Stable Higgs G-sheaves

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Abstract

For a connected reductive group G, we generalize the notion of (semi)stable Higgs G-bundles on curves to smooth projective schemes of higher dimension, allowing also Higgs G-sheaves, and construct the corresponding moduli space.

Let X be a projective and smooth scheme over $\mathbb{C}$. Recall that a Higgs bundle on X is a vector bundle E together with a Higgs field, that is, a homomorphism $\theta : E \to E \otimes \Omega_X$, where Ω_X is the cotangent bundle of X, and such that $\theta \wedge \theta = 0$ ([6, 13, 8]). If $\dim X > 1$, it is natural to consider also Higgs sheaves, that is, to allow E to be a coherent sheaf, not necessarily locally free. This has already been done by Simpson in [14]. On the other hand, it is natural to consider Higgs G-bundles, where G is a reductive algebraic group. The vector bundle is substituted by a principal G-bundle, and the Higgs field by a homomorphism $P(\mathfrak{g}) \to \Omega_X$, where $\mathfrak{g}$ is the Lie algebra of G. (Using the Killing form on the semisimple part of the Lie algebra $\mathfrak{g}'$, extended to $\mathfrak{g}$ by choosing a nondegenerate bilinear form on the center $\mathfrak{z}$, this is equivalent to a section of $P(\mathfrak{g}) \otimes \Omega_X$. See [15, §9])

In this article we will consider Higgs fields for principal G-sheaves on a smooth projective complex scheme X, which are the analog of torsion free sheaves when we work with an arbitrary reductive structure group G.

When working with principal bundles on schemes with dimension higher than one, it is natural to enlarge the category. This is analogous to the case of vector bundles, where we are led to consider also torsion free sheaves or vector bundles with singular connections. In the first case we have the moduli space of Gieseker-Maruyama, and on the second case that of Uhlenbeck. Analogously, for principal bundles there are two approaches: that of Schmitt [9, 11] and Gómez-Sols [5], which is a generalization of Gieseker-Maruyama approach, and that of Balaji [2], which is a generalizations of Uhlenbeck's method.

Recall [5] that a principal G-sheaf on E is a tuple (P, E, ξ) where E is a torsion free sheaf on X, P is a principal G-bundle on the open set U_E where E is locally free and ξ is an isomorphism

$$\xi : P(\mathfrak{g}') \xrightarrow{\cong} E|_{U_E} .$$

The group of characters $G \to \mathbb{C}^*$ of G is free. Let χ_i be a set of generators. The numerical invariants of a principal G-sheaf are the Hilbert polynomial of E and the Chern classes $d_i \in H^2(X, \mathbb{C})$ of the line bundles $P(\chi_i)$ induced by the characters χ_i.

Recall that an isomorphism between two principal G-sheaves is a pair of isomorphisms $(\beta : P \to P', \gamma : E \to E')$ such that the following diagram is commutative

$$
\begin{array}{ccc}
P(\mathfrak{g}') & \xrightarrow{\xi} & E|_{U_E} \\
\beta(\mathfrak{g}') \downarrow & & \downarrow \gamma|_{U_E} \\
P'(\mathfrak{g}') & \xrightarrow{\xi'} & E'|_{U_E}
\end{array}
$$

The isomorphism ξ and the Lie algebra structure $\mathfrak{g}' \otimes \mathfrak{g}' \to \mathfrak{g}'$ induce a homomorphism $E|_U \otimes E|_U \longrightarrow E|_U$ which uniquely extends ([5, Lemma 0.25]) to a homomorphism

$$[\,,] : E \otimes E \longrightarrow E^{\vee\vee}   .$$

Likewise, the isomorphism ξ and the Killing form $\mathfrak{g}' \otimes \mathfrak{g}' \to \mathbb{C}$ induce a homomorphism $E|_U \otimes E|_U \to \mathcal{O}_U$ which uniquely extends ([5, Lemma 0.18]) to a bilinear form

$$\kappa : E \otimes E \longrightarrow \mathcal{O}_X   .$$

Definition 0.1 *A Higgs G-sheaf is a principal G-sheaf (P, E, ξ) together with a Higgs field. By this we mean a homomorphism*

$$\Theta : E \oplus \mathfrak{z}_X \longrightarrow \Omega_X$$

($\mathfrak{z}_X := \mathfrak{z} \otimes \mathcal{O}_X$) with $\Theta \wedge \Theta = 0$, where the wedge product $\Theta \wedge \Theta$ is the section of $E^{\vee\vee} \otimes \Omega_X^2$ extending the following section on U (which is defined using the Lie algebra structure of E and its Killing form):

$$\mathcal{O}_U \xrightarrow{\Theta|_{E_U}^{\vee} \otimes \Theta|_{E_U}^{\vee}} E|_U^{\vee} \otimes \Omega_U \otimes E|_U^{\vee} \otimes \Omega_U \xrightarrow{\kappa^{-1} \otimes \kappa^{-1}} E|_U \otimes E|_U \otimes \Omega_U \otimes \Omega_U \xrightarrow{\frac{1}{2}[,] \otimes \wedge} E|_U \otimes \Omega_U^2$$

where, as usual, the last morphism is the combination of the Lie algebra structure and the wedge product.

An isomorphism of Higgs G-sheaves is an isomorphism (β, γ) of principal G-sheaves which is compatible with the Higgs field in the sense that the following diagram is commutative

$$
\begin{array}{ccc}
E \oplus \mathfrak{z}_X & \xrightarrow{\ \Theta\ } & \Omega_X \\
\scriptstyle{\gamma \oplus \mathrm{id}} \downarrow & & \downarrow \scriptstyle{\mathrm{id}} \\
E' \oplus \mathfrak{z}_X & \xrightarrow{\ \Theta'\ } & \Omega_X
\end{array}
$$

A filtration

$$
0 \subsetneq E_{-l} \subseteq E_{-l+1} \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = E
$$

is said *orthogonal* if $E_i^\perp = E_{-i-1}$ (where $E_i^\perp$ denotes the kernel of the composition $E \xrightarrow{\ \kappa\ } E^\vee \to E_i^\vee$), and it is called *algebra filtration* if $[E_i, E_j] \subset E_{i+j}{}^{\vee\vee}$ for all i, j.

