

# A finiteness theorem for the space of $L^p$ harmonic sections

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## Abstract

In this paper we give a unified and improved treatment to finite dimensionality results for subspaces of  $L^p$  harmonic sections of Riemannian or Hermitian vector bundles over complete manifolds. The geometric conditions on the manifold are subsumed by the assumption that the Morse index of a related Schrödinger operator is finite. Applications of the finiteness theorem to concrete geometric situations are also presented.

## 0. Introduction

The aim of this note is to explore some geometric aspects of the “finite Morse index” for natural Schrödinger operators on a complete manifold  $(M, \langle, \rangle)$ . Specifically, we are interested in relating the finiteness of the index with the finiteness of the dimension of the space of  $p$ -integrable harmonic sections of a Riemannian vector bundle over  $M$ . Needless to say, the operators under consideration will arise from appropriate Weitzenböck formulas on the bundle.

In the special case  $p = 2$ , finiteness results have been largely investigated by many authors under different assumptions. We limit ourselves to quote [13], [14] by P. Li and J. Wang, where Morse index assumptions are used in a way similar to the present note, and [2] by G. Carron where quantitative dimensional estimates are obtained assuming that the underlying manifold supports a global Sobolev inequality. Further references in the Kähler case are given in Section 1.2.2 below.

The present paper should be considered as a continuation of [19] where we proved  $L^p$ -vanishing results under the assumption of the vanishing of the Morse index of the operators involved. The underlying philosophy is also

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*2000 Mathematics Subject Classification:* 53C21, 35J60.

*Keywords:* Riemannian vector bundles, harmonic sections, Morse index.

similar in that a crucial step in our arguments is to combine the solutions of two differential inequalities into a single one. This approach enables us to deal with very general situations as shown by Theorem 1.1 below. See also Remark 1.3. Along the way, we prove a weak Harnack inequality for solutions of differential inequalities involving gradient terms which may be of independent interest, and we provide a new nonlinear proof of a Poincaré inequality with mixed boundary conditions. In a final Appendix we also present technical regularity results needed in the distributional computations of the paper.

Geometric situations where the main finiteness theorem applies are explicitly considered; see Corollary 1.2, Theorem 1.4 and Theorem 1.5. We also provide a new application to the reduction of the codimension of a harmonic immersion into Euclidean space; see Theorem 1.6.

## 1. Main result, spectral remarks, and geometric consequences

The main goal of the paper is the following abstract, very general, finiteness Theorem.

**Theorem 1.1.** *Let  $(M, \langle \cdot, \cdot \rangle)$  be a connected, complete,  $m$ -dimensional Riemannian manifold and  $E$  a Riemannian (Hermitian) vector bundle of rank  $l$  over  $M$ . The space of its smooth sections is denoted by  $\Gamma(E)$ . Having fixed*

$$a(x) \in C^0(M), \quad A \in \mathbb{R}, \quad H \geq p$$

*satisfying the further restrictions*

$$(1.1) \quad p \geq A + 1, \quad p > 0,$$

*let  $V = V(a, A, p, H) \subset \Gamma(E)$  be any vector space with the following property:*

**(P)** *Every  $\xi \in V$  has the unique continuation property, i.e.,  $\xi$  is the null section whenever it vanishes on some domain; furthermore the locally-Lipschitz function  $u = |\xi|$  satisfies*

$$(1.2) \quad \begin{cases} u(\Delta u + a(x)u) + A|\nabla u|^2 \geq 0 & \text{weakly on } M \\ \int_{B_r} u^{2p} = o(r^2) & \text{as } r \rightarrow +\infty. \end{cases}$$

*If there exists a solution  $0 < \varphi \in Lip_{loc}$  of the differential inequality*

$$(1.3) \quad \Delta \varphi + H a(x) \varphi \leq 0$$

*weakly outside a compact set  $K \subset M$ , then*

$$(1.4) \quad \dim V < +\infty.$$

**1.1. Spectral counterparts of assumption (1.3)**

Condition (1.3) is equivalent to the existence of a  $C^1$  function  $\varphi > 0$  satisfying

$$(1.5) \quad \Delta\varphi + Ha(x)\varphi = 0 \text{ weakly in the complement of } K,$$

and it is intimately related with the spectral properties of the Schrödinger operator  $\mathcal{L}_H = -\Delta - Ha(x)$ . Indeed, associated to  $\mathcal{L}_H$  we have the usual notions of (generalized) first eigenvalue  $\lambda_1^{\mathcal{L}_H}(M)$  and Morse index  $Ind(\mathcal{L}_H)$ . Recall that, by definition,

$$(1.6) \quad \lambda_1^{\mathcal{L}_H}(M) = \inf_{v \in C_c^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla v|^2 - Ha(x)v^2}{\int_M v^2}$$

and

$$(1.7) \quad Ind(\mathcal{L}_H) = \sup_{\Omega \subset\subset M} Ind(\mathcal{L}_H^\Omega)$$

where  $\mathcal{L}_H^\Omega$  stands for the Friedrichs extension of  $(\mathcal{L}_H, C_0^\infty(\Omega))$  in  $L^2(\Omega)$ , and  $Ind(\mathcal{L}_H^\Omega)$  denotes the (finite) number, counting multiplicity, of the strictly negative Dirichlet eigenvalues of  $\mathcal{L}_\Omega$ .

Observe that the non-negativity of the first (generalized) eigenvalue corresponds precisely to the fact that the Morse index is zero. Moreover, it is known from classical work by D. Fischer-Colbrie, [5], that  $Ind(\mathcal{L}_H) < +\infty$  implies the validity of (1.5) for some  $C^1$  function  $\varphi > 0$ ; see also [19].

As a matter of fact, assumption (1.3) seems to be slightly weaker than the request  $Ind(\mathcal{L}_H) < +\infty$ . Indeed, its validity is equivalent to saying that

$$(1.8) \quad \lambda_1^{\mathcal{L}_H}(M \setminus K) \geq 0$$

which, in turn, is implied by the finiteness of the Morse index; see [5] and [19].

Condition (1.8) is easier to handle. By way of example, suppose that  $(M, \langle, \rangle)$  supports a global,  $L^2$  Sobolev inequality of the type

$$S_\alpha^{-1} \left( \int_M v^{\frac{2}{1-\alpha}} \right)^{1-\alpha} \leq \int_M |\nabla v|^2, \forall v \in C_c^\infty(M),$$

for some  $S_\alpha > 0$  and  $0 < \alpha < 1$ . Then, it is readily seen that the validity of (1.8), hence of (1.3), follow from an  $L^{\frac{1}{\alpha}}$ -control of the potential  $a_+(x) = \max\{a(x), 0\}$ . Indeed, suppose  $\|a_+\|_{L^{\frac{1}{\alpha}}(M)} < +\infty$  so that, up to choosing the compact set  $K \subset\subset M$  large enough, we have  $\|a_+\|_{L^{\frac{1}{\alpha}}(M \setminus K)} \leq (HS_\alpha)^{-1}$ .

Then, for every  $0 \neq v \in C_c^\infty(M \setminus K)$ , using also Hölder inequality, we deduce

$$\begin{aligned} \int_{M \setminus K} |\nabla v|^2 - Ha(x) v^2 &\geq S_\alpha^{-1} \left( \int_{M \setminus K} v^{\frac{2}{1-\alpha}} \right)^{1-\alpha} - \int_{M \setminus K} Ha_+(x) v^2 \\ &\geq \left\{ S_\alpha^{-1} - H \|a_+\|_{L^{\frac{1}{\alpha}}(M \setminus K)} \right\} \left( \int_{M \setminus K} v^{\frac{2}{1-\alpha}} \right)^{1-\alpha} \\ &\geq 0, \end{aligned}$$

that is (1.8).

