# Non-uniqueness
# in a free boundary problem

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### Abstract

We show that a result of Lewis and Vogel on uniqueness in a free boundary problem for the $p$-Laplace operator is sharp in two dimensions.

## 1. Introduction

Denote points in Euclidean 2 space $\mathbb{R}^2$ by $x = (x_1, x_2)$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^2$ and let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of $x$. Set $B(x, r) = \{ y \in \mathbb{R}^2 : |x - y| < r \}$ whenever $x \in \mathbb{R}^2$ and $r > 0$. Let $dx$ denote Lebesgue measure on $\mathbb{R}^2$ and define $k$ dimensional Hausdorff measure, in $\mathbb{R}^2$, $0 < k \le 2$, as follows: For fixed $\delta > 0$ and $E \subset \mathbb{R}^2$, let $L(\delta) = \{ B(x_i, r_i) \}$ be such that $E \subset \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left( \sum \alpha(k) r_i^k \right)$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 0 < k \le 2.$$

If $O$ is open and $1 \le q \le \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions $u$ with distributional gradient $\nabla u = (u_{x_1}, u_{x_2})$, both of which are $q$ th power integrable on $O$. Let

$$\|u\|_{1,q} = \|u\|_q + \|\nabla u\|_q$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue $q$ norm in $O$. Let $C_0^\infty(O)$ be the space of infinitely differentiable functions with

compact support in $O$ and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let $\Omega$ be a domain (i. e. an open connected set) and suppose that the boundary of $\Omega$ (denoted $\partial\Omega$) is bounded and non empty. Let $N$ be a neighborhood of $\partial\Omega$, $p$ fixed, $1 < p < \infty$ and $u$ a positive weak solution to the $p$ Laplace differential equation in $\Omega \cap N$. That is $u \in W^{1,p}(\Omega \cap N)$ and

$$(1.1) \qquad \int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Observe that if $u$ is smooth and $\nabla u \neq 0$ in $\Omega \cap N$, then $\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0$ where $\nabla \cdot$ denotes divergence. We assume that $u$ has zero boundary values on $\partial\Omega$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(N)$, then $u\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend $u$ to $N \setminus \Omega$ by putting $u \equiv 0$ on $N \setminus \Omega$. Then $u \in W^{1,p}(N)$ and it follows from (1.1) as in [10] that there exists a positive finite Borel measure $\mu$ on $\mathbb{R}^2$ with support contained in $\partial\Omega$ and the property that

$$(1.2) \qquad \int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = - \int \phi \, d\mu$$

whenever $\phi \in C_0^\infty(N)$. We give a proof that $\mu$ exists provided $u$ has a continuous extension to $N$. It suffices to show

$$F(\phi) = - \int_N \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx \geq 0.$$

for $\phi \geq 0$. Then the existence follows from the Riesz representation theorem and the basic estimates listed in section 2. To see this let $\phi = \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big)\psi$ where $\psi \in C_0^\infty\big(B(z,r)\big)$ and $\psi = 1$ on $B(z, r/2)$ and supp $\psi \subset B(z,r)$ for some $z \in \partial\Omega$. Then supp $\phi \subset \Omega$ so we get

$$(1.3) \qquad 0 = \int_N \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx$$
$$= \int_{u \geq \epsilon} \eta\big(\epsilon + \max(u - \epsilon, 0)\big)^{\eta-1} |\nabla u|^p \psi \, dx$$
$$+ \int_N \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big) |\nabla u|^{p-2} \langle \nabla \psi, \nabla u \rangle \, dx$$

Note that

$$\eta \int_{u \geq \epsilon} \big(\epsilon + \max(u - \epsilon, 0)\big)^{\eta-1} |\nabla u|^p \psi \, dx \geq 0$$

so

$$0 \geq \int_N \big((\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta\big) |\nabla u|^{p-2} \langle \nabla \psi, \nabla u \rangle \, dx$$

Suppose $r$ is so small that $u < 1$ in $B(z,r)$. Then

$$\left|(\epsilon + \max(u - \epsilon, 0))^\eta - \epsilon^\eta)|\nabla u|^{p-2}\langle \nabla \psi, \nabla u \rangle \right| \le \|\nabla \psi\|_\infty |\nabla u|^{p-1}.$$

Now $|\nabla u| \in L^{p-1}(\Omega)$ so we can use the dominated convergence theorem to take the limits under the integral sign as $\epsilon$ and $\eta$ go to zero and get $F(\psi) \ge 0$. We can use a partition of unity to reduce the problem to such small $r$'s. Note that if $\partial\Omega$ is smooth enough then

$$(1.4) \qquad d\mu = |\nabla u|^{p-1} dH^{n-1}$$

Let $E$ be a compact set and $G$ an open set containing $E$. For fixed $p$, $1 < p < \infty$ set

$$K_p(E, G) = \inf \left\{ \int |\nabla \theta|^p \, dx \right\}$$

where the infimum is taken over all $\theta \in C_0^\infty(G)$ with $\theta = 1$ on $E$. $K_p(E, G)$ is called the $p$-capacity of $E$ relative to $G$.

In [17] Lewis and Vogel consider the following free boundary problem. Given $F \subset \mathbb{R}^n$ a compact convex set, $a > 0$, and $1 < p < \infty$, find a function $u$ defined on a domain $D = D(a, p) \supset F$ with

$$(1.5a) \qquad \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 \text{ weakly in } D \setminus F,$$
$$(1.5b) \qquad u(x) \to 1 \text{ whenever } x \to y \in F$$
$$\text{and } u(x) \to 0 \text{ as } x \to y \in \partial D,$$
$$(1.5c) \qquad \mu = a^{p-1}H^{n-1} \text{ on } \partial D.$$

They prove

**Theorem A.** *Suppose $K_p(F, G) > 0$ for some open $G \supset F$ and let $D, u, p, a$ be as in* (1.5a), (1.5b) *and let $\mu$ be the measure corresponding to $u$ as in* (1.2). *If $\mu$ satisfies* (1.5c) *and in addition there exists $\beta$, $0 < \beta < \infty$ and $r_0 > 0$, for which*

$$(1.6) \qquad \mu[B(x, r)] \le \beta r^{n-1}, \quad 0 < r < r_0$$

*then $u$ and $D$ are uniquely determined.*

Previously Henrot and Shahgholian had considered the classical version of this problem that is the problem obtained by replacing (1.5c) by the condition $|\nabla u(x)| \to a$ whenever $x \to y \in \partial D$. In [11] they proved

**Theorem B.** *If $K_p(F, G) > 0$ for some open $G \supset F$ then there exists a unique $\hat{u}$, $\hat{D} = \hat{D}(a, p)$ such that* (1.5a), (1.5b) *are satisfied and $|\nabla u(x)| \to a$. Moreover $\hat{D}$ is convex with a smooth ($C^\infty$) boundary.*

In this paper we show that theorem A is sharp in two dimensions, namely

**Theorem 1.** *Suppose $n = 2$ and $K_p(F, G) > 0$ for some open $G \supset F$. If $a > 0$ and $1 < p < \infty$ there exists a bounded domain $D$ which is not convex, a $p$ harmonic function $u$ and a corresponding measure $\mu$ which satisfy (1.5) but $\mu$ does not satisfy (1.6).*

The proof uses the same method as the construction of pseudospheres in [16] to construct a domain which satisfies (1.5) but is not convex and thus is not the same as the domain in [11]. To outline this method let $\Omega$ be a domain and let $u$ be a function which satisfies (1.5a), (1.5b) with $D$ replaced by $\Omega$ and suppose $a = 1$. If $p < 2$ suppose that $|\nabla u| > 1$ on $\partial\Omega$ but if $p > 2$ suppose $|\nabla u| < 1$ on $\partial\Omega$. For a given small $\epsilon$ we add smooth bumps to $\partial\Omega$ by "pushing out" or "pushing in" along certain surface elements of $\{x \in \partial\Omega : |\nabla u(x)| > 1 + \epsilon\}$ or $\{x \in \partial\Omega : |\nabla u(x)| < 1 - \epsilon\}$ depending on whether $p > 2$ or $p < 2$. In this way we obtain a new domain $\Omega' \supset \Omega$ if $p < 2$ but $\Omega' \subset \Omega$ if $p > 2$ and we choose the bumps so that for $\epsilon \le t \le 1$

$$(1.7) \qquad H^1(\partial\Omega') \ge H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| > 1 + t\}$$

if $p < 2$ but

$$(1.8) \qquad H^1(\partial\Omega') \ge H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| < 1 - t\}$$

if $p > 2$. Here $\eta$ is a positive function on $]0, \infty[$. Let $u'$ be a function in $\Omega'$ which satisfies (1.5a), (1.5b) with $D$ replaced by $\Omega'$. If $p < 2$ then $\Omega \subset \Omega'$ and it follows that $u \le u'$ in $\Omega$ and by the maximum principle $|\nabla u'| > 1$ on $\partial\Omega \cap \partial\Omega'$. In section 3 we prove that $|\nabla u'| > 1$ on the bumps. If $p > 2$ we get $|\nabla u'| < 1$ in the same way. In section 4 we will show that there exists a certain elliptic partial differential equation for which $u'$ is a solution and $\log |\nabla u'|$ is a supersolution if $1 < p < 2$ and a subsolution if $p > 2$. Then we use the divergence theorem as in [2] to prove that if $1 < p < 2$ then

$$(1.9) \qquad \int_{\partial\Omega'} |\nabla u'|^{p-1} \log |\nabla u'| \, dH^1 \le C$$

and if $p > 2$ then

$$(1.10) \qquad \int_{\partial\Omega'} |\nabla u'|^{p-1} \log |\nabla u'| \, dH^1 \ge C$$

where the constant $C$ depends only on $F$. If $1 < p < 2$ this allows us to control the size of the set where $|\nabla u'|$ is large so that by pushing out and keeping $|\nabla u'| > 1$ we in fact keep $|\nabla u'|$ close to 1 for the most part. Likewise if $p > 2$ we are able to control the size of the set where $|\nabla u'|$ is close to zero.

