Rev. Mat. Iberoamericana **24** (2008), no. 1, 267–296

Comparison of the classical BMO with the BMO spaces associated with operators and applications

Donggao Deng, Xuan Thinh Duong, Adam Sikora and Lixin Yan

Abstract

Let L be a generator of a semigroup satisfying the Gaussian upper bounds. A new BMO_L space associated with L was recently introduced in [15] and [16]. We discuss applications of the new BMO_L spaces in the theory of singular integration. For example we obtain BMO_L estimates and interpolation results for fractional powers, purely imaginary powers and spectral multipliers of self adjoint operators. We also demonstrate that the space BMO_L might coincide with or might be essentially different from the classical BMO space.

1. Introduction

The classical space of functions of bounded mean oscillation (BMO) plays a crucial role in modern harmonic analysis. See for examples [19], [22], [28] and [29]. In the case of the Euclidean space $\mathbb{R}^n$, a function f is said to in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty, \tag{1.1}$$

where f_Q denotes the average value of f on the cube Q and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

An important application of the theory of BMO spaces is the following interpolation result.

2000 Mathematics Subject Classification: 42B35, 42B25, 47B38.
Keywords: BMO space, Hardy space, Dirichlet and Neumann Laplacians, semigroup, Gaussian bounds, fractional powers, purely imaginary powers, spectral multiplier.

Proposition 1.1 *If T is a bounded sublinear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, and T is bounded from $L^\infty(\mathbb{R}^n)$ to* $\mathrm{BMO}(\mathbb{R}^n)$*, then T is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $2 < p < \infty$.*

It is well known that Calderón-Zygmund operators (such as the Hilbert transform on the real line, the Riesz transforms on $\mathbb{R}^n$, or the purely imaginary powers of the Laplacian on $\mathbb{R}^n$) do not map the space L^∞ into L^∞, but the standard conditions on their kernels ensure that they map L^∞ into the BMO space boundedly, hence we can apply Proposition 1.1 to obtain L^p boundedness of these operators for $p > 2$. In this sense, the BMO space is a natural substitute of the space L^∞ in the theory of Calderón-Zygmund singular integrals.

In this paper we are motivated by study of singular integral operators corresponding to spectral multiplier of an operator L which generates a semigroup with appropriate kernel bounds, see [15]. Such multipliers do not always map L^∞ or appropriate L^p spaces into the classical BMO space, see Example 5.4 below. Hence the classical BMO space is not necessarily a suitable space to study such singular integrals. To study these rough operators, we introduced a new BMO_L space associated with an operator L.

To explain our approach to BMO_L space associated with an operator let us recall that the space of BMO functions can be characterized by the Carleson measure estimate as follows:

Proposition 1.2 *A function f is in* BMO *if and only if f satisfies*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}dx < \infty,$$

and

$$\mu_f(x,t) = \left|t\frac{\partial}{\partial t}e^{-t\sqrt{\Delta}}f(x)\right|^2\frac{dxdt}{t}$$

is a Carleson measure.

One can see from the characterization in Proposition 1.2 that the BMO space is associated with the Laplace operator on $\mathbb{R}^n$ and it seems to be natural idea to replace the Laplace operator Δ by more general operators operator L, see also [19] and [29].

In this paper we use equivalent approach, see Definition 2.2 below. In this definition the BMO_L space associated with L is defined by using the function $e^{-t_Q L}f$ to replace the average f_Q in Definition 1.1 of BMO where the value t_Q is scaled to the length of the sides of Q. In this paper we discuss various examples which shows that Definition 2.2 is an effective tool in study of singular integrals operators associated with the operator L. We refer the reader to [2], [8] and [18] for other ideas related to generalization of the BMO space and BMO spaces associated with an operator L.

Many important features of the classical BMO space are retained by the new BMO_L spaces such as the John-Nirenberg inequality and duality between the Hardy space and the BMO_L space. See [15] and [16]. One of these important features is that the interpolation property in Proposition 1.1 is still valid if the classical space BMO is replaced by the BMO_L space associated with an operator L. Indeed, the following result is proved in [15] (Theorem 6.1).

Proposition 1.3 *Let $\mathcal{X}$ be a space of homogeneous type. If T is a bounded sublinear operator from $L^2(\mathcal{X})$ to $L^2(\mathcal{X})$, and T is bounded from $L^\infty(\mathcal{X})$ into $\mathrm{BMO}_L(\mathcal{X})$, then T is bounded from $L^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $2 < p < \infty$.*

A natural question arising from Proposition 1.3 is to compare the classical BMO space and the BMO_L space associated with an operator L. In Sections 3 and 4 we study this question systematically and we show that depending on the choice of the operator L, all the following cases are possible

Case 1: $\mathrm{BMO} \cong \mathrm{BMO}_L$;

Case 2: $\mathrm{BMO} \subseteq \mathrm{BMO}_L$ and $\mathrm{BMO} \neq \mathrm{BMO}_L$;

Case 3: $\mathrm{BMO}_L \subseteq \mathrm{BMO}$ and $\mathrm{BMO}_L \neq \mathrm{BMO}$;

Case 4: $\mathrm{BMO} \nsubseteq \mathrm{BMO}_L$ and $\mathrm{BMO}_L \nsubseteq \mathrm{BMO}$.

For other results related to Cases 1 and 2 we refer readers to Proposition 2.5 of [15], Section 6.2 of [16] and Proposition 3.1 of [23]. In Section 5 we show that if $f \in L^{n/\alpha}(\mathbb{R}^n)$ and $L^{-\alpha} f < \infty$ almost everywhere then $L^{-\alpha} f \in \mathrm{BMO}_L$. We construct an example of a function $f \in L^p(\mathbb{R})$ and an operator L such that $L^{-\frac{1}{2p}} f \in \mathrm{BMO}_L$ but $L^{-\frac{1}{2p}} f \notin \mathrm{BMO}$. This shows that the new BMO_L space does make a difference in estimates of singular integrals. Finally in Sections 6 and 7, we obtain sharp estimates of the L^∞ to BMO_L norm of the purely imaginary powers L^{is} of a self adjoint operator L. We also obtain the BMO type estimates for spectral multipliers of a self adjoint operator L and for maximal operators $\sup_{t>0} |F(tL)|$ corresponding to L and appropriate functions F. L^p boundedness of these operators, $2 < p < \infty$, then follows from Proposition 1.3.

2. Preliminaries

2.1. BMO spaces on the half spaces

Let us begin by recalling the definitions of various BMO spaces on the usual upper-half space in $\mathbb{R}^n$. For any subset $A \subset \mathbb{R}^n$ and a function $f\colon \mathbb{R}^n \to \mathbb{C}$ by $f|_A$ we denote the restriction of f to the set A. Next we set

$$\mathbb{R}^n_+ = \Big\{(x', x_n) \in \mathbb{R}^n :\ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ x_n > 0\Big\}.$$

Definition 2.1 *A function f on $\mathbb{R}^n_+$ is said to be in* $\mathrm{BMO}_r(\mathbb{R}^n_+)$ *if there exists $F \in \mathrm{BMO}(\mathbb{R}^n)$ such that $F|_{\mathbb{R}^n_+} = f$. If $f \in \mathrm{BMO}_r(\mathbb{R}^n_+)$, we set*

$$\|f\|_{\mathrm{BMO}_r(\mathbb{R}^n_+)} = \inf\Big\{\|F\|_{\mathrm{BMO}(\mathbb{R}^n)}\colon F|_{\mathbb{R}^n_+} = f\Big\}.$$

A function f on $\mathbb{R}^n_+$ belongs to $\mathrm{BMO}_z(\mathbb{R}^n_+)$ *if the function F defined by*

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n_+; \\ 0 & \text{if } x \notin \mathbb{R}^n_+ \end{cases} \tag{2.1}$$

belongs to $\mathrm{BMO}(\mathbb{R}^n)$. *If $f \in \mathrm{BMO}_z(\mathbb{R}^n_+)$, we set $\|f\|_{\mathrm{BMO}_z(\mathbb{R}^n_+)} = \|F\|_{\mathrm{BMO}_z(\mathbb{R}^n)}$.*

Compare Section 4.5.1, page 221 of [32] and Section 5.4 of [4]. In order to analyze the spaces $\mathrm{BMO}_r(\mathbb{R}^n_+)$ and $\mathrm{BMO}_z(\mathbb{R}^n_+)$, let us introduce the following notations, see [6]. For any $x = (x', x_n) \in \mathbb{R}^n$, we set $\widetilde{x} = (x', -x_n)$. If f is any function defined on $\mathbb{R}^n_+$, its even extension f_e is defined on $\mathbb{R}^n$ by

$$f_e(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n_+; \\ f(\widetilde{x}) & \text{if } x \in \mathbb{R}^n_-, \end{cases}$$

and its odd extension f_o is defined by

$$f_o(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n_+; \\ -f(\widetilde{x}) & \text{if } x \in \mathbb{R}^n_-, \end{cases}$$

where

$$\mathbb{R}^n_- = \Big\{(x', x_n) \in \mathbb{R}^n :\ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n < 0\Big\}.$$

For any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+)$, we define

$$\|f\|_{\mathrm{BMO}_e(\mathbb{R}^n_+)} = \|f_e\|_{\mathrm{BMO}(\mathbb{R}^n)} \quad \text{and} \quad \|f\|_{\mathrm{BMO}_o(\mathbb{R}^n_+)} = \|f_o\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

and we denote by $\mathrm{BMO}_e(\mathbb{R}^n_+)$ and $\mathrm{BMO}_o(\mathbb{R}^n_+)$ the corresponding Banach spaces.

We will see that $\mathrm{BMO}_e(\mathbb{R}^n_+)$ is suitable for the analysis of the Neumann Laplacian on $\mathbb{R}^n_+$ whereas $\mathrm{BMO}_o(\mathbb{R}^n_+)$ is suitable for the study of the Dirichlet Laplacian on $\mathbb{R}^n_+$. See Proposition 3.2 below.

In what follows, $Q = Q[x_Q, l_Q]$ denotes a cube of $\mathbb{R}^n$ centered at x_Q and of the side length l_Q. Given any cube Q, we denote the reflection of Q across $\partial\mathbb{R}^n_+$ by

$$\widetilde{Q} = \Big\{(x', x_n) \in \mathbb{R}^n, \quad (x', -x_n) \in Q\Big\}. \tag{2.2}$$

Let $Q_+ = Q \cap \mathbb{R}^n_+$ and $Q_- = Q \cap \overline{\mathbb{R}^n_-}$ where $\overline{\mathbb{R}^n_-} = \big\{(x', x_n) \in \mathbb{R}^n : \ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n \le 0\big\}$. If both Q_- and Q_+ are not empty, we then define

$$\begin{cases} \widehat{Q}_- = \{(x', x_n) : x' \in Q \cap \mathbb{R}^{n-1}, \ -l_Q < x_n \le 0\}, \\ \widehat{Q}_+ = \{(x', x_n) : \ x' \in Q \cap \mathbb{R}^{n-1}, \ 0 < x_n \le l_Q\}. \end{cases} \tag{2.3}$$

Obviously, we have the following properties:

(i) $Q_- \subseteq \widehat{Q}_-$, $Q_+ \subseteq \widehat{Q}_+$ and thus $Q \subseteq (\widehat{Q}_- \cup \widehat{Q}_+)$;

(ii) $|Q| = |\widehat{Q}_-| = |\widehat{Q}_+|$.

