Publ. RIMS, Kyoto Univ.
31 (1995), 725–730

Equivariant K-Theory and Maps between Representation Spheres

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

By

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§1. Introduction and Statement of Results

The equivariant K-theory has been successfully employed in the study of equivariant maps by Marzantowicz [5], Liulevicious [7] and Bartsch [3]. In the present paper, using the equivariant K-theory, we will obtain a necessary condition for the existence of G-maps $SU \to SW$, where SU and SW are the unit spheres of unitary representations U and W, respectively, of a compact Lie group G.

From Atiyah [1], [2] or Segal [8] we can see that the equivariant K-ring $K_G(SU)$ of SU is isomorphic to $R(G)/(\lambda_{-1}U)$, the complex representation ring $R(G)$ divided by the ideal $(\lambda_{-1}U)$ generated by the Euler class $\lambda_{-1}U$ of U in $K_G(\mathrm{pt}) = R(G)$. If there exists a G-map $\eta : SU \to SW$, then we obtain a ring homomorphism $\eta : R(G)/(\lambda_{-1}W) \to R(G)/(\lambda_{-1}U)$ which coincides with the homomorphism induced from the identity on $R(G)$. This implies that the condition $\lambda_{-1}W \in (\lambda_{-1}U)$ is necessary for the existence of G-maps $SU \to SW$. If G is abelian, we will reduce this condition to more exlicit form.

Let $S^1 = \{z \in \boldsymbol{C} \mid |z| = 1\}$ be the circle group of complex numbers with absolute value 1, and $\boldsymbol{Z}_n$ the cyclic group of order n considered as a subgroup of S^1. For any integer i let S^1 and $\boldsymbol{Z}_n$ act on $V_i = \boldsymbol{C}$ via $(z, v) \mapsto z^i v$ for $z \in S^1 (\text{or } \boldsymbol{Z}_n)$ and $v \in V_i$. A compact abelian group G decomposes into a cartesian product

$$G = T^k \times \boldsymbol{Z}_{n_1} \times \cdots \times \boldsymbol{Z}_{n_l},$$

where $T^k = S^1 \times \cdots \times S^1$, the cartesian product of k copies of S^1. Letting γ be a sequence $(a_1, \ldots, a_k, b_l, \ldots, b_l)$ of integers, denote by V_γ the tensor product

$$V_{a_1} \otimes \cdots \otimes V_{a_k} \otimes V_{b_1} \otimes \cdots \otimes V_{b_l},$$

Communicated by K. Saito, December 19, 1994.
1991 Mathematics Subject Classifications: 55N15, 57S99
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which can be considered as a representation of G in a natural way. Let Γ be the set of sequences

$$\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$$

with $a_1, \ldots, a_k \in \mathbf{Z}$ and $0 \le b_j \le n_j - 1$ for $1 \le j \le l$. The set $\{V_\gamma \mid \gamma \in \Gamma\}$ gives a complete set of irreducible unitary representations of G, and so any unitary representation U of G decomposes into a direct sum

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)},$$

where $u(\gamma)$ is a nonnegative integer and $V_\gamma^{u(\gamma)}$ denotes the direct sum of $u(\gamma)$ copies of V_γ. We can easily see that the fixed point set U^G of U is $\{0\}$ if and only if $u(\gamma) = 0$ for $\gamma = (0, \ldots, 0)$. Let

$$|\gamma| = |a_1| + \cdots + |a_k| + b_1 + \cdots + b_l$$

for any $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l) \in \Gamma$.

We are now in a position to state our main theorem.

Theorem 1.1. *Let U and W be unitary representations of a compact abelian group G, and decompose into*

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)} \quad \textit{and} \quad W = \bigoplus_{\gamma \in \Gamma} V_\gamma^{\mathrm{u}(\gamma)}.$$

Assume that there exists a G-map $SU \to SW$. Then

(1) *if* dim U = dim W, *then there is an integer m such that*

$$\prod_{\gamma \in \Gamma} |\gamma|^{\mathrm{u}(\gamma)} \equiv m \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \mod d,$$

where d is the greatest common divisor of $n_1, \ldots, n_l$, (if $l = 0$, then assume $d = 0$),

(2) *if* dim $U >$ dim W, *then*

$$\prod_{\gamma \in \Gamma} |\gamma|^{\mathrm{u}(\gamma)} \equiv 0 \mod d.$$

From this theorem we obtain the following two corollaries.