Finally we use (1.7)-(1.10) and induction to construct $D$. We describe the case $p < 2$ in detail, the case $p > 2$ is similar. Let $D_0$ be a domain such that $u_0$ satisfies (1.5a) and (1.5b) with $D$ replaced by $D_0$ and let $\Omega = D_0$. Modify $\Omega$ as above to get $\Omega' = D_1$ and $u' = u_1$. If $D_k$ has been constructed for $0 \le k \le m$ we put $\epsilon_m = 2^{-m}\epsilon_0$ and modify $D_m$ to obtain $D_{m+1}$. Set $D = \bigcup_0^\infty D_k$. The construction can be arranged so that $D$ is not convex (see Section 4) which shows that it is not the domain in [17]. To prove (1.5$c$) we first note

$$(1.11) \qquad C \ge \int d\mu_k = \int_{\partial D_k} |\nabla u_k|^{p-1} dH^1 \ge H^1(\partial D_k)$$

for $k = 0, 1, \dots$ because $\mu_k(\partial D_k) \le C$ for some $C$ independent of $k$ (see Section 4). Second, for each $\delta > 0$ we have

$$(1.12) \qquad \lim_{k \to \infty} H^1\{x \in \partial D_k : |\nabla u_k(x)| > 1 + \delta\} = 0$$

since otherwise (1.7) and iteration would lead to a contradiction to (1.11). Next from (1.9) and the fact that $|\nabla u_k| > 1$ on $\partial D_k$ we see that for $M > 1$ and $k = 0, 1, \dots$

$$(1.13) \quad \log M \int_{\{|\nabla u_k| > M\}} |\nabla u_k|^{p-1} dH^1 \le \int_{\partial D_k} |\nabla u_k|^{p-1} \log |\nabla u_k| \, dH^1 \le C < \infty.$$

We also show that as $k \to \infty$

$$(1.14) \qquad H^1|_{\partial D_k} \to H^1|_{\partial D} \text{ and } \mu_k \to \mu$$

weakly as measures on $\mathbb{R}^2$ in section 4. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ and $\phi \ge 0$. Then we get

$$(1.15) \qquad \int \phi \, d\mu_k = \int_{\partial D_k} \phi |\nabla u_k|^{p-1} dH^1 \ge \int_{\partial D_k} \phi \, dH^1.$$

To obtain the reverse inequality let $\delta$ be a fixed small number and $M$ be a fixed large number and put

$$(1.16) \qquad E_k = \{x \in \partial D_k : 1 \le |\nabla u_k(x)| \le 1 + \delta\}$$
$$(1.17) \qquad F_k = \{x \in \partial D_k : 1 + \delta < |\nabla u_k(x)| \le M\}$$
$$(1.18) \qquad L_k = \{x \in \partial D_k : |\nabla u_k(x)| > M\}$$

for $k = 0, 1, \dots$. Then

$$\int \phi \, d\mu_k = \int_{\partial D_k} \phi |\nabla u_k|^{p-1} dH^1 = \int_{E_k} \cdots + \int_{F_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3.$$

It is clear that

$$|I_1| \le (1 + \delta)^{p-1} \int_{\partial D_k} \phi \, dH^1.$$

Also from (1.12) we have

$$|I_2| \leq M^{p-1}\|\phi\|_\infty H^1\{x \in \partial D_k : 1 + \delta < |\nabla u_k|\} \to 0$$

as $k \to \infty$. Using (1.13) we get

$$|I_3| \leq \|\phi\|_\infty \int_{|\nabla u_k|>M} |\nabla u_k|^{p-1}\, dH^1 \leq \frac{C}{\log M}\|\phi\|_\infty$$

Letting $k \to \infty$ we obtain from the above and (1.14)

$$\int_{\partial D} \phi\, dH^1 \leq \int \phi\, d\mu \leq (1+\delta)^{p-1} \int_{\partial D} \phi\, dH^1 + \frac{C}{\log M}\|\phi\|_\infty.$$

Finally letting $\delta \to 0$ and $M \to \infty$ we obtain

$$\int \phi\, d\mu = \int_{\partial D} \phi\, dH^1$$

which is what we wanted to prove. Finally the author would like to thank J. Lewis for pointing out this problem and helpful discussions.

## 2. Basic estimates

A Jordan curve $J$ is said to be a $k$ quasicircle $0 < k < 1$ if $J = f(\partial B(0,1))$ where $f \in W^{1,2}(\mathbb{R}^2)$ is a homeomorphism of $\mathbb{R}^2$ and

(2.1) $$|f_{\bar{z}}| \leq k|f_z|, \qquad H^2 \text{ a. e. in } \mathbb{R}^2.$$

Here we use complex notation, $i = \sqrt{-1}$, $z = x_1 + ix_2$, $2f_{\bar{z}} = f_{x_1} + if_{x_2}$, $2f_z = f_{x_1} - if_{x_2}$. We call $J$ a quasicircle if $J$ is a $k$ quasicircle for some $0 < k < 1$. Let $w_1, w_2$ be distinct points on the Jordan curve $J$ and $J_1, J_2$ the arcs with endpoints $w_1, w_2$. Then $J$ is said to satisfy the Ahlfors three point condition if there exists an $1 \leq M < \infty$ such that for all $w_1, w_2 \in J$ we have

$$\min\{\operatorname{diam} J_1, \operatorname{diam} J_2\} \leq M|w_1 - w_2|.$$

A Jordan curve $J$ is a quasicircle if and only if it satisfies the Ahlfors three point condition. A domain $\Omega$ is said to be uniform provided there exists $M, 1 \leq M < \infty$ such that if $w_1, w_2 \in \Omega$, then there is a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = w_1$, $\gamma(1) = w_2$, and

(2.2a) $$H^1(\gamma) \leq M|w_1 - w_2|$$

(2.2b) $$\min\{H^1(\gamma([0,t])), H^1(\gamma([t,1]))\} \leq Md(\gamma(t), \partial\Omega)$$

where $d(E,F)$ denotes the distance between two non-empty sets $E$ and $F$. If $1 \leq \tilde{M} < \infty$ and $\Omega$ is a domain a ball $B(w,r) \subset \Omega$ is said to be $\tilde{M}$ non-tangential if

$$\tilde{M}r > d(B(w,r), \partial\Omega) > \tilde{M}^{-1}r$$

If $w_1, w_2 \in \Omega$ a Harnack chain from $w_1$ to $w_2$ in $\Omega$ is a sequence of $\tilde{M}$ non-tangential balls such that the first ball contains $w_1$ the last ball contains $w_2$ and consecutive balls intersect. The conditions (2.2) are equivalent to

(2.3a)    For any $w \in \partial\Omega, 0 < r \leq \operatorname{diam}\Omega$, there exists
    $a = a_r(w) \in \Omega$ such that $M^{-1}r < |a - r| < r$ and
    $d(a, \partial\Omega) > M^{-1}r$

(2.3b)    Given $\epsilon > 0, w_1, w_2 \in \Omega, d(w_j, \partial\Omega) > \epsilon$ and
    $|w_1 - w_2| < C\epsilon$, there is a Harnack chain from
    $w_1$ to $w_2$ whose length depends on $C$ but not on $\epsilon$.

See [9] for references.

In the sequel $c$ will denote a positive constant $\geq 1$ (not necessarily the same at each occurrence) which may depend only on $p$ unless otherwise stated. In general $c(a_1, \ldots, a_n)$ denotes a positive constant $\geq 1$ which may only depend on $p, a_1, \ldots, a_n$, not necessarily the same at each occurrence. We begin by stating some interior and boundary estimates for $u$ a positive weak solution to the $p$ Laplacian in $B(w, 4r) \cap \Omega$ with $u = 0$ on $\partial\Omega \cap B(w, 4r)$ when this set is nonempty. In this case we extend $u$ to $B(w, 4r)$ by putting $u = 0$ on $B(w, 4r) \backslash \Omega$. Let $\max_{B(z,s)} u$, $\min_{B(z,s)} u$ be the essential supremum and infimum of $u$ on $B(z, s)$ whenever $B(z, s) \subset B(w, 4r)$.

**Lemma 1.** *Let $u$ be as above. Then*

$$c^{-1} r^{p-2} \int_{B(w,r/2)} |\nabla u|^p \, dx \leq \max_{B(w,r)} u^p \leq cr^{-2} \int_{B(w,2r)} u^p \, dx.$$

*If $B(w, 2r) \subset \Omega$, then*

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

**Proof.** The first display in Lemma 1 is a standard subsolution estimate while the second display is a standard weak Harnack estimate for positive weak solutions to nonlinear partial differential equations of $p$ Laplacian type (see [20]). ∎

**Lemma 2.** *Let $u$ be as in Lemma 1. Then $u$ has a representative in $W^{1,p}\big(B(w, 4r) \cap \Omega\big)$ with Hölder continuous partial derivatives in $B(w, 4r) \cap \Omega$. That is for some $\sigma = \sigma(p) \in\, ]0, 1[$ we have*

$$c^{-1}\big|\nabla u(w_1) - \nabla u(w_2)\big| \leq \big(|w_1 - w_2|/s\big)^\sigma \max_{B(z,s)} |\nabla u|$$

$$\leq cs^{-1}\big(|w_1 - w_2|/s\big)^\sigma \max_{B(z,2s)} u$$

*whenever $w_1, w_2 \in B(z, s)$ and $B(z, 4s) \subset B(w, 4r) \cap \Omega$.*

**Proof.** The proof of Lemma 2 can be found in [4], [14] or [21] and in fact is true when $B(w, 4r) \cap \Omega \subset \mathbb{R}^n$. In $\mathbb{R}^2$ the best Hölder exponent in Lemma 2 is known when $p > 2$ while for $1 < p \leq 2$ a solution has continuous second partials (see [12]). ∎

A mapping $h : B(w, 4r) \cap \Omega \to \mathbb{R}^2$ is said to be quasiregular in $B(w, 4r) \cap \Omega$ if $h \in W^{1,2}\big(B(w, 4r) \cap \Omega\big)$ and (2.1) holds with $f$ replaced by $h$ in $B(w, 4r) \cap \Omega$. From a factorization theorem for quasiregular mappings it follows that $h = \tau \circ f$ where $f$ is quasiconformal in $\mathbb{R}^2$ and $\tau$ is an analytic function on $f\big(B(w, 4r) \cap \Omega\big)$.

