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Optimizing geometric measures for fixed minimal annulus and inradius

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Abstract

In this paper we relate the minimal annulus of a planar convex body K with its inradius, obtaining all the upper and lower bounds, in terms of these quantities, for the classic geometric measures associated with the set: area, perimeter, diameter, minimal width and circumradius. We prove the optimal inequalities for each one of those problems, determining also its corresponding extremal sets.

1. Introduction

Let K be a convex body (compact convex set) in the Euclidean plane. Associated with K there are a number of well-known functionals: the *area* $A = A(K)$ and the *perimeter* $p = p(K)$; the *diameter* $D = D(K)$ and the *minimal width* $\omega = \omega(K)$ (minimum distance between two parallel support hyperplanes of K); among all discs containing K there is exactly one with minimum radius, called the *circumradius* R_K of the set K; respectively, among all discs which are contained in K, those whose radii have maximum value, provide the *inradius* of the body, r_K. These special discs (named circumcircle and incircles) have very useful properties; some of them will be stated and used later.

Another interesting functional to be considered for a convex body K is the thickness of its *minimal annulus*. The minimal annulus of K is the annulus (the closed set consisting of the points lying between two concentric discs –concentric n-balls in $\mathbb{R}^n$) with minimal difference of radii that contains the boundary of K. Of course, the minimal annulus is uniquely determined (Bonnesen, [2], in $\mathbb{R}^2$ and $\mathbb{R}^3$, and Bárány, [1], in higher dimension). From now on, we shall denote by $A(c, r, R)$ the minimal annulus of the planar

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convex body K, where c, r and R represent, respectively, its center, radius of the inner circle, and radius of the outer circle. This object and its properties were studied mainly by Bonnesen for planar convex sets (see [2] and [3]). More recently, very interesting works have appeared, in which, the minimal annulus has been studied in a more general setting: for arbitrary dimension, replacing the ball by the boundary of a fixed smooth strictly convex body, in Minkowski space... (see, for instance, [1, 8, 9, 10, 11, 14]).

Another interesting problem would be to look for inequalities involving the classical functionals and the minimal annulus, finding the convex sets for which the equality sign is attained: the extremal sets. In [2], [5] and [4], Bonnesen and Favard studied this type of problems: in [2] and [5] the minimum and the maximum of the isoperimetric deficit $p^2/(4\pi) - A$ were obtained; in the third paper, the optimal bounds of the area and the perimeter for fixed minimal annulus were determined.

In [6], the bounds for the remaining measures (diameter, minimal width, circumradius and inradius) in terms of the minimal annulus have been obtained, as well as the *optimal* inequalities that state the best bounds for the classical magnitudes when the minimal annulus and the circumradius are fixed (see [6] and [7]): let us note that if three measures are involved, the question becomes more interesting when the equality, for a particular inequality, is not attained for a single figure, but for a continuous family of sets; in this case, that inequality, named optimal, provides the maximum or minimum value of a measure for *each pair* of possible values of the others.

In this paper, we obtain all the possible (and optimal) relations which state the maximum and minimum values of the area, the perimeter, the diameter and the minimal width of a convex body, when its minimal annulus and its inradius are given. We prove the optimal inequalities for each one of these problems, determining also their corresponding extremal sets. The circumradius case was solved in [7].

2. Some previous results

Before stating the main results of the paper, let us consider some properties of the minimal annulus of a convex body K, which will play a crucial role in the proofs of the results. Let us denote by c_r and C_R, respectively, the inner and the outer circles of the minimal annulus $A(c, r, R)$ of K.

As usual, ∂K will denote the boundary of the set K. Given two points $P, Q \in \mathbb{R}^2$, PQ will denote the straight line determined by them; $\overline{PQ}$ the line segment joining them; and $\widehat{PQ}$ any circular arc with P, Q as extreme points. Besides, if P, Q lie on a circumference (with center c), we call *central angle* of P and Q the angle $\angle(PcQ)$ determined by them with respect to the center c.

The following well-known properties were studied by Bonnesen in [2]:

(P1) *Each one of the circumferences ∂c_r and ∂C_R touches the boundary ∂K of K in, at least, two points.*

(P2) *The sets $\partial c_r \cap \partial K$ and $\partial C_R \cap \partial K$ can not be separated.*

(Two sets A and B can be separated if there exists a line ℓ such that $A \subset \ell^+$ and $B \subset \ell^-$, where ℓ^+, ℓ^- represent the halfplanes determined by ℓ).

(P3) *The minimal annulus of a convex body K is uniquely determined.*

(P4) *The minimal annulus of a convex body K is the only annulus that contains ∂K and verifies properties* **(P1)** *and* **(P2)**.

The following lemmas were obtained in [6], where we proved some properties of the minimal annulus of a convex body K, as well as its relation with the inradius of K. They will be very useful in the proofs of the results.

Lemma 1. *Let K be a convex body with minimal annulus $A(c, r, R)$. The following properties hold:*

(a) There are points $P, Q \in \partial C_R \cap \partial K$ whose central angle α verifies $\alpha \geq 2\arccos(r/R)$.

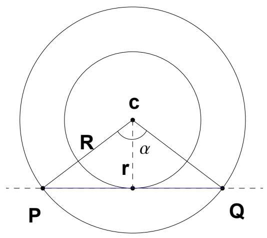


Figure 1. The limit case when the central angle of the points $P, Q \in \partial C_R \cap \partial K$ is, precisely, $\alpha = 2\arccos(r/R)$.

(b) K contains a 2-cap-body generated by the convex hull of c_r and two points of $\partial C_R \cap \partial K$, whose minimal annulus is $A(c, r, R)$ (a cap-body is the convex hull of a disc and countable many points such that the line segment joining any pair of those points intersects the disc).

(c) K is contained in a circular slice of the outer circle C_R determined by two support lines to c_r, whose minimal annulus is $A(c, r, R)$ (a circular slice is the part of a circle bounded by two straight lines, whose intersection point, if it exists, is not interior to it).

The following lemma collects some properties relating the minimal annulus of a convex body with its inradius. From now on, we shall denote by c_K an incircle of the body K, and by y_0 one of its incenters.

