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L^2 boundedness for commutator of rough singular integral with variable kernel

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Abstract

In this paper the authors prove the $L^2(\mathbb{R}^n)$ boundedness of the commutator of the singular integral operator with rough variable kernels, which is a substantial improvement and extension of some known results.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ with normalized Lebesgue measure $d\sigma$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \geq 1$, if $\Omega(x,z)$ satisfies the following conditions:

(1) for any $x,\ z \in \mathbb{R}^n$ and $\lambda > 0$, $\Omega(x, \lambda z) = \Omega(x,z)$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} := \sup_{x\in\mathbb{R}^n}\big(\int_{S^{n-1}} |\Omega(x,z')|^q\, d\sigma(z')\big)^{1/q} < \infty$, where $z' = \frac{z}{|z|}$, for any $z \in \mathbb{R}^n \setminus \{0\}$.

If $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(x,z')\, d\sigma(z') = 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.1}$$

then the singular integral operator with variable kernels is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\, dy.$$

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In 1955, Calderón and Zygmund [1] investigated the L^2 boundedness of the operator T. They found that these operators are useful in the study of second order linear elliptic equations with variable coefficients. In [1], Calderón and Zygmund obtained the following result (see also [2]):

Theorem A. *If* $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n-1)/n$, *satisfies* (1.1), *then there is a constant* $C > 0$ *such that* $\|Tf\|_{L^2} \le C\|f\|_{L^2}$.

On the other hand, it is well known that the commutator of the Calderón-Zygmund singular integral operator T and a $BMO(\mathbb{R}^n)$ function plays an important role in characterizing the Hardy space $H^1(\mathbb{R}^n)$ and in understanding the regularity of solutions of second order elliptic equations (see [3], [4], [6], for example).

To study interior $W^{2,2}$ estimates for nondivergence elliptic second order equations with discontinuous coefficients, in 1991, Chiarenza, Frasca and Longo [3] proved the $L^2(\mathbb{R}^n)$ boundedness of the commutator $T_{b,k}$ with variable kernel for $k=1$, which is defined by

$$T_{b,k}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} (b(x) - b(y))^k f(y)\, dy,$$

where $k \in \mathbb{N}$ and $b \in BMO(\mathbb{R}^n)$. That is,

$$\|b\|_* := \sup_Q \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$.

Theorem B. ([3]) *Suppose that* $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times C^\infty(S^{n-1})$ *satisfies* (1.1). *Then there is a constant* $C > 0$ *such that* $\|T_{b,1}f\|_{L^2} \le C\|b\|_*\|f\|_{L^2}$.

In 1993, Di Fazio and Ragusa [6] gave the weighted form of Theorem B, which was used to obtain the local regularity in Morrey spaces of the solutions of second order elliptic equations with discontinuous coefficients in nondivergence form.

Note that the kernel function $\Omega(x,z')$ has no any smoothness in the condition of Theorem A. However, in Theorem B, $\Omega(x,z')$ was assumed to be very smooth in its second variable. Hence, a natural problem is if the smoothness assumption of $\Omega(x,z')$ can be removed and $T_{b,1}$ is still bounded on $L^2(\mathbb{R}^n)$. The purpose of this paper is to give a positive answer to the above problem. More precisely, our result is an improvement and extension of Theorem B.

Theorem 1. *If $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q > 2(n-1)/n$, satisfies (1.1). Then for $k \in \mathbb{N}$, there is a constant $C > 0$ such that $\|T_{b,k}f\|_{L^2} \le C\|b\|_*^k\|f\|_{L^2}$.*

Remark 1. L. Tang and D. Yang [9] considered the above problem for $n = 2$ only. However, the method presented in this paper is different from the one in [9].

2. Some lemmas

We begin with some lemmas, which will be used in the proof of Theorem 1.

Lemma 2.1 ([8]) *Let $n \ge 2$, and $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ have the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree m. Then the Fourier transform of f has the form $\widehat{f} = F_0(|x|)P(x)$, where*

$$F_0(r) = 2\pi i^{-m} r^{-[(n+2m-2)/2]} \int_0^\infty f_0(s) J_{(n+2m-2)/2}(2\pi rs) s^{(n+2m)/2}\, ds,$$

$r = |x|$, and J_v is the Bessel function.

Lemma 2.2 *Suppose that $0 < \beta < 1$, $\alpha \in \mathbb{Z}$, $m \in \mathbb{N}$. Denote by $\mathcal{H}_m$ the space of surface spherical harmonics of degree m on S^{n-1}, and let D_m be the dimension of $\mathcal{H}_m$. $\{Y_{m,j}\}_{j=1}^{D_m}$ denotes the normalized complete system in $\mathcal{H}_m$. Let*

$$\sigma_{\alpha,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^n} \chi_{\{2^\alpha \le |x| \le 2^{\alpha+1}\}}(x).$$

Then

$$\big|\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C m^{-\lambda-1+\beta/2} \min\big\{2^\alpha|\xi|, |2^\alpha\xi|^{-\beta/2}\big\}\big|Y_{m,j}(\xi')\big|, \tag{2.1}$$

$$\big|\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C m^{-\lambda-1}\big|Y_{m,j}(\xi')\big|, \tag{2.2}$$

$$\big|\nabla\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C 2^\alpha, \tag{2.3}$$

where $\lambda = (n-2)/2$ and $\xi' = \frac{\xi}{|\xi|}$.

