

# $m$ -Berezin transform and compact operators

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**Abstract**

$m$ -Berezin transforms are introduced for bounded operators on the Bergman space of the unit ball. The norm of the  $m$ -Berezin transform as a linear operator from the space of bounded operators to  $L^\infty$  is found. We show that the  $m$ -Berezin transforms are commuting with each other and Lipschitz with respect to the pseudo-hyperbolic distance on the unit ball. Using the  $m$ -Berezin transforms we show that a radial operator in the Toeplitz algebra is compact iff its Berezin transform vanishes on the boundary of the unit ball.

## 1. Introduction

Let  $B$  denote the unit ball in  $n$ -dimensional complex space  $\mathbb{C}^n$  and  $dz$  be normalized Lebesgue volume measure on  $B$ . The Bergman space  $L_a^2 = L_a^2(B, dz)$  is the space of analytic functions  $h$  on  $B$  which are square-integrable with respect to Lebesgue volume measure. For  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , let  $\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$  and  $|z|^2 = \langle z, z \rangle$ .

For  $z \in B$ , let  $P_z$  be the orthogonal projection of  $\mathbb{C}^n$  onto the subspace  $[z]$  generated by  $z$  and let  $Q_z = I - P_z$ . Then

$$\phi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2} Q_z(w)}{1 - \langle w, z \rangle}$$

is the automorphism of  $B$  that interchanges 0 and  $z$ . The pseudo-hyperbolic metric on  $B$  is defined as  $\rho(z, w) = |\phi_z(w)|$ .

The reproducing kernel in  $L_a^2$  is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}},$$

for  $z, w \in B$  and the normalized reproducing kernel  $k_z$  is  $K_z(w) / \|K_z(\cdot)\|_2$ .

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If  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $L^2$ , then  $\langle h, K_z \rangle = h(z)$ , for every  $h \in L^2_a$  and  $z \in B$ . The fundamental property of the reproducing kernel  $K_z(w)$  plays an important role in this paper:

$$(1.1) \quad K_z(w) = \overline{k_\lambda(z)} K_{\phi_\lambda(z)}(\phi_\lambda(w)) k_\lambda(w).$$

Given  $f \in L^\infty$ , the Toeplitz operator  $T_f$  is defined on  $B$  by  $T_f h = P(fh)$  where  $P$  denotes the orthogonal projection  $P$  of  $L^2$  onto  $L^2_a$ .

Let  $\mathfrak{L}(L^2_a)$  be the algebra of bounded operators on  $L^2_a$ . The Toeplitz algebra  $\mathfrak{T}(L^\infty)$  is the closed subalgebra of  $\mathfrak{L}(L^2_a)$  generated by  $\{T_f : f \in L^\infty\}$ .

For  $z \in B$ , let  $U_z$  be the unitary operator given by

$$U_z f = (f \circ \phi_z) \cdot J\phi_z$$

where  $J\phi_z = (-1)^n k_z$ . For  $S \in \mathfrak{L}(L^2_a)$ , set

$$S_z = U_z S U_z.$$

Since  $U_z$  is a selfadjoint unitary operator on  $L^2$  and  $L^2_a$ ,  $U_z T_f U_z = T_{f \circ \phi_z}$  for every  $f \in L^\infty$ .

Let  $\mathcal{T}$  denote the class of trace operators on  $L^2_a$ . For  $T \in \mathcal{T}$ , we will denote the trace of  $T$  by  $tr[T]$  and let  $\|T\|_{C_1}$  denote the  $C_1$  norm of  $T$  given by ([12])

$$\|T\|_{C_1} = tr[\sqrt{T^*T}].$$

Suppose  $f$  and  $g$  are in  $L^2_a$ . Consider the operator  $f \otimes g$  on  $L^2_a$  defined by

$$(f \otimes g)h = \langle h, g \rangle f,$$

for  $h \in L^2_a$ . It is easily proved that  $f \otimes g$  is in  $\mathcal{T}$  and with norm equal to  $\|f \otimes g\|_{C_1} = \|f\|_2 \|g\|_2$  and

$$tr[f \otimes g] = \langle f, g \rangle.$$

For a nonnegative integer  $m$ , the  $m$ -Berezin transform of an operator  $S \in \mathfrak{L}(L^2_a)$  is defined by

$$(1.2) \quad \begin{aligned} B_m S(z) &= C_n^{m+n} tr \left[ S_z \left( \sum_{|k|=0}^m C_{m,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right) \right] \\ &= C_n^{m+n} tr \left[ S_z \left( \sum_{|k|=0}^m C_{m,k} u^k \otimes u^k \right) \right] \end{aligned}$$

where  $k = (k_1, \dots, k_n) \in N^n$ ,  $N$  is the set of nonnegative integers,  $|k| = \sum_{i=1}^n k_i$ ,  $u^k = u_1^{k_1} \dots u_n^{k_n}$ ,  $k! = k_1! \dots k_n!$ ,

$$C_n^{m+n} = \binom{m+n}{n} \quad \text{and} \quad C_{m,k} = C_{|k|}^m (-1)^{|k|} \frac{|k|!}{k_1! \dots k_n!}.$$

Clearly,  $B_m : \mathfrak{L}(L_a^2) \rightarrow L^\infty$  is a bounded linear operator, the norm of  $B_m$  will be given.

Given  $f \in L^\infty$ , define

$$B_m(f)(z) = B_m(T_f)(z).$$

$B_m(f)(z)$  equals the nice formula in [1]:

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u),$$

for  $z \in B$  where  $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$ .

Berezin first introduced the Berezin transform  $B_0(S)$  of bounded operators  $S$  and the  $m$ -Berezin transform of functions in [5]. Because the Berezin transform encodes operator-theoretic information in function-theory in a striking but somewhat impenetrable way, the Berezin transform  $B_0(S)$  has found useful applications in studying operators of "function-theoretic significance" on function spaces ([2], [3], [4], [6], [7], [11], and [15]). Suárez [16] introduced  $m$ -Berezin transforms of bounded operators on the Bergman space of the unit disk. We will show that our  $m$ -Berezin transform coincides with the one defined in [16] on the unit disk  $D$  by means of an integral representation of  $m$ -Berezin transform. The integral representation shows that many useful properties of the  $m$ -Berezin transforms inherit from the identity (1.1) of the reproducing kernel. On the unit ball, some useful properties of the  $m$ -Berezin transforms of functions were obtained by Ahern, Flores and Rudin [1]. Recently, Coburn [10] proved that  $B_0(S)$  is Lipschitz with respect to the pseudo-hyperbolic distance  $\rho(z, w)$ . In this paper, we will show that  $B_m S(z)$  is Lipschitz with respect to pseudo-hyperbolic distance  $\rho(z, w)$ . We will show that the  $m$ -Berezin transforms  $B_m$  are invariant under the Mobius transform,

$$(1.3) \quad B_m(S_z) = (B_m S) \circ \phi_z,$$

and commuting with each other,

$$(1.4) \quad B_j(B_m S)(z) = B_m(B_j S)(z)$$

for any nonnegative integers  $j$  and  $m$ . Properties (1.3) and (1.4) were obtained for  $S = T_f$  in [1] and for operators  $S$  on the Bergman space of the unit disk [16].

A common intuition is that for operators on the Bergman space  $L_a^2$  "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on the boundary of the unit ball  $B$ . On the

unit disk, Axler and Zheng [2] showed that if the operator  $S$  equals the finite sum of finite products of Toeplitz operators with bounded symbols then  $S$  is compact if and only if  $B_0(S)(z) \rightarrow 0$  as  $z \rightarrow \partial D$ . Englis extended this result to the unit ball even the bounded symmetric domains [11]. But the problem remains open whether the result is true if  $S$  is in the Toeplitz algebra. Recently, Suárez [17] solved the problem for radial operator  $S$  on the unit disk via the  $m$ -Berezin transform. Using the  $m$ -Berezin transform, we will show that for a radial operator  $S$  in the Toeplitz algebra on the unit ball,  $S$  is compact iff  $B_0S(z) \rightarrow 0$  as  $|z| \rightarrow 1$ .

Throughout the paper  $C(m, n)$  will denote constant depending only on  $m$  and  $n$ , which may change at each occurrence.

