Infinite groups with many permutable subgroups

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Abstract

A subgroup H of a group G is said to be *permutable in G*, if $HK = KH$ for every subgroup K of G. A result due to Stonehewer asserts that every permutable subgroup is ascendant although the converse is false. In this paper we study some infinite groups whose ascendant subgroups are permutable (AP–groups). We show that the structure of radical hyperfinite AP–groups behave as that of finite soluble groups in which the relation *to be a permutable subgroup* is transitive (PT–groups).

1. Introduction

A subgroup H of a group G is said to be *permutable in G* (or quasi-normal in G), if $HK = KH$ for every subgroup K of G. This concept arises as a generalization of that of normal subgroup. The study of the properties of the permutable subgroups started a rather long time ago (see, for example [17]). In particular, the groups (finite and infinite) in which every subgroup is permutable have been described (see [17, 2.4]). Like normality, the relation *to be a permutable subgroup* is not transitive.

A subgroup H of a group G is said to be an *ascendant subgroup of G* if there exists an *ascending series* from H to G, that is, a chain of subgroups well-ordered by inclusion and indexed by the corresponding ordinal numbers

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_\alpha \lhd H_{\alpha+1} \lhd \cdots \lhd H_\gamma = G$$

with the additional stipulation that for each limit ordinal λ the subgroup H_λ is the union of all subgroups H_β, $\beta < \lambda$. This is an evident generalization of the concept of subnormal subgroup.

According to a well-known theorem by Stonehewer ([18, Theorem A]), permutable subgroups are always ascendant. Therefore it is natural to consider the opposite situation, that is the groups whose ascendant subgroups are permutable. A group G is said to be an *AP–group* if every ascendant subgroup of G is permutable in G. These groups are very close to the groups in which the relation *to be a permutable subgroup* is transitive (studied in [3] for a class of periodic radical groups). A group G is said to be a *PT–group* if permutability is a transitive relation in G, that is, if K is a permutable subgroup of H and H is a permutable subgroup of G, then K is a permutable subgroup of G. We note that, if G is an *AP*–group, then clearly G is a *PT*–group, because the relation *to be an ascendant subgroup* is transitive. If G is a finite group, then it is known that G is a *PT*–group if and only if every subnormal subgroup is permutable. The study of finite *PT*–groups began with a paper of G. Zacher [19]. Zacher determined the structure of finite soluble *PT*–groups in a similar way to Gaschütz's [7] characterization of finite soluble groups in which normality is a transitive relation (i.e. finite soluble *T*–groups).

Theorem (Zacher) *Let G be a finite soluble group. Then G is a PT–group (and hence an AP–group) if and only if it has an abelian normal Hall subgroup L of odd order such that G/L is a nilpotent modular group and every subgroup of L is normal in G.*

As a consequence of this theorem, the class of finite soluble *PT*–groups (and therefore the class of finite soluble *AP*–groups) is subgroup-closed. There are many papers, in which several properties of finite soluble *PT*–groups have been considered (see, for example, [1, 4, 2]).

In [15, Lemma 4], it is claimed that in an arbitrary *PT*–group every ascendant subgroup is permutable. However this result is not true. An specific counterexample is the following. Let $G = A \rtimes \langle b \rangle$ the semidirect product of a Prüfer 2–group $A = \langle a_n \mid a_1^2 = 1, a_{n+1}^2 = a_n, n \in \mathbb{N} \rangle$ by a cyclic group $\langle b \rangle$ of order 2 that acts on A by $a^b = a^{-1}$ for each $a \in A$. If $x = ab$, for some $a \in A$, then $x^2 = abab = aa^{-1} = 1$. Let H be a proper subgroup of G. If H lies in A, then H is G–invariant. Otherwise, H is finite and $H = K\langle x \rangle$, where K is a proper subgroup of A, $x \notin A$, and hence $|x| = 2$. Every subgroup of A is cyclic, so $K = \langle c \rangle$, where $c \in A$. Suppose that $|c| = 2^m$, for some $m \in \mathbb{N}$. Then $|H| = 2^{m+1}$. Since A is divisible, we may choose an element $d \in A$ such that $d^t = c$, where $t = 2^5$. Put $y = dx$. Clearly $y \notin A$, and thus $|y| = 2$.

Suppose that $H\langle y \rangle = \langle y \rangle H$. Then $|H\langle y \rangle| = 2^{m+1} \cdot 2 = 2^{m+2}$. On the other hand, $(dx)x \in \langle H, y \rangle$ and $(dx)x = d$, but $|d| = 2^{m+5}$. This shows that H can not be permutable in G. In other words, if H is a permutable

subgroup of G, then $H \leq A$, and in this case every permutable subgroup of G is G–invariant. It follows that G is a PT–group. However, the subgroup $\langle b \rangle$ is ascendant in G but it is not permutable in G.