Lemma 2. *Let K be a convex body with minimal annulus $A(c,r,R)$ and incircle $c_{_K}$. The following properties hold:*

(i) $r \leq r_{_K}$.

(ii) $\mathrm{conv}(c_r \cup c_{_K}) \subset K \subset C_R$.

(iii) *c_r can not be strictly contained in $c_{_K}$, having the following possible relative positions between them (see Figure 2):*

(a) $c_r \equiv c_{_K}$.

(b) *$\partial c_r \cap \partial c_{_K}$ contains, exactly, two points.*

(c) *The boundaries ∂c_r, $\partial c_{_K}$ touch (from outside) in one point.*

(d) *There are no common points.*

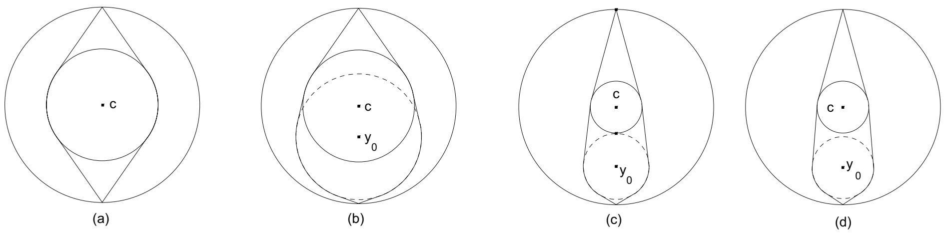


Figure 2: Some examples for the relative positions of c_r and $c_{_K}$.

(iv) *The boundary of the set $\mathrm{conv}(c_r \cup c_{_K})$ is formed by two line segments $\overline{PS}$ and $\overline{QT}$, and the corresponding circular arcs $\widehat{PQ} \subset \partial c_r$ and $\widehat{ST} \subset \partial c_{_K}$. Then, each of those arcs has, at least, one point of ∂K.*

(v) *The arc $\widehat{PQ} \subset \partial c_r$ contains two points $P', Q' \in \partial K$ (which can coincide with P and Q), determining a central angle $\alpha \geq 2\arccos(r/R)$.*

Let us add a new property to this lemma which will be needed also later:

(vi) *K contains a convex body $K^{2c} = \mathrm{conv}\{c_r \cup c_{_K}, N, M\}$, with the same minimal annulus and inradius as K, where N and M lie, respectively, on each arc of ∂C_R determined by the lines PS and QT (see figure 3).*

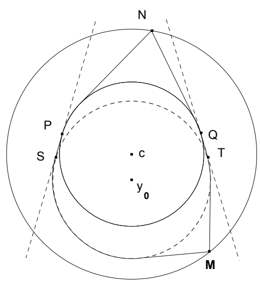
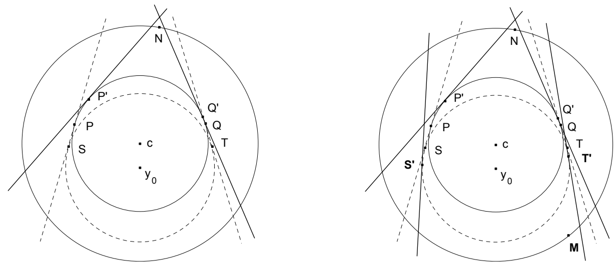


Figure 3: A set $K^{2c} \subset K$ with the same minimal annulus and inradius.

Proof of property (vi): Property (v) of lemma 2 assures the existence of, at least, two points P', Q' of ∂K, lying on the circular arc $\widehat{PQ} \subset \partial c_r$, and determining a central angle $\alpha \geq 2\arccos(r/R)$. Then, K is contained in the circular slice K^s of C_R determined by the support lines to c_r through P' and Q' (see figure 4).



(a) (b)

Figure 4: Construction of the set $K^{2c} \subset K$.

On the other hand, property (b) of lemma 1 states that K contains a 2-cap-body $K^c = \operatorname{conv}\{c_r, N, M\}$, with $N, M \in \partial C_R \cap \partial K$. Besides, this cap-body can be chosen in such a way that there is one vertex lying on *each* circular arc of the boundary of K^s; in the opposite case, property **(P2)** would be contradicted. Hence, we can suppose, for instance, that N lies on the circular arc of ∂K^s which is "closer" to $\widehat{P'Q'}$ (see figure 4(a)), and M on the opposite arc.

Now, since c_K is an incircle of K, it meets ∂K either in two diametrically opposite points, or in three points that form the vertices of an acute-angled triangle (see [3]); or equivalently, in three points which do not lie on the same half-circumference. Thus, we can always choose two points $S', T' \in \partial c_K \cap \partial K$ in such a way that $\widehat{S'T'} \subset \widehat{ST}$ is bigger than a half-circumference. It implies that the support lines to K through S' and T' do not intersect in the interior of C_R, and they determine a new circular arc on which the point M lies (see figure 4(b)). Let us note that the points P', Q', S', T' can coincide with P, Q, S, T, respectively. So, K contains the set $K^{2c} = \operatorname{conv}\{c_r \cup c_K, N, M\}$, with N, M verifying the assumptions of the result.

Finally, let us note that K^{2c} has minimal annulus $A(c, r, R)$ by property **(P4)** and inradius r_K, since its boundary contains, necessarily, diametrically opposite points of ∂c_K. ■

In the following, we are going to obtain all the possible (and optimal) relations which state the maximum and minimum values of the area, the perimeter, the diameter and the minimal width of a convex body, when its minimal annulus and its inradius are given. The circumradius case was studied and solved in [7].

3. Optimizing the area and the perimeter

In this section we state the relation between the minimal annulus, the inradius and both, the area and the perimeter of a convex body K. More precisely, we are going to obtain the best bounds (upper and lower bounds) for A and p, when we suppose that the minimal annulus of the convex body and its inradius are fixed, determining also the extremal sets in each case. We start with the upper bounds.

