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The Structure of Linear Extension Operators for C^m

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Abstract

For any subset $E \subset \mathbb{R}^n$, let $C^m(E)$ denote the Banach space of restrictions to E of functions $F \in C^m(\mathbb{R}^n)$. It is known that there exist bounded linear maps $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ such that $Tf = f$ on E for any $f \in C^m(E)$. We show that T can be taken to have a simple form, but cannot be taken to have an even simpler form.

0. Statement of Results

Fix $m, n \geq 1$, let $E \subset \mathbb{R}^n$ be given, and let $C^m(E) = \{F|_E : F \in C^m(\mathbb{R}^n)\}$, with norm

$$\| f \|_{C^m(E)} = \inf\{\| F \|_{C^m(\mathbb{R}^n)} :\ F \in C^m(\mathbb{R}^n) \text{ and } F|_E = f\}.$$

Here, as usual, $C^m(\mathbb{R}^n)$ denotes the space of m times continuously differentiable $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, for which the norm

$$\| F \|_{C^m(\mathbb{R}^n)} = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$$

is finite. A linear extension operator for $C^m(E)$ is a bounded linear map $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$, such that $Tf|_E = f$ for all $f \in C^m(E)$.

Given $E \subset \mathbb{R}^n$, there exists a linear extension operator for $C^m(E)$. See [17] for a proof, and [1,...,29] for related work going back to Whitney. In particular, Merrien [20] constructed linear extension operators for $C^m(E)$ when $E \subset \mathbb{R}^1$, and Bromberg [3] constructed linear extension operators for $C^1(E)$ when $E \subset \mathbb{R}^n$. The existence of linear extension operators for $C^m(E)$ was explicitly conjectured by Brudnyi and Shvartsman in [9].

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The purpose of this paper is to examine what a linear extension operator for $C^m(E)$ might look like. For arbitrary finite E, we showed in [11] that $C^m(E)$ admits an extension operator of bounded "depth". We recall the relevant definition from [11], in a slightly weakened form.

Let $s \geq 1$ be an integer, and let $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ be a linear map. Then we say that T has depth s if, for every $x^0 \in \mathbb{R}^n$, there exist $x_1, \ldots, x_s \in E$ and $\lambda_1, \ldots, \lambda_s \in \mathbb{R}$, such that

$$Tf(x^0) = \sum_{i=1}^{s} \lambda_i f(x_i) \text{ for all } f \in C^m(E).$$

From [11], we have the following result.

Theorem 1. *Given $m \geq 1$ and $E \subset \mathbb{R}^n$ finite, there exists an extension operator $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ with norm at most C and depth at most s; here, C and s depend only on m and n.*

One might be tempted to believe that the hypothesis of finite E can be dropped from Theorem 1. The following result dashes this hope.

Theorem 2. *There exists a countable compact set $E \subset \mathbb{R}^2$, for which $C^1(E)$ admits no extension operator of finite depth.*

We prove this result in Section 1 below, by exhibiting an explicit E. Our set E is very close to a counterexample given by Glaeser in [18].

Despite Theorem 2, we can get a positive result by modifying the notion of "depth". We prepare the way with the following definitions.

A "one-point differential operator on $C^m(\mathbb{R}^n)$" is a linear functional on $C^m(\mathbb{R}^n)$ of the form

(1) $\mathcal{D} : F \mapsto \sum_{|\alpha| \leq m} a_\alpha \, \partial^\alpha F(x^0)$, with $x^0 \in \mathbb{R}^n$ and $a_\alpha \in \mathbb{R}$ ($|\alpha| \leq m$).

Next, let $E \subset \mathbb{R}^n$, and let $\mathcal{D}$ be as in (1). We say that $\mathcal{D}$ is a "one-point differential operator on $C^m(E)$", provided we have

(2) $\mathcal{D}F = 0$ whenever $F \in C^m(\mathbb{R}^n)$ and $F|_E = 0$.

Evidently, if (1) and (2) hold, then we obtain a linear functional on $C^m(E)$, by mapping $f \in C^m(E)$ to $\mathcal{D}F$, for any $F \in C^m(\mathbb{R}^n)$ with $F|_E = f$. Abusing notation, we denote this functional by $f \mapsto \mathcal{D}f$.