This example shows that for infinite groups the classes of AP–groups and PT–groups do not coincide. In this paper we start the study of infinite AP–groups. Obviously groups with no permutable subgroups are AP–groups, but it appears unreasonable to study properties of permutable subgroups in groups with no permutable subgroups. This shows that the effective study of AP–groups must be carried out under the imposition of some natural restrictions.

The natural framework for considering AP–groups are the classes of groups that have many ascendant subgroups. If G is a nilpotent group, then every subgroup of G is subnormal. Hence every subgroup of a nilpotent AP–group is permutable. A natural extension of the class of nilpotent groups is the class of locally nilpotent groups. One of the classes of locally nilpotent groups with many ascendant subgroups is the class of N–groups. We recall that a group G is said to be an N–group, or a group with the normalizer condition, if every proper subgroup of G is properly contained in its normalizer (see [12, §63]). It is worth mentioning that G is a N–group if and only if every subgroup of G is ascendant in G ([12, §63]). A generalization of the class of N–groups is the class of Gruenberg groups. We recall that a group G is said to be a *Gruenberg* group if every finitely generated subgroup of G is ascendant in G. Every countable locally nilpotent group is a Gruenberg group, but there exists an uncountable locally nilpotent group which does not include ascendant finitely generated subgroups (see [10]). Furthermore, O. Yu. Schmidt [16] has constructed an example of a countable locally nilpotent p-group which is not an N–group. Being countable, this group is a Gruenberg group. This example shows that the classes of N-groups and Gruenberg groups are distinct. As we will see in section 2, every Sylow p–subgroup of a Gruenberg AP–group is nilpotent.

The following classes of generalized soluble groups spring from the above classes of locally nilpotent groups in same way the the class of finite soluble groups does from the class of finite nilpotent groups. Let $\mathfrak{X}$ be a class of groups. A group G is said to be a *hyper–$\mathfrak{X}$–group* if there is an ordinal γ such that G has an ascending normal series of subgroups

$$1 = D_0 \lhd D_1 \lhd \cdots \lhd D_\alpha \lhd D_{\alpha+1} \lhd \cdots \lhd D_\gamma = G$$

(where $D_\lambda = \bigcup_{\beta < \lambda} D_\beta$ if λ is a limit ordinal) whose factors are $\mathfrak{X}$–groups.

In particular, if $\mathfrak{X}$ is the class of all locally nilpotent groups, then we obtain the class of *radical* groups (so called hyper–locally nilpotent–groups) which is a natural extension of the class of finite soluble groups.

Our aim in this paper is to extend results about finite soluble PT–groups to some classes of infinite groups. We work within the class of *hyperfinite* groups. There we find some interesting subclasses as AP–hypernilpotent groups, AP–hyper–Gruenberg groups and AP–hyper–N–groups that are different extensions of the class of finite soluble AP–groups.

The layout of the paper is as follows. In section 2, some auxiliary results are proved. In section 3, we give a full characterization of radical hyperfinite AP–groups in the following way.

Theorem A (Theorem 3.5). *Let G be a radical hyperfinite AP–group. Then the following assertions hold:*

 (i) *G is metabelian;*

 (ii) *if R is the locally nilpotent radical of G, then $R = L \times Z$, where L is the locally nilpotent residual of G and Z is the upper hypercenter of G;*

 (iii) *the Sylow p–subgroup of L is the Sylow p–subgroup of G for every prime p occurring as a divisor of some element of L;*

 (iv) *every subgroup of L is G–invariant; and*

 (v) *G/L is hypercentral and every subgroup of G/L is permutable.*

Moreover, if the factor-group G/L is countable, then G splits over L. In particular, if G is countable, then G splits over L.

On the other hand, if G is a periodic group having a normal abelian subgroup L that satisfies the conditions (iii)–(v), then G is an AP–group.

In particular, a radical hyperfinite group G is an AP–group if and only if G satisfies the conditions (iii)-(v) of Theorem A.

Theorem A has the virtue of showing that *to be an AP–group* is a subgroup-closed property in the universe of all radical hyperfinite groups, a fact that is not evident from the definition. Moreover it has another not less important virtue: it allows us to construct all periodic soluble AP–groups (see Section 5).