**Lemma 3.** *If $u$ is as in Lemma 1 and $z = x_1 + ix_2$ then $u_z$ is quasiregular in $B(w, 4r) \cap \Omega$ for some $0 < k < 1$ (depending only on $p$) and consequently $\nabla u$ has only isolated zeros in $B(w, 4r) \cap \Omega$.*

**Proof.** For a proof of quasiregularity see [1], [15]. Since the zeros of an analytic function are isolated it follows from the factorization theorem that the zeros of $\nabla u$ are isolated. ∎

**Lemma 4.** *If $B(w, 4r) \subset \Omega$, $\nabla u \neq 0$ in $B(w, 4r)$ and $\max_{B(w, 2r)} |\nabla u| \leq \lambda \max_{B(w, r)} |\nabla u|$ then*

$$\max_{B(w, 2r)} |\nabla u| \leq c(\lambda) \min_{B(w, r)} |\nabla u|$$

**Proof.** Note that $v = \log |\nabla u|$ is a weak solution in $B(w, 4r)$ to the divergence form partial differential equation (see [19])

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(A_{ij}(x)v_{x_j}) = 0$$

where the $(A_{ij})$ are bounded and uniformly elliptic (with constants depending only on $p$). Using Harnacks inequality for positive solutions to partial differential equations of this type (see [20]) applied to $\max_{B(w, 2r)} v - v$ in $B(w, r)$ we obtain the lemma. ∎

**Lemma 5.** *Let $u$ be as in Lemma 1 and $w \in \partial\Omega$. If $p > 2$ there exists $\alpha = \alpha(p) \in ]0, 1[$ such that $u$ has a Hölder $\alpha$ continuous representative in $B(w, r)$ (also denoted $u$). Moreover if $x, y \in B(w, r)$ then*

$$\big|u(x) - u(y)\big| \leq c\big(|x - y|/r\big)^\alpha \max_{B(w, 2r)} u.$$

*If $1 < p \leq 2$ and $\Omega$ is simply connected, then this inequality is also valid when $1 < p \leq 2$ with $\alpha = \alpha(p)$.*

**Proof.** For $p > 2$, Lemma 5 is a consequence of Lemma 1 and Morreys inequality (see [6]). If $1 < p \leq 2$ and $\Omega$ is simply connected we deduce from the interior estimates in Lemma 2 that it suffices to consider only the case when $y \in B(w, r) \cap \partial\Omega$. We then show for some $\theta = \theta(p, k), 0 < \theta < 1$ that

$$(2.4) \quad \max_{B(z,\rho/4)} u \leq \theta \max_{B(z,\rho/2)} u \quad \text{whenever } 0 < \rho < r \text{ and } z \in \partial\Omega \cap B(w, r).$$

This inequality can then be iterated to get Lemma 5 for $x, y$ as above. To prove (2.4) we use the fact that $B(z, \rho/4) \cap \partial\Omega$ and $B(z, \rho/4)$ have comparable $p$ capacities (see [10]) and estimates for subsolutions to elliptic partial differential equations of $p$ Laplacian type (see [8], [15]). ∎

**Lemma 6.** *Let $u, \Omega, w$ be as in Lemma 5. Assume also that $\Omega$ is a uniform domain. Then there exist $c = c(M)$ and $\hat{c} = \hat{c}(M)$ with*

$$\max_{B(w,r/\hat{c})} u \leq cu(a_{r/\hat{c}}(w))$$

*where $M$ is as in (2.2) and $a_r(w)$ is as in (2.3). Hence*

$$|u(x) - u(y)| \leq c(|x - y|/r)^{\alpha}u(a_{r/\hat{c}}(w))$$

*for $x, y \in B(w, r/2\hat{c})$.*

**Proof.** The first display in Lemma 6 follows from Harnacks principle in Lemma 1, Hölder continuity of $u$ in Lemma 5 and the fact that $\Omega$ is a uniform domain and a general argument which can be found in [3]. The second display follows from the first display and Lemma 5 ∎

To proceed we consider the following scenario. Let $\Omega$ be a domain such that $\partial\Omega$ is $C^4$. Let $w \in \partial\Omega$ and let $u$ be a positive $p$ harmonic function in $\Omega \cap B(w, 2r)$ and assume that $\Omega \cap B(w, 2r)$ has only one component. We further assume that $\nabla u \neq 0$ in $\Omega \cap B(w, r)$. We have

**Lemma 7.** *Let $u$ be as above. If $x \in \Omega \cap B(w, r)$ there exists a $c \geq 1$ depending only on $k$ and $p$ such that*

$$c^{-1}d(x, \partial\Omega)^{-1}u(x) \leq |\nabla u(x)| \leq cd(x, \partial\Omega)^{-1}u(x)$$

*where $d(x, \partial\Omega)$ denotes the distance from $x$ to $\partial\Omega$*

**Proof.** Choose $y \in B(x, d(x, \partial\Omega))$ with $u(y) = u(x)/2$. Apply the mean value theorem of calculus to $u$ restricted to the line segment with endpoints $x, y$. From this and Lemma 6 it follows that there exists a constant $c \geq 4$

and $z$ such that $y \in B(x, (1 - c^{-1})d(x, \partial\Omega))$ and $z$ is on the line segment between $x$ and $y$ and

$$u(x)/2 = |u(x) - u(y)| \leq |\nabla u(z)||x - y|.$$

Using this inequality and Lemma 2 we see for some positive $\tilde{c}$ that if $t_1 = (1 - c^{-1})d(x, \partial\Omega), t_2 = (1 - (2c)^{-1})d(x, \partial\Omega)$ then

$$(2.5) \qquad \tilde{c}^{-1}u(x)/d(x, \partial\Omega) \leq \max_{B(x,t_1)} |\nabla u| \leq \max_{B(x,t_2)} |\nabla u| \leq \tilde{c}u(x)/d(x, \partial\Omega).$$

From (2.5) and Lemma 4 we conclude that Lemma 7 is valid for $u$ at $x$. ∎

Let $\theta$ be a function whose graph is after a rotation and translation $\Omega \cap B(w, r/2)$ and suppose that the $C^4$-norm of $\theta$ is bounded by $c/r$. The condition (3.1) stated in the next section is clearly sufficient. At each point $x \in \partial\Omega \cap B(w, r/2)$ we can find a tangential ball $B(z, \rho) \subset \Omega \cap B(w, r)$ with $x \in \partial B(z, \rho)$ and radius $\rho > 0$ depending only on $\lambda$ and $r$. Let $v$ be the $p$ harmonic function which is zero on $\partial B(z, \rho)$ and $\inf_{\partial B(z, \rho/2)} u$ on $\partial B(z, \rho/2)$. Then $v \leq u$ in the annulus $B(z, \rho) \setminus B(z, \rho/2)$. Therefore

$$|\nabla u(t)| \geq c^{-1}u(t)d(t, \partial\Omega)^{-1} \geq c^{-1}v(t)d(t, \partial\Omega)^{-1} \geq c^{-1} \inf_{\partial B(z, \rho/2)} u/\rho$$

for $t$ in the annulus where we used the fact that $v(x) = A|x - z|^{\frac{p-2}{p-1}} + B$ to compute $\nabla v$. Then by Harnack's inequality we get a lower bound in terms of $\max_{B(w,r)} u$. We can argue in the same way to get an upper bound so that we have

$$(2.6) \qquad c^{-1} \max_{B(w,r)} u/r \leq |\nabla u|(t) \leq c \max_{B(w,r)} u/r$$

for $t$ in $B(w, r/2) \cap \Omega$ and thus $u \in W^{1,2}(B(w, r/2))$.

Let $\tilde{u}(x) = u(rx + w)/r$. Then $\tilde{u}$ is a solution to the $p$ Laplace equation in $B(0, 2) \cap \tilde{\Omega}$ where $\tilde{\Omega} = \{x \in \mathbb{R}^2 : rx + w \in \Omega\}$. Let $\Phi$ be a differentiable mapping from $B(0, 2)$ to $B(0, 2)$ such that $0$ is mapped to $0$ and $]-1, 1[$ is mapped to $\partial\tilde{\Omega} \cap B(0, 2)$ and $\{(x, y) \in B(0, 2) : y > 0\}$ is mapped to $B(0, 2) \cap \tilde{\Omega}$. Define $v = \tilde{u} \circ \Phi$ in $\{(x, y) \in B(0, 2) : y > 0\}$ and let $v(x, y) = 0$ in $\{(x, y) \in B(0, 2) : y < 0\}$. Then $v$ satisfies an equation of the form

$$(2.7) \qquad \nabla \cdot \left( \langle A\nabla v, \nabla v \rangle^{p/2-1} A\nabla v \right) = 0$$

in $B(0, 2)^+ = B(0, 2) \cap \{(x, y) \in B(0, 2) : y > 0\}$ where $A = [A_{ij}]$ is a symmetric matrix whose coefficients are in $C^1$. From our work above it follows that

$$(2.8) \qquad c^{-1} \max_{B(0,2)} v \leq |\nabla v|(x) \leq c \max_{B(0,2)} v$$

for $x \in B(0,1)^+$ where the constant may depend on $\Phi$. If we let $A(x,\xi) = \langle A\xi, \xi \rangle^{p/2-1} A\xi$ then we have

$$(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \leq c\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle$$

and

(2.9)
$$|\nabla_x A(x,\eta)| \leq c|\eta|^{p-1}$$

where $\nabla_x$ denotes the gradient with respect to the $x$ variable.