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## 2. $m$ -Berezin transform

In this section we will show some useful properties of the  $m$ -Berezin transform. First we give an integral representation of the  $m$ -Berezin transform  $B_m(S)$ . For  $z \in B$  and a nonnegative integer  $m$ , let

$$K_z^m(u) = \frac{1}{(1 - \langle u, z \rangle)^{m+n+1}}, \quad u \in B.$$

For  $u, \lambda \in B$ , we can easily see that

$$(2.1) \quad \sum_{|k|=0}^m C_{m,k} u^k \overline{\lambda^k} = (1 - \langle u, \lambda \rangle)^m.$$

**Proposition 2.1** *Let  $S \in \mathfrak{L}(L_a^2)$ ,  $m \geq 0$  and  $z \in B$ . Then*

$$B_m S(z) = C_n^{m+n} (1 - |z|^2)^{m+n+1} \times \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} dud\lambda.$$

**Proof.** For  $\lambda \in B$ , the definition of  $B_m$  implies

$$\begin{aligned} B_m S(z) &= C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \langle S_z \lambda^k, \lambda^k \rangle \\ &= C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \int_B S(\phi_z^k k_z)(\lambda) \overline{\phi_z^k(\lambda) k_z(\lambda)} d\lambda \\ (2.2) \quad &= C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \int_B \int_B \phi_z^k(u) k_z(u) \overline{\phi_z^k(\lambda) k_z(\lambda) S^* K_\lambda(u)} dud\lambda \end{aligned}$$

where the last equality holds by  $S(\phi_z^k k_z)(\lambda) = \langle S(\phi_z^k k_z), K_\lambda \rangle = \langle \phi_z^k k_z, S^* K_\lambda \rangle$ .

Using (2.1) and (1.1), (2.2) equals

$$\begin{aligned} & C_n^{m+n} \int_B \int_B (1 - \langle \phi_z(u), \phi_z(\lambda) \rangle)^m k_z(u) \overline{k_z(\lambda)} S^* K_\lambda(u) dud\lambda \\ &= C_n^{m+n} \int_B \int_B \left( \frac{k_z(u) \overline{k_z(\lambda)}}{K_\lambda(u)} \right)^{m/(n+1)} k_z(u) \overline{k_z(\lambda)} S^* K_\lambda(u) dud\lambda \\ &= C_n^{m+n} (1 - |z|^2)^{m+n+1} \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda)} S^* K_\lambda(u) dud\lambda \end{aligned}$$

as desired. ■

Proposition 2.2 gives another form of  $B_m$ .

**Proposition 2.2** *Let  $S \in \mathfrak{L}(L_a^2)$ ,  $m \geq 0$  and  $z \in B$ . Then*

$$(2.3) \quad B_m S(z) = C_n^{m+n} (1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \langle S(u^k K_z^m), u^k K_z^m \rangle.$$

**Proof.** Since

$$\begin{aligned} & \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda)} S^* K_\lambda(u) dud\lambda \\ &= \sum_{|k|=0}^m C_{m,k} \int_B \int_B u^k \overline{\lambda^k} K_z^m(u) \overline{K_z^m(\lambda)} S^* K_\lambda(u) dud\lambda \\ &= \sum_{|k|=0}^m C_{m,k} \int_B S(u^k K_z^m)(\lambda) \overline{\lambda^k K_z^m(\lambda)} d\lambda, \end{aligned}$$

Proposition 2.1 implies (2.3). ■

For  $n = 1$ , the right hand side of (2.3) was used by Suárez in [16] to define the  $m$ -Berezin transforms on the unit disk.

Recall that given  $f \in L^\infty$ , define

$$B_m(f)(z) = B_m(T_f)(z).$$

The following proposition gives a nice formula of  $B_m(f)(z)$ . Let  $d\nu_m(u) = C_n^{m+n} (1 - |u|^2)^m du$ .

**Proposition 2.3** *Let  $z \in B$  and  $f \in L^\infty$ . Then*

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u).$$

**Proof.** By the change of variables, Theorem 2.2.2 in [14] and (2.3), we have

$$\begin{aligned} & \int_B f \circ \phi_z(u) d\nu_m(u) \\ &= C_n^{m+n} \int_B f(u) \left( \frac{(1 - |z|^2)(1 - |u|^2)}{|1 - \langle u, z \rangle|^2} \right)^m \left( \frac{(1 - |z|^2)}{|1 - \langle u, z \rangle|^2} \right)^{n+1} du \\ &= C_n^{m+n} (1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \int_B f(u) |u^k|^2 |K_z^m(u)|^2 du \\ &= C_n^{m+n} (1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \langle T_f(u^k K_z^m), u^k K_z^m \rangle = B_m(T_f)(z). \end{aligned}$$

The proof is complete. ■

The formula in the above proposition was used in [1] to define the  $m$ -Berezin transform of functions  $f$ .

Clearly, (1.2) gives  $\|B_m S\|_\infty \leq C(m, n) \|S_z\| = C(m, n) \|S\|$  for  $S \in \mathfrak{L}(L_a^2)$ . Thus,  $B_m : \mathfrak{L}(L_a^2) \rightarrow L^\infty$  is a bounded linear operator. The following theorem gives the norm of  $B_m$ .

**Theorem 2.4** *Let  $m \geq 0$ . Then*

$$\|B_m\| = C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n + |k|)!}.$$

**Proof.** From [8], we have the duality result  $\mathfrak{L}(L_a^2) = \mathcal{T}^*$ . So, the definition of  $B_m$  gives the norm of  $B_m$ . In fact,

$$\begin{aligned} \|B_m\| &= \left\| C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \frac{n!k!}{(n + |k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right\|_{C_1} \\ &= C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n + |k|)!} \end{aligned}$$

as desired. ■

The Möbius map  $\phi_z(w)$  has the following property ([14]):

$$(2.4) \quad \phi'_z(0) = -(1 - |z|^2)P_z - (1 - |z|^2)^{1/2}Q_z.$$

To show that  $m$ -Berezin transforms are Lipschitz with respect to the pseudo-hyperbolic distance we need the following lemmas.

For  $z, w \in \mathbb{C}^n$ ,  $z \hat{\otimes} w$  on  $\mathbb{C}^n$  is defined by  $(z \hat{\otimes} w)\lambda = \langle \lambda, w \rangle z$ .

**Lemma 2.5** *Let  $z, w \in B$  and  $\lambda = \phi_z(w)$ . Then*

$$\phi'_z(w) = (1 - \langle \lambda, z \rangle)(I - \lambda \hat{\otimes} z)[\phi'_z(0)]^{-1}.$$

**Proof.** Suppose that  $P_z$  and  $Q_z$  have the matrix representations as  $((P_z)_{ij})$  and  $((Q_z)_{ij})$  under the standard base of  $\mathbb{C}^n$ , respectively. In fact,

$$(P_z)_{ij} = \frac{z_i \bar{z}_j}{|z|^2} \quad \text{if } z \neq 0.$$

Let  $(a_{ij}(w)) = \phi'_z(w)$ . Write  $\phi_z(w) = (f_1(w), \dots, f_n(w))$ . Then

$$a_{ij}(w) = \frac{\partial f_i}{\partial w_j}(w).$$

Noting that

$$f_i(w) = \frac{z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i}{1 - \langle w, z \rangle},$$

we have

$$\begin{aligned} a_{ij}(w) &= \frac{(z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i)\bar{z}_j}{(1 - \langle w, z \rangle)^2} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle} \\ &= \frac{f_i(w)\bar{z}_j}{1 - \langle w, z \rangle} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle}. \end{aligned}$$

Let  $\lambda = \phi_z(w)$ . The above equality becomes

$$a_{ij}(w) = \frac{\lambda_i \bar{z}_j - ((P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij})}{1 - \langle w, z \rangle}$$

Thus

$$\phi'_z(w) = \frac{\lambda \hat{\otimes} z - (P_z + (1 - |z|^2)^{1/2}Q_z)}{1 - \langle w, z \rangle}.$$

From Theorem 2.2.5 in [14], we have

$$\frac{1}{1 - \langle w, z \rangle} = \frac{1 - \langle \lambda, z \rangle}{1 - |z|^2}.$$