Let us recall that, by lemma 1(c), if the minimal annulus of K is $A(c,r,R)$, then K is contained in a circular slice K^s of C_R determined by support lines to c_r through two points of $\partial c_r \cap \partial K$; all these sets have the same area and the same perimeter, and thus, it holds

$$A \le A(K^s) = 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right),$$
$$p \le p(K^s) = 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right).$$

The following theorem states also these ones as the upper bounds for any value r_K of the inradius.

Proposition 1. *Let K be a convex body with minimal annulus $A(c,r,R)$ and inradius r_K. Then,*

$$A \le 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right), \tag{3.1}$$

$$p \le 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right). \tag{3.2}$$

The equality holds, in both inequalities, if and only if the convex body K is the circular slice of C_R determined by the common support lines to c_r and c_K, when ∂c_K touches (in the interior) ∂C_R (see figure 5).

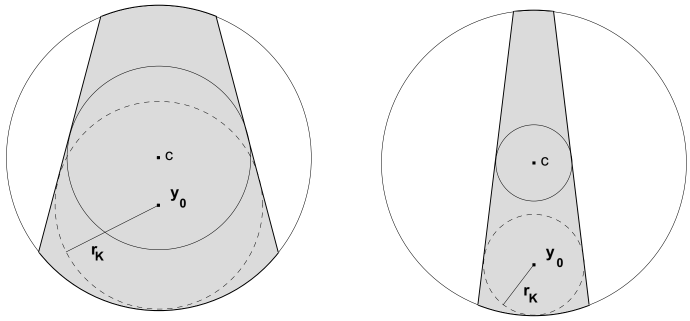


FIGURE 5: Circular slices of C_R with maximum area and perimeter.

Proof: We just have to see that, if r_K is the inradius of K, then there is a circular slice K^s of C_R (verifying the above assumptions) with inradius r_K. In order to do that, let us take the incenter y_0 in such a way that ∂c_K touches ∂C_R. Then, the circular slice K^s of C_R determined by the common support lines to c_r and c_K has inradius r_K, since ∂c_K contains three points that form an acute-angled triangle (see figure 5); and by minimal annulus $A(c,r,R)$ by property **(P4)**. Let us note that the convex body K^s so generated is always a circular slice, i.e., the lines determining it do never intersect in the interior of C_R; it holds because $r_K \leq 2Rr/(R+r)$ always (see [6, Subsect. 3.4, Prop. 7]), and just for the equality case the intersection point lies on the boundary of C_R. ■

We conclude this section stating the lower bounds for the area and the perimeter of a convex body with prescribed minimal annulus and inradius.

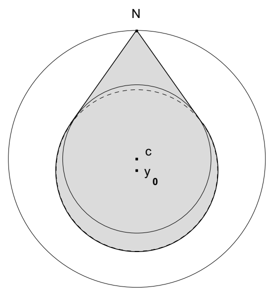


Figure 6. The set K_d.

We state some useful notation: for r_K and $A(c,r,R)$ given, let us suppose the incenter y_0 located to the suitable distance of c in order to the intersection point of the common support lines to c_r and c_K lies, precisely, on the boundary of C_R; this point will be denoted by N (see figure 6). It is an easy computation to check that this distance is given by $d(y_0,c) = R(r_K - r)/r$. We are going to denote by $K_d := \mathrm{conv}\{c_r \cup c_K, N\}$ (see figure 6), which is just the cap-body $\mathrm{conv}\{c_K, N\}$, since it always holds the relation $r_K \geq r$.

Theorem 1. *Let K be a convex body with minimal annulus $A(c,r,R)$ and inradius r_K. Then,*

$$A \geq \frac{r_K^2}{r}\left(\sqrt{R^2-r^2} + \frac{1}{r_K}\sqrt{R^2(2r-r_K)^2 - r^2 r_K^2}\right) + r_K^2\theta, \tag{3.3}$$

$$p \geq 2\left(\frac{r_K}{r}\sqrt{R^2-r^2} + \frac{1}{r}\sqrt{R^2(2r-r_K)^2 - r^2 r_K^2} + r_K\theta\right), \tag{3.4}$$

where

$$\theta = \arcsin\frac{r r_K}{R(2r-r_K)} + \arcsin\frac{r}{R}.$$

The equality holds, in both inequalities, if, and only if, $K = \mathrm{conv}\{K_d, M\}$, where M is the point of ∂C_R diametrically opposite to N (see figure 7).

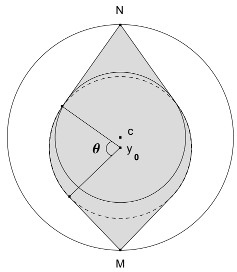


FIGURE 7: The convex body $\text{conv}\{K_d, M\}$ has minimum area and perimeter.

Proof: Let us notice that, when $r_K = r$, inequalities (3.3) and (3.4) turn into, respectively, the well-known relations

$$A \geq 2r\left(\sqrt{R^2 - r^2} + r \arcsin \frac{r}{R}\right) \quad \text{and} \quad p \geq 4\left(\sqrt{R^2 - r^2} + r \arcsin \frac{r}{R}\right)$$

(see [4]); in both cases, the extremal set is a cap-body with two vertices lying on ∂C_R. Thus, from now on we will assume that $r_K > r$ and hence, that $c_K \not\equiv c_r$.

We know (lemma 2(vi)) that K contains a set $K^{2c} = \text{conv}\{c_r \cup c_K, N, M\}$, for suitable $N, M \in \partial C_R$ (see figure 8(a)), and therefore, $A \geq A(K^{2c})$ and $p \geq p(K^{2c})$. Since K^{2c} has minimal annulus $A(c, r, R)$ and inradius r_K, we have reduce the problem to study the area and the perimeter for this particular family of sets.

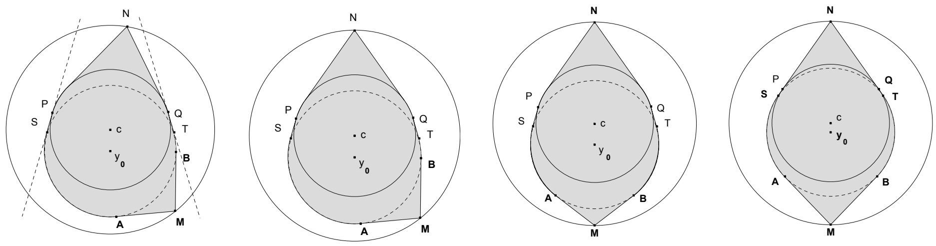


FIGURE 8: Minimizing the area and the perimeter.