Proof. We start with the estimate (2.3). Since

$$\widehat{\sigma_{\alpha,m,j}}(\xi) = \int_{2^\alpha}^{2^{\alpha+1}} \int_{S^{n-1}} Y_{m,j}(x') e^{-2\pi i r x'\cdot\xi}\, d\sigma(x') \frac{dr}{r},$$

by $\|Y_{m,j}\|_{L^2(S^{n-1})} = 1$, we get

$$\big|\nabla\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C \int_{2^\alpha}^{2^{\alpha+1}} \int_{S^{n-1}} \big|Y_{m,j}(x')\big|\, d\sigma(x') dr \le C2^\alpha.$$

To show (2.1) and (2.2), we set $P_{m,j}(x) = Y_{m,j}(x')|x|^m$. Then $P_{m,j}$ is a solid spherical harmonic of degree m and $\sigma_{\alpha,m,j}(x) = |x|^{-n-m}P_{m,j}(x)\chi_{\{2^\alpha\le|x|\le 2^{\alpha+1}\}}(x)$. Since $\psi_0(|x|) := |x|^{-n-m}\chi_{\{2^\alpha\le|x|\le 2^{\alpha+1}\}}(x)$ is a radial function in x, using Lemma 2.1 we have

$$\widehat{\sigma_{\alpha,m,j}}(\xi) = \Psi_0(|\xi|)P_{m,j}(\xi) = Y_{m,j}(\xi')|\xi|^m\Psi_0(|\xi|), \tag{2.4}$$

where

$$\begin{aligned}\Psi_0(r) &= 2\pi i^{-m} r^{-[(n+2m-2)/2]}\int_0^\infty \psi_0(s)J_{(n+2m-2)/2}(2\pi rs)s^{(n+2m)/2}\,ds\\ &= (2\pi)^{n/2}i^{-m}r^{-m}\int_{2\pi 2^\alpha r}^{2\pi 2^{\alpha+1}r}\frac{J_{(n+2m-2)/2}(t)}{t^{\frac{n-2}{2}+1}}\,dt.\end{aligned}$$

From this and (2.4) we have

$$\widehat{\sigma_{\alpha,m,j}}(\xi) = (2\pi)^{n/2}i^{-m}Y_{m,j}(\xi')\int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|}\frac{J_{(n+2m-2)/2}(t)}{t^{\frac{n-2}{2}+1}}\,dt. \tag{2.5}$$

Now we consider three cases, namely

$$1^\circ.\ 2^\alpha|\xi|\le 1,\qquad 2^\circ.\ 1<2^\alpha|\xi|<m+\lambda,\qquad 3^\circ.\ 2^\alpha|\xi|\ge m+\lambda.$$

Case 1. By a classical formula for Bessel functions (see [10, p.48]), we get

$$\begin{aligned}\big|J_{m+\lambda}(t)\big| &= \Big|\frac{(t/2)^{m+\lambda}}{\Gamma(m+\lambda+1/2)\Gamma(1/2)}\int_{-1}^1(1-r^2)^{m+\lambda-1/2}e^{itr}\,dr\Big|\\ &\le C\frac{(t/2)^{m+\lambda}}{\Gamma(m+\lambda+1/2)}.\end{aligned}$$

Applying Stirling's formula, for $x>1$

$$\sqrt{2\pi}x^{x-1/2}e^{-x}\le\Gamma(x)\le 2\sqrt{2\pi}x^{x-1/2}e^{-x}.$$

Thus, by $2^\alpha|\xi|\le 1$, we have

$$\begin{aligned}\Big|\int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|}\frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\,dt\Big| &\le \frac{C}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}\int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|}t^{m-1}dt\\ &\le \frac{C}{2^{m+\lambda}\Gamma(m+\lambda+1/2)}\cdot\frac{1}{m}(2\pi 2^{\alpha+1}|\xi|)^m\\ &\le C\frac{2^\alpha|\xi|}{m}\frac{(2\pi 2^{\alpha+1}|\xi|)^{m-1}}{2^{m+\lambda}\sqrt{2\pi}(m+\lambda+1/2)^{m+\lambda}e^{-m-\lambda}}\\ &\le C2^\alpha|\xi|m^{-\lambda-1}\frac{(4\pi)^m}{2^{m+\lambda}}\cdot\frac{e^{m+\lambda}}{(m+\lambda+1/2)^m}\\ &\le Cm^{-\lambda-1}2^\alpha|\xi|.\end{aligned}$$

Case 2. By Lemma 2 in [1], there exists $C>0$ such that for any $0 \le a, b \le \infty$

$$\left| \int_a^b \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\, dt \right| \le C m^{-\lambda-1}. \tag{2.6}$$

Thus, by (2.6) and the fact that $1 < 2^\alpha|\xi| < m+\lambda$, we get

$$\begin{aligned} \left| \int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\, dt \right| &\le C m^{-\lambda-1} \\ &\le C\Big(1+\frac{\lambda}{m}\Big)^{\beta/2} m^{-1-\lambda+\beta/2}(m+\lambda)^{-\beta/2} \le C m^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}. \end{aligned}$$

Case 3. Since $|J'_{m+\lambda}(t)| \le 1$ for $t>0$, using the second mean-value theorem and the following differential equation for $J_{m+\lambda}$ (see [10])

$$\frac{J_{m+\lambda}(t)}{t^{\lambda+1}} = -\frac{J'_{m+\lambda}(t)}{t^\lambda(t^2-(m+\lambda)^2)} - \frac{J''_{m+\lambda}(t)}{t^{\lambda-1}(t^2-(m+\lambda)^2)},$$

there exists $2\pi 2^\alpha|\xi| \le h \le 2\pi 2^{\alpha+1}|\xi|$ such that

$$\begin{aligned} \left| \int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|} \frac{J_{m+\lambda}(t)}{t^{\lambda+1}}\, dt \right| &\le C \int_{2\pi 2^\alpha|\xi|}^{2\pi 2^{\alpha+1}|\xi|} t^{-2-\lambda}\, dt + \frac{C}{(2^\alpha|\xi|)^{\lambda+1}} \left| \int_{2\pi 2^\alpha|\xi|}^{h} J''_{m+\lambda}(t)\, dt \right| \\ &\quad + \frac{C}{(2^{\alpha+1}|\xi|)^{\lambda+1}} \left| \int_h^{2\pi 2^{\alpha+1}|\xi|} J''_{m+\lambda}(t)\, dt \right| \\ &\le C(2^\alpha|\xi|)^{-\lambda-1} \le C m^{-1-\lambda+\beta/2}(2^\alpha|\xi|)^{-\beta/2}, \end{aligned}$$

where we use the assumption $2\pi 2^\alpha|\xi| \ge 2\pi(m+\lambda)$. Thus, from (2.5) and the above estimates in three cases, we get

$$\big|\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C m^{-\lambda-1+\beta/2} \min\big\{2^\alpha|\xi|, |2^\alpha\xi|^{-\beta/2}\big\}\big|Y_{m,j}(\xi')\big|.$$