Denote $\theta = \Theta|_E$. Let θ^* be the section of $E^{\vee\vee} \otimes \Omega_X$ extending the following section on U

$$
\mathcal{O}_U \xrightarrow{\ \theta^\vee\ } E^\vee|_U \otimes \Omega_U \xrightarrow{\ \kappa^{-1}\ } E|_U \otimes \Omega_U
$$

The filtration is called *compatible with the Higgs field* if the image of θ^* lies in $E_0^{\vee\vee} \otimes \Omega_X$. This is equivalent to $\theta|_{E_{-1}} = 0$. Note that, if the Higgs field θ is compatible with the filtration, then

$$
(0.1) \qquad\qquad [\theta^*, E_i] \subset E_i{}^{\vee\vee} \otimes \Omega_X
$$

for all i.

Definition 0.2 *A Higgs G-sheaf is called (semi)stable if for all orthogonal algebra filtrations $E_\bullet$ compatible with the Higgs field the following condition is satisfied*

$$
P_{E_\bullet} = \sum_{i=-l}^{l} \left(r P_{E_i} - r_i P_E \right) (\preceq) 0
$$

Now we will introduce the notion of S-equivalence. Let $(\mathcal{P}, \Theta) = (P, E, \xi, \Theta)$ be a Higgs G-sheaf, and let $E_\bullet$ be an *admissible* orthogonal algebra filtration, by which we mean $P_{E_\bullet} = 0$. In [5, Lemma 5.4] it is shown that this produces a reduction of structure group (see section §1 for definitions of standard constructions with principal G-bundles) P^Q to a parabolic subgroup $Q \subset G$ on the open set U where it is a bundle filtration ($E_0 \oplus \mathfrak{z}_X$ is the Lie algebra bundle of this reduction). Let $Q \twoheadrightarrow L$ be its Levi quotient, and $L \hookrightarrow Q \subset G$ a splitting. We call the semistable Higgs G-sheaf

$$
(P^Q(Q \twoheadrightarrow L \hookrightarrow G), \oplus E^i, \xi', \theta')
$$

the associated *admissible deformation* of $(\mathcal{P}, \Theta)$ (the homomorphisms Θ^i is the homomorphisms induced by Θ on $(\oplus E^i) \oplus \mathfrak{z}_X$ the quotients $E^i = E_i/E_{i-1}$, and ξ' is the natural isomorphism between $P^Q(Q \twoheadrightarrow L \hookrightarrow G)(\mathfrak{g}')$ and $\oplus E^i|_U$). If we iterate this process, it stops after a finite number of steps, i.e. a Higgs G-sheaf is obtained which is isomorphic to all its admissible deformations. We denote this by $\text{gr}(\mathcal{P}, \Theta)$, call it the associated polystable Higgs G-sheaf, and say that two semistable Higgs G-sheaves are S-equivalent when their associated polystable Higgs G-sheaves are isomorphic.

The main result of this article is the following

Theorem 0.3 *There is a quasi-projective coarse moduli space of S-equivalence classes of semistable Higgs G-sheaves with fixed numerical invariants.*

A. Schmitt [12, Sec. 2.9.2] has communicated us that he has constructed the moduli space of principal bundles together with a section of the vector bundle associated to a representation. In particular, he considers Higgs bundles. His proof easily extends to higher dimension, and will appear soon.

1. Preliminary definitions

Let $\rho : G_1 \to G_2$ be a group homomorphism, and P_1 a Higgs G_1-bundle. The *extension of structure group* is the principal G_2-bundle $P_2 = (P_1 \times G_2)/G_1$ (where G_1 acts on the right of P_1 by definition of principal bundle, and it acts on G_2 by left multiplication via the homomorphism ρ). It is denoted $P_1(\rho)$ or just $P_1(G_2)$ if the homomorphism is clear from the context. More generally, when G_1 acts on a scheme F (for instance, when F is the vector space of a representation o G_1), the *associated fiber bundle* is $P_1(F) := (P_1 \times F)/G_1$.

Likewise, when ρ induces an isomorphism on the semisimple part of the Lie algebra and $(\mathcal{P}_1, \Theta_1) = (P_1, E, \xi_1, \Theta)$ is a Higgs G_1-sheaf, we define $(\mathcal{P}_2, \Theta_2) = (P_1(\rho), E, \xi_2, \Theta)$ where $\xi_2 : P_1(\rho)(\mathfrak{g}'_2) = P_1(\mathfrak{g}'_1) \to E|_U$ is the isomorphism induced by ξ and ρ.

Conversely, if P_2 is a principal G_2-bundle, a *reduction of structure group* is a pair (P_1, β) where P_1 is a principal G_1-bundle and $\beta : P_1(\rho) \to P_2$ is an isomorphism. Two reductions are isomorphic when there is an isomorphism φ of principal G_1-bundles making the following diagram commutative

$$
\begin{array}{ccc}
P_1(\rho) & \xrightarrow{\ \beta\ } & P_2 \\
\downarrow{\scriptstyle\varphi(\rho)} & & \downarrow{\scriptstyle =} \\
P_1'(\rho) & \xrightarrow{\ \beta'\ } & P_2
\end{array}
$$

When ρ is injective, there is a canonical bijection between the set of isomorphism classes of reductions of P_2 and the set of sections of the associated fibration $P_2(G_2/G_1)$.