Thus (2.4) implies

$$\begin{aligned} \phi'_z(w)\phi'_z(0) &= \frac{-(1 - |z|^2)\lambda \hat{\otimes} z + (1 - |z|^2)P_z + (1 - |z|^2)Q_z}{1 - \langle w, z \rangle} \\ &= \frac{(1 - |z|^2)(-\lambda \hat{\otimes} z + I)}{1 - \langle w, z \rangle} \\ &= (1 - \langle \lambda, z \rangle)(I - \lambda \hat{\otimes} z) \end{aligned}$$

where the first equality follows from  $P_z Q_z = Q_z P_z = 0$ ,  $P_z z = z$ , and  $Q_z z = 0$ . The proof is complete. ■

**Lemma 2.6** *Suppose  $|z| > 1/2$  and  $|w| > 1/2$ . If  $|\phi_z(w)| \leq \epsilon < 1/2$ , then*

$$\|P_z - P_w\| \leq 50\epsilon(1 - |z|^2)^{1/2}.$$

**Proof.** First we will get the estimate of the distance between  $z$  and  $w$ . Since  $|\phi_z(w)| \leq \epsilon < 1/2$ ,  $w$  is in the ellipsoid:

$$\phi_z(\epsilon B) = \{w \in B : \frac{|P_z w - c|^2}{\epsilon^2 \rho^2} + \frac{|Q_z w|^2}{\epsilon^2 \rho} < 1\}$$

with center  $c = \frac{(1-\epsilon^2)z}{(1-\epsilon^2|z|^2)}$  and  $\rho = \frac{1-|z|^2}{1-\epsilon^2|z|^2}$ . Noting that  $|z| > 1/2$  and  $\epsilon < 1/2$ , we have  $\rho \leq 2(1 - |z|^2)$ . Thus

$$|Q_z w|^2 \leq \epsilon^2 \rho \leq 2\epsilon^2(1 - |z|^2), \quad |P_z w - c| \leq \epsilon \rho \leq 2\epsilon(1 - |z|^2)$$

and

$$|z - c| \leq \frac{\epsilon^2(1 - |z|^2)}{(1 - \epsilon^2|z|^2)} \leq 2\epsilon^2(1 - |z|^2).$$

So, we have

$$|P_z w - z| \leq |P_z w - c| + |z - c| \leq 3\epsilon(1 - |z|^2).$$

Because  $I = P_z + Q_z$  and  $P_z Q_z = 0$ , writing

$$(z - w) = P_z(z - w) + Q_z(z - w),$$

we have

$$\begin{aligned} |z - w|^2 &= |P_z(z - w)|^2 + |Q_z(z - w)|^2 \\ &= |P_z w - z|^2 + |Q_z w|^2 \\ (2.5) \quad &\leq 11\epsilon^2(1 - |z|^2). \end{aligned}$$

Noting that

$$\frac{z}{|z|} \hat{\otimes} \frac{z}{|z|} = \frac{(z - w)}{|z|} \hat{\otimes} \frac{z}{|z|} + \frac{w}{|z|} \hat{\otimes} \frac{(z - w)}{|z|} + \left[ \left( \frac{1}{|z|^2} - \frac{1}{|w|^2} \right) w \right] \hat{\otimes} w + \frac{w}{|w|} \hat{\otimes} \frac{w}{|w|},$$

we have

$$P_z - P_w = \frac{(z - w)}{|z|} \hat{\otimes} \frac{z}{|z|} + \frac{w}{|z|} \hat{\otimes} \frac{(z - w)}{|z|} + \left[ \left( \frac{1}{|z|^2} - \frac{1}{|w|^2} \right) w \right] \hat{\otimes} w,$$

to obtain

$$\begin{aligned} \|P_z - P_w\| &\leq \frac{|z - w|}{|z|} + \frac{2|z - w|}{|z|} + \frac{||z|^2 - |w|^2|}{|z|^2} \\ &\leq 2|z - w| + 4|z - w| + 8|z - w| \\ &\leq 14\sqrt{11}\epsilon(1 - |z|^2)^{1/2} \\ &\leq 50\epsilon(1 - |z|^2)^{1/2} \end{aligned}$$

where the last inequality holds by (2.5). ■



For given  $z, w \in B$ , set  $A(z, w) = -(1 - |z|^2)P_w - (1 - |z|^2)^{1/2}Q_w$ .

**Lemma 2.7** *Suppose  $|z| > 1/2$  and  $|w| > 1/2$ . If  $|\phi_z(w)| \leq \epsilon < 1/2$ , then*

$$\|\phi'_z(0) - A(z, w)\| \leq 150\epsilon(1 - |z|^2).$$

**Proof.** Using (2.4), we have

$$\begin{aligned} \|\phi'_z(0) - A(z, w)\| &= \|(1 - |z|^2)(P_w - P_z) + (1 - |z|^2)^{1/2}(P_z - P_w)\| \\ &\leq 3(1 - |z|^2)^{1/2}\|P_z - P_w\| \\ &\leq 150\epsilon(1 - |z|^2) \end{aligned}$$

as desired. The last inequality follows from Lemma 2.6. ■

Let  $\mathfrak{U}(n)$  be the group of  $n \times n$  complex unitary matrices.

**Lemma 2.8** *Let  $z, w \in B$ . Then  $U_z U_w = V_{\mathcal{U}} U_{\phi_w(z)}$  where*

$$(V_{\mathcal{U}} f)(u) = f(\mathcal{U}u) \det \mathcal{U}$$

for  $f \in L^2_a$  and  $\mathcal{U} = \phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$  satisfying

$$\|I + \mathcal{U}\| \leq C(n)\rho(z, w).$$

**Proof.** The map  $\phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$  is an automorphism of  $B$  that fixes 0, hence it is unitary by the Cartan theorem in [14]. Thus  $\phi_w \circ \phi_z = \phi_{\phi_w(z)} \circ \mathcal{U}$  for some  $\mathcal{U} \in \mathfrak{U}(n)$ . Since  $\phi_w$  is an involution, we have

$$\begin{aligned} U_z U_w f(u) &= (f \circ \phi_w \circ \phi_z)(u) J\phi_w(\phi_z(u)) J\phi_z(u) \\ &= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_w(\phi_w \circ \phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_w(\phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_{\phi_w(z)}(\mathcal{U}u) \det \mathcal{U} \\ &= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_{\phi_w(z)}(\mathcal{U}u) \det \mathcal{U} \\ &= V_{\mathcal{U}} U_{\phi_w(z)} f(u) \end{aligned}$$

as desired.

Now we will show that

$$\|I + \mathcal{U}\| \leq C(n)\rho(z, w).$$

Noting that  $\mathcal{U}$  is continuous for  $|z| \leq 1/2$  and  $|w| \leq 1/2$ , we need only to prove

$$\|I + \mathcal{U}\| \leq 20000\rho(z, w),$$

for  $|z| > 1/2$ ,  $|w| > 1/2$  and  $|\phi_w(z)| < 1/2$ . Let  $\lambda = \phi_w(z)$ . Then  $|\lambda| = \rho(z, w)$  and  $z = \phi_w(\lambda)$ . Since

$$\phi_w \circ \phi_z(u) = \phi_{\lambda}(\mathcal{U}u),$$

taking derivatives both sides of the above equations and using the chain rule give

$$\phi'_w(\phi_z(u)) \phi'_z(u) = \phi'_{\lambda}(\mathcal{U}u)\mathcal{U}.$$

Letting  $u = 0$ , the above equality gives

$$\mathcal{U} = [\phi'_\lambda(0)]^{-1}\phi'_w(z)\phi'_z(0).$$

By Lemma 2.5, write

$$\begin{aligned} \mathcal{U} + I &= [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)[\phi'_w(0)]^{-1}\phi'_z(0) + I \\ &= [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)[\phi'_w(0)]^{-1}[\phi'_z(0) - A(z, w)] \\ &\quad + ([\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)[\phi'_w(0)]^{-1}A(z, w) + I) \\ &:= I_1 + I_2. \end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned} \|I_1\| &\leq \|[\phi'_\lambda(0)]^{-1}\| \|1 - \langle \lambda, w \rangle\| \|I - \lambda \hat{\otimes} w\| \|[\phi'_w(0)]^{-1}\| \|\phi'_z(0) - A(z, w)\| \\ &\leq 4 \times 2 \times 2 \times \frac{3}{(1 - |w|^2)} [150|\lambda|(1 - |z|^2)]. \end{aligned}$$