These will be often used in the sequel.

2.2. Dirichlet and Neumann Laplacians

By Δ_{n,N_+} (and Δ_{n,N_-}) we denote the Neumann Laplacian on $\mathbb{R}^n_+$ (and on $\mathbb{R}^n_-$ respectively). Similarly by Δ_{n,D_+} (and Δ_{n,D_-}) we denote the Dirichlet Laplacian on $\mathbb{R}^n_+$ (and on $\mathbb{R}^n_-$ respectively).

The Dirichlet and Neumann Laplacian are positive definite self-adjoint operators. By the spectral theorem one can define the semigroups generated by these operators $\{\exp(-t\Delta_{n,D_+}) \colon \ t \ge 0\}$ and $\{\exp(-t\Delta_{n,N_+}) \colon \ t \ge 0\}$. By $p_{t,\,\Delta_{n,D_+}}(x,y)$ and $p_{t,\,\Delta_{n,N_+}}(x,y)$ we denote the heat kernels corresponding to the semigroups generated by Δ_{n,D_+} and Δ_{n,N_+} respectively.

For $n = 1$ by the reflection method (see for example [30, (6) p. 57]) we obtain

$$p_{t,\,\Delta_{1,D_+}}(x,y) = \frac{1}{(4\pi t)^{1/2}}\Big(e^{-\frac{|x_1-y_1|^2}{4t}} - e^{-\frac{|x_1+y_1|^2}{4t}}\Big).$$

Then for $n \ge 2$

$$\begin{aligned} p_{t,\,\Delta_{n,D_+}}(x,y) &= \Big(p_{t,\,\Delta_{1,D_+}}(x_n, y_n)\Big)\Big(p_{t,\,\Delta_{n-1}}(x', y')\Big) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}}\Big(e^{-\frac{|x_n-y_n|^2}{4t}} - e^{-\frac{|x_n+y_n|^2}{4t}}\Big), \end{aligned} \tag{2.4}$$

where $p_{t,\,\Delta_{n-1}}(x,y)$ is the heat kernel corresponding to the standard Laplace operator acting on $\mathbb{R}^{n-1}$. Applying the reflection method also to the Neumann Laplacian we obtain (see [30, (7) p. 57])

$$\begin{aligned} p_{t,\,\Delta_{n,N_+}}(x,y) &= \Big(p_{t,\,\Delta_{1,N_+}}(x_n,y_n)\Big)\Big(p_{t,\,\Delta_{n-1}}(x',y')\Big) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} \Big(e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}}\Big). \end{aligned} \tag{2.5}$$

In the sequel we skip the index n and we denote the Dirichlet and Neumann Laplacian by Δ_{D_+} and Δ_{N_+}. Note that by (2.4)

$$\begin{aligned} \exp(-t\Delta_{D_+})f(x) &= \int_{\mathbb{R}^n_+} p_{t,\,\Delta_{D_+}}(x,y)f(y)dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f_o(y)dy \\ &= \exp(-t\Delta)f_o(x) \end{aligned} \tag{2.6}$$

for $x \in \mathbb{R}^n_+$ and all $t > 0$. Similarly

$$\begin{aligned} \exp(-t\Delta_{N_+})f(x) &= \int_{\mathbb{R}^n_+} p_{t,\,\Delta_{N_+}}(x,y)f(y)dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f_e(y)dy \\ &= \exp(-t\Delta)f_e(x) \end{aligned} \tag{2.7}$$

for $x \in \mathbb{R}^n_+$ and all $t > 0$.

Next for any function f on $\mathbb{R}^n$, we set

$$f_- = f|_{\mathbb{R}^n_-} \quad \text{and} \quad f_+ = f|_{\mathbb{R}^n_+}.$$

Now let Δ_N be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that

$$(\Delta_N f)_+ = \Delta_{N_+} f_+ \quad \text{and} \quad (\Delta_N f)_- = \Delta_{N_-} f_- \tag{2.8}$$

for all $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f_+ \in W^{1,2}(\mathbb{R}^n_+)$ and $f_- \in W^{1,2}(\mathbb{R}^n_-)$. Then, Δ_N is a positive definite self-adjoint operator. By (2.8)

$$\begin{aligned} &(\exp(-t\Delta_N)f)_+ = \exp(-t\Delta_{N_+})f_+ \\ \text{and} \qquad &(\exp(-t\Delta_N)f)_- = \exp(-t\Delta_{N_-})f_-. \end{aligned} \tag{2.9}$$

Let $p_{t,\,\Delta_N}(x,y)$ be the heat kernel of $\exp(-t\Delta_N)$. By (2.9) and (2.5) we obtain

$$(2.10)\quad p_{t,\,\Delta_N}(x,y)=\frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|x'-y'|^2}{4t}}\Big(e^{-\frac{|x_n-y_n|^2}{4t}}+e^{-\frac{|x_n+y_n|^2}{4t}}\Big)H(x_ny_n),$$

where $H\colon\,\mathbb{R}\to\{0,1\}$ is the Heaviside function given by

$$(2.11)\qquad H(t)=\begin{cases}0 & \text{if } t<0;\\ 1 & \text{if } t\geq 0.\end{cases}$$

Similarly we define the Dirichlet Laplacian on $\mathbb{R}^n$ by the formula

$$(2.12)\qquad (\Delta_D f)_+=\Delta_{D_+}f_+\quad\text{and}\quad(\Delta_D f)_-=\Delta_{D_-}f_-$$

for all $f\colon\mathbb{R}^n\to\mathbb{R}$ such that $f_+\in W_0^{1,2}(\mathbb{R}^n_+)$ and $f_-\in W_0^{1,2}(\mathbb{R}^n_-)$. Then, Δ_D is a positive definite self-adjoint operator. By (2.12)

$$(2.13)\qquad \begin{aligned}&(\exp(-t\Delta_D)f)_+=\exp(-t\Delta_{D_+})f_+\\ \text{and}\quad&(\exp(-t\Delta_D)f)_-=\exp(-t\Delta_{D_-})f_-.\end{aligned}$$

Hence by (2.4) the kernel $p_{t,\,\Delta_D}(x,y)$ of the operator $\exp(-t\Delta_D)$ is given by

$$(2.14)\quad p_{t,\,\Delta_D}(x,y)=\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x'-y'|^2}{4t}}\Big(e^{-\frac{|x_n-y_n|^2}{4t}}-e^{-\frac{|x_n+y_n|^2}{4t}}\Big)H(x_ny_n).$$

Finally we define the Dirichlet-Neumann Laplacian by the formula

$$(2.15)\qquad (\Delta_{DN} f)_+=\Delta_{N_+}f_+\quad\text{and}\quad(\Delta_{DN} f)_-=\Delta_{D_-}f_-$$

for all $f\colon\mathbb{R}^n\to\mathbb{R}$ such that $f_+\in W^{1,2}(\mathbb{R}^n_+)$ and $f_-\in W_0^{1,2}(\mathbb{R}^n_-)$. By (2.15)

$$(2.16)\qquad \begin{aligned}&(\exp(-t\Delta_{DN})f)_+=\exp(-t\Delta_{N_+})f_+\\ \text{and}\quad&(\exp(-t\Delta_{DN})f)_-=\exp(-t\Delta_{D_-})f_-.\end{aligned}$$

Hence by (2.4) and (2.5), the kernel $p_{t,\,\Delta_{DN}}(x,y)$ of $\exp(-t\Delta_{DN})$ is given by

(2.17)
$$p_{t,\,\Delta_{DN}}(x,y)=\frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|x'-y'|^2}{4t}}\Big(e^{-\frac{|x_n-y_n|^2}{4t}}+(2H(x_n)-1)e^{-\frac{|x_n+y_n|^2}{4t}}\Big)H(x_ny_n).$$

Let us note that

(α) All the operators Δ, Δ_{N_+}, Δ_{D_+}, Δ_{N_-}, Δ_{D_-} and Δ_D, Δ_N, Δ_{DN} are self-adjoint and they generate bounded analytic positive semigroups acting on all L^p spaces for $1\leq p\leq\infty$;

(β) Suppose that $p_{t,L}(x,y)$ is the kernel corresponding to the semigroup generated by L and that L is one of the operators listed in (α). Then the kernel $p_{t,L}(x,y)$ satisfies Gaussian bounds, that is

$$|p_{t,L}(x,y)| \leq \frac{C}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}} \tag{2.18}$$

for all $x, y \in \Omega$, where $\Omega = \mathbb{R}^n$ for Δ, Δ_D, Δ_N, Δ_{DN}; $\Omega = \mathbb{R}^n_+$ for Δ_{N_+}, Δ_{D_+} and $\Omega = \mathbb{R}^n_-$ for Δ_{N_-}, Δ_{D_-}.

(γ) If L is one of the operators Δ, Δ_{N_+}, Δ_{N_-} and Δ_N, then L conserves probability, that is

$$\exp(-tL)\mathbb{1} = \mathbb{1}.$$

This conservative property does not hold for Δ_D, Δ_{D_+}, Δ_{D_-} and Δ_{DN}.

2.3. BMO spaces associated with operators

Suppose that $\Omega \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. Suppose that L is a linear operator on $L^2(\Omega)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x,y)$ satisfying Gaussian upper bound (2.18).

We define

$$\mathcal{M}(\Omega) = \left\{ f \in L^1_{\text{loc}}(\Omega) : \exists d > 0, \quad \int_\Omega \frac{|f(x)|^2}{1+|x|^{n+d}} dx < \infty \right\}.$$

Note that in virtue of the Gaussian bounds (2.18) we can extend the action of the semigroup operators $\exp(-tL)$ to the space $\mathcal{M}(\Omega)$, that is we can define $\exp(-tL)f$ for all $f \in \mathcal{M}(\Omega)$. By $B(x,r)$ we denote the ball in Ω with respect to the Euclidean distance restricted to Ω that is

$$B(x,r) = \{y \in \Omega : \ |x-y| < r\}.$$

The following $\text{BMO}_L(\Omega)$ space associated with an operator L was introduced in [15].

Definition 2.2 *We say that $f \in \mathcal{M}(\Omega)$ is of bounded mean oscillation associated with an operator L (abbreviated as $\text{BMO}_L(\Omega)$) if*

$$\|f\|_{\text{BMO}_L(\Omega)} = \sup_{B(y,r)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(x) - \exp(-r^2L)f(x)|\, dx < \infty, \tag{2.19}$$

where the supremum is taken over all balls $B(y,r)$ in Ω. The smallest bound for which (2.19) is satisfied is then taken to be the norm of f in this space, and is denoted by $\|f\|_{\text{BMO}_L(\Omega)}$.