On the other hand, by (2.5) and (2.6) we have

$$\big|\widehat{\sigma_{\alpha,m,j}}(\xi)\big| \le C m^{-\lambda-1}\big|Y_{m,j}(\xi')\big|.$$

■

Lemma 2.3 *For $0<\delta<\infty, m \in \mathbb{N}$ and $j=1,\dots,D_m$, take $B_{\delta,m,j} \in C_0^\infty(\mathbb{R}^n)$ with supp$(B_{\delta,m,j}) \subset \{\delta/2 \le |\xi| \le 2\delta\}$. Let $T_{\delta,m,j}$ be the multiplier operator defined by*

$$\widehat{T_{\delta,m,j}f}(\xi) = B_{\delta,m,j}(\xi)\widehat{f}(\xi), \quad j=1,\dots,D_m.$$

Moreover, for $b \in BMO$ and $k \in \mathbb{N}$, let $T_{\delta,m,j;b,k}f(x) = T_{\delta,m,j}\big((b(x)-b(\cdot))^k f\big)(x)$ be the k-th order commutator of $T_{\delta,m,j}$ and

$$T_{\delta,m;b,k}f(x) = \left(\sum_{j=1}^{D_m}(T_{\delta,m,j;b,k}f(x))^2\right)^{1/2}.$$

If for some constant $0 < \beta < 1$, $B_{\delta,m,j}$ *satisfies the following conditions:*

$$\big|B_{\delta,m,j}(\xi)\big| \le Cm^{-\lambda-1+\beta/2} \min\big\{\delta, \delta^{-\beta/2}\big\}\big|Y_{m,j}(\xi')\big|, \tag{2.7}$$

$$\big|B_{\delta,m,j}(\xi)\big| \le Cm^{-\lambda-1}\big|Y_{m,j}(\xi')\big|, \tag{2.8}$$

$$\big|\nabla B_{\delta,m,j}(\xi)\big| \le C, \tag{2.9}$$

then for any fixed $0 < v < 1$, *there exists a positive constant* $C = C(n, k, v)$ *such that*

$$\|T_{\delta,m;\,b,k}f\|_{L^2} \le Cm^{(-1+\beta/2)v} \min\{\delta^v, \delta^{-\beta v/2}\}\|b\|_*^k\|f\|_{L^2}.$$

Proof. We may assume that $\|b\|_* = 1$. Consider a $C_0^\infty(\mathbb{R}^n)$ radial function ϕ, such that $\mathrm{supp}\phi \subset \{1/2 \le |x| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \phi(2^{-l}|x|) = 1$ for any $|x| > 0$.

Set $\phi_0(x) = \sum_{l=-\infty}^{0} \phi(2^{-l}|x|)$ and $\phi_l(x) = \phi(2^{-l}|x|)$ for any positive integer l. Then $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\mathrm{supp}\phi_0 \subset \{x : 0 < |x| \le 2\}$. Let $K_{\delta,m,j}(x) = (B_{\delta,m,j})^\vee(x)$, the inverse Fourier transform of $B_{\delta,m,j}$. Denote $K^l_{\delta,m,j}(x) = K_{\delta,m,j}(x)\phi_l(x)$ for $l = 0, 1, \ldots$, we have

$$K_{\delta,m,j}(x) = \sum_{l=0}^{\infty} K^l_{\delta,m,j}(x).$$

Denote by $T^l_{\delta,m,j}$ and $T^l_{\delta,m,j;b,k}$ the convolution operator with kernel $K^l_{\delta,m,j}$ and the k-th order commutator of $T^l_{\delta,m,j}$ and b, respectively. Then by the Minkowski inequality

$$\begin{aligned}\|T_{\delta,m;b,k}f\|_{L^2} &= \bigg(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \Big|\sum_{l=0}^{\infty} T^l_{\delta,m,j;b,k}f(x)\Big|^2\,dx\bigg)^{1/2} \\ &\le \sum_{l=0}^{\infty} \bigg(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \Big|T^l_{\delta,m,j;b,k}f(x)\Big|^2\,dx\bigg)^{1/2} \\ &= \sum_{l=0}^{\infty} \|T^l_{\delta,m;b,k}f\|_{L^2},\end{aligned} \tag{2.10}$$

where

$$T^l_{\delta,m;b,k}f(x) = \bigg(\sum_{j=1}^{D_m} \big|T^l_{\delta,m,j;b,k}f(x)\big|^2\bigg)^{1/2}.$$

By (2.10) it is easy to see that for any fixed $0 < v < 1$, if there exists $\gamma > 0$ such that

$$\|T^l_{\delta,m;b,k}f\|_{L^2} \le Cm^{(-1+\beta/2)v}2^{-l\gamma} \min\{\delta^v, \delta^{-\beta v/2}\}\|f\|_{L^2}, \tag{2.11}$$

then Lemma 2.3 follows.