## 1.2. Geometric applications of the main theorem

The spaces of harmonic functions, and more generally, harmonic forms on a Riemannian manifold are the most typical examples of spaces of sections for which the conditions of the theorem hold. This situation can be generalized to the following setting. Let  $E$  be a Riemannian (Hermitian) vector bundle of rank  $l$  over  $M$  with a compatible connection  $D$  and let  $\Delta_E$  be a differential operator acting on the space of smooth sections  $\Gamma(E)$  of the form

$$(1.9) \quad \Delta_E = \Delta_B + \mathfrak{R}$$

where  $\Delta_B = -\text{Trace}(D^2)$  is the rough Laplacian, and  $\mathfrak{R}$  is a smooth symmetric endomorphism of  $E$ . We will refer to an operator  $\Delta_E$  as above as an admissible Laplacian.

A smooth section  $\xi \in \Gamma(E)$  is called  $\Delta_E$ -harmonic if  $\Delta_E \xi = 0$ . We define the vector spaces

$$\mathcal{H}(E) = \{\xi \in \Gamma(E) : \Delta_E \xi = 0\}$$

and

$$L^{2p}\mathcal{H}(E) = \{\xi \in \mathcal{H}(E) : |\xi| \in L^{2p}(M)\}.$$

Note that  $\Delta_E$ -harmonic sections satisfy the (strong) unique continuation property. In local coordinates, the condition  $\Delta_B \xi + \mathfrak{R}\xi = 0$  becomes a system of  $l$  elliptic differential equations satisfying the structural assumptions of Aronszajn-Cordes, see e.g. [11].

From (1.9) we deduce the Bochner-Weitzenböck formula,  $\forall \xi \in \mathcal{H}(E)$ ,

$$-\frac{1}{2}\Delta|\xi|^2 = \langle \Delta_B \xi, \xi \rangle - |D\xi|^2 = -\langle \mathfrak{R}\xi, \xi \rangle - |D\xi|^2$$

which in turn implies that the following differential inequality holds in the sense of distributions,

$$(1.11) \quad |\xi|\Delta|\xi| - \langle \mathfrak{R}\xi, \xi \rangle = |D\xi|^2 - |\nabla|\xi||^2 \geq 0.$$

The last inequality in (1.11) is known as “the first Kato inequality”. We recall for completeness that, when there exists a constant  $k > 0$  such that

$$(1.12) \quad |D\xi|^2 - |\nabla |\xi||^2 \geq k|\nabla |\xi||^2,$$

one says that a “refined Kato inequality” holds.

If we let

$$\mathfrak{R}_-(x) = \sup_{|\xi|=1} \langle -\mathfrak{R}\xi, \xi \rangle$$

then, from (1.11), we obtain

$$|\xi| (\Delta |\xi| + \mathfrak{R}_-(x) |\xi|) \geq 0.$$

and we are naturally led to considering the Schrödinger operator

$$\mathcal{L}_H = -\Delta - H\mathfrak{R}_-(x)$$

with  $H > 0$  a real number. Accordingly, from Theorem 1.1 we immediately deduce the following

**Corollary 1.2.** *Maintaining the notation introduced above, assume that, for some  $H \geq 1$*

$$Ind(\mathcal{L}_H) < +\infty.$$

*Then,*

$$\dim L^{2p}\mathcal{H}(E) < +\infty$$

*for every  $1 \leq p \leq H$ .*

As examples of bundles where the above considerations apply, we mention the space of spinors and of exterior differential  $q$ -forms. In these settings the role of the operator  $\Delta_E$  is played by the Dirac and the Hodge-De Rham Laplacians, respectively. Both operators can be written in terms of the rough Laplacian via a Bochner-type formula. In the spinorial case the endomorphism  $\mathfrak{R}$  is given by the formula

$$\langle \mathfrak{R}(x)v, v \rangle = \frac{{}^M Scal(x)}{4} |v|^2, \quad \forall v \in E_x,$$

where  ${}^M Scal$  denotes the scalar curvature of  $M$  (see [21]). For differential 1-forms  $\mathfrak{R}$  is given by

$$\langle \mathfrak{R}(x)v, v \rangle = {}^M Ricci(x)(v, v)$$

where  ${}^M Ricci$  is the Ricci tensor of  $M$ .

In the case of differential  $k$ -forms on a locally, conformally flat manifold  $M$  of even dimension  $m = 2k \geq 4$ , one has (see [1])

$$\langle \mathfrak{R}(x)v, v \rangle = \frac{k!k^M \text{Scal}(x)}{2(2k-1)} |v|^2.$$

The expression of  $\mathfrak{R}$  for the exterior bundle  $\Lambda^q(T^*M)$ ,  $q \geq 2$ , on a general manifold is quite complicated but can be estimated in terms of the sectional curvature  ${}^M \text{Sec}$  of  $M$  by (see [6])

$$\langle \mathfrak{R}(x)v, v \rangle \geq -C\lambda(x)|v|^2$$

where  $C = C(m, q) > 0$  is a constant depending on  $m$  and  $q$  and

$$\lambda(x) = \max_{\Pi \subset T_x M} |{}^M \text{Sec}_x(\Pi)|.$$

**Remark 1.3.** Comparing with [13] and [14] we see that our approach, on the one hand allows us to deal with different integrability exponents, and on the other hand it enables us to avoid the request that  $u = |\xi|$  be a solution of the more stringent inequality

$$u(\Delta u + a(x)u) \geq k|\nabla u|^2$$

for some constant  $k > 0$ . The geometric counterpart of this is that we do not need to use any refined Kato inequality a fact that seems to be essential in the Li-Wang papers cited above. Thus, we can deal with, e.g., the whole space of harmonic,  $2p$ -integrable  $q$ -forms instead of restricting ourselves to the closed and co-closed ones (for which a refined Kato inequality does hold).

### 1.2.1. Line bundles over Kähler manifolds

Suppose  $(M, \langle, \rangle, J)$  is a complete Kähler manifold of complex dimension  $m$  and Ricci form  $R_{i\bar{j}}$ . Let  $E$  be a holomorphic line bundle over  $M$  endowed with a Hermitian metric  $(\cdot, \cdot)$  with curvature form  $\Omega_{i\bar{j}}$ . The complex vector space of  $L^p$  holomorphic  $(k, 0)$ -forms with values in  $E$  is denoted by  $L^p \Lambda^{(k,0)}(E)$ . We also set  $L^p H(\otimes^k E)$  for the space of  $L^p$  holomorphic sections of tensor powers of  $E$ .

In [18], L. Ni, Y. Shi and L.F. Tam investigate geometric conditions forcing  $\dim L^{2p} \Lambda^{(k,0)}(E) = 0$ . In the  $L^2$  setting, vanishing results and corresponding quantitative finiteness theorems for  $\dim L^2 H(\otimes^k E)$  are established e.g. in works by N. Mok, [15], and by L. Ni, [17]. In this section, as a direct application of Theorem 1.1, we prove qualitative  $L^p$  finite-dimensionality results in both these situations.

To begin with, we consider the space  $L^p\Lambda^{(k,0)}(E)$ . To simplify the writings, for any fixed  $H > 0$ , let us define the Schrödinger operator

$$\mathcal{L}_H = -\Delta - 4H \left( s(x) - \min_{1 \leq i_1 < \dots < i_k \leq m} (\gamma_{i_1} + \dots + \gamma_{i_k}) \right)$$

where  $\gamma_1, \dots, \gamma_m$  are the eigenvalues of the Ricci form  $R_{i\bar{j}}$  of  $M$  and  $s(x)$  the trace, with respect to  $(\cdot, \cdot)$ , of the curvature form  $\Omega_{i\bar{j}}$  of  $E$ . We have the following

**Theorem 1.4.** *Suppose that*

$$\text{Ind}(\mathcal{L}_H) < +\infty$$

for some  $H > 0$ . Then,

$$\dim L^{4p}\Lambda^{(k,0)}(E) < +\infty$$

for every  $0 < p \leq H$ .