Corollary B (Corollary 3.6).   *Let G be a radical hyperfinite AP–group. Then every subgroup of G is a AP–group.*

The above results confirm that the structure of (infinite) radical hyperfinite AP–groups is very similar to the structure of finite soluble PT–groups.

We finally analyze in Section 4 the structure of AP–groups in some classes of groups that are larger than the class of radical hyperfinite groups.

Theorem C (Theorem 4.4). *Let G be a periodic AP–group. If G is a hyper–N–group, then G is hyperfinite.*

Corollary D. *Let G be a periodic AP–group. If G is a hyper–N–group, then G is a hypercyclic metabelian AP–group.*

Corollary E. *Let G be a periodic AP–group. If G is a hyper–Gruenberg group, then G is a hypercyclic metabelian AP–group. In particular, if G is a countable radical group, then G is a hypercyclic metabelian AP–group.*

Corollary F. *Let G be a periodic AP–group. If G is hyperabelian, then G is a hypercyclic metabelian AP–group.*

Corollary G. *Let G be a periodic AP–group. If G is soluble, then G is a hypercyclic metabelian AP–group.*

Corollary H. *Let G be a periodic AP–group. If G is residually soluble, then G is a hypercyclic metabelian AP–group.*

Our group-theoretical terminology is standard and can be found in [14] for example.

2. Preliminary properties of AP–groups

We start by establishing some useful facts. The first one is obvious.

Lemma 2.1 *Let G be an AP–group. Then G/H is an AP–group for each normal subgroup H of G.*

Lemma 2.2 *Let G be an AP–group. Then every ascendant subgroup of G is an AP–group. In particular, every subnormal subgroup of G is an AP–group.*

Proof. Indeed, let A be an ascendant subgroup of G. If B is an ascendant subgroup of A, then B is ascendant in G. Therefore B is permutable in G and hence in A.   ∎

Corollary 2.3 *Let G be an AP–group and let A be an ascendant subgroup of G. Then A/B is an AP–group for each subgroup B which is normal in A.*

Corollary 2.4 *Let G be an AP–group, A an ascendant subgroup of G and let B a normal subgroup of A. If A/B is a N–group, then every subgroup of A/B is permutable.*

As usual, if G is a group, we denote by $\Pi(G)$ the set of all primes occurring as divisors of the order of some periodic element of G.

Corollary 2.5 *Let G be a periodic AP–group. If G is a N–group, then G can be expressed as a direct product*

$$G = Dr_{p\in\Pi(G)}G_p,$$

where G_p is the Sylow p–subgroup of G and the following conditions holds:

(1) if $p \neq 2$, then either G_p is abelian or $G_p = B_p \langle a_p \rangle$, where B_p is a normal abelian subgroup of exponent p^k, and there is a positive integer t such that $t = 1 + p^m$, for some $m \leq k \leq m + d$, where $p^d = |G_p/B_p|$, and $a_p^{-1} b a_p = b^t$ for all $b \in B_p$; and

(2) if $p = 2$, then either G_p is a Dedekind group or $G_p = B_p \langle a_p \rangle$, where B_p is a normal abelian subgroup of exponent p^k, and there is a positive integer t such that $t = 1 + p^m$, for some $2 \leq m \leq k \leq m + d$, where $p^d = |G_p/B_p|$, and $a_p^{-1} b a_p = b^t$ for all $b \in B_p$.

Note that in both cases G_p is nilpotent, and bounded in the non-abelian case.

Proof. By Corollary 2.4, every subgroup of G is permutable in G and it suffices to apply the description of the periodic locally nilpotent groups, in which every subgroup is permutable (see [17, Theorem 2.4.14]).   ∎

The next result shows a local criterion of permutability, which has some fruitful consequences.

Lemma 2.6 *Suppose that a group G has a local system of subgroups $\mathcal{L}$ such that if $L \in \mathcal{L}$, then every subgroup of L is permutable in L. Then every subgroup of G is permutable in G.*

Proof. Let H be an arbitrary subgroup of G. It suffices to prove that $\langle x \rangle H = H \langle x \rangle$ for each element $x \in G$. Let $h \in H$. Then there exists a member $L \in \mathcal{L}$ such that $x, h \in L$. In particular, $U = H \cap L \neq \langle 1 \rangle$. It follows that U is a permutable subgroup of L, and thus $\langle x \rangle U = U \langle x \rangle$. In particular, $x^h = u x^k$ for some $u \in U \leq H$ and $k \in \mathbb{Z}$. Therefore $\langle x \rangle H = H \langle x \rangle$, as required.   ∎

Corollary 2.7 *Let G be a group and let $\mathcal{M}$ be a family of subgroups linearly ordered by inclusion whose union is G itself. Suppose that if $L \in \mathcal{M}$, then every subgroup of L is permutable in L. Then every subgroup of G is permutable in G.*

Corollary 2.8 *Let G be a group having an ascending series of subgroups*

$$\langle 1 \rangle = D_0 \leq D_1 \leq \cdots D_\alpha \leq D_{\alpha+1} \leq \cdots \leq D_\gamma = G$$

where γ is a limit ordinal. Suppose that every subgroup of D_α is permutable in D_α for all $\alpha < \gamma$. Then every subgroup of G is permutable in G.