To do this, for $l \geq 0$, we decompose $\mathbb{R}^n = \bigcup_{d=-\infty}^{\infty} Q_d$, where $Q_d's$ are non-overlapping cubes with side length 2^l. Set $f_d = f\chi_{Q_d}$. Then

$$f(x) = \sum_{d=-\infty}^{\infty} f_d(x), \quad a.e. \ \ x \in \mathbb{R}^n.$$

Since $\mathrm{supp}(K^l_{\delta,m,j}) \subset \{x : |x| \leq 2^{l+2}\}$, it is obvious that $\mathrm{supp}(T^l_{\delta,m,j;b,k} f_d) \subset 10nQ_d$, and that the supports of $\{T^l_{\delta,m,j;b,k} f_d\}_{d=-\infty}^{\infty}$ have bounded overlaps. So we have the following almost orthogonality property:

$$\|T^l_{\delta,m,j;b,k} f\|^2_{L^2} \leq C \sum_{d=-\infty}^{\infty} \|T^l_{\delta,m,j;b,k} f_d\|^2_{L^2}.$$

Thus

$$\begin{aligned}
\|T^l_{\delta,m;b,k} f\|^2_{L^2} &= \sum_{j=1}^{D_m} \|T^l_{\delta,m,j;b,k} f\|^2_{L^2} \leq C \sum_{d=-\infty}^{\infty} \sum_{j=1}^{D_m} \|T^l_{\delta,m,j;b,k} f_d\|^2_{L^2} \\
&= C \sum_{d=-\infty}^{\infty} \|T^l_{\delta,m;b,k} f_d\|^2_{L^2}.
\end{aligned}$$

Hence, it suffices to verify (2.11) for the function f with $\mathrm{supp} f \subset Q$, where Q has side length 2^l. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, φ is identically one on $50nQ$, and $\mathrm{supp}\varphi \subset 100nQ$. Set $\widetilde{Q} = 200nQ$, and $b_{\widetilde{Q}} = |\widetilde{Q}|^{-1} \int_{\widetilde{Q}} b(y)\,dy$. Let $\widetilde{b} = (b(x) - b_{\widetilde{Q}})\varphi(x)$. It is easy to see that

$$T^l_{\delta,m,j;b,k} f(x) = \sum_{\mu=0}^{k} C_k^\mu \widetilde{b}^\mu(x) T^l_{\delta,m,j}(\widetilde{b}^{k-\mu} f)(x).$$

Denote

$$T^l_{\delta,m} f(x) = \left(\sum_{j=1}^{D_m} \left| T^l_{\delta,m,j} f(x) \right|^2 \right)^{1/2},$$

then we have

$$\begin{aligned}
\|T^l_{\delta,m;b,k} f\|^2_{L^2} &= \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \left| \sum_{\mu=0}^{k} C_k^\mu \widetilde{b}^\mu(x) T^l_{\delta,m,j}(\widetilde{b}^{k-\mu} f)(x) \right|^2 dx \qquad (2.12) \\
&\leq C \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \sum_{\mu=0}^{k} \left| \widetilde{b}^\mu(x) T^l_{\delta,m,j}(\widetilde{b}^{k-\mu} f)(x) \right|^2 dx \\
&\leq C \sum_{\mu=0}^{k} \int_{\mathbb{R}^n} |\widetilde{b}^\mu(x)|^2 \sum_{j=1}^{D_m} |T^l_{\delta,m,j}(\widetilde{b}^{k-\mu} f)(x)|^2\,dx = C \sum_{\mu=0}^{k} \|\widetilde{b}^\mu T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|^2_{L^2}.
\end{aligned}$$

Thus by (2.12) we need only show that for any function f supported in Q with side length 2^l,

$$\|\widetilde{b}^{\mu} T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} \le C m^{(-1+\beta/2)v} 2^{-l\gamma} \min\{\delta^v, \delta^{-\beta v/2}\}\|f\|_{L^2}. \tag{2.13}$$

Let us first show that for $g \in L^{q'}(\mathbb{R}^n)$, $1 < q' \le 2$ (hence $2 \le q < \infty$), $0 < t < 1$

$$\begin{aligned}\|T^l_{\delta,m} g\|_{L^q} &\le C 2^{-\frac{2tl}{q}} m^{\frac{(-2+\beta)(1-t)}{q} - (1-\frac{2}{q}) + \frac{2t\lambda}{q}} \delta^{n(1-\frac{2}{q})} \\ &\quad \times \big(\min\{\delta, \delta^{-\beta}\}\big)^{\frac{2(1-t)}{q}} \|g\|_{L^{q'}}.\end{aligned} \tag{2.14}$$

In fact, by the definition of $T^l_{\delta,m,j}$, we have

$$\begin{aligned}|T^l_{\delta,m} g(x)| &\le \bigg(\sum_{j=1}^{D_m} \Big(\int_{\mathbb{R}^n} |K^l_{\delta,m,j}(x-y)||g(y)|\,dy\Big)^2\bigg)^{1/2} \\ &\le \int_{\mathbb{R}^n} \bigg(\sum_{j=1}^{D_m} |K^l_{\delta,m,j}(x-y)|^2\bigg)^{1/2} |g(y)|\,dy \\ &\le \int_{\mathbb{R}^n} \bigg(\sum_{j=1}^{D_m} |\widehat{K^l_{\delta,m,j}}(x)|^2\bigg)^{1/2} dx\, \|g\|_{L^1}.\end{aligned}$$

Since

$$\widehat{K^l_{\delta,m,j}}(x) = \widehat{K_{\delta,m,j}} * \widehat{\phi}_l(x) = \int_{\mathbb{R}^n} B_{\delta,m,j}(x-y)\widehat{\phi}_l(y)\,dy. \tag{2.15}$$

By (2.8) and the fact $\sum_{j=1}^{D_m} |Y_{m,j}(x')|^2 \sim m^{2\lambda}$ (see [2, p.225, (2.6)]), we get

$$\begin{aligned}|T^l_{\delta,m} g(x)| &\le \int \bigg(\sum_{j=1}^{D_m} \Big|\int_{\mathbb{R}^n} B_{\delta,m,j}(x-y)\widehat{\phi}_l(y)\,dy\Big|^2\bigg)^{1/2} dx\, \|g\|_{L^1} \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bigg(\sum_{j=1}^{D_m} |B_{\delta,m,j}(x-y)|^2\bigg)^{1/2} \widehat{\phi}_l(y)\,dy\,dx\, \|g\|_{L^1} \\ &\le \int_{\delta/2<|x|<2\delta} \bigg(\sum_{j=1}^{D_m} |B_{\delta,m,j}(x)|^2\bigg)^{1/2} dx\, \|\widehat{\phi}_l\|_{L^1}\, \|g\|_{L^1} \\ &\le m^{-\lambda-1} \int_{\delta/2<|x|<2\delta} \bigg(\sum_{j=1}^{D_m} |Y_{m,j}(x')|^2\bigg)^{1/2} dx\, \|g\|_{L^1} \\ &\le C m^{-1}\delta^n\, \|g\|_{L^1},\end{aligned}$$

i.e.