Let us note first that, for each fixed y_0, the area and the perimeter of the set $\text{conv}\{c_r \cup c_K, N\}$ are constant for any possible position of N (it moves just on the arc of ∂C_R determined by the support lines to ∂c_r through P and Q, see figure 8(a)), due to c_r and C_R are concentric. Thus, without loss of generality, we can choose N to be the intersection point of the line cy_0 with ∂C_R (see figure 8(b)).

On the other hand, if we denote by A and B the tangent points where the support lines to c_K (which passes through M) touch ∂c_K, the set limited by the circular arc $\widehat{AB}$ and the line segments $\overline{AM}$ and $\overline{BM}$, has minimum area when the distance from M to ∂c_K is as small as possible; this is, when M is the second intersection point of the straight line cy_0 with ∂C_R. It is also in this case when the length of the line segments $\overline{AM}$ and $\overline{BM}$ is the smallest possible (see figure 8(c)). So, it suffices to consider the sets K^{2c} which are symmetric with respect to the line cy_0.

Let us denote by x the distance between the centers y_0 and c, $x := d(y_0, c)$. It is clear that

$$\frac{R}{r}(r_K - r) \leq x \leq R - r_K, \tag{3.5}$$

where the lower bound corresponds to the limit case when the common support lines to c_r and c_K intersect precisely on N, whereas the upper bound is given when ∂c_K touches ∂C_R. It is a tedious calculation to compute the area and the perimeter of these figures in terms of the distance x:

$$\begin{aligned} A(x) &= (r + r_K)\sqrt{x^2 - (r_K - r)^2} + r_K\sqrt{(R-x)^2 - r_K^2} + r\sqrt{R^2 - r^2} \\ &\quad + (r_K^2 - r^2)\arcsin\frac{r_K - r}{x} + r_K^2 \arcsin\frac{r_K}{R - x} + r^2 \arcsin\frac{r}{R}, \\ \frac{1}{2}p(x) &= \sqrt{x^2 - (r_K - r)^2} + \sqrt{(R-x)^2 - r_K^2} + \sqrt{R^2 - r^2} \\ &\quad + (r_K - r)\arcsin\frac{r_K - r}{x} + r_K \arcsin\frac{r_K}{R - x} + r \arcsin\frac{r}{R}. \end{aligned}$$

So, we have just to study these functions and to obtain their minimum. It can be checked that the first derivatives are

$$\begin{aligned} A'(x) &= -r_K\sqrt{1 - \left(\frac{r_K}{R - x}\right)^2} + (r_K + r)\sqrt{1 - \left(\frac{r_K - r}{x}\right)^2}, \\ \frac{1}{2}p'(x) &= -\sqrt{1 - \left(\frac{r_K}{R - x}\right)^2} + \sqrt{1 - \left(\frac{r_K - r}{x}\right)^2}. \end{aligned}$$

But it always hold

$$\sqrt{1 - \left(\frac{r_K - r}{x}\right)^2} \geq \sqrt{1 - \left(\frac{r_K}{R - x}\right)^2} : \tag{3.6}$$

in fact, inequality (3.6) is equivalent to the easier $(r_K - r)/x \leq r_K/(R - x)$, which is, in turn, equivalent to the relation

$$x \geq \frac{R(r_K - r)}{2r_K - r}; \tag{3.7}$$

but since it holds the lower bound in (3.5), and also $r_K > r$, we can obtain easily (3.7) and hence, (3.6).

All in all, inequality (3.6) assures that both derivatives $A'(x)$ and $\frac{1}{2}p'(x)$ are (strictly) positive, and hence, that $A(x)$ and $(1/2)p(x)$ are (strictly) increasing functions on the interval $\big[R(r_{K} - r)/r, R - r_{K}\big]$. It proves the required result:

$$A \geq A\left(\frac{R}{r}(r_{K} - r)\right) = \frac{r_{K}^2}{r}\left(\sqrt{R^2 - r^2} + \frac{1}{r_{K}}\sqrt{R^2(2r - r_{K})^2 - r^2 r_{K}^2}\right) + r_{K}^2\theta,$$

$$\frac{1}{2}p \geq p\left(\frac{R}{r}(r_{K} - r)\right) = \frac{r_{K}}{r}\sqrt{R^2 - r^2} + \frac{1}{r}\sqrt{R^2(2r - r_{K})^2 - r^2 r_{K}^2} + r_{K}\theta,$$

where $\theta = \arcsin\dfrac{r r_{K}}{R(2r - r_{K})} + \arcsin\dfrac{r}{R}$.

The equality holds, in both inequalities, if and only if $x = R(r_{K} - r)/r$, i.e., when the common support lines to c_r and c_{K} intersect on N. Therefore, the extremal set is the one described in the statement of the theorem: the convex hull $\operatorname{conv}\{K_d, M\}$, where $M \in \partial C_R$ is the diametrically opposite point to N (see figure 8(d)). ■

4. Optimizing the diameter

In this section we are going to state the relation among the minimal annulus, the inradius and the diameter of a convex body K. The upper bound is almost trivial:

Proposition 2. *Let K be a convex body with minimal annulus $A(c, r, R)$ and inradius r_{K}. Then:*

$$D \leq 2R, \tag{4.1}$$

where equality holds for any set containing diametrically opposite points of ∂C_R; for instance, the convex body $K^{2c} = \operatorname{conv}\{c_r \cup c_{K}, N, M\}$, where $\{N, M\} = cy_0 \cap \partial C_R$ (see figure 9).

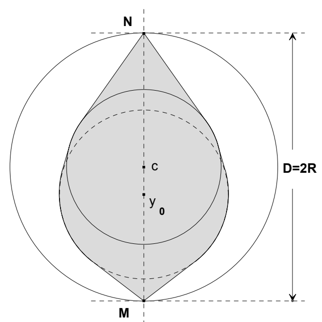


FIGURE 9: The convex body K^{2c} has maximum diameter.