Theorem 2.2.2 in [14] leads to

$$\frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |\lambda|^2}{|1 - \langle \lambda, w \rangle|^2}.$$

Thus

$$\|I_1\| \leq 4 \times 2 \times 2 \times 3 \times 2 \times 150|\lambda| = 14400|\lambda|.$$

Also, we have

$$\left| 1 - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}} \right| \leq \left| 1 - \frac{1 - |z|^2}{1 - |w|^2} \right| \leq 32|\lambda|.$$

Hence, we get

$$\left\| I - \frac{1 - |z|^2}{1 - |w|^2} P_w - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}} Q_w \right\| \leq 32|\lambda|.$$

On the other hand, clearly,

$$\|[\phi'_\lambda(0)]^{-1} + I\| \leq 4|\lambda|, \quad |(1 - \langle \lambda, w \rangle) - 1| \leq |\lambda|$$

and

$$\|(I - \lambda \hat{\otimes} w) - I\| \leq |\lambda|.$$

These give

$$\|I + [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \leq 16|\lambda|.$$

Hence, we have

$$\begin{aligned} \|I_2\| &\leq \|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)[\phi'_w(0)]^{-1}A(z, w) \\ &\quad - [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \\ &\quad + \|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w) + I\| \\ &\leq \|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \left\| I - \frac{1 - |z|^2}{1 - |w|^2}P_w - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}}Q_w \right\| \\ &\quad + 16|\lambda| \\ &\leq 4 \times 2 \times 2 \times 32|\lambda| + 16|\lambda| < 600|\lambda|. \end{aligned}$$

Combining the above estimates we conclude that

$$\|\mathcal{U} + I\| \leq 14400|\lambda| + 600|\lambda| < 20000|\lambda|. \quad \blacksquare$$

**Theorem 2.9** *Let  $S \in \mathfrak{L}(L^2_a)$ ,  $m \geq 0$  and  $z \in B$ . Then  $B_m S_z = (B_m S) \circ \phi_z$ .*

**Proof.** Proposition 2.2 and (1.2) give

$$B_m S_z(0) = C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \langle S_z u^k, u^k \rangle = B_m S(z) = (B_m S) \circ \phi_z(0).$$

For any  $w \in B$ , Proposition 2.1 and Lemma 2.8 imply

$$\begin{aligned} (B_m S_z) \circ \phi_w(0) &= B_m((S_z)_w)(0) \\ &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{U_w U_z S^* U_z U_w K_\lambda(u)} dud\lambda \\ &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{V_U U_{\phi_z(w)} S^* U_{\phi_z(w)} V_U^* K_\lambda(u)} dud\lambda \\ &= B_m S_{\phi_z(w)}(0) \end{aligned}$$

where  $V_U$  is in Lemma 2.8. Thus,  $B_m S_z(w) = (B_m S) \circ \phi_z(w)$ . ■

**Lemma 2.10** *Let  $S \in \mathfrak{L}(L^2_a)$ ,  $m \geq 1$  and  $z \in B$ . Then*

$$B_m S(z) = \frac{m+n}{m} B_{m-1} \left( S - \sum_{i=1}^n T_{(\phi_z)_i} S T_{(\phi_z)_i} \right) (z)$$

where  $(\phi_z)_i$  is  $i$ -th variable of  $\phi_z$ .

**Proof.** By Theorem 2.9, we just need to show that

$$B_m S(0) = \frac{m+n}{m} B_{m-1} \left( S - \sum_{i=1}^n T_{u_i} S T_{u_i} \right) (0).$$

Using Proposition 2.1 and (2.1), we get

$$\begin{aligned} B_m S(0) &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{S^* K_\lambda(u)} dud\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B \int_B u_i \overline{\lambda_i} u^k \overline{\lambda^k S^* K_\lambda(u)} dud\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B S(u^k u_i)(\lambda) \overline{\lambda^k \lambda_i} d\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \langle S T_{u_i}(u^k), T_{u_i}(u^k) \rangle \end{aligned}$$

as desired. ■

For  $m = 0$ , the following result was obtained in [10].

**Theorem 2.11** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $m \geq 0$ . Then there exists a constant  $C(m, n) > 0$  such that*

$$|B_m S(z) - B_m S(w)| < C(m, n) \|S\| \rho(z, w).$$

**Proof.** We will prove this theorem by induction on  $m$ . If  $m = 0$ , (1.2) gives

$$\begin{aligned} |B_0 S(z) - B_0 S(w)| &= |tr[S_z(1 \otimes 1)] - tr[S_w(1 \otimes 1)]| \\ &= |tr[S_z(1 \otimes 1) - S U_w(1 \otimes 1) U_w]| \\ &= |tr[S_z(1 \otimes 1) - S U_z(U_z U_w 1 \otimes U_z U_w 1) U_z]| \end{aligned}$$

From Lemma 2.8, the last term equals

$$\begin{aligned} |tr[S_z(1 \otimes 1 - U_{\phi_w(z)} 1 \otimes U_{\phi_w(z)} 1)]| &\leq \|S_z\| \|1 \otimes 1 - U_{\phi_w(z)} 1 \otimes U_{\phi_w(z)} 1\|_{C^1} \\ &\leq \sqrt{2} \|S_z\| (2 - 2|\langle 1, k_{\phi_w(z)} \rangle|^2)^{1/2} \\ &= 2 \|S\| [1 - (1 - |\phi_w(z)|^2)^{n+1}]^{1/2} \\ &\leq C(n) \|S\| |\phi_w(z)| \end{aligned}$$

where the second equality holds by  $\|T\|_{C^1} \leq \sqrt{l} (tr[T^* T])^{1/2}$  where  $l$  is the rank of  $T$ .

Suppose  $|B_{m-1}S(z) - B_{m-1}S(w)| < C(m, n)\|S\|\rho(z, w)$ . By Lemma 2.10, we have

$$\begin{aligned} & |B_mS(z) - B_mS(w)| \\ & \leq \frac{m+n}{m}|B_{m-1}S(z) - B_{m-1}S(w)| \\ & \quad + \frac{m+n}{m} \sum_{i=1}^n \left| B_{m-1} \left( T_{(\phi_z)_i} ST_{(\phi_z)_i} \right) (z) - B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_w)_i} \right) (w) \right|. \end{aligned}$$

Since the term in the summation is less than or equals

$$\begin{aligned} & \left| B_{m-1} \left( T_{(\phi_z)_i} ST_{(\phi_z)_i} \right) (z) - B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_z)_i} \right) (z) \right| \\ & \quad + \left| B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_z)_i} \right) (z) - B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_w)_i} \right) (z) \right| \\ & \quad + \left| B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_w)_i} \right) (z) - B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_w)_i} \right) (w) \right|, \end{aligned}$$

it is sufficient to show that

$$\left| B_{m-1} \left( T_{(\phi_z)_i} ST_{(\phi_z)_i} \right) (z) - B_{m-1} \left( T_{(\phi_w)_i} ST_{(\phi_z)_i} \right) (z) \right| < C(m, n)\|S\|\rho(z, w).$$

Lemma 2.8 gives

$$\begin{aligned} & \left| B_{m-1} \left( T_{(\phi_z)_i - (\phi_w)_i} ST_{(\phi_z)_i} \right) (z) \right| \\ & = C_n^{m+n-1} \left\| \text{tr} \left[ \left( T_{(\phi_z)_i - (\phi_w)_i} ST_{(\phi_z)_i} \right)_z \sum_{|k|=0}^{m-1} C_{m-1,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right] \right\| \\ & \leq C_n^{m+n-1} \sum_{|k|=0}^{m-1} |C_{m-1,k}| \frac{n!k!}{(n+|k|)!} \left| \left\langle S_z T_{(\phi_z)_i \circ \phi_z} \frac{u^k}{\|u^k\|}, T_{((\phi_z)_i - (\phi_w)_i) \circ \phi_z} \frac{u^k}{\|u^k\|} \right\rangle \right| \\ (2.6) \quad & \leq C(m, n)\|S_z\| \left\| T_{((\phi_z)_i - (\phi_w)_i) \circ \phi_z} \frac{u^k}{\|u^k\|} \right\|_2. \end{aligned}$$

Let  $\lambda = \phi_w(z)$ . Then

$$\begin{aligned} \left\| T_{((\phi_z)_i - (\phi_w)_i) \circ \phi_z} \frac{u^k}{\|u^k\|} \right\|_2^2 & \leq \int_B |(\phi_z \circ \phi_z)_i(u) - (\phi_w \circ \phi_z)_i(u)|^2 du \\ & = \int_B |(\mathcal{U}u)_i - (\phi_\lambda(u))_i|^2 du \\ & \leq 2 \int_B |(\mathcal{U}u)_i + u_i|^2 + |u_i + (\phi_\lambda(u))_i|^2 du \end{aligned}$$

where  $\phi_w \circ \phi_z = \phi_\lambda \circ \mathcal{U}$  for some  $\mathcal{U} \in \mathfrak{U}(n)$ .