$$\|T^l_{\delta,m} g\|_{L^\infty} \le C m^{-1}\delta^n \|g\|_{L^1}. \tag{2.16}$$

On the other hand, note that $\int_{\mathbb{R}^n} \widehat{\phi}(\eta)\, d\eta = \phi(0) = 0$, so by (2.15) and (2.9) we get

$$\begin{aligned}|\widehat{K^l_{\delta,m,j}}(x)| &\le \int_{\mathbb{R}^n} |(B_{\delta,m,j}(x-2^{-l}y) - B_{\delta,m,j}(x))||\widehat{\phi}(y)|\, dy \\ &\le C2^{-l}\|\nabla B_{\delta,m,j}\|_{L^\infty} \int_{\mathbb{R}^n} |y||\widehat{\phi}(y)|\, dy \le C2^{-l}.\end{aligned}$$

Thus, by the Plancherel theorem and the fact $D_m \sim m^{2\lambda}$ (see [2, p.226, (2.8)]), we have

$$\|T^l_{\delta,m}g\|_{L^2} \le \Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\widehat{K^l_{\delta,m,j}}(\xi)|^2 |\widehat{g}(\xi)|^2\, d\xi\Big)^{1/2} \le C2^{-l}m^{\lambda}\|g\|_{L^2}. \tag{2.17}$$

Applying the Plancherel theorem again, and by (2.15), (2.7) and

$$\sum_{j=1}^{D_m} |Y_{m,j}(x')|^2 \sim m^{2\lambda},$$

we obtain

$$\begin{aligned}\|T^l_{\delta,m}g\|^2_{L^2} &\le \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\widehat{K^l_{\delta,m,j}}(\xi)|^2 |\widehat{g}(\xi)|^2\, d\xi \\ &\le C\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |B_{\delta,m,j} * \widehat{\phi}_l(\xi)|^2 |\widehat{g}(\xi)|^2\, d\xi \\ &\le C\int_{\mathbb{R}^n} \Big\{\Big(\sum_{j=1}^{D_m} \Big|\int_{\mathbb{R}^n} B_{\delta,m,j}(\xi-y)\widehat{\phi}_l(y)\, dy\Big|^2\Big)^{1/2}\Big\}^2 |\widehat{g}(\xi)|^2\, d\xi \\ &\le C\int_{\mathbb{R}^n} \Big\{\int_{\mathbb{R}^n} \Big(\sum_{j=1}^{D_m} |B_{\delta,m,j}(\xi-y)|^2\Big)^{1/2} |\widehat{\phi}_l(y)|\, dy\Big\}^2 |\widehat{g}(\xi)|^2\, d\xi \\ &\le m^{-2\lambda-2+\beta}\big(\min\{\delta, \delta^{-\beta/2}\}\big)^2 \\ &\quad \times \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \Big(\sum_{j=1}^{D_m} |Y_{m,j}((\xi-y)')|^2\Big)^{1/2} |\widehat{\phi}_l(y)|\, dy\Big)^2 |\widehat{g}(\xi)|^2\, d\xi \\ &\le Cm^{-2+\beta}\big(\min\{\delta, \delta^{-\beta/2}\}\big)^2 \|\widehat{\phi}_l\|^2_{L^1}\|g\|^2_{L^2}.\end{aligned}$$

That is,

$$\|T^l_{\delta,m}g\|_{L^2} \le Cm^{(-1+\beta/2)} \min\{\delta, \delta^{-\beta/2}\}\|g\|_{L^2}. \tag{2.18}$$

Hence, by (2.17) and (2.18), for any $0 < t < 1$,

$$\|T^l_{\delta,m}g\|_{L^2} \le C2^{-tl}m^{t\lambda}m^{(-1+\frac{\beta}{2})(1-t)}\big(\min\{\delta, \delta^{-\beta/2}\}\big)^{1-t}\|g\|_{L^2}. \tag{2.19}$$

Thus we obtain (2.14) by interpolating between (2.16) and (2.19).

Let us return to the proof of (2.13). For $2 < q_1, q_2 < \infty$ with $1/q_1 + 1/q_2 = 1/2$, by (2.14) and the fact that

$$\|\widetilde{b}^\mu\|_{L^\sigma} \le C\|b\|_*^\mu |Q|^{1/\sigma} \le C2^{nl/\sigma} \quad \text{for} \quad 1 < \sigma < \infty,$$

we have

$$\begin{aligned} \|\widetilde{b}^\mu T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\le \|\widetilde{b}^\mu\|_{L^{q_1}} \|T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^{q_2}} \\ &\le C2^{-\frac{2tl}{q_2}} \delta^{n(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \\ &\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|\widetilde{b}^\mu\|_{L^{q_1}} \|\widetilde{b}^{k-\mu} f\|_{L^{q_2'}} \\ &\le C2^{-\frac{2tl}{q_2}} \delta^{n(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \\ &\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|\widetilde{b}^\mu\|_{L^{q_1}} \|\widetilde{b}^{k-\mu}\|_{L^{2q_2/(q_2-2)}} \|f\|_{L^2} \\ &\le C2^{-\frac{2tl}{q_2} + nl(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \delta^{n(1-\frac{2}{q_2})} \\ &\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|f\|_{L^2}. \end{aligned} \tag{2.20}$$

For any fixed $0 < v < 1$, we choose $q_2 > 2$ and sufficiently close to 2, $t > 0$ but sufficiently close to 0, such that q_2 and t satisfy:

$$2t/q_2 > n(1 - 2/q_2), \quad \beta(1-t)/q_2 > n(1 - 2/q_2) + v\beta/2 + 2t\lambda\beta/q_2.$$