Remarks. (i) Note that ($\mathrm{BMO}_L(\Omega)$, $\|\cdot\|_{\mathrm{BMO}_L(\Omega)}$) is a semi-normed vector space, with the semi-norm vanishing on the kernel space $\mathcal{K}_L$ defined by

$$\mathcal{K}_L = \Big\{ f \in \mathcal{M}(\Omega) \colon \ \exp(-tL)f = f, \qquad \forall t > 0 \Big\}.$$

The class of functions of $\mathrm{BMO}_L(\Omega)$ (modulo $\mathcal{K}_L$) is a Banach space. We refer the reader to Section 6 of [16] for a discussion on the dimension of the space $\mathcal{K}_L$ of $\mathrm{BMO}_L(\mathbb{R}^n)$ when L is a second order divergence form elliptic operator or a Schrödinger operator. In the sequel By $\mathrm{BMO}_L(\Omega)$ we always denote the space $\mathrm{BMO}_L(\Omega)$ (modulo $\mathcal{K}_L$) and we skip (modulo $\mathcal{K}_L$) to simplify notation.

(ii) Similarly to the classical BMO space, it is easy to check that $L^\infty(\Omega) \subset \mathrm{BMO}_L(\Omega)$ with $\|f\|_{\mathrm{BMO}_L(\Omega)} \le 2\|f\|_{L^\infty}$.

(iii) The classical BMO space (modulo all constant functions) and the $\mathrm{BMO}_\Delta(\mathbb{R}^n)$ space (modulo all harmonic functions) coincide, and their norms are equivalent. See Theorem 2.15 of [15].

(iv) Note that the Euclidean distance in Definition 2.2 can be replaced by any equivalent distance. That is if there exists $c > 0$ such that $c^{-1}|x-y| \le d(x,y) \le c|x-y|$ then one can take in (2.19) the supremum over all balls $B^d(x,r)$ with respect to the metric d. In particular if $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n_-$, one can take the supremum over all cubes Q such that $Q \subset \Omega$ in (2.19), i.e., we can define equivalent norm in $\mathrm{BMO}_L(\Omega)$ by the formula

$$\|f\|_{\mathrm{BMO}_L(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - \exp(-l_Q^2 L) f(x)| dx < \infty, \tag{2.20}$$

where l_Q is the side length of Q and the supremum is taken over all cubes $Q \subset \Omega$.

The following proposition is essentially equivalent to Proposition 3.1 of [23].

Proposition 2.3 *Assume that for every* $t > 0$, $e^{-tL}(\mathbb{1}) = \mathbb{1}$ *almost everywhere, that is,* $\int_{\mathbb{R}^n} p_t(x,y)dy = 1$ for almost all $x \in \mathbb{R}^n$. *Then, we have* $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_L(\mathbb{R}^n)$, *and there exists a positive constant* $c > 0$ *such that*

$$\|f\|_{\mathrm{BMO}_L(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{2.21}$$

However, the converse inequality does not hold in general.

We remark that condition $e^{-tL}(\mathbb{1}) = \mathbb{1}$, is necessary for (2.21). Indeed, (2.21) implies $\|\mathbb{1}\|_{\mathrm{BMO}_L(\mathbb{R}^n)} = 0$. Hence $e^{-tL}(\mathbb{1}) = \mathbb{1}$ almost everywhere for all $t > 0$,

3. BMO spaces on the half spaces and BMO spaces associated with the Dirichlet and Neumann Laplacian

In this section we describe the equivalence between the BMO spaces on the half space and BMO spaces corresponding to the Neumann and Dirichlet Laplacian.

Proposition 3.1 (i) *The spaces* $\mathrm{BMO}_r(\mathbb{R}^n_+)$ *and* $\mathrm{BMO}_e(\mathbb{R}^n_+)$ *coincide, and their norms are equivalent.*

(ii) *The spaces* $\mathrm{BMO}_z(\mathbb{R}^n_+)$ *and* $\mathrm{BMO}_o(\mathbb{R}^n_+)$ *coincide, and their norms are equivalent.*

Proof. Following [6], for any function $f \in L^1(\mathbb{R}^n_+)$ we set

$$\|f\|_{H^1_e(\mathbb{R}^n_+)} = \|f_e\|_{H^1(\mathbb{R}^n)} \quad \text{and} \quad \|f\|_{H^1_o(\mathbb{R}^n_+)} = \|f_o\|_{H^1(\mathbb{R}^n)} \tag{3.1}$$

and by $H^1_e(\mathbb{R}^n_+)$ and $H^1_o(\mathbb{R}^n_+)$ we denote the corresponding Banach spaces. It follows from Corollaries 1.6, 1.8 of [6] and Proposition 32 of [4] that the dual space of $H^1_e(\mathbb{R}^n_+)$ is the space $\mathrm{BMO}_r(\mathbb{R}^n_+)$ and the dual space of $H^1_o(\mathbb{R}^n_+)$ is the space $\mathrm{BMO}_z(\mathbb{R}^n_+)$. See also [3].

The inclusion $\mathrm{BMO}_e(\mathbb{R}^n_+) \subseteq \mathrm{BMO}_r(\mathbb{R}^n_+)$ is obvious. Hence to prove (i) it is enough to show that $\mathrm{BMO}_r(\mathbb{R}^n_+) \subseteq \mathrm{BMO}_e(\mathbb{R}^n_+)$. Let $f \in \mathrm{BMO}_r(\mathbb{R}^n_+)$. To see that $f \in \mathrm{BMO_e}(\mathbb{R}^n_+)$, by the definition it reduces to proving $f_e \in \mathrm{BMO}(\mathbb{R}^n)$ where f_e is the even extension of f. For any $g(x) \in H^1(\mathbb{R}^n)$, we denote by $\widetilde{g}(x) = g(\widetilde{x})$ where $\widetilde{x} = (x', -x_n)$. Since $(H^1_e(\mathbb{R}^n_+))' = \mathrm{BMO}_r(\mathbb{R}^n_+)$, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} f_e(x)g(x)dx\Big| &= \Big|\int_{\mathbb{R}^n_-} f_e(x)g(x)dx + \int_{\mathbb{R}^n_+} f_e(x)g(x)dx\Big| \\
&= |\int_{\mathbb{R}^n_+} f(x)\Big(g(\widetilde{x}) + g(x)\Big)dx| \\
&\le c\|f\|_{\mathrm{BMO}_r(\mathbb{R}^n_+)}\|(\widetilde{g}+g)\|_{H^1_e(\mathbb{R}^n_+)} \le c\|f\|_{\mathrm{BMO}_r(\mathbb{R}^n_+)}\|g\|_{H^1(\mathbb{R}^n)}.
\end{aligned}$$

This shows that $\mathrm{BMO}_r(\mathbb{R}^n_+) \subset \mathrm{BMO_e}(\mathbb{R}^n_+)$, and proves (i).

We now prove (ii). The inclusion $\mathrm{BMO}_z(\mathbb{R}^n_+) \subseteq \mathrm{BMO}_o(\mathbb{R}^n_+)$ is obvious. Let $f \in \mathrm{BMO}_o(\mathbb{R}^n_+)$ and thus $f_o \in \mathrm{BMO}(\mathbb{R}^n)$. To see that $f \in \mathrm{BMO}_z(\mathbb{R}^n_+)$, it reduces to proving $f \in (H^1_o(\mathbb{R}^n_+))'$ since $\mathrm{BMO}_z(\mathbb{R}^n_+) = (H^1_o(\mathbb{R}^n_+))'$. If $g \in H^1_o(\mathbb{R}^n_+)$, then $g_o \in H^1(\mathbb{R}^n)$. Hence

$$\begin{aligned}
|\int_{\mathbb{R}^n_+} f(x)g(x)dx\Big| &= \frac{1}{2}\Big|\int_{\mathbb{R}^n} f_o(x)g_o(x)dx\Big| \\
&\le c\|f_o\|_{\mathrm{BMO}(\mathbb{R}^n)}\|g_o\|_{H^1(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)}\|g\|_{H^1(\mathbb{R}^n_+)}.
\end{aligned}$$

This shows that $\mathrm{BMO}_o(\mathbb{R}^n_+) \subset \mathrm{BMO_z}(\mathbb{R}^n_+)$, and proves (ii). ∎

We use Proposition 3.1 to obtain the following result.

Proposition 3.2 (i) *The spaces* $\mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)$, $\mathrm{BMO}_z(\mathbb{R}^n_+)$ *and* $\mathrm{BMO}_o(\mathbb{R}^n_+)$ *coincide, and their norms are equivalent.*

(ii) *The spaces* $\mathrm{BMO}_{\Delta_{N_+}}(\mathbb{R}^n_+)$, $\mathrm{BMO}_r(\mathbb{R}^n_+)$ *and* $\mathrm{BMO}_e(\mathbb{R}^n_+)$ *coincide, and their norms are equivalent.*

Proof. We first prove (i). Let $f \in \mathrm{BMO}_z(\mathbb{R}^n_+)$. By Proposition 3.1 we have that $f \in \mathrm{BMO}_o(\mathbb{R}^n_+)$ and then $f_o \in \mathrm{BMO}(\mathbb{R}^n)$. To prove $f \in \mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)$, it suffices to show that for any cube $Q \subseteq \mathbb{R}^n_+$,

$$\int_Q |f(x) - e^{-l_Q^2\Delta_{D_+}} f(x)|dx \le c|Q|\|f\|_{\mathrm{BMO}_z(\mathbb{R}^n_+)}. \tag{3.2}$$

By (2.21) and Propositions 3.1

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x) - e^{-l_Q^2\Delta_{D_+}} f(x)|dx &= \frac{1}{|Q|}\int_Q |f(x) - e^{-l_Q^2\Delta} f_o(x)|dx \\ &\le c\|f_o\|_{\mathrm{BMO}(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}_o(\mathbb{R}^n_+)} \\ &\le c\|f\|_{\mathrm{BMO}_z(\mathbb{R}^n_+)}.\end{aligned}$$

This proves (3.2).

Next assume that $f \in \mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)$. By Proposition 3.1, $f \in \mathrm{BMO}_z(\mathbb{R}^n_+)$ or equivalently $f_o \in \mathrm{BMO}(\mathbb{R}^n)$. Note that by (2.6) it is enough to prove that for any cube $Q \subseteq \mathbb{R}^n$,

$$\int_Q |f_o(x) - e^{-l_Q^2\Delta} f_o(x)|dy \le c\,|Q|\;\|f\|_{\mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)}. \tag{3.3}$$

We now verify (3.3). Let us examine the cubes Q.