Proof: Inequality (4.1) holds trivially, independently of the value of the inradius, since $K \subset C_R$. Now, the set described in the statement of the proposition has minimal annulus $A(c, r, R)$, inradius $r_{_K}$, and diameter exactly $D(K^{2c}) = d(N, M) = 2R$. ■

Theorem 2. *Let K be a convex body with minimal annulus $A(c, r, R)$ and inradius $r_{_K}$. Then:*

$$(4.2) \qquad D \geq \begin{cases} \dfrac{r_{_K}}{r}(R+r) & \text{if } r_{_K} \geq 2r\sqrt{\frac{R-r}{R+r}}, \qquad (4.2.\text{a}) \\[2ex] 2\sqrt{R^2 - r^2} & \text{if } r_{_K} \leq 2r\sqrt{\frac{R-r}{R+r}}. \qquad (4.2.\text{b}) \end{cases}$$

The equality holds in both cases, for instance, for the set conv$\{K_d, M\}$*, where M is the second intersection point (besides N) of any of the two common support lines to c_r and $c_{_K}$ with ∂C_R (see figure 10).*

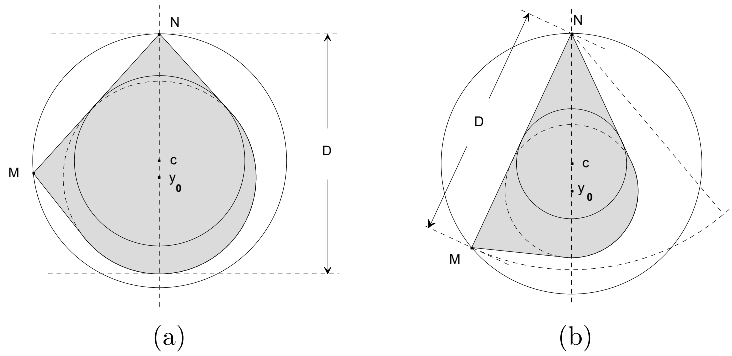


Figure 10: The convex body conv$\{K_d, M\}$ has minimum diameter.

Proof: Let us note that if $r_{_K} = r$, the relation $r_{_K} \geq 2r\sqrt{(R-r)/(R+r)}$ is equivalent to $R \leq 5r/3$. Hence, inequalities (4.2.a) and (4.2.b) turn into

$$D \geq \begin{cases} R + r & \text{if } R \leq 5r/3, \\ 2\sqrt{R^2 - r^2} & \text{if } R \geq 5r/3, \end{cases}$$

respectively, which are known (see [6, Prop. 3]); in both cases, the extremal sets are also transformed in the corresponding ones: 2-cap-bodies such that the line segment determined by the two vertices (lying on ∂C_R) is tangent to ∂c_r. Thus, from now on we will assume that $r_{_K} > r$ and hence, that $c_{_K} \not\equiv c_r$.

Lemma 2(vi) assures that K contains a set $K^{2c} = \text{conv}\{c_r \cup c_{_K}, N, M\}$, for suitable $N, M \in \partial C_R$, with the same minimal annulus and inradius as K.

Hence, $D \geq D(K^{2c})$, and it suffices to study the diameter for this particular family of sets. But because of the shape of K^{2c}, it is clear that its diameter is attained in one of the following distances:

(d_1) distance between N and M;

(d_2) distance between N and the tangent line to $\partial c_{_K}$ orthogonal to Ny_0;

(d_3) distance between M and the tangent line to $\partial c_{_K}$ orthogonal to My_0;

(d_4) distance between M and the tangent line to ∂c_r orthogonal to Mc.

Let us notice that, since $r_{_K} > r$, the last two possibilities are not feasible, since both d_3 and d_4 give values less than the one of the second option, d_2. So, we have to study just the above first two distances: d_1 and d_2.

The smallest possible distance between N and M (see lemma 1(a)) is attained when the line segment $\overline{MN}$ is tangent to ∂c_r, and hence, when the straight line NM coincides with one of the common support lines to c_r and $c_{_K}$. This distance is always $d_1 = 2\sqrt{R^2 - r^2}$, independently of the situation of the circles c_r and $c_{_K}$.

On the other hand, the distance d_2 is less as closer from c lies y_0, attaining the minimum in the limit case when $d(y_0, c) = R(r_{_K} - r)/r$; i.e., when the common support lines to c_r and $c_{_K}$ intersect on N. The value of such a distance is $r_{_K}(R + r)/r$ (see figure 11).

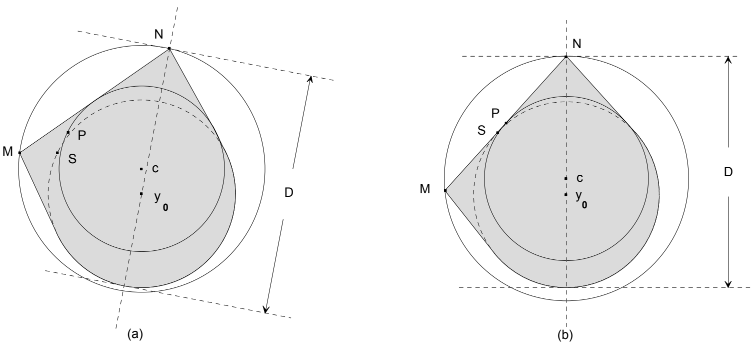


Figure 11: The distance d_2 is minimal if $d(y_0, c) = R(r_{_K} - r)/r$.