Then there exists $\gamma > 0$, independent of l, such that $2^{-2tl/q_2 + nl(1-2/q_2)} = 2^{-\gamma l}$. We also can get

$$m^{(-2+\beta)(1-t)/q_2 - (1-2/q_2) + 2t\lambda/q_2} \le m^{(-1+\beta/2)v}.$$

If $\delta \ge 1$, then by (2.20)

$$\begin{aligned} \|\widetilde{b}^\mu T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\le Cm^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{n(1-2/q_2)} \delta^{-\beta(1-t)/q_2} \|f\|_{L^2} \\ &\le Cm^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{-\beta v/2} \|f\|_{L^2}. \end{aligned}$$

If $0 < \delta < 1$, then by (2.20)

$$\begin{aligned} \|\widetilde{b}^\mu T^l_{\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\le Cm^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{n(1-2/q_2)} \delta^{2(1-t)/q_2} \|f\|_{L^2} \\ &\le Cm^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{v} \|f\|_{L^2}. \end{aligned}$$

Therefore we get (2.13) and the proof of Lemma 2.3 is complete. ■

Remark 2.1 When $k = 0$, Lemma 2.3 also holds.

Lemma 2.4 ([7]) *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that supp(ψ) $\subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \psi^3(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_l by*

$$\widehat{S_l f}(\xi) = \psi(2^{-l}\xi)\widehat{f}(\xi).$$

For $b \in BMO$ and a nonnegative integer k, denote by $S_{l;b,k}$ the k-th order commutator of S_l. Then for $1 < p < \infty$,

(i) $\Big\|\Big(\sum_{l\in\mathbb{Z}} |S_{l;b,k}f|^2\Big)^{1/2}\Big\|_{L^p} \le C(n,k,p)\|b\|_*^k \|f\|_{L^p}$;

(ii) $\Big\|\sum_{l\in\mathbb{Z}} |S_{l;b,k}f_l|\Big\|_{L^p} \le C(n,k,p)\|b\|_*^k \Big\|\Big(\sum_{l\in\mathbb{Z}} |f_l|^2\Big)^{1/2}\Big\|_{L^p}$ *where* $\{f_l\} \in L^p(l^2)$.

3. Proof of Theorem 1

As in [2], by a limit argument we may reduce the proof of Theorem 1 to the case where $f \in C_0^\infty(\mathbb{R}^n)$ and

$$\Omega(x,z') = \sum_{m\ge 0}\sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(z')$$

is a finite sum. Notice that $\Omega(x,z')$ satisfies (1.1), so $a_{0,j} \equiv 0$. Define

$$a_m(x) = \Big(\sum_{j=1}^{D_m} |a_{m,j}(x)|^2\Big)^{1/2}$$

and

$$b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}.$$

Then

$$\sum_{j=1}^{D_m} b_{m,j}^2(x) = 1, \tag{3.1}$$

and

$$\Omega(x,z') = \sum_{m\ge 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) Y_{m,j}(z').$$

If we write

$$T_{m,j;b,k}f(x) = \int_{\mathbb{R}^n} \frac{Y_{m,j}(x-y)}{|x-y|^n}(b(x)-b(y))^k f(y)dy,$$

then by using Hölder's inequality twice and (3.1), we have

$$
\begin{aligned}
(T_{b,k}f(x))^2 &= \Big(\sum_{m=1}^{\infty} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) T_{m,j;b,k}f(x)\Big)^2 \\
&\le \Big(\sum_{m=1}^{\infty} a_m^2(x) m^{-\varepsilon}\Big)\Big(\sum_{m=1}^{\infty} m^{\varepsilon} \sum_{j=1}^{D_m} b_{m,j}^2(x) \sum_{j=1}^{D_m} (T_{m,j;b,k}f(x))^2\Big) \\
&\le \Big\{\sum_{m=1}^{\infty} a_m^2(x) m^{-\varepsilon}\Big\}\Big\{\sum_{m=1}^{\infty} m^{\varepsilon} \sum_{j=1}^{D_m} (T_{m,j;b,k}f(x))^2\Big\}.
\end{aligned}
$$

By [2, p. 230], if we take $q > 2(n-1)/n$, $0 < \varepsilon < 1$ and sufficiently close to 1, then

$$
\begin{aligned}
\Big(\sum_{m\ge 1} a_m^2(x) m^{-\varepsilon}\Big)^{1/2} &\le C\Big(\int_{S^{n-1}} |\Omega(x,z')|^q d\sigma(z')\Big)^{1/q} \\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})}.
\end{aligned}
\tag{3.2}
$$

Let

$$
T_{m;b,k}f(x) = \Big(\sum_{j=1}^{D_m} |T_{m,j;b,k}f(x)|^2\Big)^{1/2}.
$$

Applying the Minkowski inequality and (3.2), for $q > 2(n-1)/n$ and $0 < \varepsilon < 1$ which is sufficiently close to 1, we get

$$
\|T_{b,k}f\|_{L^2}^2 \le C\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} \sum_{m\ge 1} m^{\varepsilon} \|T_{m;b,k}f\|^2_{L^2}. \tag{3.3}
$$

If we can show that for some $0 < \beta < (1-\varepsilon)/2$, such that

$$
\|T_{m;b,k}f\|^2_{L^2} \le C m^{-2+2\beta} \|f\|^2_{L^2}, \tag{3.4}
$$

then from (3.3) and (3.4) we immediately get the conclusion of Theorem 1. Hence, it remains to show (3.4) to prove Theorem 1.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0 \le \psi \le 1$, $\operatorname{supp}\psi \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \psi^3(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_l by $\widehat{S_l f}(\xi) = \psi(2^{-l}\xi)\hat{f}(\xi)$. Let

$$
\sigma_{\alpha,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^n} \chi_{\{2^\alpha \le |x| \le 2^{\alpha+1}\}}(x)
$$

for $\alpha \in \mathbb{Z}$, $m = 1, 2, \ldots,$ and $j = 1, \ldots, D_m$.