Case 1: If $Q \subseteq \mathbb{R}^n_-$, then for any $x \in Q$,

$$-\exp(-l_Q^2\Delta_{D_+})f(\widetilde{x}) = \exp(-l_Q^2\Delta)f_o(x)$$

and $\widetilde{x} \in \widehat{Q} \subseteq \mathbb{R}^n_+$ (here $\widetilde{Q}$ is a cube defined in (2.2)). Note also that $|\widetilde{Q}| = |Q|$. Hence

$$\begin{aligned}\int_Q |f_o(x) - e^{-l_Q^2\Delta} f_o(x)|dx &= \int_{\widetilde{Q}} |f_o(\widetilde{x}) - e^{-l_{\widetilde{Q}}^2\Delta_{D_+}} f(\widetilde{x})|dx \\ &\le c|Q|\|f\|_{\mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)}.\end{aligned}$$

Case 2: If $Q\cap\mathbb{R}^n_-\neq\emptyset$ and $Q\cap\mathbb{R}^n_+\neq\emptyset$, then let $\widehat{Q}_-$ and $\widehat{Q}_+$ be the two cubes as in (2.3). By (2.6) and Proposition 3.1,

$$\int_Q |f_o(x)-e^{-l_Q^2\Delta}f_o(x)|dx=\int_{Q_-\cup Q_+}|f_o(x)-e^{-l_Q^2\Delta}f_o(x)|dx$$
$$\leq 2\int_{\widehat{Q}_+}|f(x)-e^{-l_Q^2\Delta_{D+}}f(x)|dx\leq 2|Q|\|f\|_{\mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)}.$$

Case 3: If $Q\subseteq\mathbb{R}^n_+$, then $e^{-l_Q^2\Delta}f_o(x)=e^{-l_Q^2\Delta_{D+}}f(x)$ for any $x\in Q$. Hence

$$\int_Q |f_o(x)-e^{-l_Q^2\Delta}f_o(x)|dx\leq |Q|\|f\|_{\mathrm{BMO}_{\Delta_{D_+}}(\mathbb{R}^n_+)}.$$

The estimate (3.3) follows readily. This shows that $f_o\in\mathrm{BMO}(\mathbb{R}^n)$ so $f\in\mathrm{BMO}_z(\mathbb{R}^n_+)$.

The proof of (ii) is similar to the proof of (i) so we skip it. ■

In a similar way as for the upper-half space, we can define the space $\mathrm{BMO}_{\Delta_{D_-}}(\mathbb{R}^n_-)$ and $\mathrm{BMO}_{\Delta_{N_-}}(\mathbb{R}^n_-)$ associated with the Dirichlet and Neumann Laplacian Δ_{D_-}, Δ_{N_-} on the lower-half space $\mathbb{R}^n_-$.

The same argument as in Proposition 3.2 gives the following proposition. We leave the proof to the reader.

Proposition 3.3 (i) *The spaces* $\mathrm{BMO}_{\Delta_{D_-}}(\mathbb{R}^n_-), \mathrm{BMO}_z(\mathbb{R}^n_-)$ *and* $\mathrm{BMO}_o(\mathbb{R}^n_-)$ *coincide, and their norms are equivalent.*

(ii) *The spaces* $\mathrm{BMO}_{\Delta_{N_-}}(\mathbb{R}^n_-)$, $\mathrm{BMO}_r(\mathbb{R}^n_-)$ *and* $\mathrm{BMO}_e(\mathbb{R}^n_-)$ *coincide, and their norms are equivalent.*

4. Comparison between the classical BMO and the new BMO spaces associated with operators

In the introduction we mention that all cases of relation between the classical BMO and the new BMO spaces are possible. The following theorem provides simple example to prove this statement.

Theorem 4.1 *In the notation described above the following inclusions hold*

$$\mathrm{BMO}_{\Delta_D}(\mathbb{R}^n)\subsetneqq\mathrm{BMO}(\mathbb{R}^n)\subsetneqq\mathrm{BMO}_{\Delta_N}(\mathbb{R}^n). \tag{4.1}$$

That is, the classical BMO *space is a proper subspace of* $\mathrm{BMO}_{\Delta_N}(\mathbb{R}^n)$*, and* $\mathrm{BMO}_{\Delta_D}(\mathbb{R}^n)$ *is a proper subspace of* BMO.

Moreover, we have

$$\mathrm{BMO}(\mathbb{R}^n)\not\subseteq\mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n)\quad\textit{and}\quad\mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n)\not\subseteq\mathrm{BMO}(\mathbb{R}^n). \tag{4.2}$$

The proof of Theorem 4.1 is based on the following proposition.

Proposition 4.2 *The* BMO *spaces corresponding to the operators* Δ_N*,* Δ_D *and* Δ_{ND} *can be described in the following way:*

$$\mathrm{BMO}_{\Delta_N}(\mathbb{R}^n) = \Big\{ f \in \mathcal{M}(\mathbb{R}^n) : f_+ \in \mathrm{BMO}_r(\mathbb{R}^n_+) \text{ and } f_- \in \mathrm{BMO}_r(\mathbb{R}^n_-) \Big\};$$
$$\mathrm{BMO}_{\Delta_D}(\mathbb{R}^n) = \Big\{ f \in \mathcal{M}(\mathbb{R}^n) : f_+ \in \mathrm{BMO}_z(\mathbb{R}^n_+) \text{ and } f_- \in \mathrm{BMO}_z(\mathbb{R}^n_-) \Big\};$$
$$\mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n) = \Big\{ f \in \mathcal{M}(\mathbb{R}^n) : f_+ \in \mathrm{BMO}_r(\mathbb{R}^n_+) \text{ and } f_- \in \mathrm{BMO}_z(\mathbb{R}^n_-) \Big\}.$$

Proof. In the following proof L is one of the operators Δ_N, Δ_D or Δ_{DN}. If $L = \Delta_N$, then we denote by $L_+ = \Delta_{N_+}$ and $L_- = \Delta_{N_-}$. Similarly if $L = \Delta_D$ then $L_+ = \Delta_{D_+}$ and $L_- = \Delta_{D_-}$. Finally for $L = \Delta_{DN}$ we let $L_+ = \Delta_{N_+}$ and $L_- = \Delta_{D_-}$.

By (2.9), (2.13) and (2.16)

$$(\exp(-tL)f)_+ = \exp(-tL_+)f_+ \text{ and } (\exp(-tL)f)_- = \exp(-tL_-)f_- \tag{4.3}$$

for any of the three considered operators. Hence for any cube $Q \subset \mathbb{R}^n$ we have

$$\begin{aligned} \int_Q |f - e^{-l_Q^2 L} f(x)|dx = &\int_{Q\cap\mathbb{R}^n_-} |f_- - e^{-l_Q^2 L_-} f_-(x)|dx \\ &+ \int_{Q\cap\mathbb{R}^n_+} |f_+ - e^{-l_Q^2 L_+} f_+(x)|dx. \end{aligned} \tag{4.4}$$

In virtue of Propositions 3.2 and 3.3 it is enough to show that

$$\mathrm{BMO}_L(\mathbb{R}^n) = \Big\{ f \in \mathcal{M}(\mathbb{R}^n) : \ f_+ \in \mathrm{BMO}_{L_+}(\mathbb{R}^n_+) \text{ and } f_- \in \mathrm{BMO}_{L_-}(\mathbb{R}^n_-) \Big\}.$$

Assume now that $f \in \mathcal{M}(\mathbb{R}^n)$ such that $f_- \in \mathrm{BMO}_{L_-}(\mathbb{R}^n_-)$ and $f_+ \in \mathrm{BMO}_{L_+}(\mathbb{R}^n_+)$. In order to prove $f \in \mathrm{BMO}_L(\mathbb{R}^n)$, it suffices to prove that for any cube $Q \subseteq \mathbb{R}^n$,

$$\int_Q |f(x) - e^{-l_Q^2 L} f(x)|dy \le c|Q|\Big(\|f_-\|_{\mathrm{BMO}_{L_-}(\mathbb{R}^n_-)} + \|f_+\|_{\mathrm{BMO}_{L_+}(\mathbb{R}^n_+)}\Big).$$

As in the proof of Proposition 3.2, we consider the following three cases of Q.

Case 1: If $Q \subseteq \mathbb{R}^n_-$, then by (4.4)

$$\begin{aligned} \int_Q |f(x) - e^{-l_Q^2 L} f(x)|dx &= \int_Q |f_-(x) - e^{-l_Q^2 L_-} f_-(x)|dx \\ &\le c|Q|\|f_-\|_{\mathrm{BMO}_{L_-}(\mathbb{R}^n_-)}. \end{aligned}$$

Case 2: If $Q \cap \mathbb{R}^n_- \neq \emptyset$ and $Q \cap \mathbb{R}^n_+ \neq \emptyset$, then let $\widetilde{Q}_-$ and $\widetilde{Q}_+$ be the cubes as in (2.3). By (4.4)

$$\begin{aligned}\int_Q |f(x)-e^{-l_Q^2 L}f(x)|dx &= \int_{Q_- \cup Q_+} |f(x) - e^{-l_Q^2 L}f(x)|dx \\ &\leq \int_{\widetilde{Q}_-} |f_-(x) - e^{-l_Q^2 L_-} f_-(x)|dx + \int_{\widetilde{Q}_+} |f_+(x) - e^{-l_Q^2 L_+} f_+(x)|dx \\ &\leq c|Q|\Big(\|f_-\|_{\mathrm{BMO}_{L_-}(\mathbb{R}^n_-)} + \|f_+\|_{\mathrm{BMO}_{L_+}(\mathbb{R}^n_+)}\Big).\end{aligned}$$

Case 3: If $Q \subseteq \mathbb{R}^n_-$, then by (4.4)

$$\begin{aligned}\int_Q |f(x) - e^{-l_Q^2 L}f(x)|dx &= \int_Q |f_+(x) - e^{-l_Q^2 L_+} f_+(x)|dx \\ &\leq c|Q|\|f_+\|_{\mathrm{BMO}_{\Delta_{N_+}}(\mathbb{R}^n_+)}.\end{aligned}$$

Hence $f \in \mathrm{BMO}_L(\mathbb{R}^n)$.

We now assume that $f \in \mathrm{BMO}_L(\mathbb{R}^n)$. By (4.4), we have that

$$f_- \in \mathrm{BMO}_{L_-}(\mathbb{R}^n_-) \quad \text{and} \quad f_+ \in \mathrm{BMO}_{\Delta_{N_+}}(\mathbb{R}^n_+)\,.$$

Now Proposition 4.2 is a straightforward consequence of Propositions 3.2 and 3.3. ■

The logarithmic function is a simple example that typifies some of the essential properties of the classical space BMO. For example if we define function log: $\mathbb{R}^n \to \mathbb{R}$ by the formula $\log^e(x) = \log|x_n|$ for all $x \in \mathbb{R}^n$ and $\mathrm{Log}(x) = H(x_n)\log|x_n|$, where H is the Heaviside function then

$$\begin{aligned}&\log^e \in \mathrm{BMO}(\mathbb{R}^n)\\ &\mathrm{Log} \notin \mathrm{BMO}(\mathbb{R}^n).\end{aligned} \tag{4.5}$$

See, for examples, Chapter IV of [29] and page 217 of [31] . We will use the property (4.5) in the proof of Theorem 4.1

Proof of Theorem 4.1. It is a straightforward consequence of Definition 2.1 that if $f_+ \in \mathrm{BMO}_z(\mathbb{R}^n_+)$ and $f_- \in \mathrm{BMO}_z(\mathbb{R}^n_-)$ then $f \in \mathrm{BMO}$. It also follows from Definition 2.1 that if $f \in \mathrm{BMO}$ then $f_+ \in \mathrm{BMO}_r(\mathbb{R}^n_+)$ and $f_- \in \mathrm{BMO}_r(\mathbb{R}^n_-)$. Hence it follows from Theorem 4.1 and Propositions 3.2 and 3.3 that

$$\mathrm{BMO}_{\Delta_D}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_{\Delta_N}(\mathbb{R}^n).$$