As a trivial example, suppose E is an embedded sub-manifold in $\mathbb{R}^n$. Then any tangent vector $X \in T_{x^0}E$ is a one-point differential operator on $C^1(E)$.

The paper [2] of Bierstone-Milman-Pawłucki shows how to find all possible one-point differential operators on $C^m(E)$ for an arbitrary, given $E \subset \mathbb{R}^n$. (See also [13].)

Now let $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ be a linear map and let $s \geq 1$. Then we say that T has "breadth" s if, given any one-point differential operator $\mathcal{D}$ on $C^m(\mathbb{R}^n)$, there exist one-point differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_s$ on $C^m(E)$, such that

$$\mathcal{D}(Tf) = \sum_{i=1}^{s} \mathcal{D}_i f \text{ for all } f \in C^m(E)\,.$$

In particular, this implies that, for any $x^0 \in \mathbb{R}^n$, we can express $Tf(x^0)$ as a sum of at most s terms of the form $\mathcal{D}_i f$, where $\mathcal{D}_i$ is a one-point differential operator on $C^m(E)$.

We are ready to state our positive result.

Theorem 3. *Given $m \geq 1$ and $E \subset \mathbb{R}^n$, there exists an extension operator $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$, with norm at most C and breadth at most s; here, C and s depend only on m and n.*

The proof of Theorem 3 is accomplished by modifying the proof of the main result in [17], as explained in Section 2 below.

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1. Proof of Theorem 2

We exhibit the countable compact set $E \subset \mathbb{R}^2$ from Theorem 2. Let

(1) $P_{N,k} = (x_{N,k}, y_{N,k}) = (2^{-N} + 10^{-N-k}, (-1)^k \cdot 10^{-2N-k}) \in \mathbb{R}^2$ for $N, k \geq 1$;

and let

(2) $P_{N,\infty} = (2^{-N}, 0) \in \mathbb{R}^2$, for $N \geq 1$.

We define

(3) $E_N = \{P_{N,\infty}\} \cup \{P_{N,k} : k \geq 1\} \subset \mathbb{R}^2$ for $N \geq 1$,

and we set

(4) $E = \{(0,0)\} \cup \bigcup_{N \geq 1} E_N$.

Note that the E_N are pairwise disjoint. As promised, E is a countable compact subset of $\mathbb{R}^2$.

To show that $C^1(E)$ admits no extension operators of finite depth, we use the following three properties of E.

Lemma 1. *Let $\tilde{E} \subset E$, and suppose $\tilde{E} \cap E_N$ is finite for each $N \geq 1$. Then $\tilde{E} \subset \{(x, y) \in \mathbb{R}^2 : y = \psi(x)\}$ for some function $\psi \in C^1(\mathbb{R})$.*

Lemma 2. *Let $s \geq 1$, and let $(P_1^n, \ldots, P_s^n)$ be a sequence of s-tuples of points of E. Then there exist an integer $N_0 \geq 1$ and an increasing infinite sequence $(n_\nu)_{\nu \geq 1}$, such that $\{P_i^{n_\nu} : \nu \geq 1,\ 1 \leq i \leq s\} \cap E_N$ is finite for each $N \geq N_0$.*

Lemma 3. *Let $F \in C^1(\mathbb{R}^2)$. If $F = 0$ on E, then $\nabla F(0,0) = 0$.*

Assume these three lemmas for the moment, and suppose $T : C^1(E) \longrightarrow C^1(\mathbb{R}^2)$ is an extension operator of depth s. We will derive a contradiction.

For $n \geq 1$, let

(5) $Q^n = (0, \frac{1}{n}) \in \mathbb{R}^2$.

Since T has depth s, there exist points $P_1^n, \ldots, P_s^n \in E$ and coefficients $\lambda_1^n, \ldots, \lambda_s^n \in \mathbb{R}$, such that

$$Tf(Q^n) = \sum_{i=1}^{s} \lambda_i^n f(P_i^n) \text{ for } f \in C^1(E)\,,\ n \geq 1\,.$$

In particular, for each $n \geq 1$, we have

(6) $Tf(Q^n) = 0$ whenever $f \in C^1(E)$ with $f(P_1^n) = \cdots = f(P_s^n) = 0$.

We apply Lemma 2 to the s-tuples $(P_1^n, \ldots, P_s^n)$, $n \geq 1$.