**Proof.** As explained above, the index assumption guarantees the existence of a solution  $\varphi > 0$  of

$$\mathcal{L}_H\varphi = 0 \text{ on } M \setminus K$$

for some compact set  $K \subset M$ . Moreover, the Kodaira-Bochner formula states that, for every  $\xi \in \Lambda^{(k,0)}(E)$ , the smooth function  $u = |\xi|^2$  satisfies

$$-u\mathcal{L}_1(u) - |\nabla u|^2 \geq 0 \text{ on } M;$$

see [16] Chapter 3, Section 6. Therefore, the result follows directly from Theorem 1.1. ■

We now come to the case of  $L^p H(\otimes^k E)$ .

**Theorem 1.5.** *Let  $k \in \mathbb{N}$  and assume that, for some  $H > 0$ ,*

$$\text{Ind}(-\Delta - 2Hks(x)) < +\infty.$$

Then

$$\dim L^{2p} H(\otimes^k E) < +\infty$$

for every  $0 < p \leq H$ .

**Proof.** The proof goes as above. The only difference is that now we use the Bochner formula

$$|\xi|(\Delta|\xi| + 2ks(x)|\xi|) - |\nabla|\xi||^2 \geq 0$$

which is valid, in the sense of distributions, for every  $\xi \in H(\otimes^k E)$ ; see [17]. ■

### 1.2.2. Reduction of codimension of harmonic immersions

From a somewhat different perspective, Theorem 1.1 applies to codimensional problems for (non-isometric) harmonic immersions into Euclidean spaces. R. Greene and H. H. Wu, [7], [8], proved that any  $m$ -dimensional Riemannian manifold  $(M, \langle, \rangle)$  can be imbedded into  $\mathbb{R}^{2m+1}$  and immersed into  $\mathbb{R}^{2m}$  via a proper, harmonic immersion  $f : M \rightarrow \mathbb{R}^T$ . The properness condition insures that the induced metric  $f^* \text{can}_{\mathbb{R}^T}$  is complete. Observe that, due to e.g. volume growth restrictions, the immersion is in general non-isometric; see Remark 1.9 below. Equivalently, one in general has  $|df|^2 \neq \text{const}$ .

**Theorem 1.6.** *Let  $(M, \langle, \rangle)$  be a complete  $(m \geq 3)$ -dimensional Riemannian manifold satisfying*

$$(1.13) \quad {}^M \text{Ricci} \geq R(x) \text{ on } M$$

and assume that

$$(1.14) \quad \text{Ind}(-\Delta - HR(x)) < +\infty$$

for some  $H \geq \frac{m-2}{m-1}$ . Then, there exist a compact set  $K \subset M$  and an integer  $N \geq m$  depending on  $H$  and on the geometry of  $(M, \langle, \rangle)$  in a neighborhood of  $K$  such that the following holds.

Let  $f : M \rightarrow \mathbb{R}^{T > N}$  be a harmonic immersion whose energy density satisfies the growth condition

$$(1.15) \quad \int_{B_R} |df|^{2p} = o(R^2), \quad \text{as } R \rightarrow +\infty,$$

for some  $\frac{m-2}{m-1} \leq p \leq H$ . Then, there is an  $N$ -dimensional affine subspace  $\mathbb{A}^N \subset \mathbb{R}^T$  such that  $f(M) \subset \mathbb{A}^N$ .

**Proof.** Let  $\mathcal{H}(p)$  be the real vector space of harmonic functions  $u : M \rightarrow \mathbb{R}$  satisfying

$$\int_{B_R} |du|^{2p} = o(R^2), \quad \text{as } R \rightarrow +\infty,$$

with  $p$  as in the assumptions of the theorem. Define  $V(p) = \text{Im } d|_{\mathcal{H}(p)}$ , a vector subspace of  $\mathcal{H}(\Lambda^1(T^*M))$ , and observe that, for every  $du \in V(p)$ ,

$$|du|(\Delta |du| + R(x)|du|) - \frac{1}{m-1} |\nabla |du||^2 \geq 0.$$

This is the well known Bochner formula with refined Kato inequality for harmonic functions. Since, by (1.14), there exists a solution  $\varphi > 0$  of

$$\Delta \varphi + R(x)\varphi = 0 \text{ on } M \setminus K,$$

for some compact set  $K \subset M$ , we can apply Theorem 1.1 to deduce the existence of  $N \in \mathbb{N}$  depending on  $p$  and on the geometry of  $(M, \langle, \rangle)$  in a neighborhood of  $K$  such that,

$$(1.16) \quad \dim V(p) \leq N.$$

Let  $f = (f^A) : M \rightarrow \mathbb{R}^{T > N}$  be a harmonic immersion satisfying (1.15). Note that, for each  $A$ ,

$$df^A \in V(p)$$

and from the estimate (1.16) we deduce

$$\text{span}\{df^1, \dots, df^T\} = \text{span}\{df^{A_1}, \dots, df^{A_N}\}$$

for some  $A_1, \dots, A_N$ , where of course  $N \geq m$ . Without loss of generality, we can assume  $A_1 = 1, \dots, A_N = N$ . Thus,

$$df^\alpha = \sum_{A=1}^N \lambda_A^\alpha df^A, \quad \alpha = N+1, \dots, T,$$

for some appropriate real coefficients  $\{\lambda_A^\alpha\}$ . This latter clearly implies the existence of suitable constants  $\{c^\alpha\}$  such that

$$f^\alpha = \sum_{A=1}^N \lambda_A^\alpha f^A + c^\alpha.$$

It follows that  $f(M)$  is contained in the affine subspace  $\mathbb{A}^N$  of  $\mathbb{R}^T$  passing through  $(0, \dots, 0, c^{N+1}, \dots, c^T)^t$  and spanned by

$$\text{span}\left\{e_A + (0, \dots, 0, \lambda_A^{N+1}, \dots, \lambda_A^T)^t : A = 1, \dots, N\right\},$$

where  $\{e_A\}$  is the standard basis of  $\mathbb{R}^T$ . ■

**Remark 1.7.** The result of Theorem 1.6 is qualitative. It would be very interesting to get a quantitative version, where the dimension  $N$  of the affine ambient subspace is governed by the geometric data. In particular, forcing  $N = \dim M$  would yield a Bernstein type result.

**Remark 1.8.** The above arguments can be applied to holomorphic immersions of Kähler manifolds into  $\mathbb{C}^T$ .

**Remark 1.9.** The harmonic immersion  $f$  in the statement of Theorem 1.6 cannot be isometric. For otherwise we would have

$$\text{vol}(B_R) = o(R^2) \quad \text{as } R \rightarrow +\infty$$

and this is impossible by the monotonicity formula applied to the minimal immersion  $f : (M, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^T$ , with  $\widetilde{\langle, \rangle} = f^* \text{can}$ .

As a consequence, for dimensions  $m \geq 3$ , the immersion  $f$  cannot be conformal, because conformal harmonic maps are homotetic; see e.g. [3].

As a matter of fact, the standard monotonicity argument shows that some restriction appears even if  $f$  is a harmonic bi-Lipschitz immersion. Indeed, if

$$\sup_{x \in M} \sup_{v \in T_x M \setminus \{0\}} \frac{\widetilde{\langle v, v \rangle}_x}{\langle v, v \rangle_x} = A < +\infty; \quad \inf_{x \in M} \inf_{v \in T_x M \setminus \{0\}} \frac{\widetilde{\langle v, v \rangle}_x}{\langle v, v \rangle_x} = B > 0$$

then we must have

$$\frac{d}{dR} \left( \frac{\text{vol}(B_R)}{R\sqrt{m}BA^{-1}} \right) \geq 0.$$

It follows that a bi-Lipschitz harmonic immersions with “small” energy growth has to satisfy  $\sqrt{m}BA^{-1} < 2$ . However, we stress that, here, the quadratic-form inequality  $\widetilde{\langle, \rangle} \geq B \langle, \rangle$  plays an essential role.

Here are some special situations where Theorem 1.6 applies.