To prove that the above inclusions are proper we note that by (4.5) and Definition 2.1

$$\log_+ \notin \mathrm{BMO}_z(\mathbb{R}^n_+) \quad \text{and} \quad \log_+ \in \mathrm{BMO}_r(\mathbb{R}^n_+),$$

where $\log_+$ is the restriction of $\log^e$ to $\mathbb{R}^n_+$. Next if $\log_-$ is the restriction of $\log^e$ to $\mathbb{R}^n_-$ then

$$\log_- \notin \mathrm{BMO}_z(\mathbb{R}^n_-) \quad \text{and} \quad \log_- \in \mathrm{BMO}_r(\mathbb{R}^n_-).$$

Hence

$$\log^e \in \mathrm{BMO} \quad \text{and} \quad \log^e \notin \mathrm{BMO}_{\Delta_D}(\mathbb{R}^n).$$

Similarly

$$\mathrm{Log} \notin \mathrm{BMO} \quad \text{and} \quad \mathrm{Log} \in \mathrm{BMO}_{\Delta_N}(\mathbb{R}^n)$$

This ends the proof of (4.1). Finally to prove (4.2) we note that $\mathrm{Log} \in \mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n)$ and $\log \notin \mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n)$. ■

Remark. Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with kernels $p_t(x,y)$ satisfying upper bound (2.18). Under the additional condition that the kernel $p_t(x,y)$ of e^{-tL} has sufficient regularities on space variables x, y and $e^{-tL}(\mathbb{1}) = e^{-tL^*}(\mathbb{1}) = \mathbb{1}$, it can be proved that classical space BMO and the space $\mathrm{BMO}_L(\mathbb{R}^n)$ spaces coincide, and their norms are equivalent. See Section 6 of [16].

Next we discuss the duality of the Hardy and BMO spaces associated with operators. Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with kernels $p_t(x,y)$ satisfying Gaussian upper bound (2.18). For any $(x,t) \in \mathbb{R}^n \times (0,\infty)$, we define

$$Q_t f(x) = -t\frac{d}{dt}e^{-tL}f(x) = tLe^{-tL}f(x)$$

for any $f \in \mathcal{M}$. Following [2], given a function $f \in L^1(\mathbb{R}^n)$, the area integral function $\mathcal{S}_L(f)$ associated with an operator L is defined by

$$\mathcal{S}_L f(x) = \Big(\int_0^\infty\!\!\int_{|y-x|<t} \big|Q_{t^2} f(y)\big|^2 \,\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}, \qquad x \in \mathbb{R}^n.$$

The following definition was introduced in [2]. We say that $f \in L^1(\mathbb{R}^n)$ belongs to a Hardy space associated with L (abbreviated as $H^1_L(\mathbb{R}^n)$) if $\mathcal{S}_L f \in L^1$. If it is the case, we define its norm by

$$\|f\|_{H^1_L(\mathbb{R}^n)} = \|\mathcal{S}_L f\|_{L^1}.$$

Note that if $L = \Delta$ is the Laplacian on $\mathbb{R}^n$, then the classical Hardy space H^1 and H^1_Δ coincide, and their norms are equivalent. See [2].

Under the assumptions that L satisfies Gaussian upper bound (2.18) and has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$, it was proved in [16] that the dual space of the $H^1_L(\mathbb{R}^n)$ space is the $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$ space in which L^* is the adjoint operator of L.

Note that the operators Δ_D, Δ_N and Δ_{DN} are self-adjoint operators, hence each of them has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$. See [25]. We thus have the following corollary.

Corollary 4.3 (i) *The dual space of* $H^1_\Delta(\mathbb{R}^n)$ *is the space* $\mathrm{BMO}_\Delta(\mathbb{R}^n)$.

(ii) *The dual spaces of* $H^1_{\Delta_D}(\mathbb{R}^n)$, $H^1_{\Delta_N}(\mathbb{R}^n)$ *or* $H^1_{\Delta_{DN}}(\mathbb{R}^n)$ *are the spaces* $\mathrm{BMO}_{\Delta_D}(\mathbb{R}^n)$, $\mathrm{BMO}_{\Delta_N}(\mathbb{R}^n)$ *or* $\mathrm{BMO}_{\Delta_{DN}}(\mathbb{R}^n)$, *respectively.*

(iii) *For the Neumann Laplacian* Δ_N *on* $\mathbb{R}^n$, *we have that* $H^1_{\Delta_N}(\mathbb{R}^n) \subsetneqq H^1(\mathbb{R}^n)$ *and* $H^1_{\Delta_N}(\mathbb{R}^n) \neq \emptyset$. *That is,* $H^1_{\Delta_N}(\mathbb{R}^n)$ *is a proper subspace of the classical Hardy space* $H^1(\mathbb{R}^n)$.

Remark. In [35], it was asked if a proper subspace of the classical Hardy space exists in which the subspace is characterized by maximal functions. This question was answered positively in [33]. Our result (iii) of Corollary 4.3 gives a proper subspace of the classical Hardy space where the subspace is characterized by area integral functions.

5. Fractional powers $L^{-\alpha/2}$ and the space $\mathrm{BMO}_L(\mathbb{R}^n)$

5.1. Boundedness of fractional powers $L^{-\alpha/2}$

For any $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-tL} f(x) dt. \tag{5.1}$$

We assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (2.18) so $|L^{-\alpha/2} f(x)| \le c\mathcal{I}_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$, where

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad 0 < \alpha < n,$$

is the classical fractional powers of the Laplacian Δ on $\mathbb{R}^n$.

Let us recall that the semigroup $\{\exp(-tL)\colon t > 0\}$ acting on $L^p(\mathbb{R}^n)$ is equicontinuous on $L^p(\mathbb{R}^n)$ if $\sup_{t>0} \|e^{-tL}\|_{L^p \to L^p} < \infty$. Note that all the semigroups which we consider here are equicontinuous on all $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. In the sequel we need the following Hardy-Littlewood-Sobolev theorem. See [34, Theorem II.2.7, page 12].

Proposition 5.1 *Suppose that* e^{-tL} *is a semigroup which is equicontinuous on* $L^1(\mathbb{R}^n)$ *and* $L^\infty(\mathbb{R}^n)$. *Also suppose that*

$$p_t(x,x) \le t^{-n/2}.$$

Then for $0 < \alpha < n$,

(i) *for* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *we have*

$$\|L^{-\alpha/2} f\|_{L^q} \le c_{p,q} \|f\|_{L^p};$$

(ii) $L^{-\alpha/2}$ *is of weak-type* $(1,q)$, *that is, for any* $\lambda > 0$, *we have*

$$\Big|\{x : |L^{-\alpha/2} f(x)| > \lambda\}\Big| \le c\Big(\frac{\|f\|_{L^1}}{\lambda}\Big)^q,$$

where $q = (1 - \frac{\alpha}{n})^{-1}$.

Let us consider the limiting case $q = \infty$ in Proposition 5.1. It is well-known that for every $f \in L^{n/\alpha}(\mathbb{R}^n)$, either $\mathcal{I}_\alpha f \equiv \infty$ or $\mathcal{I}_\alpha f \in \mathrm{BMO}(\mathbb{R}^n)$ with

$$\|\mathcal{I}_\alpha f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le c\|f\|_{L^{n/\alpha}}, \tag{5.2}$$

see [31, page 221].

An example of $\mathcal{I}_\alpha f \equiv \infty$ is given by $f(x) = |x|^{-\alpha} \log^{-1}|x| \chi_{\{x:|x|\ge 2\}}$. The following result generalizes estimates (5.2).

Theorem 5.2 *Assume that the semigroup* e^{-tL} *has a kernel* $p_t(x,y)$ *which satisfies the upper bound* (2.18). *If* $f \in L^{n/\alpha}(\mathbb{R}^n)$ *and* $L^{-\alpha/2} f < \infty$ *almost everywhere, then* $L^{-\alpha/2} f \in \mathrm{BMO}_L(\mathbb{R}^n)$ *with*

$$\|L^{-\alpha/2} f\|_{\mathrm{BMO}_L(\mathbb{R}^n)} \le c\|f\|_{n/\alpha}$$

for $0 < \alpha < n$, *where the positive constant* c *depends only on* α *and* n.

Suppose that T is a bounded operator on $L^2(\Omega)$. We say that a measurable function $K_T \colon \Omega^2 \to \mathbb{C}$ is the (singular) kernel of T if

$$\langle Tf_1, f_2\rangle = \int_\Omega Tf_1(x)\overline{f_2}(x)dx = \int_\Omega \int_\Omega K_T(x,y) f_1(y)\overline{f_2(x)}dxdy. \tag{5.3}$$

for all $f_1, f_2 \in C_c(\Omega)$ (for all $f_1, f_2 \in C_c(\Omega)$ such that supp $f_1 \cap$ supp $f_2 = \emptyset$ respectively).

In order to prove Theorem 5.2, we need the following estimate on the kernel $K_{\alpha,t}(x,y)$ of the operator $(I - e^{-tL})L^{-\alpha/2}$ (see also [17, Lemma 3.1]).

Lemma 5.3 *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies upper bound (2.18). Then for $0<\alpha<n$, the difference operator $(I-e^{-tL})L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x,y)$ which satisfies*

$$|K_{\alpha,t}(x,y)| \le \frac{c}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^2} \quad \textit{for some constant } c>0. \tag{5.4}$$

Proof. Note that

$$I-e^{-tL}=\int_0^t \frac{d}{dr}e^{-rL}dr=-\int_0^t Le^{-rL}dr.$$

Hence by (5.1)

$$(I-e^{-tL})L^{-\alpha/2}=\frac{1}{\Gamma(\alpha/2)}\int_0^t\int_0^\infty \Big(v\frac{d}{du}e^{-vL}\Big)\Big|_{v=r+s}\frac{1}{r+s}\frac{dsdr}{s^{-\alpha/2+1}}.$$

By Lemma 2.5 of [9], the kernel of the operator $v\frac{d}{dv}e^{-vL}$ has Gaussian upper bound (2.18). Hence, the operator $(I-e^{-tL})L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x,y)$ which satisfies

$$\begin{aligned}
|K_{\alpha,t}(x,y)| &\le c\int_0^t\int_0^\infty \frac{1}{(r+s)^{n/2}}\,e^{-c_1\frac{|x-y|^2}{r+s}}\,\frac{1}{r+s}\frac{dsdr}{s^{-\alpha/2+1}}\\
&\le c\int_0^t\int_0^r \frac{1}{(r+s)^{n/2}}\,e^{-c_1\frac{|x-y|^2}{r+s}}\,\frac{1}{r+s}\frac{dsdr}{s^{-\alpha/2+1}}\\
&\quad +c\int_0^t\int_r^\infty \frac{1}{(r+s)^{n/2}}\,e^{-c_1\frac{|x-y|^2}{r+s}}\,\frac{1}{r+s}\frac{dsdr}{s^{-\alpha/2+1}}\\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}$$