**Lemma 8.** *Let $v$ be as above. Then $v$ has weak derivatives of second order and $v_{x_1} \in W^{1,2}(B(0,1/2))$ and we have*

$$\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_j x_i}|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_j} - a|^2 \, dx + c\rho^2 (\max_{B(0,1)} v)^2$$

*if $B(z,\rho) \subset B(0,1/2)^+ = \{(x_1, x_2) \in B(0,1/2) : x_2 > 0\}$ and $a \in \mathbb{R}$. In addition we have*

$$\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_1 x_i}|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_1}|^2 \, dx + c\rho^2 (\max_{B(0,1)} v)^2$$

*for any $z \in B(0,1/2)$ and $\rho \leq 1/4$*

**Proof.** Let

$$D_k^h v(x) = \frac{v(x + he_k) - v(x)}{h}$$

where $e_k$ denotes the $k$-th unit vector. Let $\zeta$ be a smooth function such that $\zeta = 1$ on $B(z,\rho/2)$, supp $\zeta \subset B(z,\rho)$ and $|\nabla \zeta| \leq c/\rho$ for some constant $c$. Since $v \in W^{1,2}(B(0,1)^+)$ and $v = 0$ on $\{(x_1, x_2) : x_2 = 0\}$ the function $\phi = D_k^{-h}(\zeta^2(D_k^h v - a))$ belongs to $W_0^{1,2}(B(0,1)^+)$ if $B(z,\rho) \subset B(0,1/2)^+$ and if $a = 0$ and $k = 1$ we have $\phi \in W_0^{1,2}(B(0,1)^+)$ for any $z \in B(0,1/2)$ and $\rho \leq 1/4$. This function is therefore an admissible test function. We obtain

$$0 = \int \left\langle A(x, \nabla v), \nabla\big(D_k^{-h}(\zeta^2(D_k^h v - a))\big) \right\rangle dx$$

$$= \int \left\langle D_k^h A(x, \nabla v), \nabla\big(\zeta^2(D_k^h v - a)\big) \right\rangle dx$$

$$= \int \left\langle \frac{A(x + he_k, \nabla v(x + he_k)) - A(x + he_k, \nabla v(x))}{h}, \nabla\big(\zeta^2(D_k^h v - a)\big) \right\rangle dx$$

$$+ \int \left\langle \frac{A(x + he_k, \nabla v(x)) - A(x, \nabla v(x))}{h}, \nabla\big(\zeta^2(D_k^h v - a)\big) \right\rangle dx = \mathrm{I} + \mathrm{II}$$

The first of these integrals is

$$
\int \left\langle \frac{A(x + he_k, \nabla v(x + he_k)) - A(x + he_k, \nabla v(x))}{h}, \zeta^2 D_k^h \nabla v \right\rangle dx
$$
$$
+ \int \left\langle \frac{A(x + he_k, \nabla v(x + he_k)) - A(x + he_k, \nabla v(x))}{h}, 2\zeta \nabla \zeta (D_k^h v - a) \right\rangle dx
$$

The first term in this expression can be bounded below by

$$
c^{-1} \int \left( |\nabla v(x + he_k)| + |\nabla v(x)| \right)^{p-2} \zeta^2 |D_k^h \nabla v|^2 \, dx
$$

and the second term can be bounded above by

$$
\int \left( |\nabla v(x + he_k)| + |\nabla v(x)| \right)^{p-2} |D_k^h \nabla v||\nabla \zeta| |2\zeta (D_k^h v - a)| \, dx
$$
$$
\leq \epsilon \int \left( |\nabla v(x + he_k)| + |\nabla v(x)| \right)^{p-2} |D_k^h \nabla v|^2 \zeta^2 \, dx
$$
$$
+ \frac{c}{\epsilon} \int \left( |\nabla v(x + he_k)| + |\nabla v(x)| \right)^{p-2} |\nabla \zeta|^2 |D_k^h v - a|^2 \, dx
$$

by Youngs inequality. As for II we get

$$
\text{II} \leq c \int |\nabla v|^{p-1} \zeta^2 |D_k^h \nabla v| \, dx + c \int |\nabla v|^{p-1} |\zeta||\nabla \zeta||D_k^h v - a| \, dx
$$
$$
\leq \epsilon \int |\nabla v|^{p-2} \zeta^2 |D_k^h \nabla v|^2 dx + \frac{c}{\epsilon} \int |\nabla v|^p \zeta^2 dx + \frac{c}{\rho^2} \int |\nabla v|^{p-2} |D_k^h v - a|^2 dx.
$$

Choosing $\epsilon$ small enough and using (2.8) to estimate $|\nabla v|^{p-2}$ we get

$$
\int_{B(z,\rho/2)} |D_k^h \nabla v|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_k} - a|^2 \, dx + \rho^2 (\max_{B(0,1)} v)^2.
$$

We conclude that $(D_k^h v)\zeta \in W_0^{1,2}(B(z,\rho))$ with a norm independent of $h$. It now follows from a weak compactness argument that $v_{x_k}\zeta \in W_0^{1,2}(B(z,\rho))$ and

$$
\int_{B(z,\rho/2)} \sum_{i=1}^{2} |v_{x_k x_i}|^2 \, dx \leq \frac{c}{\rho^2} \int_{B(z,\rho)} |v_{x_k} - a|^2 \, dx + \rho^2 (\max_{B(0,1)} v)^2.
$$

This is what we wanted to prove.                                    ■

Recall that if $\psi \in W^{1,2}(B(z,\rho))$ and $\psi_{B(z,\rho)} = \frac{1}{|B(z,\rho)|} \int_{B(z,\rho)} \psi \, dx$ then

$$
|\psi(x) - \psi_{B(z,\rho)}| \leq C \int_{B(z,\rho)} \frac{|\nabla \psi(y)|}{|x - y|} \, dy
$$

Let $\psi(x) = v_{x_1}(x)$ and $1/4 > \rho$. It follows if $x \in B(0, 1/4)$

$$|\psi(x) - \psi_{B(z,\rho)}| \leq \int_{B(z,\rho)} \frac{|\nabla\psi(y)|}{|x-y|}\, dy$$
$$= \int_{B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\, dy + \int_{B(z,\rho)\backslash B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\, dy$$

and by Hölders inequality

$$(2.10) \quad \int_{B(z,2\rho)\backslash B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\, dy$$
$$\leq c \left( \int_{B(z,2\rho)} |\nabla\psi(y)|^q\, dy \right)^{1/q} \left( \int_{B(z,2\rho)\backslash B(x,\delta)} \frac{1}{|x-y|^{\frac{q}{q-1}}}\, dy \right)^{\frac{q-1}{q}}$$
$$\leq c\|\nabla\psi\|_q \delta^{(q-2)/q}$$

and for the other integral we have the estimate

$$(2.11) \quad \int_{B(x,\delta)} \frac{|\nabla\psi(y)|}{|x-y|}\, dy \leq \sum_{n=0}^{\infty} 2\left(\frac{2^k}{\delta}\right) \int_{\{2^{-(k+1)}\delta \leq |x-y| \leq 2^{-k}\delta\}} |\nabla\psi(y)|\, dy$$
$$\leq 2 \sum_{k=0}^{\infty} \frac{\delta}{2^k} M(|\nabla\psi|) \leq 2\delta M(|\nabla\psi|)$$

Here $M(f)$ denotes the maximal function of $f$. We conclude

$$|\psi(x) - \psi_{B(z,\rho)}| \leq 2\delta M(|\nabla\psi|) + \delta^{(q-2)/q}\|\nabla\psi\|_q$$

and if we choose

$$\delta = \left( \frac{\|\nabla\psi\|_q}{2M(|\nabla\psi|)} \right)^{q/2}$$

we get

$$|\psi(x) - \psi_{B(z,\rho)}|^2 \leq cM(|\nabla\psi|)^{2-q}\|\nabla\psi\|_q^q$$

Integrating and applying Hölder's inequality yields for $1 < q < \frac{3}{2}$

$$(2.12) \quad \|\psi(x) - \psi_{B(z,\rho)}\|_2^2 \leq \rho \left( \int_{B(z,2\rho)} M(|\nabla\psi|)^{4-2q}\, dx \right)^{1/2} \|\nabla\psi\|_q^q$$
$$\leq \rho \left( \int_{B(z,2\rho)} |\nabla\psi|^{4-2q}\, dx \right)^{1/2} \|\nabla\psi\|_q^q$$

If $B(z, \rho) \subset B(0, 1/2)^+$ Lemma 8 and (2.12) yield with $\psi = v_{x_1}$, $\psi_{B(z,\rho)} = a$

(2.13)

$$
\int_{B(z,\rho/2)} \sum_i |v_{x_1 x_i}|^2 \, dx
$$
$$
\leq \frac{c}{\rho} \left( \int_{B(z,\rho)} \left( \sum_{i,j} |v_{x_j x_i}| \right)^{4-2q} dx \right)^{1/2} \int_{B(z,\rho)} \left( \sum_{i,j} |v_{x_j x_i}| \right)^q dx + c\rho^2 (\max_{B(0,1)} v)^2
$$

If $B(z, \rho) \cap B(0, 1/2)^- \neq \emptyset$ then we take $x = (x_1, x_2) \in B(z, \rho) \cap B(0, 1/2)^+$ and let $x^* = (x_1, -x_2)$. Note that if $x, y \in B(0, 1)^+$ then $|x - y| \leq |x^* - y|$. Since $\psi = 0$ in $B(0, 1)^-$ we get

$$
|\psi_{B(z,\rho)}| \leq \int_{B(z,\rho)} \frac{|\nabla \psi(y)|}{|x^* - y|} \, dy \leq c \int_{B(z,\rho)} \frac{|\nabla \psi(y)|}{|x - y|} \, dy
$$

since $\psi(x^*) = 0$. This allows us to get rid of $\psi_{B(z,\rho)}$ in our work above and we see that (2.13) holds in this case as well.