**Corollary 1.10.** *Let  $(M, \langle, \rangle)$  be a complete,  $m$ -dimensional Riemannian manifold satisfying both*

$$\text{vol}(B_R) = o(R^2) \quad \text{as } R \rightarrow +\infty.$$

and

$${}^M \text{Ricci} \geq 0 \quad \text{on } M \setminus K$$

for some compact set  $K \subset M$ . Then, there exists  $N \in \mathbb{N}$  depending on the geometry of  $(M, \langle, \rangle)$  in a neighborhood of  $K$  such that, any (non-isometric) harmonic, Lipschitz immersion  $f : M \rightarrow \mathbb{R}^{T>N}$  must satisfy  $f(M) \subset \mathbb{A}^N$  for some  $N$ -dimensional affine subspace  $\mathbb{A}^N \subset \mathbb{R}^T$ .

We say that a Riemannian metric  $(,)$  on the smooth manifold  $M$  is dominated by the metric  $\langle, \rangle$  if  $(, ) \leq C^2 \langle, \rangle$ , in the sense of quadratic forms, for some constant  $C > 0$ .

**Corollary 1.11.** *Let  $(M, \langle, \rangle)$  be a complete,  $m$ -dimensional Riemannian manifold satisfying*

$${}^M \text{Ricci} \geq 0 \text{ on } M \setminus K$$

*for some compact set  $K \subset M$ . Then, there exists  $N \in \mathbb{N}$  depending on the geometry of  $(M, \langle, \rangle)$  in a neighborhood of  $K$  such that the following holds.*

*Let  $f : M \rightarrow \mathbb{R}^{T>N}$  be a harmonic immersion whose induced metric  $f^*\text{can}$  is dominated by a metric  $\widetilde{\langle, \rangle}$  in the conformal class of  $\langle, \rangle$  satisfying*

$$\widetilde{\text{vol}}(M) < +\infty.$$

*Then,  $f$  is in fact a harmonic immersion into some  $N$ -dimensional affine subspace  $\mathbb{A}^N \subset \mathbb{R}^T$ .*

**Proof.** We set  $\widetilde{\langle, \rangle} = u^2(x) \langle, \rangle$  and we note that

$$\int_M |df|^m d\text{vol} \leq C \int_M u^m d\text{vol} = C \widetilde{\text{vol}}(M)$$

for some constant  $C > 0$ . ■

Suppose  $f : (M, \langle, \rangle_M) \rightarrow (N, \langle, \rangle_N)$  be a smooth map between Riemannian manifolds of dimensions  $m$  and  $n$  respectively. Denote by

$$\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_m(x) \geq 0$$

the eigenvalues of the quadratic form  $f_x^* \langle, \rangle_N$ . We say that  $f$  has bounded  $k^{\text{th}}$  dilation if

$$\lambda_1(x) \leq C_k \lambda_k(x) \text{ on } M$$

for some constant  $C_k \geq 1$ . When  $k = m$  we (perhaps improperly) say that  $f$  is of bounded distortion (or, equivalently, quasi-regular).

**Corollary 1.12.** *Let  $(M, \langle, \rangle)$  be a complete,  $m$ -dimensional Riemannian manifold satisfying*

$${}^M \text{Ricci} \geq 0 \text{ on } M \setminus K$$

*for some compact set  $K \subset M$ . Then, there exists  $N \in \mathbb{N}$  depending on the geometry of  $(M, \langle, \rangle)$  in a neighborhood of  $K$  such that, any (non-isometric) harmonic immersion  $f : M \rightarrow \mathbb{R}^{T>N}$  of bounded distortion and satisfying*

$$\text{vol}_{f^*\text{can}}(M) < +\infty,$$

*is in fact a harmonic immersion into some  $N$ -dimensional affine subspace  $\mathbb{A}^N \subset \mathbb{R}^T$ .*

**Proof.** We set  $\widetilde{\langle, \rangle} = f^*\text{can}$  and we observe that

$$\frac{|df|^m}{d\widetilde{\text{vol}}} = \frac{\{\text{tr}(f^*\text{can})\}^{\frac{m}{2}}}{\{\det(f^*\text{can})\}^{\frac{1}{2}}} = \frac{(\sum \lambda_i)^{\frac{m}{2}}}{(\prod \lambda_i)^{\frac{1}{2}}} \leq C \left( \frac{\lambda_1}{\lambda_m} \right)^{\frac{m}{2}} \leq CC_k,$$

for a suitable constant  $C = C(m) > 0$  ■

## 2. Proof of Theorem 1.1

The proof of the main theorem is based on a (suitable version of the) classical estimating lemma due to Peter Li which we now recall (see [12, Lemma 11]).

**Lemma 2.1.** *Let  $E$  be a Riemannian (Hermitian) vector bundle of rank  $l$  over a Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$  and let  $T$  be a finite dimensional subspace of  $L^2\Gamma(E|_\Omega)$ , the space of square-integrable sections of  $E$  on  $\Omega \subset\subset M$ . Then, there exists a (non-zero) section  $\bar{\xi} \in T$  such that, for any  $p > 0$ ,*

$$(2.1) \quad (\dim T)^{\min(1,p)} \int_{\Omega} |\bar{\xi}|^{2p} \leq \text{vol}(\Omega) \min\{l, \dim T\}^{\min(1,p)} \sup_{\Omega} |\bar{\xi}|^{2p},$$

**Proof.** The original version was stated for  $p = 1$ . To deduce the validity of (2.1) simply note that, if  $p > 1$ , then

$$\begin{aligned} \dim T \int_{\Omega} |\bar{\xi}|^{2p} &= \dim T \int_{\Omega} |\bar{\xi}|^{2(p-1)} |\bar{\xi}|^2 \\ &\leq \left( \dim T \int_{\Omega} |\bar{\xi}|^2 \right) \sup_{\Omega} |\bar{\xi}|^{2(p-1)} \\ &\leq \left\{ \text{vol}(\Omega) \min(l, \dim T) \sup_{\Omega} |\bar{\xi}|^2 \right\} \sup_{\Omega} |\bar{\xi}|^{2(p-1)}. \end{aligned}$$

On the other hand, if  $p < 1$ , we can use Hölder inequality to obtain

$$\begin{aligned} \dim T \int_{\Omega} |\bar{\xi}|^{2p} &\leq \dim T \left( \int_{\Omega} |\bar{\xi}|^2 \right)^p \text{vol}(\Omega)^{1-p} \\ &= (\dim T)^{1-p} \left( \dim T \int_{\Omega} |\bar{\xi}|^2 \right)^p \text{vol}(\Omega)^{1-p} \\ &\leq (\dim T)^{1-p} \left( \text{vol}(\Omega) \min\{l, \dim T\} \sup_{\Omega} |\bar{\xi}|^2 \right)^p \text{vol}(\Omega)^{1-p} \end{aligned}$$

which implies

$$(\dim T)^p \int_{\Omega} |\bar{\xi}|^{2p} \leq \text{vol}(\Omega) \min\{l, \dim T\}^p \sup_{\Omega} |\bar{\xi}|^{2p}. \quad \blacksquare$$

In view of Peter Li's lemma, the strategy of the proof consists of showing that there are a geodesic ball  $B_{\bar{R}} \subset M$  and a constant  $C > 0$ , such that the following a-priori local estimate holds true

$$(2.2) \quad \sup_{B_{\bar{R}}} |\xi|^{2p} \leq C \int_{B_{\bar{R}}} |\xi|^{2p}$$

for every  $\xi \in V$ . This is obtained in Lemma 2.7 below combining two main ingredients: the annuli-estimate technique contained in [13], [14] and a local, weak Harnack inequality for solutions of (1.2), see Proposition 2.4 below.

Our technique, in the spirit of [19], is based on the interaction of the two differential inequalities (1.2) and (1.3) and represents the crucial step towards the extension of the result to situations where a refined Kato inequality does not hold (see Remark 1.3 above).

The following Caccioppoli type inequality will play an important rôle.