In summary, the diameter will take either the value $2\sqrt{R^2 - r^2}$, when it is attained in the distance $d_1 = d(N, M)$, or $r_{_K}(R + r)/r$, if it is attained in the distance d_2 from N to the support line to $c_{_K}$ orthogonal to Ny_0; it depends on the relation between R, r and $r_{_K}$. It is easy to check that $D \geq r_{_K}(R + r)/r$ if $r_{_K} \geq 2r\sqrt{(R - r)/(R + r)}$ (see figure 10(a)), and that $D \geq 2\sqrt{R^2 - r^2}$ when $r_{_K} \leq 2r\sqrt{(R - r)/(R + r)}$ (see figure 10(b)). ∎

5. Optimizing the minimal width

In this section we state the relation between the minimal annulus, the inradius and the minimal width of a convex body K. The lower bound is almost trivial:

Proposition 3. *Let K be a convex body with minimal annulus $A(c, r, R)$ and inradius r_K. Then:*

$$\omega \geq 2r_K, \tag{5.1}$$

where equality holds for any set containing in its boundary diametrically opposite points of ∂c_K; for instance, the convex body $K^{2c} = \mathrm{conv}\{c_r \cup c_K, N, M\}$, where $\{N, M\} = cy_0 \cap \partial C_R$ (see figure 12).

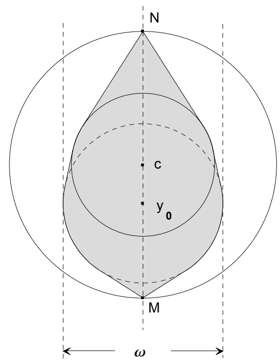


FIGURE 12: The convex body K^{2c} has minimum width.

Proof: Inequality (5.1) holds trivially, independently of the minimal annulus, since $K \supset c_K$. Now, the set described in the statement of the proposition has minimal annulus $A(c, r, R)$, inradius r_K, and minimal width exactly $\omega(K^{2c}) = 2r_K$. ■

Theorem 3. *Let K be a convex body with minimal annulus $A(c, r, R)$ and inradius r_K. Then:*

(5.2)

$$\omega \leq \begin{cases} R + r & \text{if } R \leq 2r, \quad (5.2.a) \\ \dfrac{4r}{R^2}(R^2 - r^2) & \text{if } R \geq 2r \text{ and } r_K \geq \dfrac{2r(R-r)}{R}, \quad (5.2.b) \\ \dfrac{4r}{R^2}(R^2 - r^2)\dfrac{\sin\beta}{\sin(\alpha+\beta)} & \text{if } R \geq 2r \text{ and } r_K \leq \dfrac{2r(R-r)}{R}, \quad (5.2.c) \end{cases}$$

where

$$\alpha = 2\arcsin\frac{r}{R} \qquad \text{and} \qquad \beta = 2\arctan\frac{rr_K}{(2r - r_K)\sqrt{R^2 - r^2}}.$$

The equality holds, in inequalities (5.2.a) and (5.2.b), for instance, for the convex bodies K^B shown in figures 13(a,b): let us consider the circular slice of C_R determined by the common support lines to c_r and $c_{_K}$, ℓ' and ℓ'', when they intersect on the point $N \in \partial C_R$; and let us take the halfplane delimited by the support line to $c_{_K}$, orthogonal to cy_0, containing $c_{_K}$; the intersection of both sets gives K^B.

In inequality (5.2.c), the equality holds only for the triangle determined by ℓ', ℓ'', and the support line to $c_{_K}$ passing through the point $M' \in \ell' \cap \partial C_R$, $M' \neq N$ (see figure 13(c)).

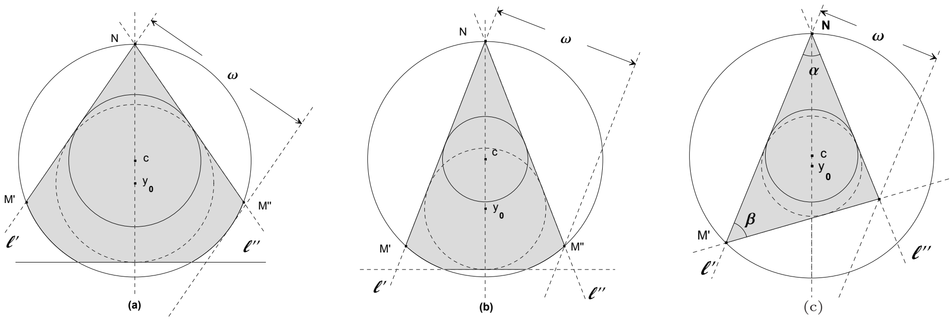


Figure 13: The convex bodies with maximum minimal width.

Let us note that the triangle obtained as extremal set of inequality (5.2.c) verifies that its angles α and β (as previously defined, see figure 13 (c)) are always less or equal than the third one, $\pi - \alpha - \beta$. It implies, in particular, that $2\beta + \alpha \leq \pi$, i.e., $\alpha + \beta \leq \pi - \beta$. Consequently, $\sin\beta \leq \sin(\alpha + \beta)$, which assures that the upper bound in (5.2.c) is always less (or equal) than the upper bound in (5.2.b), as was to be expected. And they will be equal precisely if $\beta = \pi - \alpha - \beta$, i.e., when the triangle is isosceles, which holds only if $r_{_K} = 2r(R - r)/R$.

Proof: When $R \leq 2r$, it is known that inequality (5.2.a) always holds (for given minimal annulus), independently of the value of the inradius (see [6, Subsect. 3.2, Prop. 2]); hence, we just have to find a convex body with inradius $r_{_K}$ verifying the equality. Thus, if we consider the convex body K^B defined in the statement of the theorem, it is clear that it has minimal annulus $A(c, r, R)$ and inradius $r_{_K}$ (in the particular case $r_{_K} = r$, it is enough to consider $c_{_K} \equiv c_r$ and ℓ', ℓ'' to be the support lines to c_r which intersect on $N \in \partial C_R$). Let us denote by M' and M'' the intersection points of, respectively, ℓ' and ℓ'' with ∂C_R (different from N, see figure 13 (a)). Let us note that, if $R = 2r$, then the triangle $NM'M''$ is equilateral and circumscribes c_r; and since $r_{_K} \geq r$, the minimal width of K^B is precisely the distance $d(M'', \ell')$. Therefore, if $R \leq 2r$, the minimal width of K^B is attained in the distance between the straight line ℓ' and its parallel one supporting K^B; and this touching point will lie on the circular arc of $\partial C_R \cap \partial K^B$ starting in M'' (see figure 13(a)). This distance is, clearly, $R+r$, which proves the case.