**Lemma 9.** *Let $u$ be defined as above Lemma 7 and $v$ be defined as above Lemma 8. Then $v \in C^4(\overline{B(0, 1/4)}^+)$ and we have*

(2.14)
$$
|D^2 v|(x) \leq c \max_{B(0,1)} v
$$

*for $x$ in $B(0, 1/4)^+$. For the function $u$ we have $u \in C^4(\overline{\Omega \cap B(w, r/8)})$ and*

(2.15)
$$
|\nabla u|(x) \leq \frac{c}{r} \max_{B(w,r)} u
$$

(2.16)
$$
|D^2 u|(x) \leq \frac{c}{r^2} \max_{B(w,r)} u
$$

*for $x \in B(w, r/8) \cap \Omega$.*

**Proof.** It follows from lemma 8 that $v$ is a strong solution of (2.7). Writing the equation in nondivergence form we obtain

$$
|v_{x_2 x_2}|^2 \leq c \left( \sum_{i=1}^2 |v_{x_1 x_i}| \right)^2 + c (\max_{B(0,1)} v)^2
$$

Let $g = \sum |v_{x_i x_j}|$. We obtain

(2.17)

$$
\fint_{B(z,\rho)} g^2 \, dx \leq \fint_{B(z,2\rho)} g^q \, dx \left( \fint_{B(z,2\rho)} g^{4-2q} \, dx \right)^{1/2} + \rho^2 (\max_{B(0,1)} v)^2
$$

$$
\leq \epsilon \left( \fint_{B(z,2\rho)} g^q \, dx \right)^{2/q} + C \left( \fint_{B(z,2\rho)} g^{4-2q} \, dx \right)^{1/(2-q)} + \rho^2 (\max_{B(0,1)} v)^2
$$

$$
\leq \epsilon \fint_{B(z,2\rho)} g^2 \, dx + C \left( \fint_{B(z,2\rho)} g^{4-2q} \, dx \right)^{1/(2-q)} + \rho^2 (\max_{B(0,1)} v)^2
$$

where we first used Youngs inequality and then Jensens inequality. In a ball $B(x, 2t) \subset B(0, 1)$ we define

$$
f(y) = \frac{\delta(y)}{2t} g(y)
$$

where $\delta(y)$ is the distance from $y$ to $\partial B(x, 2t)$ and note

(2.18)    $2f(y) \geq g(y)$ for $y \in B(x, t)$ and $f(y) \leq g(y)$ for $y \in B(x, 2t)$

If $z \in B(x, 2t)$ then

(2.19)    $$
\fint_{B(z,\delta(z)/2)} f^2(y) \, dy \leq \left( \frac{2}{\delta(z)} \right)^2 \int_{B(z,\delta(z)/2)} \left( \frac{\delta(y)}{2t} \right)^2 g^2(y) \, dy
$$

$$
\leq \frac{4}{t^2} \int_{B(x,2t)} g^2(y) \, dy = \lambda_0^2
$$

Let $\mu_0^2 = \lambda_0^2 + 2t^2 (\max_{B(0,1)} v)^2$, take $\lambda \geq \lambda_0$, let $\mu^2 = \lambda^2 + 2t^2 (\max_{B(0,1)} v)^2$ and $F(\mu) = \{z \in B(x, 2t) : f(z) > \mu\}$. Then it follows from differentiation theory that for almost every $z \in F(\mu)$ there exists $\rho > 0$ such that

$$
\fint_{B(z,\rho)} f^2 \, dx > \mu^2
$$

If $z \in F(\mu)$ and $\rho$ is sufficiently small it follows from (2.19) that we can select $\rho$ such that $10\rho < \delta(z)/2$ and

(2.20)    $$
\fint_{B(z,10\rho)} f^2 \, dx < \mu^2
$$

(2.21)    $$
\fint_{B(z,\rho)} f^2 \, dx > \mu^2
$$

Then we obtain

$$(2.22) \quad \int_{B(z,2\rho)} g^2 \, dy \le \int_{B(z,10\rho)} g^2 \, dy \le (10\rho)^2 \left(\frac{2t}{\delta(z)}\right)^2 \fint_{B(z,10\rho)} f^2(y) \, dy$$

$$\le (10\rho)^2 \left(\frac{2t}{\delta(z)}\right)^2 \fint_{B(z,\rho)} f^2 \, dy \le c \int_{B(z,\rho)} g^2 \, dy.$$

Along with (2.17) this gives the estimate

$$(2.23) \quad \fint_{B(z,\rho)} g^2 \, dx \le C \left(\fint_{B(z,2\rho)} g^{4-2q} \, dx\right)^{1/(2-q)} + 2t^2 (\max_{B(0,1)} v)^2$$

Since $10\rho < \delta(z)/2$ we have $\delta(z)/4 < \delta(y)/2 < \delta(z)$ for all $y \in B(z, 2\rho)$. Therefore

$$\fint_{B(z,\rho)} f^2 \, dx \le C \left(\fint_{B(z,2\rho)} f^{4-2q} \, dx\right)^{1/(2-q)} + 2t^2 (\max_{B(0,1)} v)^2$$

From (2.21) it now follows

$$(2.24) \quad \lambda^{4-2q} \le C \fint_{B(z,2\rho)} f^{4-2q} \, dx$$

so

$$(2.25) \quad \fint_{B(z,10\rho)} f^2 \, dx \le \mu^2 = \lambda^2 + 2t^2 (\max_{B(0,1)} v)^2$$

$$\le C\lambda^{2q-2} \left(\fint_{B(z,2\rho)} f^{4-2q} \, dx\right) + 2t^2 (\max_{B(0,1)} v)^2$$

Let $E(\mu) = \{y \in B(x, 2t) : f(y) < \mu\}$ and note

$$(2.26) \quad \int_{E(\delta\mu) \cap B(z,2\rho)} f^{4-2q} \, dx \le (\delta\mu)^{4-2q} m(B(z, 2\rho))$$

where $m$ denotes two dimensional Lebesgue measure. By a well known covering theorem we can find a sequence of balls $\{B(z_i, \rho_i)\}$ such that (2.21), (2.20) and (2.25) hold and

$$(2.27) \quad m(F(\mu) \setminus \bigcup_i B(z_i, 10\rho_i)) = 0$$

$$(2.28) \quad B(z_i, 2\rho_i) \cap B(z_j, 2\rho_j) = \emptyset \quad i \ne j$$

Now we have

$$(2.29) \qquad \int_{F(\mu)} f^2 \, dx \leq \sum_i \int_{B(z_i, 10\rho_i)} f^2 \, dx$$

$$\leq \lambda^{2q-2} \left( \sum_i \int_{B(z_i, 2\rho_i)} f^{4-2q} \, dx \right) + 2t^2 (\max_{B(0,1)} v)^2$$

$$\leq C\lambda^{2q-2} \left( \int_{F(\delta\mu)} f^{4-2q} \, dx \right) + 2t^2 (\max_{B(0,1)} v)^2.$$

Let $M$ be a large number and put

$$(2.30) \qquad\qquad\qquad \tilde{f} = \min\{f, M\}$$

$$(2.31) \qquad\qquad\qquad \tilde{F}(\mu) = \{z \in B(x, 2t) : \tilde{f}(z) > \mu\}$$

Then it follows that

$$(2.32) \qquad \int_{\tilde{F}(\mu)} \tilde{f}^2 \, dx \leq C\lambda^{2q-2} \left( \int_{\tilde{F}(\delta\mu)} \tilde{f}^{4-2q} \, dx \right) + 2t^2 (\max_{B(0,1)} v)^2$$

Now we get with integration by parts and Fubini's theorem

$$(2.33)$$

$$\int_{\tilde{F}(\mu_0)} \tilde{f}^{2+\gamma} \, dx = \gamma \int_{\tilde{F}(\mu_0)} \tilde{f}^2 \int_0^{\tilde{f}} \mu^{\gamma-1} \, d\mu \, dx = \gamma \int_{\mu_0}^\infty \mu^{\gamma-1} \int_{\tilde{F}(\mu)} \tilde{f}^2 \, dx \, d\mu$$

$$\leq \gamma \int_{\mu_0}^\infty \mu^{\gamma+2q-3} \left( \int_{\tilde{F}(\delta\mu)} \tilde{f}^{4-2q} \, dx \right) + 2t^2 (\max_{B(0,1)} v)^2 \, d\mu$$

$$= \frac{(4-2q)\gamma\delta^{4-2q}}{\gamma+2q-2} \int_{\mu_0}^\infty \mu^{1+\gamma} m(\tilde{F}(\delta\mu)) \, d\mu$$

$$+ \frac{\gamma}{\gamma+2q-2} \mu_0^{\gamma+2q-2} \left( \int_{\tilde{F}(\delta\mu_0)} \tilde{f}^{4-2q} \, dx \right) + 2t^2 (\max_{B(0,1)} v)^2$$

By choosing $\delta$ small enough this gives

$$(2.34) \qquad \int_{\tilde{F}(\mu_0)} \tilde{f}^{2+\gamma} \, dx \leq C\mu_0^{\gamma+2q-2} \left( \int_{B(x,2t)} \tilde{f}^{4-2q} \, dx \right) + t^2 (\max_{B(0,1)} v)^2$$

By the monotone convergence theorem we see that this inequality holds for $f$ and by (2.18), (2.19) and Jensen's inequality that

$$(2.35) \qquad \left( \fint_{B(x,t)} g^{2+\gamma} \, dx \right)^{1/(2+\gamma)} \leq C \left( \fint_{B(x,2t)} g^2 \, dx \right)^{1/2} + c (\max_{B(0,1)} v)^2$$

This implies that $v \in W^{2,2+\gamma}(B(0,1/4)^+)$ and from Morrey's inequality we see that $v \in C^{1,\alpha}(\overline{B(0,1/4)}^+)$ and the $C^{1,\alpha}$ norm of $v$ is bounded by the $W^{2,2+\gamma}$ norm of $v$. If we write (2.7) in nondivergence form we obtain an equation

$$\sum_{i,j} a_{ij}(x, \nabla v) v_{x_i x_j} + b(x, \nabla v) = 0.$$

Since the matrix $A$ in (2.7) is smooth and the function $v \in C^{1,\alpha}(\overline{B(0,1/4)}^+)$ it follows that $a_{ij} \in C^{\alpha}(\overline{B(0,1/4)}^+)$. Also (2.8) gives us that the equation is strictly elliptic. Then lemma 9 follows from boundary Schauder estimates (see [18, chapter 6]). ∎