Corollary 1.2 (cf. Liulevicious [7], Bartsch [4], Marzantowicz [6]). *Let U and W be representations of $G = T^k$ with $W^G = \{0\}$. If there exists a G-map $SU \to SW$, then* dim $U \le$ dim W.

Corollary 1.3 (Liulevicious [7], Marzantowicz [6]). *Let U and W be representations of $G = \mathbf{Z}_n$ with n any. If G acts freely on SW and if there exists a G-map $SU \to SW$, then* dim $U \le$ dim W.

Remark 1.4. If U is an orthogonal representation of $G = T^k$ or $\boldsymbol{Z}_n$ with n odd and if $U^G = \{0\}$, then U can be considered a unitary representation. In general, if U is orthogonal then $U \oplus U$ becomes unitary. Since the join of two G-maps $SU \to SW$ gives a G-map

$$S(U \oplus U) = SU * SU \to SW * SW = S(W \oplus W),$$

Corollaries 1.2 and 1.3 follow for orthogonal representations U and W.

Remark 1.5. We should refer to a recent paper [6] of Marzantowicz. Using the Borel cohomology theory, he also studies equivariant maps between representation spheres, and obtains a necessary condition for the existence of such maps. A detailed study is done for the case of $G = T^k$ or $\boldsymbol{Z}_p^k (= \boldsymbol{Z}_p \times \cdots \times \boldsymbol{Z}_p)$. It is also shown that his condition is sufficient in some case.

§2. A Necessary Condition in Terms of the Euler Classes

Let U be a unitary representation of a compact Lie group G. The sequence

(2.1) $$\cdots \to K_G^n(DU, SU) \to K_G^n(DU) \to K_G^n(SU) \to K_G^{n+1}(DU, SU) \to \cdots$$

is the long exact sequence of the equivariant K-theory K_G^{*} for the pair (DU, SU) of the unit disk DU and the unit sphere SU of U. Segal [8; Proposition 3.2] or Atiyah [2] gives the Thom isomorphism

$$\varphi_! : K_G(\mathrm{pt}) \to K_G(U) = K_G(DU, SU)$$

such that $\varphi \varphi (\xi) = \xi \cdot \lambda_{-1} U$ for $\xi \in K_G(\mathrm{pt})$, where $\varphi : K_G(U) \to K_G(\mathrm{pt})$ is the homomorphism induced from the inclusion map $\varphi : \{\mathrm{pt}\} \to U$,

$$\lambda_{-1} U = \sum_i (-1)^i \Lambda^i U \in K_G^0(\mathrm{pt}),$$

and $\Lambda^i U$ is the i-th exterior algebra of U. Since $K_G^1(DU, SU) = K_G^1(U) \cong K_G^1(\mathrm{pt}) = 0$ and $K_G^0(\mathrm{pt}) \cong R(G)$, the sequence (2.1) yields the exact sequence

(2.2) $$R(G) \to R(G) \to K_G(SU) \to 0,$$

where the first homomorphism is given by multiplication by $\lambda_{-1} U$. This argument is done in the same manner as in Atiyah [1; Lemma 2.7.4, Corollary 2.7.5] where G is finite abelian.

From the exact sequence (2.2) we obtain

Proposition 2.3. $K_G(SU) \cong R(G)/(\lambda_{-1} U)$.

Let $\eta : SU \to SW$ be a G-map for representations U and W of G. Since the sequence (2.1) is functorial, we see that the composite

$$R(G)/(\lambda_{-1}W) \cong K_G(SW) \xrightarrow{\eta} K_G(SU) \cong R(G)/(\lambda_{-1}U)$$

coincides with the homomorphism induced from the identity on $R(G)$. This implies the following.

Proposition 2.4. *If there exists a G-map $SU \to SW$, then $\lambda_{-1}W \in (\lambda_{-1}U)$ in $R(G)$.*

§3. Calculation of $K_G(SU)$

In this section we will calculate the ring $K_G(SU)$ for the case where G is abelian.

We first recall the following facts about the complex representation rings of G:

(1) $R(S^1) \cong \mathbf{Z}[x, x^{-1}]/(1 - xx^{-1})$, in which the representation V_i corresponds to x^i if $i \geq 0$ and to $(x^{-1})^{-i}$ if $i \leq 0$.