Now we suppose that $R \geq 2r$. Then, inequality (5.2.b) always holds, independently of the value of the inradius (see [6, Subsect. 3.2, Prop. 2]). Let us state the range of r_K for which (5.2.b) keeps its validity. In order to do that, we consider again the set K^B previously defined. Let us notice that, if $r_K = r$, the relation $r \geq 2r(R-r)/R$ would be equivalent to $R \leq 2r$, and hence, inequality (5.2.b) would be nonsense. Therefore, for this case, $r_K > r$, and $c_K \not\equiv c_r$. If r_K is large enough for the line segment $\overline{M'M''}$ intersects ∂c_K (see figure 13(b)), then its width is attained in the distance $d(M'', \ell')$, i.e., $4r(R^2 - r^2)/R^2$. It is an easy computation to check that it happens when $r_K \geq 2r(R-r)/R$ (if $r_K = 2r(R-r)/R$, $\overline{M'M''}$ just touches ∂c_K). This proves inequality (5.2.b).

Finally, let us suppose that $R \geq 2r$ and $r_K \leq 2r(R-r)/R$. Under these assumptions, the line segment $\overline{M'M''}$ will never intersect c_K; at most, it will touch its boundary precisely when $r_K = 2r(R-r)/R$ (see figure 14).

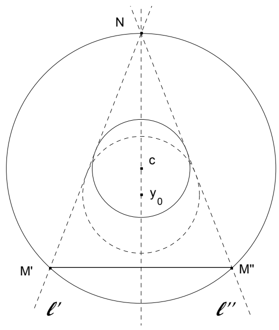
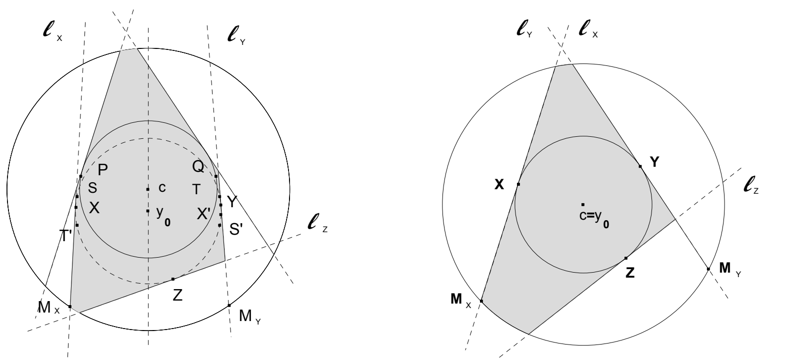


FIGURE 14. If $R \geq 2r$ and $r_K \leq 2r(R-r)/R$, the line segment $\overline{M'M''}$ never intersects the incircle c_K.

Lemma 1(c) states that K is contained in a circular slice K^s of C_R, determined by two support lines to c_r, and with minimal annulus $A(c, r, R)$; of course, since $K \subset K^s$, both $\omega(K) \leq \omega(K^s)$ and $r_K \leq r_{K^s}$. On the other hand, since c_K is the incircle of K, ∂K contains two diametrically opposite points of ∂c_K, or three points $X, Y, Z \in \partial c_K$ forming the vertices of an acute-angled triangle. In the first case, $\omega = 2r_K$, which can be excluded, because it gives the minimum value of the width, and we want to maximize it. So, we suppose the existence of $X, Y, Z \in \partial c_K$ under the above conditions, which also implies the uniqueness of the incircle.

Let us start assuming that $r_K > r$, and consequently, that $c_K \not\equiv c_r$. Lemma 2(ii) assures that $\mathrm{conv}(c_r \cup c_K) \subset K$; hence, the support lines determining K^s touch ∂c_r in the circular arc $\widehat{PQ} \subset \partial c_r \subset \mathrm{conv}(c_r \cup c_K)$, whereas the points X, Y, Z lie on the circular arc $\widehat{ST} \subset \partial c_K \subset \mathrm{conv}(c_r \cup c_K)$.

From now on, we are going to use a prime, $'$, for denoting the symmetral of a point of ∂c_K with respect to y_0. Following this notation, we can assure that one of the points X, Y, Z lies on the circular arc, either $\widehat{ST'}$, or $\widehat{S'T}$: in the opposite case, the three of them would lie on the $\widehat{S'T'}$, and thus, on a half-circumference, which is impossible (see figure 15(a)). Without loss of generality, let us suppose that $X \in \widehat{ST'}$. Then, one of the two remaining points, say Y, lies on the circular arc $\widehat{TX'}$ (again because otherwise, the three of them would be located on the half-circumference $\widehat{XX'}$). Finally, since X, Y, Z determine an acute-angled triangle, $Z \in \widehat{X'Y'} \in \partial c_K$. Besides, K is contained in the intersection set of K^s with the triangle T_{XYZ} determined by the support lines ℓ_X, ℓ_Y, ℓ_Z, to c_K through X, Y, Z, respectively (see figure 15).



(a) (b)

Figure 15: The problem is reduced to study the sets $K^s \cap T_{XYZ}$.

On the other hand, if $r_K = r$, then $c_K \equiv c_r$; in the opposite case, it would be impossible to find three points of $\partial c_K \cap \partial K$ determining an acute-angled triangle. Now, there is no arcs $\widehat{PQ}$ or $\widehat{ST}$, because $\mathrm{conv}(c_r \cup c_K) \equiv c_K$. So, in this case, we will just suppose that $Z \in \widehat{XY}$ and that lies "below" both points, X and Y (as shown in figure 15(b)). Besides, the set $\bar{K}$ will be just the intersection $\bar{K} = T_{XYZ} \cap C_R$. In the following, almost all the arguments will be valid in both cases, $c_K \equiv c_r$ or not. So, the remainder part of the proof is the same, and we will distinguish the cases just in the precise moment when it is necessary.