## 3. Preliminary reductions

Assume $\Omega$ is a bounded domain of class $C^4$. This means that for each $y \in \partial\Omega$ there exists $s > 0$ such that $B(y,s) \cap \partial\Omega$ is a part of the graph of a four times continuously differentiable function defined on a line in $\mathbb{R}^2$ and $B(y,s) \cap \Omega$ lies above the graph. We use compactness and a standard covering argument to obtain $y^1, \ldots, y^N \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{i=1}^N B(y^i, 100r) \quad \text{and} \quad B(y^i, 10r) \cap B(y^j, 10r) = \emptyset, \quad i \neq j$$

If $r$ is sufficiently small and $y = y^i$ then it follows from the implicit function theorem that there exists a function $\theta = \theta(\cdot, y)$ four times continuously differentiable on $\mathbb{R}$ with $\theta(0) = 0$ and $\theta_x(0) = 0$ such that after a rotation of the axes, if necessary:

$$\partial\Omega \cap B(y, 1000r^{1/2}) \subset \left\{ (x_1 + y_1, \theta(x_1) + y_2) : x_1 \in \mathbb{R} \right\}$$
$$\Omega \cap B(y, 1000r^{1/2}) \subset \left\{ (x_1 + y_1, x_2) : x_2 - y_2 > \theta(x_1), x_1 \in \mathbb{R} \right\}$$

Let

$$K_1 = \max_{y \in \{y^i\}_1^N} \left( \max_{x \in \partial\Omega \cap B(y, 100r^{1/2})} \sum_{k=1}^4 |\theta^{(k)}(\cdot, y)| \right)$$

and for $0 < \epsilon < \sigma_0 \leq 10^{-3}$ choose $r_0 > 0$ so small that for $0 < r \leq r_0$

$$(3.1) \qquad\qquad K_1 r^{1/2} \leq 10^{-3} r^{1/4} \leq 10^{-9} \epsilon^4$$

which is possible since $K_1 < +\infty$ by compactness of $\partial\Omega$. Let $u$ be a function satisfying (1.5a)–(1.5b) with $D$ replaced by $\Omega$ and assume that $u \in C^4(\overline{\Omega})$

and $|\nabla u| > 1$ on $\partial\Omega$. Let

$$K_2 = \max_{y \in \{y^i\}_1^N} \left( \max_{x \in \bar{\Omega} \cap B(y, 100r^{1/2})} \sum |\partial_\alpha u(x)| \right)$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multiindex and $0 \le |\alpha| \le 4$. Choose $r_0$ even smaller so that if $0 < r \le r_0$ then

$$(3.2) \qquad K_2 r^{1/2} \le 10^{-3} r^{1/4} \le 10^{-9} \epsilon^4$$

Let $l$ be the largest nonnegative integer such that $2^{-l} \sigma_0 > \epsilon$ and let $\sigma_k = 2^{-k} \sigma_0$ for $k = 0, 1, \ldots$. Put

$$(3.3) \qquad E_k = \left\{ x \in \partial\Omega : 1 + \sigma_k < |\nabla u(x)| \le 1 + \sigma_{k-1} \right\},$$

for $1 \le k \le l+1$ and

$$(3.4) \qquad E_0 = \left\{ x \in \partial\Omega : |\nabla u(x)| > 1 + \sigma_0 \right\}$$

Let $\psi \ge 0$ be a $C^\infty$ function on $\mathbb{R}$ with $\max \psi = 1$ and support in the unit interval. Let $L$ be the set of all $y \in \{y^i\}_1^N$ for which

$$B(y, 100r) \cap \bigcup_{k=0}^{l+1} E_k \ne \emptyset$$

For a fixed $y = (y_1, y_2) \in L$ let $j$ be the smallest nonnegative integer with

$$(3.5) \qquad B(y, 100r) \cap E_j \ne \emptyset$$

Put

$$\xi(x_1) = \theta(x_1) - \sigma_j^4 r \psi(x_1 / r \sigma_j^2) + y_2 \quad x_1 \in \mathbb{R}$$

Now we define $\Omega'$ as follows

(i) $\qquad \Omega \setminus \bigcup_{y \in L} B(y, 10r) = \Omega' \setminus \bigcup_{y \in L} B(y, 10r)$

(ii) $\qquad \partial\Omega' \cap B(y, 10r) = \{(x_1 + y_1, \xi(x_1)) : x_1 \in \mathbb{R}\} \cap B(y, 10r)$

(iii) $\qquad \Omega' \cap B(y, 10r) = \{(x_1 + y_1, x_2) : x_2 > \xi(x_1)\}\} \cap B(y, 10r)$.

Clearly $\Omega'$ is of class $C^4$.

**Lemma 10.** *Let $u'$ be defined by (1.5a)–(1.5b) with $D$ replaced by $\Omega'$. Then $u' \in C^4(\bar{\Omega}')$ and if $r_0$ is small enough*

$$(3.6) \qquad |\nabla u'(x)| > 1, \quad x \in \partial\Omega'.$$

**Proof.** First $u' \in C^4(\overline{\Omega})$ follows from lemma 9 since $\nabla u' \neq 0$ in $\Omega'$ (see [15]). If $x \in \partial\Omega' \cap \partial\Omega$ then it follows from the maximum principle that (3.6) is true. Let $Z(y, t) = \{(x_1, x_2) : |x_i - y_i| < t, i = 1, 2\}$. If $x \in \partial\Omega' \setminus \partial\Omega$ we first note that since $\psi$ has support in the unit interval

$$(3.7) \qquad\qquad (\partial\Omega' \setminus \partial\Omega) \cap B(y, 10r) \subset Z(y, r)$$

whenever $y \in L$. From the maximum principle and (3.7) it follows that to prove (3.6) it suffices to show that

$$(3.8) \qquad\qquad |\nabla u^*(x)| > 1 \quad x \in Z(y, r) \cap \partial\Omega^*$$

where $\Omega^*$ is obtained by adding just one bump to $\Omega$ at the point $y$ and $u^*$ satisfies (1.5a)-(1.5b) with $D$ replaced by $\Omega^*$.

We note that since $|\nabla u(x)| > 1$ on $\partial\Omega$ it follows from (3.2) that $u_{x_2} > 1/2$ when $x \in Z(y, r)$. Let $t_0 = \min_Y u$ where $Y = \{(x_1, x_2) \in \partial Z(y, r) \cap \Omega : |y_2 - x_2| = r\}$. Note that $ct_0 \geq \max_{\Omega \cap Z(y,r)} u$ by Harnack's inequality. Let $U = \Omega \cap Z(y, r) \cap \{u < t_0\}$ and note that $u$ is increasing on $\partial U \cap \partial Z(y, r)$. Let $U^* = \Omega^* \cap Z(y, r) \cap \{u(x) < t_0\}$. Define $v$ to be the $p$ harmonic function in $U^*$ such that $v = 0$ on $\partial\Omega^*$ and $v = u$ on $\partial U^* \setminus \partial\Omega^*$. Note that $v \leq u^*$ in $U^*$ by the boundary maximum principle so it suffices to show $|\nabla v| > 1$ on $\partial\Omega^*$. In order to do this we need to apply the estimates in section 2 to the function $v$. This requires us to show that $\nabla v \neq 0$.

Consider the function $v^\epsilon$ in $U^*$ which solves the equation

$$(3.9) \qquad\qquad \nabla \cdot ((|\nabla v^\epsilon|^2 + \epsilon)^{p/2-1} \nabla v^\epsilon)$$

and satisfies $v^\epsilon = v$ on $\partial U^*$. This equation is strictly elliptic so it follows from Schauder estimates (see [13] or [18]) that $v^\epsilon$ is real analytic in the interior of $U^*$ and continous in the closure of $U^*$ (see [13]). If $t < t_0$ the set $\partial U \cap \{u = t\}$ contains exactly two points. Since $v^\epsilon = u$ on $\partial U^* \setminus \partial\Omega^*$ the set $\{v^\epsilon(x) > s\}$ is connected in $U^*$ ($s < t_0$) since each component must intersect the boundary of $U$ by the maximum principle for $v^\epsilon$. We note that it follows from [15] that if $|\nabla v^\epsilon(x_0)| = 0$ then $\{v^\epsilon(x) > v^\epsilon(x_0)\}$ can not be connected. Since we have already concluded that these sets are connected we see that $\nabla v^\epsilon \neq 0$ in $U^*$. Now one can argue as in [15] to obtain $\nabla v \neq 0$ in $U^*$. Since $u$ is Hölder continous there exists a $\lambda$ which depends only on $p$ so that $u < t_0$ in $Z(y, \lambda r)$ so $Z(y, \lambda r) \cap \Omega^* \subset U^*$. Thus we have $\nabla v \neq 0$ in $Z(y, \lambda r)$.

Now we can apply lemma 9 to $v$ and obtain

$$\max_{Z(y, \lambda r/8) \cap \Omega^*} |D^2 v| \leq \frac{c}{r^2} \max_{Z(y, \lambda r)} v \leq \frac{c}{r^2} \max_{Z(y, r)} u \leq \frac{c}{r} |\nabla u|(t)$$

for $t \in Z(y, r)$.