Let N_0 and $(n_\nu)_{\nu \geq 1}$ be as in Lemma 2. We define sets

(7) $\hat{E} = \{P_i^{n_\nu} : \nu \geq 1, 1 \leq i \leq s\}\,,$

(8) $E^{\#} = \{P \in \hat{E} : P \in E_N$ for some $N < N_0\}$, and

(9) $\tilde{E} = \hat{E} \setminus E^{\#}$.

The set $\tilde{E} \cap E_N$ is finite for $N \geq N_0$ (by Lemma 2), and empty for $N < N_0$ (by (8) and (9)). Hence, Lemma 1 applies, and there exists $\psi \in C^1(\mathbb{R})$ such that

(10) $y = \psi(x)$ for all $(x, y) \in \tilde{E}$.

Now let $\theta(x, y)$ be a smooth cutoff function on $\mathbb{R}^2$, equal to one in a neighborhood of the origin, and equal to zero on E_N for $N < N_0$. We define

(11) $F(x, y) = \theta(x, y) \cdot [y - \psi(x)]$ for $(x, y) \in \mathbb{R}^2$, and

(12) $f = F|_E \in C^1(E)$.

The functions F and Tf both belong to $C^1(\mathbb{R}^2)$, and are both equal to f on E. Hence, Lemma 3 gives

(13) $\nabla(Tf)\,(0,0) = \nabla F(0,0)$.

On the other hand, we can compute $\frac{\partial}{\partial y}(Tf)(0,0)$ and $\frac{\partial}{\partial y}F(0,0)$, and they will turn out to be unequal.

In fact, we have $F = 0$ on $\tilde{E}$ thanks to (10), (11); and $F = 0$ on $E^{\#}$, since $\theta = 0$ on E_N for $N < N_0$. (See (8), (11).) Thus, $F = 0$ on $\hat{E}$, hence $f = 0$ on $\hat{E}$, and therefore $Tf(Q^{n_\nu}) = 0$ for $\nu \geq 1$, thanks to (6) and (7).

Recalling (5), we conclude that

$$(14)\quad \frac{\partial}{\partial y}(Tf)(0,0) \;=\; 0.$$

However, since $\theta = 1$ in a neighborhood of the origin, the definition (11) yields

$$(15)\quad \frac{\partial}{\partial y}F(0,0) \;=\; 1.$$

Thus, $\frac{\partial}{\partial y}(Tf)(0,0)$ and $\frac{\partial}{\partial y}F(0,0)$ are distinct, as claimed.

This contradicts (13), showing that $C^1(E)$ cannot have an extension operator of depth s.

To complete the proof of Theorem 2, it remains to establish Lemmas 1, 2 and 3. We begin with the following elementary result, which will be used in the proof of Lemma 1.

Proposition. *Given $M \geq 1$, there exists $\psi_M \in C^1(\mathbb{R})$, with*

(16) $\mathsf{supp}\,\psi_M \subset (0,1)$,

(17) $\psi_M(10^{-k}) \;=\; (-1)^k \cdot 10^{-k}$ *for* $1 \leq k \leq M$, *and*

(18) $\| \psi_M \|_{C^1(\mathbb{R})} \leq C$, *with C independent of M.*

Proof. Fix smooth functions $\theta, \tilde{\theta}$ on $\mathbb{R}$, with $\theta(x) = 0$ for $x \leq 1/2$, $\theta(x) = 1$ for $x \geq 1$, $\tilde{\theta}(x) = 1$ for $|x| \leq 1/2$, $\tilde{\theta}(x) = 0$ for $|x| \geq 2/3$. One checks easily that

$$\psi_M(x) \;=\; \theta(10^M x) \cdot \tilde{\theta}(x) \cdot x \cos(\pi \log_{10}|x|)$$

satisfies all the conditions asserted in the proposition. ∎

Proof of Lemma 1. For each $N \geq 1$, pick

$$(19)\quad M_N \;>\; \max\{k : P_{N,k} \in \tilde{E}\}.$$

We can do this, since $\tilde{E} \cap E_N$ is assumed finite. Define

$$(20)\quad \psi(x) \;=\; \sum_{N \geq 1} 10^{-2N}\, \psi_{M_N}(10^N \cdot [x - 2^{-N}]) \text{ for } x \in \mathbb{R},$$

with ψ_{M_N} as in the Proposition.