**Lemma 2.2.** *Let  $(M, \langle, \rangle)$  be a Riemannian manifold and  $\Omega \subset\subset M$  any relatively compact domain. Let  $0 < w \in C^0(\bar{\Omega})$  and  $v \in Lip_{loc}(\Omega)$  satisfy the differential inequality*

$$(2.3) \quad v \operatorname{div}(w \nabla v) \geq 0$$

weakly on  $\Omega$ . Then, for any fixed  $q \geq 0$ ,

$$(2.4) \quad D_q \int_{\Omega} w |v|^q |\nabla v|^2 \eta^2 \leq \int_{\Omega} w |v|^{q+2} |\nabla \eta|^2, \quad \forall \eta \in C_0^\infty(\Omega)$$

where

$$D_q = \frac{(1+q)^2}{4}.$$

**Proof.** We assume that  $v > 0$ . The general case can be handled using the function  $v_\delta = (v^2 + \delta^2)^{1/2}$  and letting  $\delta \rightarrow 0+$  (see [19]).

Inequality (2.3) means that, for each  $0 \leq \rho \in Lip_0(\Omega)$ ,

$$-\int \langle w \nabla v, \nabla(v\rho) \rangle = -\int w \rho |\nabla v|^2 - \int w v \langle \nabla v, \nabla \rho \rangle \geq 0.$$

Choosing

$$\rho = v^q \eta^2$$

with  $\eta \in C_0^\infty(\Omega)$  and using the Schwarz and Young inequalities we get

$$\begin{aligned} 0 &\geq \int w v^q \eta^2 |\nabla v|^2 + \int q w v^q \eta^2 |\nabla v|^2 + \int 2 w v^{q+1} \eta \langle \nabla v, \nabla \eta \rangle \\ &\geq (1+q) \int w v^q \eta^2 |\nabla v|^2 - 2 \int w v^{q+1} |\eta| |\nabla \eta| |\nabla v| \\ &= (1+q) \int w v^q \eta^2 |\nabla v|^2 - 2 \int \frac{w^{\frac{1}{2}} v^{\frac{q}{2}} |\eta| |\nabla v|}{\varepsilon} \cdot \varepsilon w^{\frac{1}{2}} v^{\frac{q}{2}+1} |\nabla \eta| \\ &\geq (1+q - \varepsilon^{-2}) \int w v^q \eta^2 |\nabla v|^2 - \varepsilon^2 \int w v^{q+2} |\nabla \eta|^2. \end{aligned}$$

The required conclusion now follows optimizing with respect to  $\varepsilon$ . ■

We note that (2.4) implies the usual  $L^2$ -Caccioppoli inequality

$$\frac{\inf_{\Omega} w}{4 \sup_{\Omega} w} \int_{\Omega} |\nabla v|^2 \eta^2 \leq \int_{\Omega} v^2 |\nabla \eta|^2, \quad \forall \eta \in C_0^\infty(\Omega)$$

It is known that once we have both a Caccioppoli and a Sobolev type inequality, the Moser iteration procedure gives the validity of a weak Harnack inequality (see e.g. [20, pp.486-487]). Since, locally, an  $L^2$ -Sobolev inequality is always available, from Lemma 2.2 we deduce the following

**Corollary 2.3.** *Let  $B_{R+1}(o)$  be a relatively compact geodesic ball in a Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$  of dimension  $m \geq 2$ , and let  $w$  be a positive continuous function. Then, for any fixed  $0 < \delta < 1$  there exists a constant  $C > 0$  depending only on  $w|_{\bar{B}_{R+1}(o)}$ ,  $R$ ,  $\delta$  and the geometry of  $B_{R+1}(o)$  such that*

$$(2.5) \quad \sup_{B_R(o)} v^2 \leq C \int_{B_{R+\delta}(o)} v^2$$

for every non-negative function  $v \in Lip_{loc}(B_{R+1}(o))$  satisfying,

$$v \operatorname{div}(w \nabla v) \geq 0$$

weakly on  $B_{R+1}(o)$ .

Our next step is to obtain the validity of an  $L^{2p}$ -version of (2.5) for non-negative, weak solutions of differential inequalities of type (1.2).

**Proposition 2.4.** *Let  $B_{R+1}(o)$  be a relatively compact geodesic ball in a Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$  of dimension  $m \geq 2$ . Let*

$$(2.6) \quad a(x) \in C^0(B_{R+2}(o)), \quad A \in \mathbb{R}$$

and

$$(2.7) \quad p \geq A + 1, \quad p > 0$$

Then, there exists a constant  $C > 0$  depending on the above data and the geometry of  $B_{R+1}(o)$ , such that

$$(2.8) \quad \sup_{B_R(o)} u^{2p} \leq C \int_{B_{R+1}(o)} u^{2p}$$

for every non-negative, locally Lipschitz, weak solution  $u$  of

$$(2.9) \quad u(\Delta u + a(x)u) + A|\nabla u|^2 \geq 0.$$

**Proof.** We shall show that, for every  $x \in \bar{B}_R(o)$ , there exists  $0 < \varepsilon \ll 1$  and a constant  $C' > 0$  independent of  $u$  such that

$$(2.10) \quad \sup_{B_\varepsilon(x)} u^{2p} \leq C' \int_{B_{2\varepsilon}(x)} u^{2p}.$$

Since  $\bar{B}_R(o)$  is compact, the desired inequality (2.8) will follow from (2.10) using a covering argument.

Let us consider the Schrödinger operator

$$\mathcal{L} = -\Delta - pa(x)$$

on  $L^2(B_{3\varepsilon}(x))$ . Since the first Dirichlet eigenvalue  $\lambda_1^{-\Delta}(B_r(x))$  of  $-\Delta$  on  $L^2(B_r(x))$  grows like  $r^{-2}$  as  $r \rightarrow 0+$ , we can choose  $\varepsilon > 0$  so small that

$$\lambda_1^{\mathcal{L}}(B_{3\varepsilon}(x)) > 0.$$

Let  $w$  be the corresponding, positive, first eigenfunction, i.e., a solution of the eigenvalue problem

$$(2.11) \quad \begin{cases} \Delta w + pa(x)w = -\lambda_1^{\mathcal{L}}(B_{\varepsilon+2}(x))w \leq 0 & \text{on } B_{3\varepsilon}(x) \\ w > 0 & \text{on } B_{3\varepsilon}(x) \\ w \equiv 0 & \text{on } \partial B_{3\varepsilon}(x). \end{cases}$$

The regularity theory for elliptic equations implies that  $w \in C^1(B_{3\varepsilon}(x))$ . Combining  $u$  and  $w$ , we define a new function

$$v = w^{-1}u^p.$$

Arguing as in the proof of [19], Theorem 1.4, a computation that uses (2.11), (2.9), (2.6) and (2.7) and the results in the Appendix, shows that

$$v \operatorname{div}(w^2 \nabla v) \geq 0$$

weakly on  $B_{3\varepsilon}(x)$ . Therefore Corollary 2.3 applies and we have

$$\sup_{B_\varepsilon(x)} v^2 \leq C \int_{B_{2\varepsilon}(x)} v^2,$$

for some constant  $C > 0$  depending on  $w|_{\bar{B}_{2\varepsilon}(x)}$  and the geometry of  $B_{2\varepsilon}(x)$ . Whence, the validity of (2.10) with

$$C' = \left( \frac{\sup_{B_{2\varepsilon}} w}{\inf_{B_\varepsilon} w} \right)^2 C. \quad \blacksquare$$

In order to obtain the integral estimate (2.2) above, we shall use a local Poincaré inequality on annuli for functions which vanish only on the interior boundary component. Although the result is well known, we take the opportunity to give an elementary nonlinear proof of the inequality in the form we are going to use. In Euclidean setting, more general and sophisticated forms of these type of inequalities can be found in some papers by L. Hedberg. In particular, we refer the interested reader to Lemma 2.1 in [9] and Theorem 4.1 in [10].