Let $\sigma_0$ be so small that $\sigma_0 < \lambda/8$. By the maximum principle $|\nabla v| \geq |\nabla u|$ on $\partial\Omega \cap \partial\Omega^*$ and from our construction we know that there exists some point $x \in \partial\Omega \cap B(y, 100r)$ such that $1 + \sigma_j \leq |\nabla u|(x)$. From (3.2) it follows that $|\nabla u|(x) \geq 1 + \sigma_j/2$ for all $x \in \partial\Omega \cap B(y, 100r)$. Pick a point $z \in \partial\Omega \cap \partial\Omega^* \cap B(y, \sigma_j^2 r)$. By (3.2) we see that

$$|\nabla u(t)| \leq |\nabla u(z)| + 10^{-9}\epsilon^4 r^{1/2}$$

for $t \in Z(y, r)$. Choosing $\sigma_0$ smaller so that $C\sigma_0 < 10^{-3}$ and using the mean value theorem and (3.2) we obtain for $x \in \partial\Omega^* \cap B(y, \sigma_j^2 r)$

(3.10)

$$|\nabla v(z) - \nabla v(x)| \leq \max_{Z(y,r)\cap\Omega^*} |D^2 v||z - x|$$

$$\leq c|\nabla u|(t)\frac{|z - x|}{r} \leq 10^{-3}\sigma_j|\nabla u|(z) + 10^{-12}\epsilon^4 r^{1/2}\sigma_j$$

and since $|\nabla v|(z) \geq |\nabla u|(z)$

(3.11)
$$|\nabla v|(x) \geq (1 - 10^{-3}\sigma_j)|\nabla v|(z) - 10^{-12}\epsilon^4 r^{1/2}\sigma_j$$

$$\geq (1 - 10^{-3}\sigma_j)(1 + \frac{1}{2}\sigma_j) - 10^{-12}\epsilon^4 r^{1/2}\sigma_j > 1.$$

Which is what we needed to prove. ∎

**Lemma 11.** *Let $\Omega$, $\Omega'$ be as above. If $\epsilon \leq t \leq 1$*

(3.12) $\qquad H^1(\partial\Omega') \geq H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| > 1 + t\}$

*if $p < 2$ but*

(3.13) $\qquad H^1(\partial\Omega') \geq H^1(\partial\Omega) + \eta(t)H^1\{x : |\nabla u(x)| < 1 - t\}$

*if $p > 2$. Here $\eta$ is a positive function on $]0, \infty[$.*

**Proof.** To prove (3.12) let

$$c_2 = \int_{\mathbb{R}} |\psi'(x)|^2 \, dx$$

and choose $\sigma_0$ even smaller so that

(3.14) $\qquad\qquad \sigma_0 \leq c_2 \leq 2(\max_{\mathbb{R}} |\psi'|)^2 \leq \sigma_0^{-1} 10^{-6}$

Then it follows from (3.1) and the definition of $\sigma_j$

$$(3.15) \quad H^1(Z(y,r) \cap \partial\Omega') = \int_{-r}^{r} \sqrt{1 + |\xi'|^2} \, dx$$

$$\geq \int_{-r}^{r} \sqrt{1 + \sigma_j^4 |\psi'(x/r)|^2} \, dx - 2\epsilon^8 r$$

$$= r \int_{-1}^{1} \sqrt{1 + \sigma_j^4 |\psi'(x)|^2} \, dx - 2\epsilon^8 r$$

$$\geq (1 + \frac{1}{4}\sigma_j^4 c_2 - \epsilon^8)2r \geq \frac{1}{8}\sigma_j^4 c_2 2r + H^1(Z(y,r) \cap \partial\Omega).$$

Take $t \geq \epsilon$ and let $k$ be the least nonnegative integer such that $t \geq \sigma_k$, $0 \leq k \leq l+1$. Let $J = J(k)$ be the set of all $i$ such that (3.5) holds with $y = y^i$ and $j \leq k$. From (3.1) it is clear that

$$(3.16) \quad H^1\{x \in \partial\Omega : |\nabla u(x)| \geq 1 + t\} \leq H^1\left(\bigcup_{i \in J} B(y^i, 100r) \cap \partial\Omega\right)$$

$$\leq 2 \sum_{i \in J} 200r$$

and we conclude that

$$(3.17) \quad H^1(\partial\Omega') \geq H^1(\partial\Omega) + c_3 \sigma_k^4 H^1\{x \in \partial\Omega : |\nabla u(x) > 1 + t\}$$

Let

$$(3.18) \quad \eta(t) = \begin{cases} c_3\sigma_0^4 & \text{if } \sigma_0 \leq t \\ c_3\sigma_k^4 & \text{if } \sigma_k \leq t < \sigma_{k-1}, k = 1, 2, \ldots \end{cases}$$

Since $\eta$ does not depend on $\Omega$ this proves (3.12). The case when $p > 2$ is similar. ∎

## 4. Proof of Theorem 1

**Lemma 12.** *Let $u$, $\Omega$ be as above. If $1 < p < 2$ then*

$$(4.1) \qquad \int_{\partial\Omega} |\nabla u|^{p-1} \log |\nabla u| \, dH^1 \leq C$$

*and if $p > 2$ then*

$$(4.2) \qquad \int_{\partial\Omega} |\nabla u|^{p-1} \log |\nabla u| \, dH^1 \geq C$$

*where the constant $C$ depends only on $F$.*

**Proof.** We proceed as in [2]. Note that if $\eta \in \mathbb{R}^2$ and $|\eta| = 1$ then $\zeta = \langle \nabla u, \eta \rangle$ is a strong solution to

$$L\zeta = \nabla \cdot \left( (p-2)|\nabla u|^{p-4}\langle \nabla u, \nabla \zeta \rangle \nabla u + |\nabla u|^{p-2}\nabla \zeta \right) = 0$$

in $\Omega \cap N$ since $\nabla u \neq 0$. In other words

$$(4.3) \qquad L\zeta = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left( a_{ik}(x)\zeta_{x_k}(x) \right) = 0$$

where

$$(4.4) \qquad a_{ik}(x) = |\nabla u|^{p-4}\left( (p-2)u_{x_i}u_{x_k} + \delta_{ik}|\nabla u|^2 \right)(x)$$

and $\delta_{ij}$ is the Kronecker $\delta$. Note that

$$(4.5) \qquad Lu = (p-1)\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$$

Since the equation is rotationally invariant we can assume that $\nabla u(x) = (|\nabla u(x)|, 0)$. Let $v = \log|\nabla u(x)|$. Then

$$v_{x_k} = |\nabla u|^{-2}\sum_{l=1}^{2} u_{x_l}u_{x_l x_k}$$

and so

$$Lv = \sum_{i,k=1}^{2} \frac{\partial(a_{ik}v_{x_k})}{\partial x_i} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left( |\nabla u|^{-2}\sum_{k,l=1}^{2} a_{ik}u_{x_l}u_{x_l x_k} \right).$$

Using (4.3) on the righthand side we get

$$(4.6) \qquad Lv = -2|\nabla u|^{-4}\sum_{i,k,l,m=1}^{2} a_{ik}\left( u_{x_l}u_{x_l x_k}u_{x_m}u_{x_m x_i} \right)$$

$$+ |\nabla u|^{-2}\sum_{i,k,l=1}^{2} a_{ik}u_{x_l x_i}u_{x_l x_k} = T_1 + T_2.$$

From the definiton of the $a_{ik}$'s and our assumption that $\nabla u(x) = (|\nabla u(x)|, 0)$ we see at $x$

$$(4.7) \qquad a_{11} = (p-1)|\nabla u|^{p-2}, a_{22} = |\nabla u|^{p-2} \quad \text{and} \quad a_{12} = a_{21} = 0$$

and also from (4.5)

$$(4.8) \qquad (p-1)u_{x_1 x_1} + u_{x_2 x_2} = 0.$$

Using this in the definitions of $T_1, T_2$ we obtain at $x$

$$T_1 = -2|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2)$$

and

$$T_2 = p|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2)$$

and we conclude

(4.9)    $$Lv = (p-2)|\nabla u|^{p-4}((p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2)$$

so $Lv \leq 0$ when $1 < p < 2$ and $Lv \geq 0$ when $p > 2$. Since $u$ is smooth and $\nabla u \neq 0$ and $\partial\Omega$ is smooth we can apply the divergence theorem to the vector field whose $i$th component is

$$u \sum_{k=1}^{2} a_{ik} v_{x_k} - v \sum_{k=1}^{2} a_{ik} u_{x_k}$$

in the region $\Omega \backslash G$ where $G$ is a region with smooth boundary which contains the set $F$ in its interior. If $1 < p < 2$ we obtain

(4.10)

$$0 \geq \int_{\Omega \backslash G} u \left( \sum_{k=1}^{2} \frac{\partial}{\partial x_i}(a_{ik}v_{x_k}) \right) - v \left( \sum_{k=1}^{2} \frac{\partial}{\partial x_i}(a_{ik}u_{x_k}) \right) dx$$

$$= \int_{\partial\Omega} |\nabla u|^{p-1} \log|\nabla u| \, dH^1 + \int_{\partial G} \sum_{i=1}^{2} \left( u \sum_{k=1}^{2} a_{ik}v_{x_k} - v \sum_{k=1}^{2} a_{ik}u_{x_k} \right) \eta_i \, dH^1$$

where $\eta$ is the outward unit normal for $\Omega \backslash G$ on $\partial G$ and we used the fact that $u = 0$ on $\partial\Omega$ and $\eta = -\frac{\nabla u}{|\nabla u|}$ on $\partial\Omega$. This gives (1.9) and (1.10) where the constant is determined by the integral over $\partial G$ which is independent of $\Omega$. ∎