Each summand in (20) is a C^1 function on $\mathbb{R}$, with the N^{th} summand having C^1 norm at most $C \cdot 10^{-N}$. (This follows easily from (18).) Hence, $\psi \in C^1(\mathbb{R})$.

From (1) and (17), we have

$$10^{-2N}\,\psi_{M_N}(10^N \cdot [x_{N,k} - 2^{-N}]) \;=\; 10^{-2N}\,\psi_{M_N}(10^{-k}) = (-1)^k \cdot 10^{-2N-k}$$

for $1 \le k \le M_N$. Hence, (1) and (19) yield

(21) $10^{-2N}\,\psi_{M_N}(10^N \cdot [x_{N,k} - 2^{-N}]) \;=\; y_{N,k}$ whenever $P_{N,k} \in \tilde{E}$.

On the other hand, (16) implies easily that

(22) $10^{-2N'}\psi_{M_{N'}}(10^{N'}\cdot[x_{N,k}-2^{-N'}]) \;=\; 0$ whenever $N' \neq N$ $(N, N', k \ge 1)$.

Putting (21), (22) into (20), we see that

(23) $\psi(x_{N,k}) \;=\; y_{N,k}$ whenever $P_{N,k} = (x_{N,k}, y_{N,k}) \in \tilde{E}$.

For $(x, y) = P_{N,\infty}$ or $(0, 0)$, we have $y = 0$, and all the summands in (20) are equal to zero, thanks to (16). Hence,

(24) $\psi(x) = y$ whenever $(x, y) \;=\; P_{N,\infty}$ or $(0, 0)$.

From (23), (24) and (3), (4), we conclude that $\psi(x) = y$ for all $(x, y) \in \tilde{E}$, since $\tilde{E} \subset E$. The proof of Lemma 1 is complete. ■

Proof of Lemma 2. For $n \ge 1$, let $\mathcal{P}^n$ be the set

(25) $\mathcal{P}^n \;=\; \{P_1^n, \ldots, P_s^n\}$.

Suppose $\mathcal{N}$ is any set of positive integers. We say that $\mathcal{N}$ is a "sink" if there are infinitely many $n \ge 1$ for which $\mathcal{P}^n$ intersects E_N for each $N \in \mathcal{N}$. The empty set is a sink. On the other hand, no sink can have more than s elements, since the E_N are pairwise disjoint and the $\mathcal{P}^n$ have cardinality at most s. Hence there exists a sink $\bar{\mathcal{N}}$ of maximal cardinality. Thus,

(26) The set $A = \{n \ge 1 : \mathcal{P}^n$ intersects E_N for each $N \in \bar{\mathcal{N}}\}$ is infinite (since $\bar{\mathcal{N}}$ is a sink)

and

(27) Given $N \ge 1$ not belonging to $\bar{\mathcal{N}}$, there are at most finitely many $n \in A$ for which $\mathcal{P}^n$ intersects E_N. (Otherwise, $\bar{\mathcal{N}} \cup \{N\}$ would be a sink, contradicting the maximal cardinality of $\bar{\mathcal{N}}$.)

In view of (26), we can write

(28) $A = \{n_1, n_2, n_3, \ldots\}$

for an infinite increasing sequence $(n_\nu)_{\nu \ge 1}$.

Since $\bar{\mathcal{N}}$ is a sink, it has at most s elements. Hence we can pick an integer $N_0 \ge 1$ such that

(29) $N_0 \;>\; N$ for all $N \in \bar{\mathcal{N}}$.

From (27), (28), (29), we learn the following:

(30) Given $N \geq N_0$, there are at most finitely many ν for which $\mathcal{P}^{n_\nu}$ intersects E_N.

From (25) and (30), we obtain the conclusion of Lemma 2. ■

Proof of Lemma 3. Let $F \in C^1(\mathbb{R}^2)$, with $F = 0$ on E. Fix $N \geq 1$, and note that $F(P_{N,\infty}) = F(P_{N,k}) = 0$ for $k \geq 1$. Consequently,

(31) $$0 = \lim_{\substack{k\to\infty \\ (k \text{ even})}} \left[\frac{F(P_{N,k}) - F(P_{N,\infty})}{10^{-N-k}}\right] = \left(\frac{\partial F}{\partial x} + 10^{-N}\frac{\partial F}{\partial y}\right)(P_{N,\infty}) \quad \text{and}$$

(32) $$0 = \lim_{\substack{k\to\infty \\ (k \text{ odd})}} \left[\frac{F(P_{N,k}) - F(P_{N,\infty})}{10^{-N-k}}\right] = \left(\frac{\partial F}{\partial x} - 10^{-N}\frac{\partial F}{\partial y}\right)(P_{N,\infty}).$$

(See (1)...(4).)