Let us notice that the straight line ℓ_Z intersects the circular arc $\widehat{M_X M_Y}$, where $M_X, M_Y \in \partial K^s$ denote the intersection points of ℓ_X and ℓ_Y with ∂C_R: otherwise, there would be no point of $\partial C_R \cap \partial K$ on the arc $\widehat{M_X M_Y}$, which would be a contradiction to property **(P2)** ($\partial C_R \cap \partial K$ and $\partial c_r \cap \partial K$ could be separated).

It is clear that the set K' has minimal annulus $A(c,r,R)$ and inradius $r_{_K}$. Since $K\subset\bar{K}$, $\omega(K)\leq\omega(\bar{K})$, and the problem has been reduced to study the width of the convex bodies belonging to this particular family. Let us note the following:

1. The angle determined by ℓ_X and ℓ_Y is always less or equal than the one corresponding to the straight lines that determine K^s (when $r_{_K}>r$; if $r_{_K}=r$, K^s plays no role, and this observation is not considered).
2. The angle determined by ℓ_X and ℓ_Y can not vanish, i.e., ℓ_X and ℓ_Y will never be parallel, since in this case the points X,Y,Z would lie on the same half-circumference.
3. Since X,Y,Z are not on a half-circumference, the intersection point $\ell_X\cap\ell_Y$ lies on the upper halfplane determined by the orthogonal line to cy_0 passing through c; besides, $\ell_X\cap\ell_Y$ is not an interior point of the circle C_R.
4. The triangle T_{XYZ} has inradius $r_{_K}$ and verifies that, at most, just one of its vertices (either $\ell_Y\cap\ell_Z$ or $\ell_X\cap\ell_z$) lies in the interior of C_R.

From these observations we can conclude that the minimal width of $\bar{K}$ will be attained in the distance of the only vertex of T_{XYZ} which lies in the interior of C_R to the opposite side, if it exists, or it will be, in the opposite case, the minimum of the distances $d(M_X,\ell_Y)$, $d(M_Y,\ell_X)$. But in any of these two cases, the minimal width will increase, as bigger is the angle θ determined by the lines ℓ_X and ℓ_Y: in fact, both the only "interior" vertex of T_{XYZ} (if it exists), as well as the points M_X,M_Y increase their distance to their corresponding opposite sides (either ℓ_X or ℓ_Y) when θ increases.

Thus, let us first note that, if $r_{_K}>r$ and we fix $c_{_K}$, the angle θ will be maximal when the points X and Y coincide, respectively, with the extreme points S and T of the circular arc of $\partial c_{_K}\subset\partial(c_r\cup c_{_K})$. Now, after this observation, and thus assuming that $X\equiv S$ and $Y\equiv T$, it is clear that θ will also increase (and hence the minimal width) as closer is the incenter y_0 to c (see figure 16). The limit case will be when $d(y_0,c)=R(r_{_K}-r)/r$, i.e., when the lines ℓ_X,ℓ_Y are, precisely, the common support lines, ℓ',ℓ', to c_r and $c_{_K}$ which intersect on $N\in\partial C_R$.

Let us notice that if $r_{_K}=r$, the above argument is trivial: we just have to increase the angle θ till $\ell_X\cap\ell_Y\in\partial C_R$, because $y_0\equiv c$.

On the other hand, for any possible position of the incircle $c_{_K}$ (even, in the case $c_{_K}\equiv c_r$), and fixing the lines ℓ_X and ℓ_Y, it is clear that the minimal width of $\bar{K}$ is larger as closer is Z to the intersection point $cy_0\cap\partial c_{_K}$ ($Ny_0\cap\partial c_{_K}$ if $c\equiv y_0$). But let us recall that we are assuming $R\geq 2r$ and $r_{_K}\leq 2r(R-r)R$, which implies that, in the limit situation for the maximum minimal width (i.e., $\ell_X \equiv \ell'$, $\ell_Y \equiv \ell''$ and $\ell_X \cap \ell_Y = N \in \partial C_R$), the line segment $\overline{M_X M_Y} \equiv \overline{M'M''}$ does not intersect $c_{_K}$; at most it will touch $\partial c_{_K}$ when $R = 2r$ and $r_{_K} = r$. Hence, the real limit position for the point Z is the one when ℓ_Z passes, precisely, through M_X (or equivalently M_Y, because of the symmetry of the figure), see figure 16).

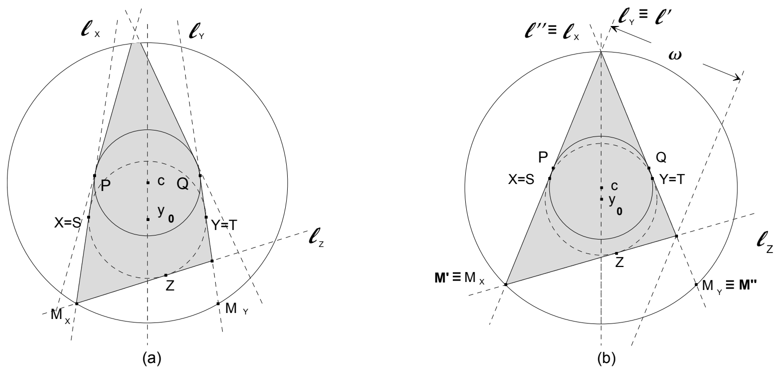


Figure 16: Moving the incenter in order to increase the minimal width.

In this way, we can conclude that under the assumptions of the theorem, the convex body with maximum minimal width is the triangle determined by the straight lines ℓ', ℓ'', and the support line to $c_{_K}$ passing through M' (see figure 13). An easy computation shows that the minimal width of this triangle is given by

$$\omega = 2\sqrt{R^2 - r^2}\frac{\sin\alpha \sin\beta}{\sin(\alpha+\beta)} = \frac{4r}{R^2}(R^2 - r^2)\frac{\sin\beta}{\sin(\alpha+\beta)},$$

where

$$\alpha = 2\arcsin\frac{r}{R} \quad \text{and} \quad \beta = 2\arctan\frac{r r_{_K}}{(2r - r_{_K})\sqrt{R^2 - r^2}},$$

which proves inequality (5.2.c). ■

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