Noting that

$$\phi_\lambda(u) + u = \frac{\lambda - \langle u, \lambda \rangle u + [1 - (1 - |\lambda|^2)^{1/2}]Q_\lambda(u)}{1 - \langle u, \lambda \rangle},$$

we have that for  $|\lambda| \leq 1/2$ ,

$$|\phi_\lambda(u) + u| \leq 2(|\lambda| + |\lambda| + |\lambda|^2) \leq 6|\lambda|.$$

By Lemma 2.8 we also have

$$\int_B |(\mathcal{U}u)_i + u_i|^2 du = \int_B |((\mathcal{U} + I)u)_i|^2 du \leq C\|\mathcal{U} + I\|^2 \leq C|\lambda|^2.$$

Thus (2.6) is less than or equal to

$$C(m, n)\|S_z\|[36|\lambda|^2 + C|\lambda|^2]^{1/2} \leq C(m, n)\|S\||\lambda|.$$

The proof is complete. ■

**Lemma 2.12** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $m, j \geq 0$ . If  $|S^*K_\lambda(z)| \leq C$  for any  $z \in B$  then  $(B_m B_j)(S) = (B_j B_m)(S)$ .*

**Proof.** By Theorem 2.9, it is enough to show that  $(B_m B_j)S(0) = (B_j B_m)S(0)$ . From Proposition 2.3, Proposition 2.1 and Fubini’s Theorem, we have

$$\begin{aligned} B_m(B_j S)(0) &= B_m(T_{B_j S})(0) \\ &= C_n^{m+n} \int_B B_j S(z)(1 - |z|^2)^m dz \\ &= C_n^{m+n} C_n^{j+n} \int_B \int_B \int_B (1 - |z|^2)^{m+j+n+1} (1 - \langle u, \lambda \rangle)^j \times \\ &\quad K_z^j(u) \overline{K_z^j(\lambda)} \overline{S^* K_\lambda(u)} dud\lambda dz \\ &= C_n^{m+n} C_n^{j+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^j \int_B (1 - |z|^2)^{m+j+n+1} \times \\ &\quad K_z^j(u) \overline{K_z^j(\lambda)} dz S^* \overline{K_\lambda(u)} dud\lambda. \end{aligned}$$

Let

$$F_{m,j}(u, \lambda) = (1 - \langle u, \lambda \rangle)^j \int_B (1 - |z|^2)^{m+j+n+1} K_z^j(u) \overline{K_z^j(\lambda)} dz.$$

Then  $F_{m,j}(u, \lambda) = \sum_{i=1}^l H_i(u) \overline{G_i(\lambda)}$  where  $H_i$  and  $G_i$  are holomorphic functions and for some  $l \geq 0$ . Thus, from Lemma 9 in [9], we just need to show  $F_{m,j}(\lambda, \lambda) = F_{j,m}(\lambda, \lambda)$  for  $\lambda \in B$ .

The change of variables implies

$$\begin{aligned} F_{m,j}(\lambda, \lambda) &= (1 - |\lambda|^2)^j \int_B (1 - |z|^2)^{m+j+n+1} |K_\lambda^j(z)|^2 dz \\ &= (1 - |\lambda|^2)^j \int_B (1 - |\phi_\lambda(z)|^2)^{m+j+n+1} |K_\lambda^j(\phi_\lambda(z))|^2 |k_\lambda(z)|^2 dz \\ &= (1 - |\lambda|^2)^m \int_B (1 - |z|^2)^{m+j+n+1} |K_\lambda^m(z)|^2 dz \\ &= F_{j,m}(\lambda, \lambda) \end{aligned}$$

as desired. ■

**Lemma 2.13** *For any  $S \in \mathfrak{L}(L_a^2)$ , there exists sequences  $\{S_\alpha\}$  satisfying*

$$|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$$

*such that  $B_m(S_\alpha)$  converges to  $B_m(S)$  pointwise.*

**Proof.** Since  $H^\infty$  is dense in  $L_a^2$  and the set of finite rank operators is dense in the ideal  $\mathcal{K}$  of compact operators on  $L^2$ , the set  $\{\sum_{i=1}^l f_i \otimes g_i : f_i, g_i \in H^\infty\}$  is dense in the ideal  $\mathcal{K}$  in the norm topology. Since  $\mathcal{K}$  is dense in the space of bounded operators on  $L_a^2$  in strong operator topology, (2.3) gives that for any  $S \in \mathfrak{L}(L_a^2)$ , there exists a finite rank operator sequences  $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$  such that  $B_m(S_\alpha)$  converges to  $B_m(S)$  pointwise for some  $f_i, g_i$  in  $H^\infty$ . Also, for  $l \geq 0$ , for such  $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$ , we have

$$\begin{aligned} |S_\alpha^* K_\lambda(u)| &= \left| \sum_{i=1}^l (g_i \otimes f_i) K_\lambda(u) \right| = \left| \sum_{i=1}^l \langle K_\lambda(u), f_i(u) \rangle g_i(u) \right| \\ &\leq \sum_{i=1}^l |f_i(\lambda)| |g_i(u)| \leq \sum_{i=1}^l \|f_i\|_\infty \|g_i\|_\infty < C. \end{aligned}$$

The proof is complete. ■

**Proposition 2.14** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $m, j \geq 0$ . Then*

$$(B_m B_j)(S) = (B_j B_m)(S).$$

**Proof.** Let  $S \in \mathfrak{L}(L_a^2)$ . Then Lemma 2.13 implies that there exists a sequence  $\{S_\alpha\}$  satisfying  $|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$ , hence  $B_m(B_j S_\alpha)(z) = B_j(B_m S_\alpha)(z)$  by Lemma 2.12. From Proposition 2.3, we know

$$B_m(B_j S_\alpha)(z) = \int_B (B_j S_\alpha) \circ \phi_z(u) d\nu_m(u)$$

and  $\|(B_j S_\alpha) \circ \phi_z\|_\infty \leq C(j, n) \|S\|$ . Also,  $(B_j S_\alpha) \circ \phi_z(u)$  converges to  $(B_j S) \circ \phi_z(u)$ . Therefore  $B_m(B_j S_\alpha)(z)$  converges to  $B_m(B_j S)(z)$ . By the uniqueness of the limit, we have  $(B_m B_j)(S) = (B_j B_m)(S)$ . ■

**Proposition 2.15** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $m \geq 0$ . If  $B_0S(z) \rightarrow 0$  as  $z \rightarrow \partial B$  then  $B_mS(z) \rightarrow 0$  as  $z \rightarrow \partial B$ .*

**Proof.** Suppose  $B_0S(z) \rightarrow 0$  as  $z \rightarrow \partial B$ . Then we will prove that  $S_z \rightarrow 0$  in the  $\mathcal{T}^*$ -norm as  $z \rightarrow \partial B$ . Suppose it is not true. Then for some net  $\{w_\alpha\} \in B$  and an operator  $V \neq 0$  in  $\mathfrak{L}(L_a^2)$ , there exists a sequence  $\{S_{w_\alpha}\}$  such that  $S_{w_\alpha} \rightarrow V$  in the  $\mathcal{T}^*$ -norm as  $w_\alpha \rightarrow \partial B$ , hence  $tr[S_{w_\alpha}T] \rightarrow tr[VT]$  for any  $T \in \mathcal{T}$ . Let  $T = k_\lambda \otimes k_\lambda$  for fixed  $\lambda \in B$ . Then Theorem 2.9 implies

$$\begin{aligned} tr[S_{w_\alpha}T] &= tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] = \langle S_{w_\alpha}k_\lambda, k_\lambda \rangle \\ &= B_0S_{w_\alpha}(\lambda) = (B_0S) \circ \phi_{w_\alpha}(\lambda) \rightarrow 0 \end{aligned}$$

as  $w_\alpha \rightarrow \partial B$ . Since  $tr[VT] = B_0V(\lambda)$  and  $B_0$  is one-to-one mapping,  $V = 0$ . This is the contradiction. Thus  $S_z \rightarrow 0$  as  $z \rightarrow \partial B$  in the  $\mathcal{T}^*$ -norm. (1.2) finishes the proof of this proposition. ■

### 3. Operators $S$ approximated by Toeplitz operators $T_{B_m(S)}$

In this section we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their  $m$ -Berezin transforms. The main result in this section is Theorem 3.7. It extends and improves Theorem 2.4 in [17]. Even on the unit disk, we will show an example that the result in the theorem is sharp on the unit disk.