Set

$$B_{\alpha,m,j}(\xi) = \widehat{\sigma_{\alpha,m,j}}(\xi), \quad B^l_{\alpha,m,j}(\xi) = B_{\alpha,m,j}(\xi)\psi(2^{\alpha-l}\xi).$$

Define the operator $T_{\alpha,m,j}$ and $T^l_{\alpha,m,j}$ by

$$\widehat{T_{\alpha,m,j}f}(\xi) = B_{\alpha,m,j}(\xi)\widehat{f}(\xi), \quad \widehat{T^l_{\alpha,m,j}f}(\xi) = B^l_{\alpha,m,j}(\xi)\widehat{f}(\xi).$$

Denote by $T_{\alpha,m,j;b,k}$ and $T^l_{\alpha,m,j;b,k}$ the k-th order commutator of $T_{\alpha,m,j}$ and $T^l_{\alpha,m,j}$, respectively. Define the operator $V_{l,j}$ by

$$V_{l,j}f(x) = \sum_{\alpha\in\mathbb{Z}}((S_{l-\alpha}T^l_{\alpha,m,j}S_{l-\alpha})_{b,k}f)(x).$$

We claim that

$$T_{m,j;b,k}f(x) = \sum_{l\in\mathbb{Z}} V_{l,j}f(x). \tag{3.5}$$

Once (3.5) holds, then by the Minkowski inequality, we get

$$\begin{aligned}\|T_{m;b,k}f\|_{L^2} &= \left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big|\sum_{l\in\mathbb{Z}}V_{l,j}f(x)\Big|^2\,dx\right)^{1/2} \\ &\le \sum_{l\in\mathbb{Z}}\left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{l,j}f(x)|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.6}$$

So by (3.6), to prove (3.4), it suffices to show (3.5) and

$$\sum_{l\in\mathbb{Z}}\left(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{l,j}f(x)|^2\,dx\right)^{1/2} \le Cm^{-1+\beta}\|f\|_{L^2}. \tag{3.7}$$

where C is independent of l and f. Let us first consider (3.5). By the definition of $\sigma_{\alpha,m,j}$,

$$T_{m,j}f(x) = \sum_{\alpha\in\mathbb{Z}}\sigma_{\alpha,m,j} * f(x).$$

For a fixed cube Q, write

$$T_{m,j;b,k}f(x) = \sum_{s=0}^{k} C^s_k(b(x)-b_Q)^s T_{m,j}\big((b_Q-b(\cdot))^{k-s}f\big)(x).$$

Thus as in [5, p. 545], it follows that

$$\begin{aligned}
T_{m,j;b,k}f(x) &= \sum_{s=0}^{k} C_k^s (b(x)-b_Q)^s \sum_{\alpha\in\mathbb{Z}} \sigma_{\alpha,m,j} * \Big(\sum_{l\in\mathbb{Z}} S^3_{l-\alpha}\big((b_Q-b(\cdot))^{k-s}f\big)\Big)(x) \\
&= \sum_{s=0}^{k} C_k^s (b(x)-b_Q)^s \\
&\qquad \times \sum_{l\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}} \big(S_{l-\alpha}\sigma_{\alpha,m,j} * S_{l-\alpha}S_{l-\alpha}\big((b_Q-b(\cdot))^{k-s}f\big)\big)(x) \\
&= \sum_{s=0}^{k} C_k^s (b(x)-b_Q)^s \sum_{l\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}} \big(S_{l-\alpha}T^l_{\alpha,m,j}S_{l-\alpha}\big((b_Q-b(\cdot))^{k-s}f\big)\big)(x) \\
&= \sum_{l\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}} \big((S_{l-\alpha}T^l_{\alpha,m,j}S_{l-\alpha})_{b,k}f\big)(x).
\end{aligned}$$

This establishes (3.5). Hence it remains to show (3.7). With the aid of the formula

$$\big(b(x)-b(y)\big)^k = \sum_{s=0}^{k} C_k^s \big(b(x)-b(z)\big)^s \big(b(z)-b(y)\big)^{k-s}, \quad x,y,z\in\mathbb{R}^n,$$

we can write

$$\begin{aligned}
V_{l,j}f(x) &= \sum_{\alpha\in\mathbb{Z}} (S_{l-\alpha}T^l_{\alpha,m,j}S_{l-\alpha})_{b,k}f(x) \\
&= \sum_{s=0}^{k} C_k^s \sum_{\alpha\in\mathbb{Z}} S_{l-\alpha;b,k-s}\big((T^l_{\alpha,m,j}S_{l-\alpha})_{b,s}f\big)(x).
\end{aligned}$$

This together Lemma 2.4 (ii) now tells us that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V_{l,j}f(x)|^2\,dx \qquad (3.8)\\
&\qquad \le C\sum_{s=0}^{k}\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\Big(\sum_{\alpha\in\mathbb{Z}}\big|S_{l-\alpha;b,k-s}\big((T^l_{\alpha,m,j}S_{l-\alpha})_{b,s}f\big)(x)\big|\Big)^2 dx \\
&\qquad \le C\sum_{s=0}^{k}\|b\|_*^{2(k-s)}\int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{Z}}\sum_{j=1}^{D_m}\big|\big((T^l_{\alpha,m,j}S_{l-\alpha})_{b,s}f\big)(x)\big|^2\,dx.
\end{aligned}$$

Write

$$(T^l_{\alpha,m,j}S_{l-\alpha})_{b,s}f(x) = \sum_{u=0}^{s} C_s^u T^l_{\alpha,m,j;b,u}(S_{l-\alpha;b,s-u}f)(x), \qquad (3.9)$$

then by (3.8) and (3.9)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V_{l,j} f(x)|^2 \, dx \qquad (3.10) \\
&\le C \sum_{s=0}^{k} \|b\|_*^{2(k-s)} \int_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}} \sum_{j=1}^{D_m} \Big| \sum_{u=0}^{s} C_s^u T^l_{\alpha,m,j;b,u}(S_{l-\alpha;b,s-u} f)(x) \Big|^2 dx \\
&\le C \sum_{s=0}^{k} \|b\|_*^{2(k-s)} \int_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}} \sum_{j=1}^{D_m} \sum_{u=0}^{s} |T^l_{\alpha,m,j;b,u}(S_{l-\alpha;b,s-u} f)(x)|^2 \, dx \\
&\le C \sum_{s=0}^{k} \sum_{u=0}^{s} \sum_{\alpha\in\mathbb{Z}} \|b\|_*^{2(k-s)} \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |T^l_{\alpha,m,j;b,u}(S_{l-\alpha;b,s-u} f)(x)|^2 \, dx.
\end{aligned}$$