Let us estimate term I. Note that $0<s<r$. We have

$$\begin{aligned}
\mathrm{I} &\le c\int_0^t\int_0^r r^{-n/2}e^{-c_2\frac{|x-y|^2}{r}}\frac{dsdr}{rs^{-\alpha/2+1}}\\
&= \frac{c}{|x-y|^{n-\alpha}}\int_0^{t/|x-y|^2} r^{(\alpha-n-2)/2}e^{-c_2r^{-1}}dr \le \frac{c}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^2},
\end{aligned}$$

where the last inequality follows from $r^{(\alpha-n-2)/2}e^{-cr^{-1}}\le c$ for some positive constant c. On the other hand, using the condition $0<\alpha<n$ we obtain

$$\begin{aligned}
\mathrm{II}\le c\int_0^t\int_r^\infty s^{-\frac{n}{2}}e^{-c_2\frac{|x-y|^2}{s}}\frac{dsdr}{s^{-\alpha/2+2}} &\le \frac{ct}{|x-y|^{n+2-\alpha}}\int_0^\infty s^{(\alpha-n-4)/2}e^{-c_2s^{-1}}ds\\
&\le \frac{c}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^2}.
\end{aligned}$$

Therefore, condition (5.4) is satisfied and the proof of Lemma 5.3 is complete. ∎

Proof of Theorem 5.2. In virtue of the definition of $\mathrm{BMO}_L(\mathbb{R}^n)$, it suffices to prove there exists a constant $C > 0$ such that that for any ball $B(x,r)$ with radius r centered at x

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |(I-e^{-r^2L})L^{-\alpha/2}f(y)|dy \le C\|f\|_{L^{n/\alpha}} \tag{5.5}$$

for all $f \in L^{n/\alpha}(\mathbb{R}^n)$. Set $f_1(y) = f(y)$ if $|x-y| \le 2r$ and $f_1(y) = 0$ otherwise. Next, put $f_2 = f - f_1$. Note that

$$\begin{aligned}
&\frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{-\alpha/2}f(y)|dy\\
&\le \frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{-\alpha/2}f_1(y)|dy + \frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{-\alpha/2}f_2(y)|dy\\
&= \mathrm{I} + \mathrm{II},
\end{aligned}$$

where $|B| = |B(x,r)|$. To estimate the first term note that, by Hölder's inequality $\|f_1\|_{L^p} \le c|B(x,r)|^{1/p-\alpha/n}\|f\|_{L^{n/\alpha}}$. for all $1 < p < n/\alpha$. Next, set $1/q = 1/p - \alpha/n$. By Proposition 5.1

$$\begin{aligned}
\mathrm{I} &\le \frac{1}{|B|^{1/q}}\|(I-e^{-r^2L})L^{-\alpha/2}f_1\|_{L^q} \le c\frac{1}{|B|^{1/q}}\|L^{-\alpha/2}f_1\|_{L^q}\\
&\le c\frac{1}{|B|^{1/q}}\|f_1\|_{L^p} \le c\|f\|_{L^{n/\alpha}}.
\end{aligned}$$

To estimate the second term note that if $y \in B(x,r)$, then by Lemma 5.3

$$\begin{aligned}
\left|(I-e^{-r^2L})L^{-\alpha/2}f_2(y)\right| &\le \int_{B(x,2r)^c} |K_{\alpha,r^2}(y,z)||f(z)|dz\\
&\le c\sum_{k=1}^{\infty}\int_{2^kr\le|x-z|<2^{k+1}r} \frac{1}{|x-z|^{n-\alpha}}\frac{r^2}{|x-z|^2}|f(z)|dz\\
&\le c\sum_{k=1}^{\infty} 2^{-2k}\frac{1}{|B(x,r2^{k+1})|^{1-\alpha/n}}\int_{B(x,r2^{k+1})} |f(z)|dz\\
&\le c\sum_{k=1}^{\infty} 2^{-2k}\|f\|_{L^{n/\alpha}} \le c\|f\|_{L^{n/\alpha}}.
\end{aligned}$$

Combining the above estimates, we obtain (5.5). ■

Remarks. (i) Under the extra assumption that for each $t > 0$, the kernel $p_t(x,y)$ of e^{-tL} is a Hölder continuous function in x, it can be proved that for $f \in L^{n/\alpha}(\mathbb{R}^n)$, either $L^{-\alpha/2}f \equiv \infty$ or $L^{-\alpha/2}f \in \mathrm{BMO}_L(\mathbb{R}^n)$ with

$$\|L^{-\alpha/2}f\|_{\mathrm{BMO}_L(\mathbb{R}^n)} \le c\|f\|_{L^{n/\alpha}}.$$

We leave the details of the proof to the reader.

(ii) We now give a list of examples of operators L satisfying the assumptions in Proposition 5.1 and Theorem 5.2.

(α) The operator Δ_N, Δ_D or Δ_{DN} as in Section 2.3;

(β) Let $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a nonnegative function on $\mathbb{R}^n$ $(n \geq 3)$. The Schrödinger operator with potential V is defined by

$$L = -\Delta + V(x) \quad \text{ on } \mathbb{R}^n. \tag{5.6}$$

From the Feynman-Kac formula, it is well-known that the kernels $p_t(x, y)$ of the semigroup e^{-tL} satisfy the estimate

$$0 \leq p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} \, e^{-\frac{|x-y|^2}{4t}}. \tag{5.7}$$

However, unless V satisfies additional conditions, the heat kernel can be a discontinuous function of the space variables and the Hölder continuous estimates may fail to hold. See, for example, [11].

We note that the corresponding result in Theorem 1 of [18] is a special case of Theorem 5.2.

(γ) Let $A = (a_{ij}(x))_{1\leq i,j\leq n}$ be an $n \times n$ matrix with complex entries $a_{ij} \in L^\infty(\mathbb{R}^n)$ satisfying $\lambda|\xi|^2 \leq \text{Re} \sum a_{ij}(x)\xi_i\xi_j$ for all $x \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Let T be the divergence form operator

$$Lf \equiv -\text{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

It is known that Gaussian bound (2.18) on the heat kernel e^{-tL} is true when A has real entries, or when $n = 1, 2$ in the case of complex entries. See, for example, [5].

5.2. Properties of fractional powers of Neumann Laplacian on $\mathbb{R}$

The following example complements Theorems 4.1 and 5.2. It also provides a convincing justification of introduction of the BMO_L spaces.

Example 5.4 *Let Δ_N be the Neumann Laplacian on $\mathbb{R}$. Then, there exists a function $f \in L^{1/\alpha}(\mathbb{R})$ such that $\Delta_N^{-\alpha/2} f(x) < \infty$ for almost every $x \in \mathbb{R}$, $\Delta_N^{-\alpha/2} f \in \text{BMO}_{\Delta_N}(\mathbb{R})$ and*

$$\|\Delta_N^{-\alpha/2} f\|_{\text{BMO}_{\Delta_N}(\mathbb{R})} \leq c\|f\|_{L^{n/\alpha}}. \tag{5.8}$$

However, $\Delta_N^{-\alpha/2} f \notin \text{BMO}(\mathbb{R})$.

Proof. For any $0<\alpha<1$, we let

$$f(x)=-\frac{1}{x^\alpha \log x}\chi_{\{0<x\le 1/2\}}(x). \tag{5.9}$$

Then

$$\int_{\mathbb{R}}|f(y)|^{1/\alpha}dy=\int_0^{1/2}\frac{1}{y(\log y^{-1})^{1/\alpha}}dy=(1-\alpha)\alpha^{-1}\big(\log 2\big)^{1/\alpha-1}<\infty.$$

This proves that $f\in L^{1/\alpha}(\mathbb{R})$. It can be verified that $\mathcal{I}_\alpha f(x)<\infty$ a.e.. Also, we have that $\Delta_N^{-\alpha/2}f<\infty$ a.e.. Hence,

(a) $\mathcal{I}_\alpha f\in \mathrm{BMO}(\mathbb{R})$ with $\|\mathcal{I}_\alpha f\|_{\mathrm{BMO}(\mathbb{R})}\le c\|f\|_{L^{n/\alpha}}$. See [31, page 221].

(b) By Theorem 5.2, we have that $\Delta_N^{-\alpha/2}f\in \mathrm{BMO}_{\Delta_N}$ with estimate (5.8).

We now prove $\Delta_N^{-\alpha/2}f\notin \mathrm{BMO}(\mathbb{R})$. Denote by $k_\alpha^N(x,y)$ the kernel of the fractional powers $\Delta_N^{-\alpha/2}$ of Δ_N. By (2.10) and (5.1)

$$k_\alpha^N(x,y)=\frac{1}{\gamma(\alpha)}\Big(\frac{1}{|x-y|^{1-\alpha}}+\frac{1}{|x+y|^{1-\alpha}}\Big)H(xy), \tag{5.10}$$

where H is the Heaviside function (2.11). By (5.10)

$$\Delta_N^{-\alpha/2}f(x)=\begin{cases}0 & \text{if } x\le 0;\\ \mathcal{I}_\alpha(f_e)(x) & \text{if } x>0,\end{cases} \tag{5.11}$$

where $f_e\in L^{1/\alpha}(\mathbb{R})$ is given by the formula $f_e(x)=-\frac{1}{|x|^\alpha\log|x|}\chi_{\{|x|\le 1/2\}}(x)$.

For any $k\ge 5$, we denote $Q_k=[-1/k,1/k]$. Next if $0<x<y<1/2$, then $|x-y|<|y|$. Hence

$$\begin{aligned}\Delta_N^{-\alpha/2}f(x) &= \frac{1}{\gamma(\alpha)}\int_{-1/2}^{1/2}\frac{1}{|x-y|^{1-\alpha}}f_e(y)dy\\ &\ge -\frac{1}{\gamma(\alpha)}\int_x^{1/2}\frac{1}{|x-y|^{1-\alpha}}\frac{1}{y^\alpha\log y}dy\\ &\ge -\frac{1}{\gamma(\alpha)}\int_x^{1/2}\frac{1}{y\log y}dy\\ &\ge \frac{1}{\gamma(\alpha)}\Big(\log(\log\frac{1}{x})-\log(\log 2)\Big),\end{aligned}$$

which yields

$$\begin{aligned}m_{Q_k}(\Delta_N^{-\alpha/2}f) &= \frac{1}{|Q_k|}\int_{Q_k}\Delta_N^{-\alpha/2}f(y)dy\\ &\ge \frac{k}{2\gamma(\alpha)}\int_0^{1/k}\Big(\log(\log\frac{1}{y})-\log(\log 2)\Big)dy\\ &\ge \frac{1}{2\gamma(\alpha)}\Big(\log(\log k)-\log(\log 2)\Big).\end{aligned}$$

Therefore, from (5.11) we obtain

$$\begin{aligned}
\frac{1}{|Q_k|} & \int_{Q_k} |\Delta_N^{-\alpha/2} f(x) - m_{Q_k}(\Delta_N^{-\alpha/2} f)|dx \\
&= \frac{k}{2}\int_0^{1/k} |\Delta_N^{-\alpha/2} f(x) - m_{Q_k}(\Delta_N^{-\alpha/2} f)|dx + \frac{k}{2}\int_{-1/k}^0 |m_{Q_k}(\Delta_N^{-\alpha/2} f)|dx \\
&\geq \frac{1}{2}|m_{Q_k}(\Delta_N^{-\alpha/2} f)| \\
&\geq \frac{1}{4\gamma(\alpha)}\Big(\log(\log k) - \log(\log 2)\Big).
\end{aligned}$$

Note that the last term in the above inequality tends to ∞ as $k \to \infty$. Hence

$$\sup_Q \frac{1}{|Q|}\int_Q |\Delta_N^{-\alpha/2} f(x) - m_Q(\Delta_N^{-\alpha/2} f)|dx = \infty,$$

where the supremum is taken over all cubes Q of $\mathbb{R}$. Therefore $\Delta_N^{-\alpha/2} f \notin \mathrm{BMO}(\mathbb{R})$. ■

Remark. Example 5.4 shows that for the Neumann Laplacian Δ_N on the real line $\mathbb{R}$, the $\mathrm{BMO}_{\Delta_N}(\mathbb{R})$ space is considered as a natural substitute for classical BMO space to study the end-point boundedness of the fractional powers $\Delta_N^{-\alpha/2}$.