From Proposition 1.4.10 in [14], we have the following lemma

**Lemma 3.1** *Suppose  $a < 1$  and  $a + b < n + 1$ . Then*

$$\sup_{z \in B} \int_B \frac{d\lambda}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^b} < \infty.$$

This lemma gives the following lemma which extends Lemma 4.2 in [13].

Let  $1 < q < \infty$  and  $p$  be the conjugate exponent of  $q$ . If we take  $p > n + 2$ , then  $q < (n + 2)/(n + 1)$ .

**Lemma 3.2** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $p > n + 2$ . Then there exists  $C(n, p) > 0$  such that  $h(z) = (1 - |z|^2)^{-a}$  where  $a = (n + 1)/(n + 2)$  satisfies*

$$(3.1) \quad \int_B |(SK_z)(w)|h(w)dw \leq C(n, p)\|S_z1\|_p h(z)$$

for all  $z \in B$  and

$$(3.2) \quad \int_B |(SK_z)(w)|h(z)dz \leq C(n, p)\|S_w^*1\|_p h(w)$$

for all  $w \in B$ .



**Proof.** Fix  $z \in B$ . Since

$$U_z 1 = (-1)^n (1 - |z|^2)^{(n+1)/2} K_z,$$

we have

$$\begin{aligned} SK_z &= (-1)^n (1 - |z|^2)^{-(n+1)/2} S U_z 1 \\ &= (-1)^n (1 - |z|^2)^{-(n+1)/2} U_z S_z 1 \\ &= (1 - |z|^2)^{-(n+1)/2} (S_z 1 \circ \phi_z) k_z. \end{aligned}$$

Thus, letting  $\lambda = \phi_z(w)$ , the change of variables implies

$$\begin{aligned} \int_B \frac{|(SK_z)(w)|}{(1 - |w|^2)^a} dw &= \frac{1}{(1 - |z|^2)^{(n+1)/2}} \int_B \frac{|(S_z 1 \circ \phi_z)(w)| |k_z(w)|}{(1 - |w|^2)^a} dw \\ &= \frac{1}{(1 - |z|^2)^a} \int_B \frac{|S_z 1(\lambda)|}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^{n+1-2a}} d\lambda \\ &\leq \frac{\|S_z 1\|_p}{(1 - |z|^2)^a} \left( \int_B \frac{1}{(1 - |\lambda|^2)^{aq} |1 - \langle \lambda, z \rangle|^{(n+1-2a)q}} d\lambda \right)^{\frac{1}{q}}. \end{aligned}$$

The last inequality comes from Holder’s inequality. Since  $aq < 1$  and  $aq + (n + 1 - 2a)q < n + 1$ , Lemma 3.1 implies (3.1).

To prove (3.2), replace  $S$  by  $S^*$  in (3.1), interchange  $w$  and  $z$  in (3.1) and then use the equation

$$(3.3) \quad (S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, SK_z \rangle = \overline{SK_z}(w)$$

to obtain the desired result. ■

**Lemma 3.3** *Let  $S \in \mathfrak{L}(L_a^2)$  and  $p > n + 2$ . Then*

$$\|S\| \leq C(n, p) \left( \sup_{z \in B} \|S_z 1\|_p \right)^{1/2} \left( \sup_{z \in B} \|S_z^* 1\|_p \right)^{1/2}$$

where  $C(n, p)$  is the constant of Lemma 3.2.

**Proof.** (3.3) implies

$$(Sf)(w) = \langle Sf, K_w \rangle = \int_B f(z) \overline{(S^* K_w)(z)} dz = \int_B f(z) (SK_z)(w) dz$$

for  $f \in L_a^2$  and  $w \in B$ . Thus, Lemma 3.2 and the classical Schur’s theorem finish the proof. ■

**Lemma 3.4** *Let  $S_m$  be a bounded sequence in  $\mathfrak{L}(L_a^2)$  such that*

$$\|B_0 S_m\|_\infty \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

*Then*

$$(3.4) \quad \sup_{z \in B} |\langle (S_m)_z 1, f \rangle| \rightarrow 0$$

*as  $m \rightarrow \infty$  for any  $f \in L_a^2$  and*

$$(3.5) \quad \sup_{z \in B} |(S_m)_z 1| \rightarrow 0$$

*uniformly on compact subsets of  $B$  as  $m \rightarrow \infty$ .*

**Proof.** To prove (3.4), we only need to have

$$(3.6) \quad \sup_{z \in B} |\langle (S_m)_z 1, w^k \rangle| \rightarrow 0$$

as  $m \rightarrow \infty$  for any multi-index  $k$ .

Since

$$(3.7) \quad K_z(w) = \sum_{|\alpha|=0}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \bar{z}^\alpha w^\alpha,$$

we have

$$\begin{aligned} B_0 S_m(\phi_z(\lambda)) &= B_0 (S_m)_z(\lambda) \\ &= (1 - |\lambda|^2)^{n+1} \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n + |\alpha|)! (n + |\beta|)!}{n! \alpha! n! \beta!} \langle (S_m)_z w^\alpha, w^\beta \rangle \bar{\lambda}^\alpha \lambda^\beta \end{aligned}$$

where  $\alpha, \beta$  are multi-indices.

Then for any fixed  $k$  and  $0 < r < 1$ ,

$$\begin{aligned} &\int_{rB} \frac{B_0 S_m(\phi_z(\lambda)) \bar{\lambda}^k}{(1 - |\lambda|^2)^{n+1}} d\lambda \\ &= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n + |\alpha|)! (n + |\beta|)!}{n! \alpha! n! \beta!} \langle (S_m)_z w^\alpha, w^\beta \rangle \int_{rB} \bar{\lambda}^{\alpha+k} \lambda^\beta d\lambda \\ &= r^{2n+2|k|} \left( \langle (S_m)_z 1, w^k \rangle + \sum_{|\alpha|=1}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \langle (S_m)_z w^\alpha, w^{\alpha+k} \rangle r^{2|\alpha|} \right). \end{aligned}$$

Since  $S_m$  is bounded sequence, we have

$$\begin{aligned} |\langle (S_m)_z 1, w^k \rangle| &\leq r^{-2n-2|k|} \left| \int_{rB} \frac{B_0 S_m(\phi_z(\lambda)) \bar{\lambda}^k}{(1-|\lambda|^2)^{n+1}} d\lambda \right| + \\ &\quad \sum_{|\alpha|=1}^{\infty} \frac{(n+|\alpha|)!}{n! \alpha!} \| (S_m)_z \| \| w^\alpha \| \| w^{\alpha+k} \| r^{2|\alpha|} \\ &\leq r^{-2n-2|k|} \| B_0 S_m \|_\infty \int_{rB} \frac{|\lambda|^k}{(1-|\lambda|^2)^{n+1}} d\lambda + C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}, \end{aligned}$$

hence, by assumption

$$\limsup_{m \rightarrow \infty} \sup_{z \in B} |\langle (S_m)_z 1, w^k \rangle| \leq C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}.$$

Letting  $r \rightarrow 0$ , we have (3.6).