From (31) and (32), we learn that $\nabla F(P_{N,\infty}) = 0$. Taking the limit as $N \to \infty$, we conclude that $\nabla F(0,0) = 0$, proving Lemma 3. ■

We have now established Lemmas 1, 2 and 3. Since we reduced Theorem 2 to those lemmas, the proof of Theorem 2 is complete. ■

It is an amusing exercise to construct a linear extension operator for $C^1(E)$ with $E \subset \mathbb{R}^2$ given by (1)...(4).

2. Sketch of Proof of Theorem 3

We recall the main result of [17], then explain how to modify it to prove Theorem 3. We begin with some notation and definitions.

We write $\mathcal{R}_x$ for the ring of m-jets of smooth real-valued functions at $x \in \mathbb{R}^n$. For $F \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we write $J_x(F)$ to denote the m-jet of F at x.

Let $E \subset \mathbb{R}^n$ be compact. For each $x \in E$, suppose we are given an m-jet $f(x) \in \mathcal{R}_x$ and an ideal $I(x)$ in $\mathcal{R}_x$. Then $(f(x)+I(x))_{x\in E}$ is called a "family of cosets". (We allow the possibilities $I(x) = \{0\}$ and $I(x) = \mathcal{R}_x$.) The family of cosets $(f(x)+I(x))_{x\in E}$ is called "Glaeser stable" if it satisfies the following condition: Given $x_0 \in E$ and $P_0 \in f(x_0) + I(x_0)$, there exists $F \in C^m(\mathbb{R}^n)$ such that $J_{x_0}(F) = P_0$, and $J_x(F) \in f(x) + I(x)$ for all $x \in E$.

More generally, suppose Ξ is a vector space, and again let $E \subset \mathbb{R}^n$ be compact. For each $x \in E$, suppose we are given a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, and an ideal $I(x)$ in $\mathcal{R}_x$. Then we call $(f_\xi(x)+I(x))_{x\in E,\xi\in\Xi}$ a "family of cosets depending linearly on $\xi \in \Xi$". We say that $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ is "Glaeser stable" if, for each fixed $\xi \in \Xi$, the family of cosets $(f_\xi(x) + I(x))_{x\in E}$ is Glaeser stable.

These notions arise naturally in [16, 17], and we refer the reader to those papers for the motivation. The main result of [17] is as follows.

Theorem 4. *Let Ξ be a vector space, with seminorm $|\cdot|$. Let $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.*

Assume that for each $\xi \in \Xi$ with $|\xi| \le 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \le 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, such that

(A) *$J_x(F_\xi) \in f_\xi(x) + I(x)$ for all $x \in E$, $\xi \in \Xi$; and*

(B) *$\| F_\xi \|_{C^m(\mathbb{R}^n)} \le C|\xi|$ for all $\xi \in \Xi$, with C depending only on m and n.*

This result easily implies the existence of extension operators for $C^m(E)$. To prove Theorem 3, we modify Theorem 4 by introducing the notion of "s-admissible" operators, which we now explain.

Let $\hat{\Xi}$ be a set of (real) linear functionals on the linear space Ξ, and let $s \ge 1$ be an integer. Then:

- A linear functional on Ξ will be called "s-admissible" (with respect to $\hat{\Xi}$) if it can be written as a linear combination of at most s elements of $\hat{\Xi}$.
- A linear map T from Ξ to a finite-dimensional vector space V is called "s-admissible" (with respect to $\hat{\Xi}$) if, for every linear functional λ on V, the linear functional $\lambda \circ T$ on Ξ is s-admissible.
- A linear map $T : \Xi \longrightarrow C^m(\mathbb{R}^n)$ will be called "s-admissible" (with respect to $\hat{\Xi}$) if, for every $x \in \mathbb{R}^n$, the map $\xi \mapsto J_x(T\xi)$ is s-admissible as a map from Ξ to $\mathcal{R}_x$.