**Proposition 2.5.** *Let  $B_{\bar{R}}(o)$  be a relatively compact geodesic ball in the Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$ . Assume that*

$${}^M \text{Ricci} \geq -(m-1)k \text{ on } B_{\bar{R}}(o)$$

for some  $k \geq 0$ . Having fixed  $q > 1$  and  $0 < R_1 < R_2 < \bar{R}$ , there exists a constant

$$C = q(R_2 - R_1) \left( \frac{\sinh(\sqrt{k}R_2)}{\sinh(\sqrt{k}R_1)} \right)^{\frac{m-1}{q-1}}$$

such that

$$C^{-q} \int_{B_{R_2} \setminus B_{R_1}} |u|^q \leq \int_{B_{R_2} \setminus B_{R_1}} |\nabla u|^q$$

for every  $u \in C^0(\overline{B_{R_2}} \setminus \overline{B_{R_1}}) \cap W^{1,q}(B_{R_2} \setminus \overline{B_{R_1}})$  satisfying

$$u = 0 \text{ on } \partial B_{R_1}.$$

The proof of the Theorem is based on the the next (non-linear) lemma. For the sake of completeness we recall that, for any  $p > 1$ , the  $p$ -Laplacian of a function  $u \in W_{loc}^{1,p}$  is defined by the expression

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$$

where the divergence has to be understood in the weak sense.

**Lemma 2.6.** *Let  $\Omega_i$ ,  $i = 1, 2$ , be open relatively compact domains in  $M$  with  $\overline{\Omega}_1 \subset \Omega_2$  and let  $A$  be the annular domain  $\Omega_2 \setminus \Omega_1$  with compact boundary  $\partial\Omega_1 \cup \partial\Omega_2$ . Let  $q > 1$  and  $0 \leq \phi \in \text{Lip}(\bar{A})$  be a non-null solution of the problem*

$$(2.12) \quad \begin{cases} \Delta_q \phi \geq 0 & \text{weakly on } A \\ \phi = 0 & \text{on } \partial\Omega_2. \end{cases}$$

Suppose also that

$$(2.13) \quad |\nabla \phi| > 0 \text{ on } \bar{A}.$$

Then, there exists an explicit constant

$$C = \frac{\inf_{\bar{A}} |\nabla \phi|^q}{q^q \sup_{\bar{A}} \phi^q} > 0$$

such that

$$(2.14) \quad C \int_A |u|^q \leq \int_A |\nabla u|^q$$

for every  $u \in C^0(\bar{A}) \cap W^{1,q}(A)$  satisfying

$$(2.15) \quad u = 0 \text{ on } \partial\Omega_1.$$

**Proof.** By assumption,

$$(2.16) \quad - \int_A \langle |\nabla\phi|^{q-2} \nabla\phi, \nabla\rho \rangle \geq 0,$$

for every  $0 \leq \rho \in C_0^\infty(A)$ . As a matter of fact, since  $\phi \in Lip(\bar{A}) \subset W^{1,q}(A)$ , we have the validity of (2.16) for every  $0 \leq \rho \in W_0^{1,q}(A)$ . Note that  $\phi|u|^q \in W_0^{1,q}(A)$ . Indeed  $\phi|u|^q$  lies in  $W^{1,q}(A)$  and vanishes continuously on  $\partial A$ . Therefore, we can use  $\rho = \phi|u|^q$  in (2.16). Using Schwarz and Hölder inequalities we get

$$(2.17) \quad \begin{aligned} 0 &\leq - \int_A \langle |\nabla\phi|^{q-2} \nabla\phi, \nabla(\phi|u|^q) \rangle \\ &= - \int_A |\nabla\phi|^q |u|^q - \int_A q|u|^{q-1} \phi |\nabla\phi|^{q-2} \langle \nabla\phi, \nabla|u| \rangle \\ &\leq - \int_A |\nabla\phi|^q |u|^q + q \int_A \phi |u|^{q-1} |\nabla|u|| |\nabla\phi|^{q-1} \\ &\leq - \int_A |u|^q |\nabla\phi|^q + q \left( \int_A \phi^q |\nabla u|^q \right)^{1/q} \left( \int_A |u|^q |\nabla\phi|^q \right)^{(q-1)/q} \end{aligned}$$

proving the Caccioppoli type inequality

$$(2.18) \quad \int_A u^q |\nabla\phi|^q \leq q^q \int_A \phi^q |\nabla u|^q.$$

Whence, we conclude the validity of (2.14). ■

**Proof (of Theorem 2.5).** We simply have to choose the test function  $\phi$  in Lemma 2.6. One observes that, in case of model manifolds, the  $q$ -equilibrium potential of the condenser  $E = (B_{R_2}, \overline{B_{R_1}})$  is suitable for the purpose. Thus, the general case can be obtained by a model-comparison argument. More precisely, up to renormalizing the metric, we can suppose  $k = 1$ . Set  $r(x) = \text{dist}_M(x, o)$  and define

$$\phi(r(x)) = \int_{r(x)}^{R_2} \frac{dt}{\sinh(t)^{\frac{m-1}{q-1}}}$$

Then  $\phi \geq 0$ ,  $\phi = 0$  on  $\partial B_{R_2}$ ,  $\phi > 0$  on  $\partial B_{R_1}$  and

$$|\nabla\phi|(x) = \sinh(r(x))^{\frac{1-m}{q-1}} > 0 \quad \text{on } \overline{B_{R_2}} \setminus B_{R_1}.$$

Moreover, since  $\phi' \leq 0$ , using the Laplacian comparison theorem, we obtain, pointwise outside of  $\text{cut}(o)$ ,

$$\begin{aligned} \Delta_q \phi &= (q-1)(-\phi')^{q-2} \phi'' - (-\phi')^{q-1} \Delta r \\ &\geq (q-1)(-\phi')^{q-2} \phi'' - (-\phi')^{q-1} (m-1) \coth r = 0. \end{aligned}$$

As usual, this latter extends weakly on all of the annulus. Therefore, Lemma 2.6 applies and the desired inequality follows. ■

Here is the local a-priori estimate alluded to above.

**Lemma 2.7.** *Keeping notation and assumptions of Theorem 1.1, having fixed an origin  $o \in M$ , there exist  $\bar{R} > 0$  and a constant  $C > 0$  depending on  $p, H$  and the geometry of  $B_{\bar{R}}(o)$  such that*

$$(2.19) \quad \sup_{B_{\bar{R}}(o)} u^{2p} \leq C \int_{B_{\bar{R}}(o)} u^{2p}$$

for every  $u = |\xi|$ ,  $\xi \in V$ .

**Proof.** From now on, we assume that all the geodesic balls under consideration are centered at the point  $o \in M$  and so, to simplify the notation, we omit it from the writing.

We choose  $R_0 > 0$  so large that  $K \subset B_{R_0}$ . We shall show that (2.19) is met with  $\bar{R} = R_0 + 1$ . To this end, let us note that by Proposition 2.4 there exists a constant  $D > 0$  independent of  $u$  such that

$$\sup_{B_{R_0+1}} u^{2p} \leq D \int_{B_{R_0+2}} u^{2p} = D \left( \int_{B_{R_0+2} \setminus B_{R_0+1}} + \int_{B_{R_0+1}} \right) u^{2p}.$$

The goal is to prove that

$$(2.20) \quad \int_{B_{R_0+2} \setminus B_{R_0+1}} u^{2p} \leq E \int_{B_{R_0+1}} u^{2p}$$

for some constant  $E > 0$  independent of  $u$ .