6. BMO_L estimates of imaginary powers and maximal functions.

In this section we apply the technique of BMO_L spaces to discuss optimal L^p estimates for the imaginary powers of the operator L. We refer readers to [10, 20] for related results concerning imaginary powers of self-adjoint operators.

Let us recall that if L is a self-adjoint positive definite operator on $L^2(\mathbb{R}^n)$. Then L admits the spectral resolution:

$$L = \int_0^\infty \lambda dE_L(\lambda),$$

where the $E_L(\lambda)$ are spectral projectors. For any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$, we define the operator $F(L)$ by the formula

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda). \tag{6.1}$$

In particular

$$L^{is} = \int_0^\infty t^{is} dE(t).$$

By spectral theory $\|L^{is}\|_{L^2\to L^2} = 1$ for all $s \in \mathbb{R}$. In the following theorem we obtain sharp estimates for the $L^\infty \to \mathrm{BMO}_L$ norm of the operators L^{is}.

Theorem 6.1 *Assume that the heat kernel $p_t(x,y)$ corresponding to the self-adjoint operator L satisfies upper bound* (2.18). *Then*

$$\|L^{is}f\|_{\mathrm{BMO}_L(\mathbb{R}^n)} \le c(1+|s|)^{n/2}\|f\|_{L^\infty}$$

for all $s \in \mathbb{R}$.

Proof. It is enough to show that for any ball $B(x,r)$ with radius r centered at x, there exists a constant $C>0$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |(I-e^{-r^2L})L^{is}f(y)|dy \le c(1+|s|)^{n/2}\|f\|_{L^\infty}. \tag{6.2}$$

To prove (6.2), for any $f \in L^\infty(\mathbb{R}^n)$, we set $\theta = (1+|s|)^{-1/2}$, $f_1(y) = f(y)$ if $|x-y| \le \theta^{-1}r$ and $f_1(y) = 0$ otherwise. Next, we put $f_2 = f - f_1$. Note that

$$\begin{aligned}\frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{is}f(y)|dy &\le \frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{is}f_1(y)|dy\\ &\quad+\frac{1}{|B|}\int_{B(x,r)} |(I-e^{-r^2L})L^{is}f_2(y)|dy\\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where $|B| = |B(x,r)|$. To estimate the term I we note that, by Hölder's inequality

$$\begin{aligned}\|f_1\|_{L^2} &\le |B(x,\theta^{-1}r)|^{1/2}\|f\|_{L^\infty}\\ &\le \frac{|B(x,r)|^{\frac12}}{\theta^{n/2}}\|f\|_{L^\infty} = |B|^{1/2}(1+|s|)^{n/2}\|f\|_{L^\infty}.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{I} &\le |B|^{-1/2}\|(I-e^{-r^2L})L^{is}f_1\|_{L^2} \le c|B|^{-1/2}\|L^{is}f_1\|_{L^2}\\ &\le c|B|^{-1/2}\|f_1\|_{L^2} \le c(1+|s|)^{n/2}\|f\|_{L^\infty}.\end{aligned}$$

To estimate the term II we note that if $y \in B(x,r)$, then

$$\begin{aligned}\left|(I-e^{-r^2L})L^{is}f_2(y)\right| &\le \int_{B(x,\theta^{-1}r)^c} |K_{is,r^2}(y,z)||f(z)|dz\\ &\le \|f\|_{L^\infty}\sup_{x\in\Omega,\,r>0}\int_{B(x,\theta^{-1}r)^c}|K_{is,r^2}(x,z)|dz,\end{aligned}$$

where $K_{is,r^2}(y,z)$ is the kernel of the operator $(I-e^{-r^2L})L^{is}$. Hence the proof of Theorem 6.1 reduces to the following Lemma. ■

Lemma 6.2 *Assume that L is a self-adjoint operator and its heat kernel $p_t(x,y)$ satisfies the Gaussian bound (2.18). Then the associated kernel $K_{is,r^2}(x,z)$ of the operator $(I-e^{-r^2L})L^{is}$ satisfies*

$$\int_{B(x,\theta^{-1}r)^c} |K_{is,r^2}(x,z)|dz \le c(1+|s|)^{n/2}$$

for all $s\in\mathbb{R}$ and $r>0$.

The proof of Lemma 6.2 is a minor modification of the proof of estimates (17) of [27]. We leave the details to the reader.

Theorem 6.1 applied to the standard Laplace operator gives the following estimates.

Corollary 6.3 *If Δ is the standard Laplace operator acting on $\mathbb{R}^n$ then*

$$\|\Delta^{is}f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le c(1+|s|)^{n/2}\|f\|_{L^\infty} \tag{6.3}$$

for all $s\in\mathbb{R}$.

Proof. Corollary 6.3 is a straightforward consequence of Theorem 6.1 and the equivalence of the classical BMO space and BMO_Δ. ■

Remark. For the standard Laplace operator one can explicitly compute the kernel $|K_{is,r^2}(x,z)|$ and check that

$$\int_{B(x,r)^c} |K_{is,r^2}(x,z)|dz \ge c(1+|s|)^{n/2}\log(1+|s|)\,.$$

See [27]. Hence one has to replace $B(x,2r)$ by $B(x,\theta^{-1}r)^c$ to obtain estimates without the additional logarithmic term. As in [27] (Theorem 1) one can show that the norm of $\|\Delta^{is}\|_{L^\infty\to\mathrm{BMO}(\mathbb{R}^n)} \ge c(1+|s|)^{n/2}$. Hence the estimates in Theorem 6.1 and Corollary 6.3 are sharp. Even for the Laplace operator, our estimate (6.3) is stronger than any other known estimates of $L^\infty\to\mathrm{BMO}$ norm of the imaginary powers of the Laplace operator.

Theorem 2 of [27] says that if L satisfies assumption of Theorem 6.1 then the following estimates of the weak type $(1,1)$ norm of the imaginary powers of L holds

$$\|L^{is}\|_{L^1\to L^{1,\infty}} \le c(1+|s|)^{n/2} \tag{6.4}$$

Note, however, that the week type $(1,1)$ norm is not subadditive so despite its name is not a norm. Whereas $\|\cdot\|_{L^\infty\to\mathrm{BMO}_L}$, the norm of linear operators form L^∞ to BMO_L, is a proper norm. This difference is crucial for the results which we discuss next.

Suppose that $F\colon \mathbb{R} \to \mathbb{C}$. Let us recall that the Mellin transform of the function F is defined by

$$m(u) = \frac{1}{2\pi}\int_0^\infty F(\lambda)\lambda^{-1-iu}d\lambda, \quad u \in \mathbb{R}.$$

Moreover the inverse transform is given by the following formula

$$F(\lambda) = \int_{\mathbb{R}} m(u)\lambda^{iu}du, \quad \lambda \in [0,\infty).$$

Next we define the maximal operator $F^*(L)$ by the formula

$$F^*(L)f(x) = \sup_{t>0}|F(tL)f(x)|,$$

where $f \in L^p(\Omega)$ for some $1 \leq p \leq \infty$.

Corollary 6.4 *Assume that L is a self-adjoint operator acting on $L^2(\mathbb{R}^n)$ and that the heat kernel $p_t(x,y)$ of the operator L satisfies upper bound* (2.18). *Suppose also that $F\colon \mathbb{R} \to \mathbb{C}$ is a bounded Borel function such that*

$$\int_{\mathbb{R}} |m(u)|(1+|u|)^{n/2}du = C_{F,n} < \infty$$

where m is the Mellin transform of F. Then $F(L)$ and $F^*(L)$ are bounded operators from L^∞ to BMO_L *and*

$$\|F(L)\|_{L^\infty \to \mathrm{BMO}_L} \leq \|F^*(L)\|_{L^\infty \to \mathrm{BMO}_L} \leq cC_{F,n}.$$

Proof. Note that

$$\begin{aligned} F(tL) &= \int_0^\infty F(t\lambda)dE_L(\lambda) = \int_0^\infty \int_{\mathbb{R}} m(u)(t\lambda)^{iu}du\, dE_L(\lambda) \\ &= \int_{\mathbb{R}}\int_0^\infty m(u)(t\lambda)^{iu}dE_L(\lambda)du = \int_{\mathbb{R}} m(u)t^{iu}L^{iu}du. \end{aligned}$$

Hence

$$\sup_{t>0}|F(tL)f(x)| \leq \int_{\mathbb{R}} |m(u)||L^{iu}f(x)|du$$

and

$$\begin{aligned} \|F^*(L)f\|_{\mathrm{BMO}_L} &\leq \int_{\mathbb{R}} |m(u)|\|f\|_{L^\infty}\|L^{iu}\|_{L^\infty \to \mathrm{BMO}_L}du \\ &\leq c\|f\|_{L^\infty}\int_{\mathbb{R}} |m(u)|(1+|u|)^{n/2}du. \end{aligned}$$

The inequality $\|F(L)\|_{L^\infty \to \mathrm{BMO}_L} \leq \|F^*(L)\|_{L^\infty \to \mathrm{BMO}_L}$ is an obvious consequence of the definition of $F^*(L)$. ■

7. BMO_L estimates for spectral multipliers of self-adjoint operators

In this section we discuss an application of $\mathrm{BMO}_L(\Omega)$ technique to the theory of Hörmander spectral multipliers. In the sequel if $F(L)$ is the operator defined by (6.1) then by $K_{F(L)}$ we denote the kernel associated with $F(L)$. See (5.3) of [15].