Remember that $\psi$ is a $C^\infty$ function on $\mathbb{R}$ with $\max \psi = 1$ and support in the unit interval. Also, in section 3 $\sigma_0$, $0 < \sigma_0 \leq 10^{-3}$ was chosen so that (3.14) was true. Finally, for a given $\epsilon$, $0 < \epsilon \leq \sigma_0$ $r_0$ was chosen so small that the estimates in section 3 are true for $0 < r \leq r_0$. We describe the construction of $D$ in more detail. We only describe the case of "pushing out" since the other case is similar. Let $D_0$ be a domain such that $F \subset D_0$ and the function $u_0$ which satisfies (1.5a)-(1.5b) for $D_0$ also satisfies $|\nabla u_0| > 1$ on $\partial D_0$. Let $\rho = d(\partial\Omega, F)$ . Let $\epsilon_0 = \sigma_0$ and $\epsilon_k = 2^{-k}\epsilon_0$ for $k = 1, 2, \ldots$. Choose a covering $L_1 = \{B(z_0^i, t_0^i)\}$, $1 \leq i \leq k_0$ of $\partial D_0$ such that $t_0^i \leq 1/2$ for all $i$ and

$$2 \sum_{i=1}^{k_0} t_0^i \leq H^1(\partial D_0) + \frac{1}{2}$$

Since $D_0$ is compact we can assume $k_0 < \infty$. Let $2r_1' > 0$ be the distance from $\partial D_0$ to $\mathbb{R}^2 \setminus \bigcup_1^{k_0} B(z_0^i, t_0^i)$. Set $\Omega = D_0$, $\epsilon = \epsilon_1$ and $r_1 = \min\{r_1', r_0(\epsilon_1, K_1, K_2), 10^{-9}\rho\}$ where $K_1$ and $K_2$ are defined relative to $D_0$, $u_0$ as in section 3. Then we do as in section 3 to obtain $D_1 = \Omega'$. Now suppose for some $m \geq 1$ we have defined $\{D_k\}_0^m$, $\{L_k\}_0^m$, $\{r_k'\}_0^m$ and $\{r_k\}_0^m$. Let $L_{m+1} = \{B(z_m^i, t_m^i)\}_1^{k_m}$ be a covering of $\partial D_m$ such that $t_m^i \leq 2^{-(m+1)}$, $1 \leq i \leq k_m$ and

$$(4.11) \qquad 2\sum_{i=1}^{k_m} t_m^i \leq H^1(\partial D_m) + 2^{-(m+1)}$$

Let $2r_m' > 0$ be the distance between $\partial D_m$ and $\mathbb{R}^2 \setminus \bigcup_1^{k_m} B(z_m^i, t_m^i)$. Let $\Omega = D_m$, $\epsilon = \epsilon_m$ and $r = r_{m+1} = \min\{r_m', r_0(\epsilon_{m+1}, K_1, K_2), 10^{-4m}r_m\}$ where $K_1$ and $K_2$ are defined relative to $D_m$, $u_m$ as in Section 3. Then we do as in Section 3 to obtain $D_{m+1} = \Omega' \supset D_m$. By induction we get $\{D_k\}_0^\infty$, $\{L_k\}_1^\infty$, $\{r_k'\}_1^\infty$ and $\{r_k\}_1^\infty$. Finally define $D$ to be the union of the sets $D_k$

**Lemma 13.** *Let $D$, $D_k$, $k = 1, 2, \ldots$ be as above. Then $D$ is a quasicircle which is not convex. For $D_k$ we have $\mu_k(\partial D_k) \leq C$ where $C$ is inedependent of $k$ and $\mu_k$ is the measure corresponding to $u_k$ as in (1.2).*

**Proof.** To prove that $D$ is a quasicircle it suffices to show that $\partial D_m$ satisfies the Ahlfors three point condition for $m = 1, 2, \ldots$ with constant independent of $m$. Once we have proved this we get a sequence $\{f_m\}$ of quaisconformal mappings of $\mathbb{R}^2$ with

$$(4.12) \qquad f_m(\partial B(0,1)) = \partial D_m \text{ and } |(f_m)_{\bar{z}}| \leq k|(f_m)_z|$$

where $0 < k < 1$ is independent of $m$. Since a subsequence of $\{f_m\}$ converges uniformly on compact subsets of $\mathbb{R}^2$ to a quasiconformal $f : \mathbb{R}^2 \to \mathbb{R}^2$ we see that (4.12) holds with $f_m, D_m$ replaced by $f, D$. To show that $\partial D_m$ satisfies the Ahlfors three point condition independent of $m$ we first find a constant $C$ such that $|z_1 - z_3| < C|z_1 - z_2|$ for $z_1, z_2, z_3$ on the graph of $\psi$ and $z_3$ between $z_1$ and $z_2$. Now suppose $z_1, z_2, z_3$ lie on $\partial D_m$ and $|z_1 - z_2| < 10r_m$. Let $\xi$ be a function whose graph is after a rotation and translation $\partial D_m \cap B(z_1, 10r_m)$. By (3.1) the distance $|\xi(x) - \psi(x)|$ is less than $10^{-9}\epsilon_m^4|x|$ which implies that the graph of $\xi$ and therefore $\partial D_m \cap B(z_1, 10r_m)$ satisfies the Ahlfors three point condition with a slightly larger constant $C$ but still independent of $m$. If $|z_1 - z_2| > 10r_m$ we find $k < m$ such that $|z_1 - z_2| < 10r_k$ but $|z_1 - z_2| > 100r_{k+1}$. Let $z^*$ be the projection of $z \in \partial D_m$ on $\partial D_k$. Then $|z_1 - z_2| > |z_1^* - z_2^*| - \eta r_k$ where $\eta$ is small and likewise $|z_1 - z_3| < |z_1^* - z_3^*| + \eta r_k$. From this it follows that $|z_1 - z_3| < 2C|z_1 - z_3|$ for all $m$.

To prove that $\mu_k(\partial D_k) \leq C$ where $C$ is independent of $k$ we recall that $\mu(B(x, r)) \leq cr^{2-p}(\max_{B(x, 2r)} u)^{p-1}$ for any measure defined by (1.2). This estimate is proved in [5] and our claim follows immediately by covering the boundaries of the domains $D_k$ with balls and then applying the estimate in each ball since $u_k(x) \leq 1$ for all $x \in D_k$. To see that the domain is not convex note that the function $\psi$ can be chosen so that $D_m$ has the property that there exist points $x, y \in D_m$ such that

$$\max_{t \in [0,1]} d(tx + (t-1)y, D_m) > \frac{\epsilon_m^4 r_m}{8} > 5^{4m} r_{m+1} \frac{\epsilon_0}{8} > r_{m+1}$$

if $m$ is large enough. It is clear from the construction described above that if $z \in D$ then $d(z, D_m) < r_{m+1}$ so the line segment between $x$ and $y$ does not lie in $D$. However $x, y \in D$ so $D$ is not convex. ∎

The proof of Theorem 1 follows from the above lemmas and the argument at the end of section 1 once we prove (1.14). The proof that $H^1|_{\partial D_m} \to H^1|_{\partial D}$ in [16] applies to our case without change. For completeness we give a brief outline. First show that there exists a mapping $h_m$ from $\partial D_m$ to $\partial D_{m+1}$ which satisfies

$$|h_m(x) - h_m(z)| \geq (1 - cr_m^{1/2})|x - z|.$$

Then let

$$p_j(x) = \lim_{k \to \infty} h_k \circ \cdots \circ h_{j+1}(x) \text{ for } x \in \partial D_j.$$

If

$$e_j = \prod_{m=j+1}^{\infty} (1 - cr_m^{1/2})$$

it follows that

$$e_j|x - y| \leq |p_j(x) - p_j(y)|, \qquad x, y \in \partial D_j,$$

and if $q_j$ is the inverse of $p_j$ we have

(4.13) $$|q_j(x) - q_j(y)| \leq e_j^{-1}|x - y|$$

when $x, y \in \partial D$. Next we use Kirsbraun's Theorem (see [7]) to obtain an extension of $q_j$ to $\mathbb{R}^2$ such that (4.13) holds whenever $x, y \in \mathbb{R}^2$. Let $\nu(E) = H^1(q_j^{-1}(E) \cap \partial D)$. Then we have

$$H^1(E \cap \partial D_j) \leq e_j \nu(E)$$

Also note that it follows from the definition of the $r_m$'s that $e_j \to 1$ when $j \to \infty$. Let $g \geq 0$ be a continuous function. Then it follows from the

change of variables formula that

$$(4.14) \qquad e_j \int_{\partial D_j} g \, dH^1 \le \int_{\mathbb{R}^n} g \, d\nu = \int_{\partial D} g \circ q_j \, dH^1$$

If we let $j \to \infty$ then $q_j(x) \to x$ uniformly on compact subsets of $\mathbb{R}^n$ so

$$\int_{\partial D} g \circ q_j \, dH^1 \to \int_{\partial D} g \, dH^1$$

Hence from (4.14) we have

$$\limsup_{k \to \infty} \int_{\partial D_n} g \, dH^1 \le \int_{\partial D} g \, dH^1$$

From our construction of $D$ it follows that

$$H^1(\partial D) \le \liminf_{m \to \infty} H^1(\partial D_m)$$

If $0 \le g \le 1$ then it follows that

$$H^1(\partial D) \le \liminf_{k \to \infty} H^1(\partial D_{n_k})$$

$$\le \liminf_{k \to \infty} \int_{\partial D_k} g \, dH^1 + \limsup_{k \to \infty} \int_{\partial D_k} (1 - g) \, dH^1$$

$$\le \limsup_{k \to \infty} \int_{\partial D_n} g \, dH^1 + \limsup_{k \to \infty} \int_{\partial D_n} (1 - g) \, dH^1$$

$$\le \int_{\partial D} g \, dH^1 + \int_{\partial D} (1 - g) \, dH^1 = H^1(\partial D)$$

Thus equality holds everywhere so

$$\lim_{k \to \infty} \int_{\partial D_n} g \, dH^1 = \int_{\partial D} g \, dH^1$$

which is what we wanted to prove.

To show that $\mu_k \to \mu$ we note that if we are pushing out then $u(x) < \epsilon$ on $\partial D_n$ for $n$ large enough. Therefore $u(x) < u_n(x) + \epsilon$ in $D_n$ in other words $u(x) - u_n(x) < \epsilon$ in $D_n$. Elsewhere $u_n(x) = 0$ and $u(x) < \epsilon$ so $u_n \to u$ uniformly. Since the measures $\mu_n$ are bounded we have a subsequence which is weakly convergent to some measure $\nu$. Now

$$(4.15) \qquad \int \phi \, d\nu = \lim_{n \to \infty} \int \phi \, d\mu_n = \lim_{n \to \infty} \int_N |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle \, dx$$

$$= \int_N \lim_{n \to \infty} |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla \phi \rangle \, dx$$

$$= \int_N |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = \int \phi \, d\mu$$

where $N$ is some neighborhood containing $\partial D$ and $\partial D_n$ if $n$ is large enough and $\phi \in C_0^\infty(N)$. It follows that $\nu = \mu$ which is what we wanted to show.

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