A reduction of structure group of a Higgs G_2-sheaf $(\mathcal{P}_2, \Theta_2)$ under the homomorphism ρ (again assuming that it induces an isomorphism between the semisimple part of the Lie algebras) is a Higgs G_1-sheaf $(\mathcal{P}_1, \Theta_1)$ together with an isomorphism between its extension by ρ and $(\mathcal{P}_2, \Theta_2)$. The notion of isomorphism between reductions is analogously defined. Note that such a reduction is just a reduction of the principal bundle part.

A family of Higgs G-sheaves parameterized by a complex scheme S is a family of principal G-sheaves (P_S, E_S, ξ_S) (see [5, definition 0.5]) together with a Higgs field $\Theta_S : E_S \oplus p_X^* \mathfrak{z}_X \to p_X^* \Omega_X$, and an isomorphism between two such families is an isomorphism as principal G-sheaves which is compatible with the Higgs fields.

We define the functor of semistable Higgs G-sheaves as the sheafification of the functor which associates to each complex scheme S, locally of finite type, the set of isomorphism classes of families of semistable Higgs G-sheaves with fixed numerical invariants. As usual, on morphisms it is defined as pullback.

Given a family of Higgs G_2-sheaves $(\mathcal{P}_S, \Theta_S)$ parameterized by a scheme S, we defined the functor of reductions as the sheafification of the functor which associates to each morphism $f : S' \to S$ the set of isomorphism classes of families of reductions of $(\mathcal{P}_{S'}, \Theta_{S'})$.

Definition 1.1 *A Lie algebra sheaf is a pair (E, φ) where E is a torsion free sheaf and φ is a homomorphism*

$$E \otimes E \longrightarrow E^{\vee\vee}$$

which induces a Lie algebra structure on the fibers where E is locally free. If this Lie algebra is isomorphic to $\mathfrak{g}'$, we call it a $\mathfrak{g}'$-sheaf.

Definition 1.2 *A non-zero homomorphism $\xi : (\otimes^a E)^{\oplus b} \to L \otimes (\det E)^c$ is called a tensor of type (a, b, c, L).*

The moduli space for these tensors was constructed in [4], where they were called decorated tuples of type I. See also [10].

We have already introduced filtrations of the form

$$0 \subsetneq E_{-l} \subseteq E_{-l+1} \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = E$$

If we delete, from 0 onwards, all the non-strict inclusions, we obtain a strict filtration

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E \qquad \lambda_1 < \cdots < \lambda_{t+1}$$

Conversely, from the strict filtration $E_{\lambda_\bullet}$ we can recover the original one $E_\bullet$ by defining $E_m = E_{\lambda_{i(m)}}$, where $i(m)$ is the maximum index with $\lambda_{i(m)} \leq m$.

Definition 1.3 *A balanced filtration is a filtration with $\sum i \operatorname{rk} E_i / E_{i-1} = 0$. In terms of $E_{\lambda_\bullet}$, this is $\sum \lambda_i \operatorname{rk} E_{\lambda_i} / E_{\lambda_{i-1}} = 0$.*

Fix a polynomial δ with $\deg \delta \leq \dim X - 1$ and positive leading coefficient. A tensor is called δ-(semi)stable if for every balanced filtration $E_{\lambda_\bullet}$ it is

$$\sum_{i=-l}^{l} (\lambda_{i+1} - \lambda_i)\big(r P_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) + \mu(\psi, E_{\lambda_\bullet})\delta \ (\preceq) \ 0$$

where

$$\mu(\psi, E_\bullet) = \min_{I \in \mathcal{I}} \left\{ \lambda_{i_1} + \cdots + \lambda_{i_a} : \psi|_{(E_{\lambda_{i_1}} \otimes \cdots \otimes E_{\lambda_{i_a}})^{\oplus b}} \neq 0 \right\}$$

and $\mathcal{I}$ is the set of multi-indexes $I = (i_1, \ldots, i_a)$.

Given a Lie algebra sheaf, using the canonical isomorphism $(\wedge^{r-1} E)^\vee \otimes \det E \cong E^{\vee\vee}$ we obtain a homomorphism

$$E \otimes E \otimes \wedge^{r-1} E \longrightarrow \det E \ ,$$

hence a Lie tensor, by which we mean a tensor ψ of type $(r+1, 1, 1, \mathcal{O}_X)$, i.e.

$$\psi : E^{\otimes r+1} \longrightarrow \det E$$

which factors through $E \wedge E \otimes \wedge^{r-1} E$ and satisfies the Jacobi identity (see [5, Definition 0.14]). From now on, we write $a = r + 1$.

(Note that in [5] a Lie tensor was defined as a tensor $F^{\otimes a} \to \mathcal{O}_X$, i.e. with values in $\mathcal{O}_X$, because E denoted $E = F \otimes \det F$. In this article we change the definition, because it is more convenient when we have to deal with the Higgs field.)

As we have already seen, the isomorphism ξ in a Higgs G-sheaf induces a Lie algebra sheaf, and hence a Lie tensor we denote $\psi_1 : E^{\otimes a} \to \det E$. On the other hand, recall that $\theta : E \to \Omega_X$ is the restriction $\Theta|_E$ of the Higgs field $\Theta : E \oplus \mathfrak{z}_X \to \Omega_X$. It induces a homomorphism

$$\psi_2 : E^{\otimes a} = E \otimes E^{\otimes r} \xrightarrow{\operatorname{id} \otimes \det} E \otimes \det E \xrightarrow{\theta \otimes \operatorname{id}} \Omega_X \otimes \det E \ ,$$

The direct sum of these morphisms defines a tensor

$$(1.1) \qquad \psi = \psi_1 \oplus \psi_2 : E^{\otimes a} \oplus E^{\otimes a} \longrightarrow (\mathcal{O}_X \oplus \Omega_X) \otimes \det E$$

which is a *Higgs Lie tensor*, by which we understand a tensor of type $(a, 2, 1, \mathcal{O}_X \oplus \Omega_X)$ which satisfies the following closed conditions:

1. It is "block-diagonal", i.e., of the form

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix},$$

where

$$\psi_1 : E^{\otimes a} \longrightarrow \det E$$

and

$$\psi_2 : E^{\otimes a} \longrightarrow \Omega_X \otimes \det E$$

2. the first summand ψ_1 is a Lie tensor,

3. the second summand ψ_2 factors as follows

$$E \otimes E^{\otimes r} \overset{\mathrm{id}\,\otimes\,\det}{\longrightarrow} E \otimes \det E \overset{\theta\otimes\mathrm{id}}{\longrightarrow} \Omega_X \otimes \det E ,$$

2. Boundedness

In this section we prove that the set of semistable Higgs G-sheaves is bounded. We do this by considering the Higgs sheaf induced by the adjoint representation.