We set

$$\alpha = \frac{p}{H}$$

and consider the function

$$v = \varphi^{-\alpha} u^p \text{ on } M \setminus B_{R_0}.$$

As in the proof of Proposition 2.4, a direct computation which uses the results of the Appendix shows that

$$(2.21) \quad v \operatorname{div} (\varphi^{2\alpha} \nabla v) \geq 0$$

weakly on  $M \setminus B_{R_0}$ . Moreover, since

$$\begin{aligned} \int_{B_{R_0+2} \setminus B_{R_0+1}} u^{2p} &= \int_{B_{R_0+2} \setminus B_{R_0+1}} \varphi^{2\alpha} (\varphi^{-\alpha} u^p)^2 \\ &\leq \left( \sup_{B_{R_0+2} \setminus B_{R_0+1}} \varphi^{2\alpha} \right) \int_{B_{R_0+2} \setminus B_{R_0+1}} v^2 \end{aligned}$$

the desired inequality (2.20) will follow once we prove that

$$(2.22) \quad \int_{B_{R_0+2} \setminus B_{R_0+1}} v^2 \leq E \int_{B_{R_0+1}} u^{2p}.$$

Towards this aim, we consider the family of compactly supported, Lipschitz functions  $\{\phi_k\}$  defined by

$$\phi_k(x) = \begin{cases} 0 & \text{on } B_{R_0} \\ r(x) - R_0 & \text{on } B_{R_0+1} \setminus B_{R_0} \\ 1 & \text{on } B_{R_0+2} \setminus B_{R_0+1} \\ \frac{R_k - r(x)}{R_k - R_0 - 2} & \text{on } B_{R_k} \setminus B_{R_0+2} \\ 0 & \text{on } M \setminus B_{R_k}. \end{cases}$$

Furthermore, we set

$$\phi_\infty = \begin{cases} 0 & \text{on } B_{R_0} \\ r(x) - R_0 & \text{on } B_{R_0+1} \setminus B_{R_0} \\ 1 & \text{on } M \setminus B_{R_0+1}. \end{cases}$$

According to (2.21) we can apply Lemma 2.2 with  $q = 0$  to obtain

$$\begin{aligned} & D_0 \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 |\nabla v|^2 \\ & \leq D_0 \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha} \int_{B_{R_0+2} \setminus B_{R_0}} \varphi^{2\alpha} \phi_\infty^2 |\nabla v|^2 \\ & \leq D_0 \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha} \int_{M \setminus B_{R_0}} \varphi^{2\alpha} \phi_k^2 |\nabla v|^2 \\ & \leq \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha} \int_{M \setminus B_{R_0}} \varphi^{2\alpha} v^2 |\nabla \phi_k|^2 \\ & \leq \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha} \left\{ \int_{B_{R_0+1} \setminus B_{R_0}} \varphi^{2\alpha} v^2 + \int_{B_{R_k} \setminus B_{R_0+2}} \varphi^{2\alpha} v^2 |\nabla \phi_k|^2 \right\} \\ & \leq \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha} \left\{ \int_{B_{R_0+1} \setminus B_{R_0}} u^{2p} + \frac{1}{(R_k - R_0 - 2)^2} \int_{B_{R_k} \setminus B_{R_0}} u^{2p} \right\}. \end{aligned}$$

Letting  $k \rightarrow +\infty$  we deduce

$$(2.23) \quad \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 |\nabla v|^2 \leq \tilde{D} \int_{B_{R_0+1} \setminus B_{R_0}} u^{2p}$$

where we have set

$$\tilde{D} = \frac{1}{D_0 \sup_{B_{R_0+2} \setminus B_{R_0}} \varphi^{-2\alpha}} > 0.$$

On the other hand, applying Proposition 2.5 with  $q = 2$  to the function  $\phi_\infty v$ , and using the Schwarz and Young inequalities, we get

$$\begin{aligned} C_1 \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 v^2 &\leq \\ &\leq \int_{B_{R_0+2} \setminus B_{R_0}} |\nabla(\phi_\infty v)|^2 \\ &= \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 |\nabla v|^2 + v^2 |\nabla \phi_\infty|^2 + 2v\phi \langle \nabla v, \nabla \phi_\infty \rangle \\ &\leq \int_{B_{R_0+2} \setminus B_{R_0}} 2\phi_\infty^2 |\nabla v|^2 + 2v^2 |\nabla \phi_\infty|^2 \\ &\leq 2 \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 |\nabla v|^2 + 2 \int_{B_{R_0+1} \setminus B_{R_0}} v^2 \\ &\leq 2 \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 |\nabla v|^2 + 2 \sup_{B_{R_0+1} \setminus B_{R_0}} \varphi^{-2\alpha} \int_{B_{R_0+1} \setminus B_{R_0}} u^{2p}, \end{aligned}$$

whence, using (2.23), we conclude

$$\begin{aligned} C_1 \int_{B_{R_0+2} \setminus B_{R_0+1}} v^2 &\leq C_1 \int_{B_{R_0+2} \setminus B_{R_0}} \phi_\infty^2 v^2 \\ &\leq 2 \left( \tilde{D} + \sup_{B_{R_0+1} \setminus B_{R_0}} \varphi^{-2\alpha} \right) \int_{B_{R_0+1} \setminus B_{R_0}} u^{2p} \\ &\leq 2 \left( \tilde{D} + \sup_{B_{R_0+1} \setminus B_{R_0}} \varphi^{-2\alpha} \right) \int_{B_{R_0+1}} u^{2p}, \end{aligned}$$

as required to prove (2.22). ■

We are now in the position to prove Theorem 1.1.

**Proof (of Theorem 1.1).** Let  $B_{\bar{R}} \subset M$  and  $C > 0$  be as in Lemma 2.7. From the unique continuation property we have that the restriction map

$$\xi \longmapsto \xi|_{B_{\bar{R}}}$$

defines an injective homomorphism of  $V$  into  $L^2\Gamma(E|_{B_{\bar{R}}})$ , the space of square-integrable sections of  $E$  on  $B_{\bar{R}}$ .

Let  $T$  be any finite dimensional subspace of  $V$ . We have to prove that  $t = \dim T$  is bounded from above by an absolute constant. To this end we apply Peter Li's Lemma to deduce that there exists  $\bar{\xi} \in T$  such that

$$t^{\min(1,p)} \int_{B_{\bar{R}}} |\bar{\xi}|^{2p} \leq \text{vol}(B_{\bar{R}}) \min\{l, t\}^{\min(1,p)} \sup_{B_{\bar{R}}} |\bar{\xi}|^{2p}.$$

On the other hand, using  $u = |\bar{\xi}|$  in (2.19) of Lemma 3 we see that

$$\sup_{B_{\bar{R}}} |\bar{\xi}|^{2p} \leq C \int_{B_{\bar{R}}} |\bar{\xi}|^{2p}.$$

As a consequence

$$t^{\min(1,p)} \leq \text{vol}(B_{\bar{R}}) \min\{l, t\}^{\min(1,p)} C$$

which in turn implies

$$t = \dim T \leq l \max \left\{ C^{\frac{1}{\min(1,p)}} \text{vol}(B_{\bar{R}})^{\frac{1}{\min(1,p)}}, 1 \right\}.$$

■

### 3. Appendix

This section gives the technical support for the distributional computations needed in the proofs of Proposition 2.4 and Lemma 2.7. First, we present a regularity result.