Let

$$T^l_{\alpha,m;b,u} h(x) = \Big(\sum_{j=1}^{D_m} |T^l_{\alpha,m,j;b,u} h(x)|^2 \Big)^{1/2}.$$

Thus by (3.10), we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V_{l,j} f(x)|^2 \, dx \le C \sum_{s=0}^{k} \sum_{u=0}^{s} \sum_{\alpha\in\mathbb{Z}} \|b\|_{BMO}^{2(k-s)} \qquad (3.11) \\
\times \int_{\mathbb{R}^n} |T^l_{\alpha,m;b,u}(S_{l-\alpha;b,s-u} f)(x)|^2 \, dx.
\end{aligned}$$

For $0 < \beta < (1-\varepsilon)/2$, if we can prove that there exists $0 < v_0 < 1$ such that

$$\|T^l_{\alpha,m;b,u} h\|_{L^2} \le C m^{-1+\beta} 2^{-\beta v_0 |l|/2} \|b\|_*^u \|h\|_{L^2}, \qquad (3.12)$$

then we have (3.7). In fact, by (3.12) and Lemma 2.4 (i), we have

$$\begin{aligned}
&\Big\| \Big(\sum_{\alpha\in\mathbb{Z}} |T^l_{\alpha,m;b,u}(S_{l-\alpha;b,s-u} f)|^2 \Big)^{1/2} \Big\|_{L^2}^2 \qquad (3.13) \\
&\qquad \le C m^{-2+2\beta} 2^{-\beta v_0 |l|} \|b\|_*^{2u} \sum_{\alpha\in\mathbb{Z}} \|S_{l-\alpha;b,s-u} f\|_{L^2}^2 \\
&\qquad \le C m^{-2+2\beta} 2^{-\beta v_0 |l|} \|b\|_*^{2s} \|f\|_{L^2}^2.
\end{aligned}$$

Then from (3.11) and (3.13), we get

$$\Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V_{l,j} f(x)|^2 \, dx \Big)^{1/2} \le C m^{-1+\beta} 2^{-\beta v_0 |l|/2} \|b\|_*^k \|f\|_{L^2}. \qquad (3.14)$$

So we get

$$\sum_{l\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|V_{l,j}f(x)|^2\,dx\Big)^{1/2}\le Cm^{-1+\beta}\|f\|_{L^2}.$$

This is (3.7). So to prove (3.7), it suffices to show (3.12).

Define $\widehat{\widetilde{T}^l_{\alpha,m,j}f}(\xi)=B^l_{\alpha,m,j}(2^{-\alpha}\xi)\widehat{f}(\xi)$. Denote by $\widetilde{T}^l_{\alpha,m,j;b,u}$ the u-th order commutator of $\widetilde{T}^l_{\alpha,m,j}$. Let

$$\widetilde{T}^l_{\alpha,m;b,u}f(\xi)=\Big(\sum_{j=1}^{D_m}|\widetilde{T}^l_{\alpha,m,j;b,u}f(\xi)|^2\Big)^{1/2}.$$

Recall $B_{\alpha,m,j}(\xi)=\widehat{\sigma_{\alpha,m,j}}(\xi)$, by Lemma 2.2, we get for any $0<\beta<1$,

$$\begin{aligned}\big|B_{\alpha,m,j}(\xi)\big|&\le Cm^{-\lambda-1+\beta/2}\min\big\{2^\alpha|\xi|,(2^\alpha|\xi|)^{-\beta/2}\big\}\big|Y_{m,j}(\xi')\big|,\\ \big|B_{\alpha,m,j}(\xi)\big|&\le Cm^{-\lambda-1}\big|Y_{m,j}(\xi')\big|,\\ \big|\nabla B_{\alpha,m,j}(\xi)\big|&\le C2^\alpha.\end{aligned}$$

Thus $\mathrm{supp}(B^l_{\alpha,m,j}(2^{-\alpha}\cdot))\subset\{2^{l-1}\le|\xi|\le2^{l+1}\}$, and

$$\begin{aligned}\big|B^l_{\alpha,m,j}(2^{-\alpha}\xi)\big|&\le Cm^{-\lambda-1+\beta/2}\min\big\{2^l,2^{-\beta l/2}\big\}\big|Y_{m,j}(\xi')\big|,\\ \big|B^l_{\alpha,m,j}(2^{-\alpha}\xi)\big|&\le Cm^{-\lambda-1}\big|Y_{m,j}(\xi')\big|,\\ \big|\nabla B^l_{\alpha,m,j}(2^{-\alpha}\xi)\big|&\le C.\end{aligned}$$

Using Lemma 2.3 and Remark 2.1 with $\delta=2^l$, we know that for any fixed $0<v<1$ and nonnegative integer u, we have

$$\|\widetilde{T}^l_{\alpha,m;b,u}f\|_{L^2}\le Cm^{(-1+\beta/2)v}2^{-\beta|l|v/2}\|b\|_*^u\|f\|_{L^2}.$$

For $0<\beta<(1-\varepsilon)/2$, we can find $0<v_0<1$, such that $v_0(-1+\beta/2)\le -1+\beta$. Then

$$\|\widetilde{T}^l_{\alpha,m;b,u}f\|_{L^2}\le Cm^{-1+\beta}2^{-\beta|l|v_0/2}\|b\|_*^u\|f\|_{L^2},$$

which, by dilation-invariance, implies

$$\|T^l_{\alpha,m;b,u}f\|_{L^2}\le Cm^{-1+\beta}2^{-\beta|l|v_0/2}\|b\|_*^u\|f\|_{L^2}.$$

So (3.12) is proved.

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