Let E be a coherent $\mathcal{O}_X$-module. Endowing it with a Higgs field θ is equivalent to endowing it with the structure of a $\mathcal{O}_{T^\vee X}$-module structure, where $T^\vee X$ is the total space of the cotangent vector bundle Ω_X on X. In other words, a Higgs sheaf (E, θ) (with E torsion free) can be thought of as a coherent sheaf $\mathcal{E}$ on $T^\vee X$ (of pure dimension $\dim X$), and this gives an equivalence of categories, called the *spectral construction*. Under this correspondence, $E = p_* \mathcal{E}$, where $p : T^\vee X \to X$ is the natural projection. Therefore, the Hilbert polynomials coincide $P_E = P_{\mathcal{E}}$ if we define the polarization on $T^\vee X$ as the pullback of the polarization on X. For more details on this, see [3] or [15, Lemma 6.8].

This point of view is very useful, because it allows us to reduce problems for Higgs sheaves to problems for coherent sheaves. For instance, the proof of the following lemma becomes straightforward (see also [14, Lemma 3.1]):

Lemma 2.1 (Higgs-Harder-Narasimhan) *Let (E, θ) be a Higgs sheaf. There exists a unique filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_l = E$$

of Higgs sheaves, i.e. inducing $\theta_i : E_i \to E_i \otimes \Omega_X$, such that the induced Higgs sheaf $(E^i = E_i/E_{i+1}, \theta^i : E^i \to E^i \otimes \Omega_X)$ is semistable for all i, and

$$(2.1) \qquad \mu_{\max}(E, \theta) := \mu(E^1) > \mu(E^2) > \cdots > \mu(E^l) =: \mu_{\min}(E, \theta)$$

Given two torsion free sheaves (E_1, θ_1) and (E_2, θ_2), we define their tensor product

$$(E_1, \theta_1) \otimes' (E_2, \theta_2) = (E_1 \otimes' E_2, \theta_1 \otimes \mathrm{id} + \mathrm{id} \otimes \theta_2),$$

where $E_1 \otimes' E_2$ is the torsion free part of the tensor product.

Lemma 2.2 *Let (E_1, θ_1) and (E_2, θ_2) be two Higgs sheaves.*

1. *If they are semistable, then their tensor product is also semistable.*

2. *In general we have*

$$\mu_{\min}((E_1, \theta_1) \otimes' (E_2, \theta_2)) = \mu_{\min}(E_1, \theta_1) + \mu_{\min}(E_2, \theta_2).$$

3. *If there exists a nonzero homomorphism $(E_1, \theta_1) \to (E_2, \theta_2)$, then*

$$\mu_{\min}(E_1, \theta_1) \le \mu_{\max}(E_2, \theta_2)$$

Proof.   The first statement is [13, Corollary 3.8] (we remark that it is reduced to the case $\dim X = 1$, using the restriction to ample hypersurfaces, and then using the relationship between stability and existence of solutions to certain differential equations). The second statement is given without Higgs field in [1, Proposition 2.9], but, using the first statement, their proof also works when the Higgs field is non-zero. The third statement follows at once from the spectral construction and the corresponding result for coherent sheaves [7, Lemma 1.3.3]   ∎

Let (P, E, ξ, Θ) be a Higgs G-sheaf. Let $j : U \hookrightarrow X$ be the open subset where E is locally free. As usual, we denote $\theta = \Theta|_E$. The homomorphism

$$f : E|_U \xrightarrow{\mathrm{id} \otimes \theta^*} E|_U \otimes E|_U \otimes \Omega_X|_U \xrightarrow{[,] \otimes \mathrm{id}} E|_U \otimes \Omega_X|_U$$

induces a Higgs field

$$ad\theta = j_* f : E^{\vee\vee} \longrightarrow E^{\vee\vee} \otimes \Omega_X$$

on $E^{\vee\vee} \cong j_* E|_U$ (this isomorphism holds because U is big, i.e., the codimension of its complement is at least 2. Cfr. [5, Lemma 0.11]). The resulting Higgs sheaf $(E^{\vee\vee}, ad\theta)$ is called the adjoint Higgs sheaf.

Proposition 2.3 *A Higgs G-sheaf (P, E, ξ, Θ) is slope semistable if and only if the associated adjoint Higgs sheaf $(E^{\vee\vee}, ad\theta)$ is slope semistable.*

Proof. We may and shall assume that G is semisimple. $\Leftarrow$) If (P, E, ξ, Θ) is not slope semistable, then there exists an orthogonal algebra filtration $E_\bullet \subset E$, compatible with Θ and such that

$$\sum_{i=-l}^{l} (\lambda_{i+1} - \lambda_i)\big(r \deg E_{\lambda_i} - r_{\lambda_i} \deg E\big) \; > \; 0$$

This filtration induces a filtration $E_\bullet^{\vee\vee}$ on $E^{\vee\vee}$, and since the degree of a torsion free sheaf is equal to the degree of its double dual, it still holds

$$(2.2) \qquad \sum_{i=-l}^{l} (\lambda_{i+1} - \lambda_i)\big(r \deg E_{\lambda_i}^{\vee\vee} - r_{\lambda_i} \deg E^{\vee\vee}\big) \; > \; 0$$

Using (0.1),

$$ad\theta(E_{\lambda_i}^{\vee\vee}) = (j_* f)(E_{\lambda_i}^{\vee\vee}) = j_*[\theta^*|_U, E_{\lambda_i}|_U] \subset j_*(E_{\lambda_i} \otimes \Omega_X|_U) = E_{\lambda_i}^{\vee\vee} \otimes \Omega_X \,,$$

and therefore, (2.2) implies that $(E^{\vee\vee}, ad\theta)$ is unstable.