(2) $R(\mathbf{Z}_n) \cong \mathbf{Z}[x]/(1 - x^n)$, in which V_i corresponds to x^i.

(3) $R(G_1 \times G_2) \cong R(G_1) \otimes R(G_2)$.

From these facts we obtain

Proposition 3.1. *If $G = T^k \times \mathbf{Z}_{n_1} \times \cdots \times \mathbf{Z}_{n_l}$ is a compact abelian group, then*

$$R(G) \cong \mathbf{Z}[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, y_1, \ldots, y_l]/(X, Y),$$

where

$$X = \{1 - x_i x_i^{-1} \mid 1 \leq i \leq k\},$$
$$Y = \{1 - y_j^{n_j} \mid 1 \leq j \leq l\},$$

and (X, Y) is the ideal generated by $X \cup Y$. The isomorphism sends the representation V_γ to the monomial $x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}$ if $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$.

Since λ_{-1} is multiplicative, i.e., $\lambda_{-1}(U_1 \oplus U_2) = \lambda_{-1}U_1 \cdot \lambda_{-1}U_2$, Propositions 2.3 and 3.1 give the following.

Proposition 3.2. *Let $U = \oplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)}$ be a unitary representation of $G = T^k \times \mathbf{Z}_{n_1} \times \cdots \times \mathbf{Z}_{n_l}$. Then*

$$K_G(SU) \cong \mathbf{Z}[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, y_1, \ldots, y_l]/(X, Y, z_U),$$

where $z_U = \Pi_\gamma (1 - (\mathbf{xy})^\gamma)^{u(\gamma)}$, $(\mathbf{xy})^\gamma = x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}$ if $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$, and (X, Y, z_U) is the ideal generated by $X \cup Y \cup \{z_U\}$.

§4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Let $G=T^k\times \mathbf{Z}_{n_1}\times\cdots\times\mathbf{Z}_{n_l}$ be a compact abelian group, and

$$U=\bigoplus_{\gamma\in\Gamma}V_\gamma^{u(\gamma)},\quad W=\bigoplus_{\gamma\in\Gamma}V_\gamma^{w(\gamma)}$$

its unitary representations. Assume that there exists a G-map $\eta: SU\to SW$. If $W^G\neq\{0\}$, then the theorem is trivially valid. So we assume $W^G=\{0\}$.

For the representation

$$V_\gamma=V_{a_1}\otimes\cdots\otimes V_{a_k}\otimes V_{b_1}\otimes\cdots\otimes V_{b_l},$$

let

$$\overline{V}_\gamma=V_{|a_1|}\otimes\cdots\otimes V_{|a_k|}\otimes V_{b_1}\otimes\cdots\otimes V_{b_l},$$

and

$$\overline{U}=\bigoplus_{\gamma\in\Gamma}\overline{V}_\gamma^{u(\gamma)},\quad \overline{W}=\bigoplus_{\gamma\in\Gamma}\overline{V}_\gamma^{w(\gamma)}.$$

Since $V_a\cong V_{|a|}$ as real representations, we see $U\cong\overline{U}$ and $W\cong\overline{W}$. Therefore $\eta: SU\to SW$ induces a G-map $\overline{\eta}: S\overline{U}\to S\overline{W}$, and then $\overline{\eta}$ induces a ring homomorphism $\overline{\eta}^{!}: K_G(S\overline{W})\to K_G(S\overline{U})$. From Proposition 3.2 we obtain a ring homomorphism

$$\overline{\eta}^{!}:\mathbf{Z}[x_1,x_1^{-1},\dots,x_k,x_k^{-1},y_1,\dots,y_l]/(X,Y,\bar{z}_W)\to\mathbf{Z}[x_1,x_1^{-1},\dots,x_k,x_k^{-1},y_1,\dots,y_l]/(X,Y,\bar{z}_U),$$

where X and Y are as given in Proposition 3.1,

$$\bar{z}_U=\prod_{\gamma\in\Gamma}(1-\overline{\boldsymbol{xy}}^\gamma)^{u(\gamma)},\qquad \bar{z}_W=\prod_{\gamma\in\Gamma}(1-\overline{\boldsymbol{xy}}^\gamma)^{w(\gamma)},$$

and

$$\overline{\boldsymbol{xy}}^\gamma=x_1^{|a_1|}\cdots x_k^{|a_k|}y_1^{b_1}\cdots y_l^{b_l}.$$