Now we prove (3.5). From (3.7), we get

$$\begin{aligned} |(S_m)_z 1(\lambda)| &= |\langle (S_m)_z 1, K_\lambda \rangle| \\ &\leq \sum_{|\alpha|=0}^{\infty} \frac{(n+|\alpha|)!}{n! \alpha!} |\langle (S_m)_z 1, w^\alpha \rangle| |\lambda^\alpha| \\ &\leq \sum_{|\alpha|=0}^{l-1} \frac{(n+|\alpha|)!}{n! \alpha!} |\langle (S_m)_z 1, w^\alpha \rangle| + \sum_{|\alpha|=l}^{\infty} \frac{(n+|\alpha|)!}{n! \alpha!} \| S_m \| \| w^\alpha \| |\lambda^\alpha| \end{aligned}$$

for  $z \in B$ ,  $\lambda \in rB$  and  $l \geq 1$ . Since the second summation is less than or equals to

$$\begin{aligned} \sum_{j=l}^{\infty} \left( \frac{(n+j)!}{n! j!} \right)^{1/2} \sum_{|\alpha|=j} \left( \frac{j!}{\alpha!} \right)^{1/2} |\lambda^\alpha| &\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n! j!} \left[ \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha|^2 \right]^{1/2} \\ &\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n! j!} r^j, \end{aligned}$$

for any  $\epsilon > 0$ , we can find sufficiently large  $l$  such that the second summation is less than  $\epsilon$ . Thus, (3.6) imply  $\sup_{z \in B} |(S_m)_z 1| \rightarrow 0$  uniformly on compact subsets of  $B$  as  $m \rightarrow \infty$ . ■

**Lemma 3.5** *Let  $\{S_m\}$  be a sequence in  $\mathfrak{L}(L_a^2)$  such that for some  $p > n + 2$ ,  $\|B_0 S_m\|_\infty \rightarrow 0$  as  $m \rightarrow \infty$ ,*

$$\sup_{z \in B} \| (S_m)_z 1 \|_p \leq C \quad \text{and} \quad \sup_{z \in B} \| (S_m^*)_z 1 \|_p \leq C$$

where  $C > 0$  is independent of  $m$ , then  $S_m \rightarrow 0$  as  $m \rightarrow \infty$  in  $\mathfrak{L}(L_a^2)$ -norm.

**Proof.** Lemma 3.3 implies

$$\|S_m\| \leq C(n, p) \left( \sup_{z \in B} \|(S_m)_z 1\|_p \right)^{1/2} \left( \sup_{z \in B} \|(S_m^*)_z 1\|_p \right)^{1/2} \leq C(n, p),$$

hence, Lemma 3.4 gives

$$(3.8) \quad \sup_{z \in B} |(S_m)_z 1| \rightarrow 0$$

uniformly on compact subsets of  $B$  as  $m \rightarrow \infty$ .

Here, for  $n + 2 < s < p$ , Holder's inequality gives

$$\begin{aligned} \sup_{z \in B} \|(S_m)_z 1\|_s^s &\leq \sup_{z \in B} \int_{B \setminus r\bar{B}} |(S_m)_z 1(w)|^s dw + \sup_{z \in B} \int_{r\bar{B}} |(S_m)_z 1(w)|^s dw \\ &\leq C \sup_{z \in B} \|(S_m)_z 1\|_p^s (1 - r)^{1-s/p} + \sup_{z \in B} \int_{r\bar{B}} |(S_m)_z 1(w)|^s dw \end{aligned}$$

and (3.8) implies the second term tends to 0 as  $m \rightarrow \infty$ . Also, the first term is less than or equals to  $C^s(1 - r)^{1-s/p}$  which can be small by taking  $r$  close to 1. Consequently, Lemma 3.3 gives

$$\begin{aligned} \|S_m\| &\leq C(n, s) \left( \sup_{z \in B} \|(S_m)_z 1\|_s \right)^{1/2} \left( \sup_{z \in B} \|(S_m^*)_z 1\|_s \right)^{1/2} \\ &\leq C(n, s) \left( \sup_{z \in B} \|(S_m)_z 1\|_s \right)^{1/2} \rightarrow 0 \end{aligned}$$

■

**Corollary 3.6** *Let  $S \in \mathfrak{L}(L_a^2)$  such that for some  $p > n + 2$ ,*

$$(3.9) \quad \sup_{z \in B} \|S_z 1 - (T_{B_m S})_z 1\|_p \leq C \quad \text{and} \quad \sup_{z \in B} \|S_z^* 1 - (T_{B_m(S^*)})_z 1\|_p \leq C,$$

*where  $C > 0$  is independent of  $m$ . Then  $T_{B_m S} \rightarrow S$  as  $m \rightarrow \infty$  in  $\mathfrak{L}(L_a^2)$ -norm.*

**Proof.** Let  $S_m = S - T_{B_m S}$ . Then Proposition 2.14 and Theorem 2.11 imply

$$B_0(S_m) = B_0 S - B_0(T_{B_m S}) = B_0 S - B_0(B_m S) = B_0 S - B_m(B_0 S)$$

which tends uniformly to 0 as  $m \rightarrow \infty$ , hence  $\|B_0(S_m)\|_\infty \rightarrow 0$ . Consequently, by Lemma 3.5 we complete the proof. ■

**Theorem 3.7** *Let  $S \in \mathfrak{L}(L_a^2)$ . If there is  $p > n + 2$  such that*

$$(3.10) \quad \sup_{z \in B} \|T_{(B_m S) \circ \phi_z} 1\|_p < C \quad \text{and} \quad \sup_{z \in B} \|T_{(B_m S) \circ \phi_z}^* 1\|_p < C$$

where  $C > 0$  is independent of  $m$ , then  $T_{B_m S} \rightarrow S$  as  $m \rightarrow \infty$  in  $\mathfrak{L}(L_a^2)$ -norm.

**Proof.** By Corollary 3.6, we only need to show that (3.10) implies (3.9). Since  $T_{(B_m S) \circ \phi_z} = (T_{B_m S})_z$  and

$$T_{(B_m S) \circ \phi_z}^* = T_{\overline{B_m S_z}} = T_{B_m(S_z^*)} = T_{(B_m(S^*)) \circ \phi_z},$$

it is sufficient to show that

$$\sup_{z \in B} \|S_z 1\|_p < \infty.$$

By Lemma 3.3, we get

$$\|T_{B_m S}\| \leq C(n, p) \left( \sup_{z \in B} \|T_{B_m S \circ \phi_z} 1\|_p \right)^{1/2} \left( \sup_{z \in B} \|T_{B_m S \circ \phi_z}^* 1\|_p \right)^{1/2} < C$$

where  $C$  is independent of  $m$ , hence writing  $S_m = S - T_{B_m S}$ , we have  $\|S_m\| \leq C$  where  $C$  is independent of  $m$ . Also, the proof of Corollary 3.6 implies

$$\|B_0 S_m\|_\infty \rightarrow 0$$

as  $m \rightarrow \infty$ .

Let  $f$  be a polynomial with  $\|f\|_q = 1$ . Then Lemma 3.4 implies

$$\sup_{z \in B} |\langle (S_m)_z 1, f \rangle| \rightarrow 0$$

as  $m \rightarrow \infty$ . Thus, for any  $\epsilon > 0$  and  $z_0 \in B$ , we have

$$|\langle S_{z_0} 1, f \rangle| \leq \sup_{z \in B} |\langle (S_m)_z 1, f \rangle| + |\langle (T_{B_m S})_{z_0} 1, f \rangle| \leq \epsilon + C$$

for sufficiently large  $m$ , where  $C$  is independent of  $m$ . Since  $\epsilon$  is arbitrary, we get

$$\sup_{z \in B} \|S_z 1\|_p < \infty$$

as desired. ■

### 4. Compact Radial operator

Given  $\mathcal{U} \in \mathfrak{U}(n)$ , define  $V_{\mathcal{U}}f(w) = f(\mathcal{U}w) \det \mathcal{U}$  for  $f \in L^2_a$ . Then  $V_{\mathcal{U}}$  is a unitary operator on  $L^2_a$ . We say that  $S \in \mathfrak{L}(L^2_a)$  is a radial operator if  $SV_{\mathcal{U}} = V_{\mathcal{U}}S$  for any  $\mathcal{U} \in \mathfrak{U}(n)$ .