$\Rightarrow$) If $(E^{\vee\vee}, ad\theta)$ is unstable, let $F_\bullet^{\vee\vee} \subset E^{\vee\vee}$ be its Higgs-Harder-Narasimhan filtration (Lemma 2.1), and let $F_\bullet \subset E$ be the induced filtration on E defined as $F_i = F_i^{\vee\vee} \cap E$. Denote

$$(2.3) \qquad\qquad\qquad \lambda_i = -r!\mu(F^i)$$

and $E_{\lambda_i} = F_i$, thus giving a filtration

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_{t+1}} = E$$

Using $\deg E = 0$, it is easy to check that the filtration is balanced.

We claim that it is an algebra filtration. Indeed, let $(\lambda_i, \lambda_j, \lambda_k)$ be a triple with $\lambda_i + \lambda_j < \lambda_k$. We need to show

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k-1}}^{\vee\vee}$$

Let k' be the minimum integer such that $E_{\lambda_{k'-1}}^{\vee\vee}$ contains $[E_{\lambda_i}, E_{\lambda_j}]$. The morphism

$$E_{\lambda_i} \otimes E_{\lambda_j} \xrightarrow{[,]} E_{\lambda_{k'-1}}^{\vee\vee}/E_{\lambda_{k'-2}}^{\vee\vee}$$

is nonzero. It follows from (2.3), Lemma 2.2 (2) and (3) that $\lambda_i + \lambda_j \geq \lambda_{k'-1}$ which in turn implies $\lambda_{k'-1} < \lambda_k$, proving the claim.

Therefore, $E_\bullet$ is a balanced algebra filtration, or equivalently (by corollary [5, Lemma 5.10]), an orthogonal filtration.

We claim that Θ is compatible with this reduction. Let $i(\Theta)$ be the minimum index i such that $\Theta^*(\mathcal{O}_X) \subset E_{\lambda_i}^{\vee\vee} \otimes \Omega_X$ It follows that $ad\theta(E_\lambda^{\vee\vee}) \subset E_{\lambda+\lambda_{i(\theta)}}^{\vee\vee} \otimes \Omega_X$. Since the Higgs-Harder-Narasimhan filtration is a filtration of Higgs sheaves, this forces $\lambda_{i(\Theta)} \leq 0$, thus proving the claim.

Finally, the inequality

$$\sum(\lambda_{i+1} - \lambda_i)\big(r \deg E_{\lambda_i} - r_{\lambda_i} \deg E\big) \;>\; 0$$

follows from (2.1), proving that (P, E, ξ, Θ) is not slope semistable.   ∎

Proposition 2.4 *The set of isomorphism classes of sheaves E occurring in semistable Higgs G-bundles (E, P, ξ, Θ), and having fixed numerical invariants, is a bounded set.*

Proof. Let $F \subset E$ be a subsheaf. A semistable Higgs sheaf (P, E, ξ, Θ) is slope semistable, therefore $(E^{\vee\vee}, ad\theta)$ is a slope semistable Higgs sheaf by proposition 2.3. Since $F \subset E^{\vee\vee}$, it follows from [14, Lemma 3.3] that $\mu(F) \leq \mu(E) + K$, where K is a constant depending only on the rank of E and the chosen polarization of X. The well-known theorem of Simpson [14, Thm 1.1] says that the set in the statement is then bounded.   ∎

Proposition 2.5 *There exists a polynomial δ_0 of degree $\dim X - 1$ and positive leading coefficient such that for all $\delta > \delta_0$, a Higgs G-sheaf is (semi)stable if and only if the associated Higgs $\mathfrak{g}'$-tensor is δ-(semi)stable.*

Proof. We have seen (1.1) that the Higgs Lie tensor associated to a Higgs G-sheaf is of the form $\psi = \psi_1 + \psi_2 = \psi_1 + \theta \otimes \det$. For a multi-index $I = (i_1, \ldots, i_a)$, denote $\lambda_I = \lambda_{i_1} + \cdots + \lambda_{i_a}$ and $E_I = E_{\lambda_{i_1}} \otimes \cdots \otimes E_{\lambda_{i_a}}$. Using this notation,

$$\mu(\psi, E_{\lambda_\bullet}) = \min_{I \in \mathcal{I}} \left\{ \lambda_I : \psi|_{E_I^{\oplus 2}} \neq 0 \right\}$$

$$= \min_{I \in \mathcal{I}} \left\{ \lambda_I : \psi_1|_{E_I} \neq 0 \text{ or } \psi_2|_{E_I} \neq 0 \right\}$$

$$= \min \left(\min_{I \in \mathcal{I}} \left\{ \lambda_I : \psi_1|_{E_I} \neq 0 \right\}, \min_{I \in \mathcal{I}} \left\{ \lambda_I : \theta|_{E_{i_1}} \neq 0 \text{ and } \det |_{E_{i_2} \otimes \cdots \otimes E_{i_a}} \neq 0 \right\} \right)$$

$$= \min \left(\min_{I \in \mathcal{I}} \left\{ \lambda_I : \psi_1|_{E_I} \neq 0 \right\}, \min_{I \in \mathcal{I}} \left\{ \lambda_{i_1} : \theta|_{E_{i_1}} \neq 0 \right\} \right)$$

$$=: \min \left(\mu_{21}, \mu_{10} \right)$$

Note that if θ is identically zero, μ_{10} is not defined, and then $\mu(\psi, E_{\lambda_\bullet}) = \mu_{21}$. The notation μ_{21} and μ_{10} reminds us that the tensors come from a homomorphism $\varphi : E \otimes E \to E^{\vee\vee}$ and a homomorphism $\theta : E \to \Omega_X$.

