let τ be a permutation on $\{1,\ldots,d\}$. Let $I := (x_{\tau(1)}^{e_1},\ldots,x_{\tau(s)}^{e_s})$ be a generalized-parameter monomial ideal of R. Then for any $n \in \mathbb{N}$, I^n and $\widetilde{I^n}$ are $(x_{\tau(1)},\ldots,x_{\tau(s)})$-primary monomial ideals of R and $(I^n)_a = \widetilde{I^n}$.*

Proof. By virtue of Remark 3.1 and Lemma 3.3, it will suffice to show that $(I^n)_a = \widetilde{I^n}$ for each $n \in \mathbb{N}$. To this end, let $\mathfrak{p} := (x_{\tau(1)},\ldots,x_{\tau(s)})$. Then since $\mathfrak{p}$ is generated by a regular R-sequence it follows that $R_\mathfrak{p}$ is a regular local ring and $I^n R_\mathfrak{p}$ is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal. Hence by [3, (2.2.5)] it yields that $(I^n R_\mathfrak{p})_a = \widetilde{I^n R_\mathfrak{p}}$. Now the assertion follows easily from [10, Lemma 2.3]. ■

For any monomial ideal I of R with respect to the regular R-sequence $\mathbf{x} := x_1, \ldots, x_d$, we shall use $\widetilde{A}(I)$ to denote the ultimate constant values of the sequence $\{\mathrm{Ass}_R\, R/\widetilde{I^n}\}_{n\geq 1}$.

Proposition 3.5. *Let T be a faithfully flat Noetherian extension of R such that $\mathbf{x} := x_1, \ldots, x_d$ is contained in the Jacobson radical of T. Then for any monomial ideal I of R the following conditions hold:*

(i) *IT is a monomial ideal of T and $\widetilde{IT} \cap R = \widetilde{I}$.*

(ii) *For any $\mathfrak{p} \in \widetilde{A}(I)$ there exists $\mathfrak{q} \in \widetilde{A}(IT)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. In order to prove (i), let m be a monomial element of T with respect to $\mathbf{x} := x_1, \ldots, x_d$ (recall that $\mathbf{x}$ is a T-sequence by faithfully flatness) such that $m \in \widetilde{IT} \cap R$. Then $m^l \in I^l T$ for some $l \in \mathbb{N}$. Since T is faithful over R, it follows that $m^l \in I^l$. Hence $\widetilde{IT} \cap R \subseteq \widetilde{I}$. As the opposite inclusion is obvious, the result follows. Finally, in order to show (ii), let $\mathfrak{p} \in \widetilde{A}(I)$. Then $\mathfrak{p} \in \mathrm{Ass}_R\, R/\widetilde{I^n}$ for large n. Since T is a Noetherian ring and $\widetilde{I^n T} \cap R = \widetilde{I^n}$ by (i), it follows that there exists $\mathfrak{q} \in \widetilde{A}(IT)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. This completes the proof. ■

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Monireh Sedghi
Department of Mathematics
University of Tabriz
Tabriz, Iran

and

Department of Mathematics
Azarbaidjan University of Tarbiat Moallem
Azarshar, Tabriz, Iran
m.sedghi@tabrizu.ac.ir

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