Our modification of Theorem 4 is as follows.

Theorem 5. *Let Ξ be a vector space, with seminorm $|\cdot|$, let $\hat{\Xi}$ be a set of linear functionals on Ξ, and let $s \ge 1$ be an integer. Let $(f_\xi(x)+I(x))_{x\in E,\xi\in\Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.*

Assume that the map $\xi \mapsto f_\xi(x)$, from Ξ into $\mathcal{R}_x$, is s-admissible with respect to $\hat{\Xi}$, for each $x \in E$.

Assume also that, for each $\xi \in \Xi$ with $|\xi| \le 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

$$\| F \|_{C^m(\mathbb{R}^n)} \le 1\,, \text{ and } J_x(F) \in f_\xi(x) + I(x) \text{ for all } x \in E\,.$$

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, such that

(A) *$J_x(F_\xi) \in f_\xi(x) + I(x)$ for all $x \in E$, $x \in \Xi$;*

(B) *$\| F_\xi \|_{C^m(\mathbb{R}^n)} \le C|\xi|$ for all $\xi \in \Xi$, with C depending only on m and n;*

(C) *The map $\xi \mapsto F_\xi$ is s'-admissible, with s' depending only on s, m and n.*

We indicate briefly why Theorem 5 implies Theorem 3 and then we explain how the proof of Thm. 4 in [17] may be modified to prove Theorem 5.

Reduction of Theorem 3 to Theorem 5. To prove Theorem 3, we may assume that the set E is compact. (In fact, for a general E, we may pass without difficulty to the closure of E, and then reduce matters to the case of closed, bounded E by a partition of unity.)

For $E \subset \mathbb{R}^n$ compact, we make the following definitions.

- $\Xi = C^m(E)$.
- $|\xi| = 2 \parallel \xi \parallel_{C^m(E)}$ for $\xi \in \Xi$.
- $\widehat{\Xi}$ is the set of all one-point differential operators on $C^m(E)$.

For each $x \in E$:

- $I(x) = \{J_x(F) : F \in C^m(\mathbb{R}^n) \text{ and } F = 0 \text{ on } E\}$.
- $V(x) =$ any complementary subspace to $I(x)$ in $\mathcal{R}_x$.
- $\pi_x : \mathcal{R}_x \longrightarrow V(x)$ is the natural projection arising from the direct sum $\mathcal{R}_x = V(x) \oplus I(x)$.

Suppose $x \in E$ and $\xi \in \Xi$. Since $\xi \in \Xi$, there exists $F \in C^m(\mathbb{R}^n)$ with $F|_E = \xi$. We define $f_\xi(x) = \pi_x(J_x(F))$. This is independent of the choice of F. (In fact, suppose $F_1, F_2 \in C^m(\mathbb{R}^n)$, with $F_i|_E = \xi$. Then $F_1 - F_2 \in C^m(\mathbb{R}^n)$ and $(F_1 - F_2)|_E = 0$. Hence, $J_x(F_1 - F_2) \in I(x)$, and therefore $\pi_x(J_x(F_1) - J_x(F_2)) = 0$.)

One checks easily that the above Ξ, $|\cdot|$, $\widehat{\Xi}$, $I(x)$, $f_\xi(x)$ satisfy the hypotheses of Theorem 5, with $s = 1$. Hence, applying Theorem 5, we obtain a linear map $\xi \mapsto F_\xi$ from Ξ into $C^m(\mathbb{R}^n)$, satisfying (A), (B), (C).

From (A), we see that $\xi \mapsto F_\xi$ is an extension operator for $C^m(E)$. Conclusion (B) controls the norm of this extension operator, and conclusion (C) tells us that it has breadth s', with s' depending only on m and n. Thus, Theorem 3 is reduced to Theorem 5.

Sketch of Proof of Theorem 5. We assume that the reader is familiar with our previous papers [11,...,17]. It is a long, routine exercise to follow the proof of Theorem 4, as given in [17], and note that at each step, we preserve s'-admissibility (although s' may increase). ("Admissibility" will always be defined with respect to $\widehat{\Xi}$, given in the hypotheses of Theorem 5.) The highlights of this tedious exercise are as follows.