We claim that, *for any filtration $E_{\lambda_\bullet}$, $\mu(\psi, E_{\lambda_\bullet}) \leq 0$, with equality if and only if it is an orthogonal algebra filtration compatible with the Higgs field.* This is proved in two steps: first note that $\mu_{10} \geq 0$ if and only if the filtration is compatible with the Higgs field ($\theta|_{E_{-1}} = 0$) and recall from [5, Lemma 1.3] that $\mu_{21} \geq 0$ is equivalent to being an algebra filtration.

Therefore, $\mu(\psi, E_{\lambda_\bullet}) \geq 0$ if and only if both conditions apply, i.e. if and only if it is an orthogonal algebra filtration. On the other hand, it was shown in the same place that $\mu_{21} \leq 0$ for all filtration, because $\mathfrak{g}'$ is semisimple, and therefore $\mu(\psi, E_{\lambda_\bullet}) = \min(\mu_{21}, \mu_{10}) \leq 0$. Combining both observations, the claim follows.

$\Rightarrow$) To check (semi)stability, consider an orthogonal algebra filtration $E_{\lambda_\bullet}$ which is compatible with the Higgs field. This implies $\mu(\psi, E_{\lambda_\bullet}) = 0$, and the result follows.

$\Leftarrow$) By proposition 2.4, the set $\mathcal{S}_1$ of torsion free sheaves E occurring as Higgs Lie tensors associated to (semi)stable Higgs G-sheaves of given numerical invariants is a bounded set. On the other hand, to check δ-(semi)stability of a tensor, it suffices to consider filtrations $E_{\lambda_\bullet}$ such that $|\lambda_i|$ is bounded with a constant B independent of the polynomial δ ([4, Lemma 3.4.4]). This implies the boundedness of the set of balanced filtrations $\mathcal{S}_2 = \{E_{\lambda_\bullet} \subset E : E \in \mathcal{S}_1, \ |\lambda_i| < B\}$ which we need to consider when checking the semistability of the tensor. Therefore, there is a polynomial δ_0 such that

$$\sum (\lambda_{i+1} - \lambda_i)\big(r P_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) \prec \delta_0$$

for all filtrations in $\mathcal{S}_2$. Let $E_{\lambda_\bullet}$ be a filtration in $\mathcal{S}_2$. If it is an algebra filtration, then $\mu_{21} = 0$, and hence $\mu(\psi, E_{\lambda_\bullet}) = 0$ and the result follows.

If it is not an algebra filtration, then $\mu_{21} < 0$, thus $\mu(\psi, E_{\lambda_\bullet}) \leq -1$. Therefore, for all $\delta \succ \delta_0$, it is

$$\sum (\lambda_{i+1} - \lambda_i)\big(r P_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) + \delta\mu(\psi, E_{\lambda_\bullet}) \prec$$

$$\delta_0 + \delta\mu(\psi, E_{\lambda_\bullet}) \preceq \delta_0 - \delta \prec 0 \ . \qquad \blacksquare$$

3. Construction of the moduli space

3.1. Construction of the schemes T and T_1

Proposition 2.4 says that the set of isomorphism classes of sheaves E occurring in semistable Higgs G-sheaves of fixed numerical invariants is bounded, and so we can choose an integer m large enough so that $E(m)$ and $\Omega_X((r+1)m)$ are generated by global sheaves and their higher cohomology vanish for all such E.

Definition 3.1 *A based Higgs G-sheaf (q, E, P, ξ, Θ) is a Higgs G-sheaf (E, P, ξ, Θ) together with a quotient $q : V \otimes \mathcal{O}_X(-m) \to E$ which induces an isomorphism between V and $H^0(E(m))$. We analogously define based tensors, based Lie tensors, etc...*

Let $(q, E, \varphi : E \otimes E \to E^{\vee\vee}, \theta : E \to \Omega_X)$ be a based semistable Higgs $\mathrm{Aut}(\mathfrak{g}')$-sheaf. The quotient q corresponds to a point in the Hilbert scheme $\mathcal{H}$ of quotients of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial P and trivial determinant. Let l be an integer such that $l > m$, and denote $W = H^0(\mathcal{O}_X(l-m))$. We obtain, from $q(l) : V \otimes \mathcal{O}_X(l-m) \twoheadrightarrow E(l)$, a linear map $V \otimes W \to H^0(E(l))$, and, from this,

$$\bigwedge\nolimits^{P(l)}(V \otimes W) \;\to\; \bigwedge\nolimits^{P(l)} H^0(E(l)) \quad \cong \; \mathbb{C}$$

hence a point in $\mathbb{P}\big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\big)$. For l large enough, this gives the Grothendieck embedding of $\mathcal{H}$ in a projective space.

As we have already seen, a Higgs G-sheaf (E, P, ξ, Θ) induces a tensor

$$\psi = \psi_1 \oplus \psi_2 : E^{\otimes a} \oplus E^{\otimes a} \longrightarrow (\mathcal{O}_X \oplus \Omega_X) \otimes \det E$$

with $a = r + 1$. Choosing an isomorphism $\alpha : \det E \to \mathcal{O}_X$, we obtain a homomorphism

$$
\begin{array}{ccc}
V^{\otimes a} \oplus V^{\otimes a} & & B_{\mathcal{O}_X} \oplus B_{\Omega_X} \\
\downarrow & & \uparrow{\scriptstyle =} \\
H^0(E(m))^{\otimes a} \oplus H^0(E(m))^{\otimes a} & & H^0(\mathcal{O}_X(am)) \oplus H^0(\Omega_X(am)) \\
\downarrow & & \uparrow{\scriptstyle \cong} \\
H^0(E^{\otimes a}(am) \oplus E^{\otimes a}(am)) & \longrightarrow & H^0((\mathcal{O}_X(am) \oplus \Omega_X(am)) \otimes \det E)
\end{array}
$$