As in Proposition 2.4, we see $\bar{z}_W\in(X,Y,\bar{z}_U)$. Then there are polynomials $f_j\ (1\le j\le l+1)$ in $\mathbf{Z}[x_1,x_1^{-1},\dots,x_k,x_k^{-1},y_1,\dots,y_l]$ such that

$$\bar{z}_W=\sum_{j=1}^{l}f_j\cdot(1-y_j^{n_j})+f_{l+1}\cdot\bar{z}_U \tag{4.1}$$

in $\mathbf{Z}[x_1,x_1^{-1},\dots,x_k,x_k^{-1},y_1,\dots,y_l]/(X)$. Multiplying (4.1) by $x_1^{m_1}\cdots x_k^{m_k}$ for sufficiently large $m_1,\cdots,m_k>0$, we obtain

$$x_1^{m_1}\cdots x_k^{m_k}\bar{z}_W=\sum_{j=1}^{l}\tilde{f}_j\cdot(1-y_j^{n_j})+\tilde{f}_{l+1}\cdot\bar{z}_U \tag{4.2}$$

in $\mathbb{Z}[x_1,\dots,x_k,y_1,\dots,y_l]$, where $\tilde{f}_j(1\le j\le l+1)$ are polynomials in $\mathbb{Z}[x_1,\dots,x_k,y_1,\dots,y_l]$. Substituting x for all of $x_1,\dots,x_k,y_1,\dots,y_l$ in (4.2), we obtain

$$x^m\prod_{\gamma\in\Gamma}(1-x^{|\gamma|})^{w(\gamma)}=\sum_{j=1}^{l}g_j(x)(1-x^{n_j})+g_{l+1}(x)\prod_{\gamma\in\Gamma}(1-x^{|\gamma|})^{u(\gamma)}, \tag{4.3}$$

where $m=m_1+\cdots+m_k$, $g_j(x)\in\mathbb{Z}[x]\,(1\le j\le l+1)$ and $|\gamma|=|a_1|+\cdots+|a_k|+b_1+\cdots+b_l$ if $\gamma=(a_1,\dots,a_k,b_1,\dots,b_l)$. If dim $U\ge$ dim W, we can divide the both sides of (4.3) by $(1-x)^{\sum w(\gamma)}$, and obtain

$$\begin{aligned}x^m\prod_{\gamma\in\Gamma}&(1+x+\cdots+x^{|\gamma|-1})^{w(\gamma)}\\&=h(x)+g_{l+1}(x)(1-x)^{\sum u(\gamma)-\sum w(\gamma)}\prod_{\gamma\in\Gamma}(1+x+\cdots+x^{|\gamma|-1})^{u(\gamma)},\end{aligned} \tag{4.4}$$

where $h(x)=\sum_{j=1}^{l}g_j(x)(1-x^{n_j})/(1-x)^{\sum w(\gamma)}\in\mathbb{Z}[x]$. Since

$$1-x^{n_j}=(1-x)(1+x+\cdots+x^{d_j-1})p_j(x)$$

for any divisor d_j of n_j and some $p_j(x)\in\mathbb{Z}[x]$, we see

$$\sum_{j=1}^{l}g_j(x)(1-x^{n_j})=(1-x)(1+x+\cdots+x^{d-1})\sum_{j=1}^{l}g_j(x)p_j(x),$$

where d is the greatest common divisor of $n_1,\dots,n_l$. Since $1-x$ and $1+x+\cdots+x^{d-1}$ are prime to each other, $h(x)=(1+x+\cdots+x^{d-1})q(x)$ for some $q(x)\in\mathbb{Z}[x]$. Therefore, substituting 1 for x in (4.4), we obtain

$$\prod_{\gamma\in\Gamma}|\gamma|^{w(\gamma)}=d\cdot q(1)+g_{l+1}(1)\prod_{\gamma\in\Gamma}|\gamma|^{u(\gamma)}$$

if $\sum_\gamma u(\gamma)=\sum_\gamma w(\gamma)$, and

$$\prod_{\gamma\in\Gamma}|\gamma|^{w(\gamma)}=d\cdot q(1)$$

if $\sum_\gamma u(\gamma)>\sum_\gamma w(\gamma)$. This completes the proof of Theorem 1.1.

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