**Lemma 3.1.** *Let  $a(x) \in L^\infty_{loc}(M)$  and  $A \in \mathbb{R}$ . Let  $\psi \in Lip_{loc}(M)$  be a weak solution of*

$$\psi \Delta \psi + a(x) \psi^2 + A |\nabla \psi|^2 \geq 0 \text{ on } M.$$

*Then*

$$(3.1) \quad |\psi|^{p-1} \psi \in W_{loc}^{1,2}(M)$$

*provided*

$$\begin{cases} p \geq 1 & \text{if } A \geq 1 \\ p > \max\left\{0, \frac{A+1}{2}\right\} & \text{if } A < 1 \end{cases}$$

*and, furthermore,*

$$(3.2) \quad \nabla \left( (\psi^2 + \varepsilon)^{(p-1)/2} \psi \right) \xrightarrow{L^2} \nabla (|\psi|^{p-1} \psi) \quad \text{as } \varepsilon \rightarrow 0+$$

**Proof.** We suppose  $A < 1$  and  $p < 1$ , which is the delicate case. Consider the family of functions  $(\psi^2 + \varepsilon)^{(p-1)/2}\psi$  and note that, as  $\varepsilon \rightarrow 0+$

$$(\psi^2 + \varepsilon)^{(p-1)/2}\psi \rightarrow |\psi|^{p-1}\psi \quad \text{in } L^2_{loc}$$

We are going to use the fact that if a sequence  $\{f_n\}$  is uniformly bounded in  $W^{1,2}_{loc}$  and converges to  $f$  strongly in  $L^2_{loc}$ , then the limit function  $f$  is in  $W^{1,2}_{loc}$  and  $\nabla f_n$  converges to  $\nabla f$  weakly in  $L^2_{loc}$  (see [4, Lemma 6.2, page 16]). Since

$$\begin{aligned} |\nabla((\psi^2 + \varepsilon)^{(p-1)/2}\psi)| &= (\psi^2 + \varepsilon)^{(p-1)/2} \frac{p\psi^2 + \varepsilon}{\psi^2 + \varepsilon} |\nabla\psi| \\ &\leq (\psi^2 + \varepsilon)^{(p-1)/2} |\nabla\psi| \end{aligned}$$

it suffices to show that the right hand side is uniformly bounded in  $L^2_{loc}$  as  $\varepsilon \rightarrow 0+$ .

By assumption, for any  $0 \leq \rho \in Lip_c(M)$ , we have

$$-\int \langle \nabla\psi, \nabla(\rho\psi) \rangle \geq -\int a(x)\psi^2\rho - A\int |\nabla\psi|^2\rho,$$

that is,

$$(3.3) \quad -\int \psi \langle \nabla\psi, \nabla\rho \rangle \geq -\int a(x)\psi^2\rho + (-A+1)\int |\nabla\psi|^2\rho.$$

Fix  $\varepsilon > 0$  and choose

$$\rho = (\psi^2 + \varepsilon)^{p-1}\phi^2$$

where  $0 \leq \phi \in C_c^\infty(M)$ . Then,

$$\nabla\rho = 2(p-1)\phi^2(\psi^2 + \varepsilon)^{p-2}\psi\nabla\psi + 2\phi(\psi^2 + \varepsilon)^{p-1}\nabla\phi,$$

so that, using the Cauchy-Schwarz and Young inequalities and the fact that  $p-1 < 0$ , we estimate

$$\begin{aligned} &LHS \text{ of (3.3)} \\ &= -2\int \phi(\psi^2 + \varepsilon)^{p-1}\psi \langle \nabla\psi, \nabla\phi \rangle - 2(p-1)\int \phi^2(\psi^2 + \varepsilon)^{p-2}\psi^2|\nabla\psi|^2 \\ &\leq 2\int \phi(\psi^2 + \varepsilon)^{p-1/2}|\nabla\psi||\nabla\phi| - 2(p-1)\int \phi^2(\psi^2 + \varepsilon)^{p-1}|\nabla\psi|^2 \\ &\leq \frac{4}{\eta}\int (\psi^2 + \varepsilon)^p|\nabla\phi|^2 - (2p-2-\eta)\int \phi^2(\psi^2 + \varepsilon)^{p-1}|\nabla\psi|^2. \end{aligned}$$

Moreover

$$\begin{aligned} & \text{RHS of (3.3)} \\ &= - \int a(x)\psi^2(\psi^2 + \varepsilon)^{p-1}\phi^2 + (-A + 1) \int \phi^2(\psi^2 + \varepsilon)^{p-1}|\nabla\psi|^2 \\ &\geq - \int |a(x)|(\psi^2 + \varepsilon)^p\phi^2 + (-A + 1) \int \phi^2(\psi^2 + \varepsilon)^{p-1}|\nabla\psi|^2, \end{aligned}$$

for  $\eta > 0$ . Combining the two inequalities and rearranging we obtain

$$\begin{aligned} (2p - A - 1 - \eta) \int \phi^2(\psi^2 + \varepsilon)^{p-1}|\nabla\psi|^2 \\ \leq \frac{4}{\eta} \int (\psi^2 + \varepsilon)^p|\nabla\phi|^2 + \int |a(x)|(\psi^2 + \varepsilon)^p\phi^2. \\ \leq \int \max\{1, |\psi|^{2p}\} \left( \frac{4}{\eta}|\nabla\phi|^2 + |a(x)|\phi^2 \right). \end{aligned}$$

Since  $2p - A - 1 > 0$ , we may choose  $\eta > 0$  small enough that  $(2p - A - 1 - \eta) > 0$ , and conclude that the left hand side is uniformly bounded as  $\varepsilon \rightarrow 0+$ , as required to conclude.  $\blacksquare$

Next we prove that, in the above assumptions, one can use the ordinary chain rule to compute the weak gradient of  $\psi^p$  even if  $p < 1$ . Note that, in this situation, the function  $x \mapsto x^p$  is not Lipschitz so that standard results in the literature do not apply directly.

**Lemma 3.2.** *Let  $0 < \delta (< 1)$  and assume that  $0 \leq \psi \in Lip_{loc}(M)$  satisfies (3.1) and (3.2), for every  $\delta < p$ . Then*

$$(3.4) \quad \frac{\nabla\psi}{\psi^{1-p}} \in L^2_{loc}(M)$$

and

$$(3.5) \quad \nabla(\psi^p) = p \frac{\nabla\psi}{\psi^{1-p}}, \text{ a.e. on } M,$$

the LHS of this latter being understood in the sense of distribution.

**Proof.** Let  $\delta < p' (< 1)$  be any real number, and  $\Omega \subset\subset M$  a fixed domain. Using  $\psi \in Lip(\Omega)$  as a test function in (3.2) we have

$$\int_{\Omega} \langle p'(\psi + \varepsilon)^{p'-1} \nabla\psi, \nabla\psi \rangle \rightarrow \int_{\Omega} \langle \nabla(\psi^{p'}), \nabla\psi \rangle, \text{ as } \varepsilon \searrow 0.$$

On the other hand, by monotone convergence,

$$\int_{\Omega} \left\langle p' (\psi + \varepsilon)^{p'-1} \nabla \psi, \nabla \psi \right\rangle \nearrow \int_{\Omega} p' \frac{|\nabla \psi|^2}{\psi^{1-p'}}, \text{ as } \varepsilon \searrow 0$$

proving that

$$\int_{\Omega} p' \frac{|\nabla \psi|^2}{\psi^{1-p'}} = \int_{\Omega} \left\langle \nabla (\psi^{p'}), \nabla \psi \right\rangle < +\infty.$$

Therefore

$$\frac{\nabla \psi}{\psi^{\frac{1-p'}{2}}} \in L^2(\Omega).$$

We now use this function in (3.2) to get, as above,

$$\frac{\nabla \psi}{\psi^{\frac{3(1-p')}{4}}} \in L^2(\Omega).$$

Iterating this procedure  $n$ -times finally gives

$$(3.6) \quad \frac{\nabla \psi}{\psi^{(1-p')\frac{2^n-1}{2^n}}} \in L^2(\Omega)$$

which readily implies the validity of (3.4). Indeed, having fixed  $\delta < p (< 1)$  we can choose  $p' \in (\delta, p)$  and  $n \in \mathbb{N}$  sufficiently large so that

$$1 - p = (1 - p') \frac{2^n - 1}{2^n}.$$

Then, according to (3.6),

$$\frac{\nabla \psi}{\psi^{1-p}} = \frac{\nabla \psi}{\psi^{(1-p')\frac{2^n-1}{2^n}}} \in L^2(\Omega)$$

as desired.

Finally, to conclude the validity of (3.5), we observe that (3.4) enable us to apply the dominated convergence theorem in (3.2).  $\blacksquare$

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*Recibido:* 7 de marzo de 2006  
*Revisado:* 8 de noviembre de 2006

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