If  $S \in \mathfrak{L}(L^2_a)$ , the radialization of  $S$  is defined by

$$S^\sharp = \int_{\mathfrak{U}} V_{\mathcal{U}}^* S V_{\mathcal{U}} d\mathcal{U}$$

where  $d\mathcal{U}$  is the Haar measure on the compact group  $\mathfrak{U}(n)$  and the integral is taken in the weak sense. Then  $S^\sharp = S$  if  $S$  is radial and  $\mathfrak{U}$ -invariance of  $d\mathcal{U}$  shows that  $S^\sharp$  is indeed a radial operator.

If  $f \in L^\infty$  and  $g, h \in L^2_a$  then

$$\langle V_{\mathcal{U}}^* T_f V_{\mathcal{U}} g, h \rangle = \int_B f(w) V_{\mathcal{U}} g(w) \overline{V_{\mathcal{U}} h(w)} dw = \int_B f(\mathcal{U}^* w) g(w) \overline{h(w)} dw.$$

Thus  $V_{\mathcal{U}}^* T_f V_{\mathcal{U}} = T_{f \circ \mathcal{U}^*}$  and

$$V_{\mathcal{U}}^* T_{f_1} \cdots T_{f_l} V_{\mathcal{U}} = T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*}$$

for  $f_1, \dots, f_l \in L^\infty, l \geq 0$ .

**Lemma 4.1** *Let  $S \in \mathfrak{L}(L^2_a)$  be a radial operator. Then  $T_{B_m(S)} = \int_B S_w d\nu_m(w)$ .*

**Proof.** Let  $z \in B$ . By (2.3) and Lemma 2.8, we obtain

$$\begin{aligned} B_0 \left( \int_B S_w d\nu_m(w) \right) (z) &= \left\langle \left( \int_B S_w d\nu_m(w) \right)_z, 1, 1 \right\rangle \\ &= \int_B \langle U_z U_w S U_w U_z^* 1, 1 \rangle d\nu_m(w) \\ &= \int_B \langle U_{\phi_z(w)} V_{\mathcal{U}}^* S V_{\mathcal{U}} U_{\phi_z(w)} 1, 1 \rangle d\nu_m(w) \end{aligned}$$

where  $V_{\mathcal{U}}$  is in Lemma 2.8. Since  $S$  is a radial operator, Theorem 2.9, Proposition 2.3 and Proposition 2.14 imply that the last integral equals

$$\begin{aligned} \int_B \langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1 \rangle d\nu_m(w) &= \int_B B_0 S \circ \phi_z(w) d\nu_m(w) \\ &= B_m B_0 S(z) \\ &= B_0 B_m S(z) \\ &= B_0(T_{B_m(S)})(z). \end{aligned}$$

Since  $B_0$  is one-to-one mapping, the proof is complete. ■

**Theorem 4.2** *Let  $S \in \mathfrak{T}(L^\infty)$  be a radial operator. Then  $S$  is compact if and only if  $B_0S \equiv 0$  on  $\partial B$ .*

**Proof.** Suppose  $B_0S \equiv 0$  on  $\partial B$ . Then  $B_mS \equiv 0$  on  $\partial B$  by Proposition 2.15, hence  $T_{B_mS}$  is compact for all  $m \geq 0$ .

Let

$$Q = \int_{\mathfrak{U}} T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*} d\mathfrak{U}$$

with  $f_1, \dots, f_l \in L^\infty$  for some  $l \geq 0$ . Then  $Q \in \mathfrak{L}(L_a^2)$ . By Lemma 4.1, for any  $z \in B$ , we have

$$\begin{aligned} T_{(B_m(Q)) \circ \phi_z} &= \int_B ((Q)_z)_w d\nu_m(w) \\ &= \int_B \int_{\mathfrak{U}} T_{f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} \cdots T_{f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} d\mathfrak{U} d\nu_m(w). \end{aligned}$$

Consequently,

$$\begin{aligned} \|T_{(B_m(Q)) \circ \phi_z}\| &\leq C(l) \|f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \cdots \|f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \\ &= C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty. \end{aligned}$$

Similarly, we have

$$\|T_{(B_m(Q)) \circ \phi_z}^*\| \leq C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Thus, Theorem 3.7 gives that

$$(4.1) \quad T_{B_m(Q)} \rightarrow Q$$

in  $\mathfrak{L}(L_a^2)$ -norm.

Since  $S \in \mathfrak{T}(L^\infty)$ , there exists a sequence  $\{S_k\}$  such that  $S_k \rightarrow S$  in  $\mathfrak{L}(L_a^2)$ -norm where each  $S_k$  is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and  $S$  is radial,  $S_k^\sharp \rightarrow S^\sharp = S$ . From Lemma 4.1, we have

$$\|T_{B_mS}\| = \left\| \int_B S_w d\nu_m(w) \right\| \leq \int_B \|S_w\| d\nu_m(w) = \|S\|.$$

Thus

$$\begin{aligned} \|S - T_{B_mS}\| &\leq \|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| + \|T_{B_m(S_k^\sharp)} - T_{B_mS}\| \\ &\leq 2\|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| \end{aligned}$$

and (4.1) imply  $T_{B_m(S)} \rightarrow S$  as  $m \rightarrow \infty$  in  $\mathfrak{L}(L_a^2)$ -norm, hence  $S$  is compact.

The other direction is trivial. ■

**Example.** This example shows that for  $n = 1$ , the number  $n + 2 = 3$  in Theorem 3.7 is sharp. We show that there is a bounded operator  $S$  on  $L_a^2$  such that

$$\sup_{z \in D} \max\{\|T_{(B_m S) \circ \phi_z} 1\|_3, \|T_{(B_m S) \circ \phi_z}^* 1\|_3\} < \infty,$$

and for each  $m \geq 0$ ,  $B_m(S)(z) \rightarrow 0$  as  $z \rightarrow \partial D$ , but  $S$  is not compact on  $L_a^2$ .

The following operator  $S$  was constructed in [3] to show that  $B_0(S)(z) \rightarrow 0$  as  $z \rightarrow \partial D$ , but  $S$  is not compact on  $L_a^2$ . Let  $S$  be defined on  $L_a^2$  by

$$S\left(\sum_{l=0}^{\infty} a_l w^l\right) = \sum_{l=0}^{\infty} a_{2^l} w^{2^l}.$$

It is clear that  $S$  is a self-adjoint projection with infinite-dimensional range. Thus  $S$  is not compact on  $L_a^2$ . From

$$B_0(S)(z) = \langle S k_z, k_z \rangle = \|S k_z\|_2^2 = (1 - |z|^2)^2 \sum_{l=0}^{\infty} (2^l + 1) (|z|^2)^{2^l},$$

it is easy to see that

$$B_0(S)(z) \rightarrow 0 \quad \text{as } z \rightarrow \partial D.$$

By Proposition 2.15, we see that

$$B_m(S)(z) \rightarrow 0 \quad \text{as } z \rightarrow \partial D.$$

This gives that  $T_{B_m(S)}$  is compact. Hence  $T_{B_m(S)}$  does not converge to  $S$  in the norm topology.

By means of the Zygmund theorem on gap series [18], it was proved in [13] that

$$C = \sup_{z \in D} \max\{\|S_z 1\|_3, \|S_z^* 1\|_3\} < \infty.$$

Clearly,  $S$  is a radial operator. By Lemma 4.1, we have

$$T_{(B_m S) \circ \phi_z} 1 = \int_D (S_w)_z 1 d\nu_m(w) = \int_D S_{\phi_z(w)} 1 d\nu_m(w) = \int_D S_\lambda 1 d\nu_m \circ \phi_z(\lambda).$$

Noting that for each  $z \in D$ ,  $d\nu_m \circ \phi_z$  is a probability measure on  $D$ , we have

$$\|T_{(B_m S) \circ \phi_z} 1\|_3 \leq \int_D \|S_\lambda 1\|_3 d\nu_m \circ \phi_z(\lambda) \leq C.$$

Similarly, we also have

$$\|T_{(B_m S) \circ \phi_z}^* 1\|_3 \leq C.$$



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