Theorem 7.1 *Suppose that* $\|F\|_{L^\infty} \le C_1$*, and that*

$$\sup_{r>0}\sup_{y\in\Omega}\int_{B(y,r)^c}|K_{F(L)(I-e^{-r^2L})}(x,y)|dx \le C_1. \tag{7.1}$$

Then

$$\|F(L)\|_{L^\infty\to \mathrm{BMO}_L} \le cC_1.$$

Proof. We note again that it is enough to show that for any ball $B(x,r)$ with radius r centered at x, there exists a constant $C > 0$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|(I-e^{-r^2L})F(L)f(y)|dy \le cC_1\|f\|_{L^\infty}. \tag{7.2}$$

To prove (7.2) for any $f \in L^\infty(\mathbb{R}^n)$ we set $f_1(y) = f(y)$ if $|x-y| \le 2r$ and $f_1(y) = 0$ otherwise. Next, we put $f_2 = f - f_1$. Note that

$$\begin{aligned}\frac{1}{|B|}\int_{B(x,r)}&|(I-e^{-r^2L})F(L)f(y)|dy \le\\ &\le \frac{1}{|B|}\int_{B(x,r)}|(I-e^{-r^2L})F(L)f_1(y)|dy\\ &\quad+\frac{1}{|B|}\int_{B(x,r)}|(I-e^{-r^2L})F(L)f_2(y)|dy\\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where $|B| = |B(x,r)|$. To estimate the term I we note that, by Hölder's inequality

$$\|f_1\|_{L^2} \le |B(x,2r)|^{1/2}\|f\|_{L^\infty} \le c|B(x,2r)|^{1/2}\|f\|_{L^\infty}.$$

Then

$$\begin{aligned}\mathrm{I} &\le |B|^{-1/2}\|(I-e^{-r^2L})F(L)f_1\|_{L^2}\\ &\le c|B|^{-1/2}\|F(L)f_1\|_{L^2}\\ &\le c|B|^{-1/2}C_1\|f_1\|_{L^2}\\ &\le cC_1\|f\|_{L^\infty}.\end{aligned}$$

To estimate the term II we note that if $y \in B(x,r)$, then

$$\begin{aligned}\left|(I-e^{-r^2L})F(L)f_2(y)\right| &\leq \int_{B(y,r)^c} |K_{(I-e^{-r^2L})F(L)}(y,z)||f(z)|dz \\ &\leq \|f\|_{L^\infty} \sup_{x\in\Omega,\, r>0} \int_{B(y,r)^c} |K_{(I-e^{-r^2L})F(L)}(y,z)|dz \\ &\leq cC_1\|f\|_{L^\infty}\end{aligned}$$

■

In the standard theory of Hörmander spectral multipliers one usually begins with proving weak type $(1,1)$ estimates for a spectral multiplier $F(L)$. Next $F(L)$ is bounded on L^2 by the spectral theorem so continuity of the operator $F(L)$ on L^p spaces for $1 < p < \infty$ follows from the Marcinkiewicz interpolation theorem. One can use Theorem 7.1 and Proposition 1.3 to obtain an alternative proof of boundedness of $F(L)$ on an L^p space for $1<p<\infty$. Of course continuity of $F(L)$ as an operator from L^∞ to BMO_L is of independent interest even if we already know that $F(L)$ is of weak type $(1,1)$.

The Hörmander type spectral multipliers is a very broad subject. For example such multipliers were studied in [1, 7, 13, 21, 24, 26]. One can use Theorem 7.1 to show that all spectral multipliers of weak type $(1,1)$ which are discussed in [1, 7, 13, 21, 24, 26] are also bounded from L^∞ to BMO_L. As an example we discuss the following BMO_L versions of Theorem 3.1 of [13]. Let us recall that if $F\colon \mathbb{R} \to \mathbb{C}$ then

$$\|F\|_{W^p_s} = \|(I+\Delta)^{n/2}F\|_{L^p(\mathbb{R})}.$$

Theorem 7.2 *Suppose that L is a self-adjoint operator acting on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ and that the heat kernel $p_t(x,y)$ of L satisfies the Gaussian bound (2.18) and that $\eta \in C_c^\infty(\mathbb{R}_+)$. Then for every $s > n/2$ and for all Borel bounded function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s} < \infty$ the operator $F(L)$ is bounded on $L^p(\Omega)$ for all $1 < p < \infty$. Moreover*

$$\|F(L)\|_{L^\infty \to \mathrm{BMO}_L} \leq C_s\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s}\Big) \quad \textit{for all } s > n/2. \tag{7.3}$$

Proof. Note that by [13] ((4.19) and Remark 1), we have

$$\sup_{r>0}\sup_{y\in\Omega} \int_{B(y,r)^c} |K_{F(L)(I-e^{-r^2L})}(x,y)|dx \leq C_s\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^\infty_s}\Big).$$

Hence Theorem 7.2 is a straightforward consequence of Theorem 7.1. ■

Acknowledgments. The authors thank A. McIntosh for helpful suggestions and discussions.

References

[1] Alexopoulos, G.: Spectral multipliers for Markov chains. *J. Math. Soc. Japan* **56** (2004), no. 3, 833–852.

[2] Auscher, P., Duong, X. T. and McIntosh, A.: Boundedness of Banach space valued singular integral operators and Hardy spaces. Preprint, 2004.

[3] Auscher, P. and Russ, E.: Hardy spaces and divergence operators on strongly Lipschitz domain of $\mathbb{R}^n$. *J. Funct. Anal.* **201** (2003), 148–184.

[4] Auscher, P., Russ, E. and Tchamitchian, P.: Hardy Sobolev spaces on strongly Lipschitz domain of $\mathbb{R}^n$. *J. Funct. Anal.* **218** (2005), 54–109.

[5] Auscher, P. and Tchamitchian, P.: *Square root problem for divergence operators and related topics*. Astérisque **249**. Soc. Math. France, 1998.

[6] Chang, D.-C., Krantz, S. G. and Stein, E. M.: H^p theory on a smooth domain in $\mathbb{R}^N$ and elliptic boundary value problems. *J. Funct. Anal.* **114** (1993), 286–347.

[7] Christ, M.: L^p bounds for spectral multipliers on nilpotent groups. *Trans. Amer. Math. Soc.* **328** (1991), 73–81.

[8] Coifman, R. and Weiss. G.: Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[9] Coulhon, T. and Duong, X. T.: Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss. *Adv. Differential Equations* **5** (2000), 343–368.

[10] Cowling, M. and Meda, S.: Harmonic analysis and ultracontractivity. *Trans. Amer. Math. Soc.* **340** (1993), 733–752.

[11] Davies, E. B.: Heat kernels and spectral theory. Cambridge Tracts in Mathematics **92**. Cambridge Univ. Press, Cambridge, 1989.

[12] Duong, X. T. and McIntosh, A.: Singular integral operators with nonsmooth kernels on irregular domains. *Rev. Mat. Iberoamericana* **15** (1999), 233–265.

[13] Duong, X. T., Ouhabaz, E. M. and Sikora, A.: Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.* **196** (2002), 443–485.

[14] Duong, X. T., Ouhabaz, E. M. and Yan, L.: Endpoint estimates for Riesz transforms of magnetic Schrödinger operators. *Ark. Mat.* **44** (2006), 261–275.

[15] Duong, X. T. and Yan, L.: New function spaces of BMO type, the John-Nirenberg inequality, interpolation and applications. *Comm. Pure Appl. Math.* **58** (2005), 1375–1420.

[16] Duong, X. T. and Yan, L.: Duality of Hardy and BMO spaces associated with operators with heat kernels bounds. *J. Amer. Math. Soc.* **18** (2005), 943–973.

[17] Duong, X. T. and Yan, L.: On commutators of fractional integrals. *Proc. Amer. Math. Soc.* **132** (2004), 3549–3557.

[18] Dziubański, J., Garrigós, G., Martínez, T., Torrea, J. L. and Zienkiewicz, J.: BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. *Math. Z.* **249** (2005), 329–356.

[19] Fefferman, C. and Stein, E. M.: H^p spaces of several variables. *Acta Math.* **129** (1972), 137–193.

[20] Gunawan, H.: On weighted estimates for Stein's maximal function. *Bull. Austral. Math. Soc.* **54** (1996), 35–39.

[21] Hebisch, W.: A multiplier theorem for Schrödinger operators. *Colloq. Math.* **60/61** (1990), no. 2, 659–664.

[22] John, F. and Nirenberg, L.: On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **14** (1961) 415–426.

[23] Martell, J. M.: Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications. *Studia Math.* **161** (2004), 113–145.

[24] Mauceri, G. and Meda, S.: Vector-valued multipliers on stratified groups. *Rev. Mat. Iberoamericana* **6** (1990), 141–154.

[25] McIntosh, A.: Operators which have an H_∞ functional calculus. In *Miniconference on operator theory and partial differential equations (North Ryde, 1986)*, 210–231. Proc. Centre Math. Anal. Austral. Nat. Univ. **14**. Austral. Nat. Univ., Canberra, 1986.

[26] Müller, D. and Stein, E. M.: On spectral multipliers for Heisenberg and related groups. *J. Math. Pures Appl. (9)* **73** (1994), 413–440.

[27] Sikora, A. and Wright, J.: Imaginary powers of Laplace operators. *Proc. Amer. Math. Soc.* **129** (2001), 1745–1754

[28] Stein, E. M.: *Singular integral and differentiability properties of functions*. Princeton Mathematical Series **30**. Princeton University Press, Princeton, N.J., 1970.

[29] Stein, E. M.: *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*. Princeton Mathematical Series **43**. Princeton Univ. Press, 1993.

[30] Strauss, W. A.: *Partial differential equation: An introduction*. John Wiley& Sons, Inc., New York, 1992.

[31] Torchinsky, A.: *Real-variable methods in harmonic analysis*. Pure and Applied Mathematics **123**. Academic Press, Inc. Orlando, FL, 1986.

[32] Triebel, H.: *Theory of function spaces*. Monographs in Mathematics **78**. Birkhauser verlag, Basel, Boston, Stuggart, 1983.

[33] Uchiyama, A. and Wilson, J. M.: Approximate identities and $H^1(\mathbb{R})$. *Proc. Amer. Math. Soc.* **88** (1983), 53–58.

[34] Varopoulos, N., Saloff-Coste, L. and Coulhon, T.: *Analysis and geometry on groups*. Cambridge Tracts in Mathematics **100**. Cambridge Univ. Press, London, 1992.

[35] Weiss, G.: Some problems in the theory of Hardy spaces. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, 1978)*, Part 1, 189–200. *Proc. Sympos. Pure Math.* **35**. Amer. Math. Soc., Providence, R.I., 1979.

Recibido: 2 de octubre de 2006

Donggao Deng
Department of Mathematics
Zhongshan University
Guangzhou, 510275, P. R. China
stsddg@mail.sysu.edu.cn

Xuan Thinh Duong
Department of Mathematics
Macquarie University
NSW 2109, Australia
duong@ics.mq.edu.au

Adam Sikora
Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, ACT 0200, Australia
and Department of Mathematical Sciences
New Mexico State University
P.O. BOX 30001, Las Cruces, NM, 88003-8001
sikora@maths.anu.edu.au

Lixin Yan
Department of Mathematics
Macquarie University
NSW 2109, Australia
and Department of Mathematics
Zhongshan University
Guangzhou, 510275, P. R. China
mcsylx@mail.sysu.edu.cn

D. G. Deng and L. X. Yan are supported by NNSF of China (Grant No. 10371134/10571182) and the Foundation of Advanced Research Center, Zhongshan University. X. T. Duong and L. X. Yan are supported by a grant from Australia Research Council. Also A. Sikora participated in the collaboration with the support of the ARC.