If we change the isomorphism α this homomorphism will just be multiplied by a scalar, so we get a well defined point in

$$\mathbb{P}' = \mathbb{P}(\mathrm{Hom}(V^{\otimes a} \oplus V^{\otimes a}, B_{\mathcal{O}_X} \oplus B_{\Omega_X}))$$

Using the polarization induced on $\mathcal{H}$ by Grothendieck's embedding, we endow the product $\mathcal{H} \times \mathbb{P}'$ with a polarization $\mathcal{O}_X(b, b')$, where b and b' are integers with

$$\frac{b'}{b} = \frac{P(l)\delta(m) - \delta(l)P(m)}{P(m) - a\delta(m)}$$

Let Z be the closure in $\mathcal{H} \times \mathbb{P}'$ of the points corresponding to δ-semistable based tensors, and let $Z^{ss} \subset Z$ be the open subset corresponding to those that are δ-semistable. Let $T \subset Z^{ss}$ be the closed subscheme of those corresponding to Higgs Lie tensors.

On $X \times T$ there is a tautological sheaf F_T and homomorphisms

$$F_T \otimes F_T \otimes \wedge^{r-1} F_T \longrightarrow p_{\mathbb{P}'}^* \mathcal{O}_X(1)$$
$$F_T \otimes \det F_T \longrightarrow p^* \Omega_X \otimes p_{\mathbb{P}'}^* \mathcal{O}_X(1)$$

Defining $E_T = F_T \otimes \det F_T \otimes p_{\mathbb{P}'}^* \mathcal{O}_X(-1)$ these give homomorphisms

$$E_T \otimes E_T \longrightarrow E_T^{\vee\vee}$$

$$E_T \longrightarrow p_X^* \Omega_X$$

Proposition 3.2 *The scheme T represents the functor of based δ-semistable Higgs Lie tensors.*

Call an open subset of T GIT-saturated when it is $\mathrm{SL}(V)$-invariant and, if an orbit $\mathrm{SL}(V) \cdot p$ lies on it, all the orbits in the closure of $\mathrm{SL}(V) \cdot p$ also lie on it. If an open subset is GIT-saturated, its good quotient is an open subset of the GIT quotient of T.

Recall that, for a "high" polynomial δ, a Higgs $\mathfrak{g}'$-sheaf is (semi)stable if and only if the corresponding Higgs Lie tensor is δ-(semi)stable (proposition 2.5).

Proposition 3.3 *There is a GIT-saturated open subset U of T corresponding to Higgs Lie sheaves whose Lie algebra structure is semisimple. The subscheme $T' \subset U$, corresponding to those whose Lie algebra structure is isomorphic to $\mathfrak{g}'$ at points where E is locally free, is a union of connected components.*

Proof. The subset of points corresponding to Higgs Lie tensors whose Lie algebra structure is semisimple is open, because the non-vanishing of the determinant of the Killing form is an open condition. Assume this open set were not GIT-saturated. This would imply there is a point p in U and a 1-PS subgroup λ of $\mathrm{SL}(V)$, such that the limit $\lim_{t\to0} \lambda(t) \cdot p$ exists and corresponds to a Lie tensor whose Lie structure is not semisimple. This translates into the existence of an orthogonal algebra filtration $E_\bullet$ with

$$\sum (\lambda_{i+1} - \lambda_i)\big(r P_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) = 0$$

such the the graded Lie algebra structure induced on the graded sheaf $\mathrm{gr}\, E_\bullet$ is identically zero (cf. [5, §4]). But this is not possible, because, given a semisimple Lie algebra and an orthogonal algebra filtration of it, the associated graded Lie algebra is also semisimple (cf. [5, Lemma 5.2]). The second assertion follows from the rigidity of semisimple Lie algebras. ∎

Consider the restricted family $(E_{T'}, \varphi_{T'} : E_{T'} \otimes E_{T'} \longrightarrow E_{T'}^{\vee\vee}, \theta_{T'} : E_{T'} \longrightarrow p_X^* \Omega_X)$ parameterized by T'. Let T_1 be the closed subset defined by the condition that $[\theta, \theta]$ is identically zero. The restricted family $(E_{T_1}, \varphi_{T_1} : E_{T_1} \otimes E_{T_1} \longrightarrow E_{T_1}^{\vee\vee}, \theta_{T_1} : E_{T_1} \longrightarrow p_X^* \Omega_X)$ can be thought of as a family of Higgs $\mathrm{Aut}(\mathfrak{g}')$-sheaves. The group $\mathrm{PGL}(V)$ acts on T_1, and, arguing as before proposition 1.8 in [5], the action lifts to this family.

Proposition 3.4 *With this action, the family becomes a universal family with group* $\mathrm{PGL}(V)$ *for the functor of Higgs* $\mathrm{Aut}(\mathfrak{g}')$*-sheaves.*

Proof. Analogous to [5, Prop 1.8]. See [5, Def. 0.29] for the definition of universal family.   ■

3.2. Construction of the schemes T_2 and T_2'

The rest of the construction of the moduli space is very similar to that in [5], so we will just give a sketch of it. The next step is to construct a scheme $T_2 \to T_1$, finite and étale, which parameterizes reductions of structure group under the homomorphism $\rho_2 : G/Z \to \mathrm{Aut}(\mathfrak{g}')$. Note that G/Z is just the connected component of $\mathrm{Aut}(\mathfrak{g}')$, so if P is a principal $\mathrm{Aut}(\mathfrak{g}')$ bundle, the set of reductions under ρ_2 is the set of sections of the associated fibration $P(F)$, where F is the finite group $\mathrm{Aut}(\mathfrak{g}')/(G/Z)$. If $P(F)$ is a trivial filtration then, this set is an F-torsor, and if it is not trivial, this set is empty. For this reason $T_2 \to T_1$ is finite and étale.