- For $E \subset \mathbb{R}^n$ compact, let $C^m_{\text{jet}}(E)$ be the space of families of jets $\vec{f} = (f_x)_{x \in E}$, with $f_x \in \mathcal{R}_x$ for each $x \in E$, such that there exists $F \in C^m(\mathbb{R}^n)$ satisfying

  (1) $J_x(F) = f_x$ for each $x \in E$.

The norm $\| \vec{f} \|_{C^m_{\text{jet}}(E)}$ is defined as the infimum of $\| F \|_{C^m(\mathbb{R}^n)}$ over all $F \in C^m(\mathbb{R}^n)$ satisfying (1).

The proof of the standard Whitney extension theorem [19, 24, 25] gives an operator $T : C^m_{\text{jet}}(E) \longrightarrow C^m(\mathbb{R}^n)$, with the following properties.

(a) $\| T \| \leq C$, with C depending only on m and n.

(b) For $\vec{f} = (f_x)_{x \in E} \in C^m_{\text{jet}}(E)$, we have $J_x(T\vec{f}) = f_x$ for each $x \in E$.

(c) For each $x_0 \in \mathbb{R}^n$ there exist $x_1, \ldots, x_k \in E$ such that, as $\vec{f} = (f_x)_{x \in E}$ varies over $C^m_{\text{jet}}(E)$, the jet $J_{x_0}(T\vec{f})$ depends only on $f_{x_1}, \ldots, f_{x_k}$. Here, k depends only on m and n.

In view of (c), we have the following result.
Let $\xi \mapsto \vec{f_\xi} = (f_{x,\xi})_{x \in E}$ be a linear map from Ξ into $C^m_{\text{jet}}(E)$. Assume that $\xi \mapsto f_{x,\xi}$ is s'-admissible, for each $x \in E$.

Then the map $\xi \mapsto T\vec{f_\xi}$ is s''-admissible from Ξ into $C^m(\mathbb{R}^n)$, where T is as above, and s'' depends only on s', m, n.

- Suppose we add to the hypotheses of Lemma 3.3 in [16] the assumption that $\xi \mapsto f_\xi(x)$ is s'-admissible for each $x \in E$. (Here, $s' \geq 1$ is given.) Then the map $\xi \mapsto \tilde{f}_\xi(x_0)$ in the conclusion of that lemma may be taken to be s''-admissible, with s'' depending only on $s', m, n, k^\#$. (That's because the $\tilde{f}_\xi(x_0)$ constructed in the proof of Lemma 3.3 in [16] depends on ξ only through the $f_\xi(x)$ for $x \in \bar{S}$, where $\bar{S} \subset E$ has cardinality less than $k^\#$.) When we apply the above lemma in [17], we take $k^\#$ to depend only on m and n.

  Therefore, the property of s'-admissibility (for some s' depending only on m, n, s) is preserved when we apply Lemma 3.3 from [16].

- Whenever we applied Theorem 5 from [16] in the proof of Theorem 4, we now apply instead Theorem 8 from [16]. Note that the notion of "depth" in [16] differs from our present notion.

- For suitable $x \in E$, let $\text{proj}_x : \mathcal{R}_x \longrightarrow \mathcal{R}_x$ be the linear map defined in Section 10 of [17]. If $\xi \mapsto g_\xi(x)$ is an s'-admissible linear map from Ξ into $\mathcal{R}_x$, then also $\xi \mapsto \text{proj}_x(g_\xi(x))$ is s'-admissible. (This follows trivially from the definition of s'-admissibility.)

- Suppose $F_\xi = \sum_\nu \theta_\nu \cdot F^\nu_\xi$ for $\xi \in \Xi$; and suppose that, for each $x \in \mathbb{R}^n$, we are given a finite set $\Omega(x)$, such that

$$J_x(F_\xi) \;=\; \sum_{\nu \in \Omega(x)} J_x(\theta_\nu \;\cdot\; F^\nu_\xi) \text{ for all } \xi \in \Xi\,.$$

  If $\xi \mapsto F^\nu_\xi$ is s'-admissible for each ν, and if $\Omega(x)$ has cardinality at most k for each $x \in \mathbb{R}^n$, then $\xi \mapsto F_\xi$ is s''-admissible, with $s'' = k \cdot s'$.

Finally, we can prove Theorem 5 by following the proof of Theorem 4 in [17], and using the above observations to keep track of s'-admissibility of every operator and functional that enters the argument. We dispense with further details.

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