Now we consider reductions under the homomorphism $\rho_2' : G/Z' \to G/Z$, where Z' is the center of $G' = [G, G]$. Note that $G/Z' \cong G/Z \times G/G'$, and ρ_2' is just projection to the first factor. For a principal G/Z-bundle (on a big open set U of X), giving a reduction of structure group to G/Z' is just giving a principal G/G'-bundle. Since $G/G' \cong \mathbb{C}^{*q}$, this is equivalent to giving a set of q line bundles on U, or equivalently on X, whose Chern classes $d_i \in H^2(X, \mathbb{C})$ are fixed by the numerical invariants. On the other hand, the Lie algebra of the group G/Z' is $\mathfrak{z}$. A Higgs field for a principal G/Z-sheaf is a homomorphism $E \to \Omega_X$, but a Higgs field for a principal G/Z'-sheaf is a homomorphism $E \oplus \mathfrak{z}_X \to \Omega_X$, i.e. we have to give a section of $\mathfrak{z}^\vee \otimes \Omega_X$. Therefore, these reductions are parameterized by the scheme

$$T_2' = H^0(X, \mathfrak{z}^\vee \otimes \Omega_X) \times J^{d_1} \times \cdots \times J^{d_q} \times T_2 \longrightarrow T_2$$

The details of the construction of T_2 and T_2' are the same as the construction of R_2 and R_2' in [5, §2], so we just state the final result.

Proposition 3.5 *There is a family of Higgs* G/Z'*-sheaves parameterized by a scheme* T_2' *and a natural action of* $G/G' \times \mathrm{PGL}(V)$*, providing it with a structure of universal family with group* $G/G' \times \mathrm{PGL}(V)$ *for the functor of Higgs* G/Z'*-sheaves.*

We remark that the action of G/G' on T_2' is trivial, but its lift to the family is not.

3.3. Construction of the scheme T_3

Finally we consider reductions under the homomorphism $\rho_3 : G \to G/Z'$. If H is a group, we denote by $\underline{H}$ the sheaf of sets of sections of the trivial H-bundle $X \times H$. We denote by $\check{H}^i_{\mathrm{et}}(X, \underline{H})$ the Czech étale cohomology set. The short exact sequence of groups

$$1 \longrightarrow Z' \longrightarrow G \longrightarrow G/Z' \longrightarrow 1$$

gives rise to an exact sequence of pointed sets

$$\check{H}^1_{\mathrm{et}}(X, \underline{Z'}) \longrightarrow \check{H}^1_{\mathrm{et}}(X, \underline{G}) \longrightarrow \check{H}^1_{\mathrm{et}}(X, \underline{G/Z'}) \longrightarrow \check{H}^2_{\mathrm{et}}(X, \underline{Z'})$$

It is $\check{H}^i_{\mathrm{et}}(X, \underline{Z'}) \cong H^i(X; Z')$, $i = 1, 2$, the singular cohomology with coefficients in Z'. A principal G/Z'-bundle gives a point in $\check{H}^1_{\mathrm{et}}(X, G/Z')$, and it admits a reduction of structure group to G if and only if its image in $H^2(X; Z')$ is zero. In this case, the set of reductions is an $H^1(X; Z')$-torsor. In general we have to deal with principal bundles which are defined only on a big open set of X, but, as shown in [5, §3], this does not affect the first cohomology group, thus obtaining the following result.

Proposition 3.6 *There is a scheme $R_3 \to R_2'$, finite and étale, which is a universal space with group $\mathrm{PGL}(V)$ for the functor of Higgs G-sheaves.*

We remark that, in general, there is no tautological Higgs G-sheaf parameterized by R_3. This is why we do not get a universal family, but just a universal space (cfr. [5, Def. 0.30]) with group $\mathrm{PGL}(V)$ (recall that in proposition 3.5 the action of G/G' on T_2' was trivial).

3.4. Construction of quotient

Now the same arguments as in [5, §4] provide the following theorem.

Theorem 3.7 *There is a quasi-projective moduli scheme of semistable Higgs G-sheaves with fixed numerical invariants. There is an open subscheme whose closed points are in canonical bijection with isomorphism classes of stable Higgs G-sheaves.*

This moduli space is not expected to be projective, but it would be very interesting to prove that the fibers of the Hitchin map are projective, as it happens with Higgs bundles in dimension one.

In this article we have defined Higgs fields for principal sheaves. Another approach is to start with a singular principal bundle, as defined in [9, 4], and define Higgs fields on them. This could be specially interesting if we wanted to consider a specific group G (orthogonal, symplectic, ...) because then we can choose a representation which is well adapted to the group.

For instance, if G is the orthogonal group and if we choose the standard representation, a Higgs singular principal bundle would be a torsion free sheaf with a non-degenerate symmetric bilinear form (by this we mean that the induced homomorphism $E \to E^\vee$ is injective) together with a homomorphism $\theta : E \to E \otimes \Omega$ respecting the bilinear form (i.e., skew-symmetric with respect to the bilinear form). The condition of stability would then be defined in terms of orthogonal filtrations of E which are respected by the homomorphism. To follow these ideas, [12, Sec. 2.9.2] is relevant. If $\dim X = 1$ these approaches are equivalent, but they are not if $\dim X > 1$. Under good conditions, we can expect that these two approaches give birational moduli spaces.

The moduli space of Higgs principal bundles on curves plays a central role in the Geometric Langlands Program. One can expect that the moduli spaces constructed in this article could be used to generalize that program to projective varieties of dimension higher than one.

It would be interesting to study the structure of these moduli spaces. Since the moduli spaces of torsion free sheaves on a projective scheme can already be quite complicated (non-reduced, ...), likewise these moduli spaces are expected to be difficult to describe in general.

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