Corollary 2.9 *Let G be a periodic AP–group. If G is a Gruenberg group, then every subgroup of G is permutable. In particular, G is the direct product of nilpotent Sylow subgroups.*

Proof. Let K be an arbitrary finitely generated subgroup of G. Then K is ascendant in G and hence it is permutable in G. By the same reason every subgroup of K is permutable in G, and hence in K. By Lemma 2.6 every subgroup of G is permutable in G. ∎

Lemma 2.10 *Let G be a group and let H be a periodic subgroup of G. If H is ascendant in G, then $H \leq O_\pi(G)$, where $\pi = \Pi(H)$.*

Proof. It suffices to apply the formula (10) of [14, p. 20]. ∎

The following result is devoted to establish sufficient conditions for a group to be an AP–group.

Lemma 2.11 *Let G be a periodic group and let L be a normal subgroup satisfying the following conditions:*

(i) every subgroup of L is G–invariant;

(ii) G/L is an AP–group; and

(iii)] $\Pi(L) \cap \Pi(G/L) = \emptyset$.

Then G is an AP–group.

Proof. Let H be an ascendant subgroup of G. Let $D = L \cap H$; by (i), D is normal in G. Therefore it suffices to prove that H/D is permutable in G/D. There is no loss if we assume that $D = \langle 1 \rangle$, that is $H \cap L = \langle 1 \rangle$. By Lemma 2.10, $H \leq O_\pi(G)$, where $\pi = \Pi(H)$. Let $\sigma = \Pi(L)$. Then $[H, L] \leq O_\pi(G) \cap O_\sigma(G) = \langle 1 \rangle$ since $\pi \cap \sigma = \emptyset$ by (iii). Then $[H, L] = \langle 1 \rangle$ and it suffices to show that $\langle g \rangle H = H \langle g \rangle$, where g is a p–element and p is an arbitrary prime.

If $p \in \sigma$, then $g \in L$ and $\langle g \rangle H = \langle g \rangle \times H$. Suppose now that $p \notin \sigma$, and put $T = \langle H, g \rangle$. By Lemma 2.10, $H \leq O_\pi(G)$. Let $\omega = \pi \cup \{p\}$, then $\Pi(T) = \omega$ and, by (iii), $T \cap L = \langle 1 \rangle$. By (ii), G/L is an AP–group. Since the subgroup HL/L is ascendant in G/L, it is permutable in G/L. In particular, $(HL/L)\langle gL \rangle = \langle gL \rangle(HL/L)$. In other words, for each element $h \in H$ and every positive integer k, there are an element $u \in H$ and a positive integer m such that $hg^k L = g^m u L$. It follows that $u^{-1} g^{-m} h g^k \in L$. On the other hand $u^{-1} g^{-m} h g^k \in T$, that is $u^{-1} g^{-m} h g^k \in L \cap T = \langle 1 \rangle$. Therefore $hg^k = g^m u$ and hence $H \langle g \rangle = \langle g \rangle H$, which proves the result. ∎

The next results are mainly devoted to prove that a radical hyperfinite AP–group is hypercyclic.

Proposition 2.12 *Let P be a locally finite p–group, where p is a prime, and let G be a finite subgroup of $\mathrm{Aut}(P)$. If $p \notin \Pi(G)$, then $P = C_P(G)[P, G]$. Moreover, if P is abelian, then $P = C_P(G) \times [P, G]$.*

Proof. Let $\mathcal{L}$ be the local system of finite subgroups of P and put $\mathcal{M} = \{L^G \mid L \in \mathcal{L}\}$. Since G is finite, every member $B \in \mathcal{M}$ is also finite. By [8, Theorem 5.3.5],

$$B = C_B(G)[B, G].$$

If $C \in \mathcal{M}$ satisfies $B \leq C$, then $C_B(G) \leq C_C(G)$ and $[B, G] \leq [C, G]$. These inclusions give that

$$C_P(G) = \bigcup_{B \in \mathcal{M}} C_B(G) \text{ and } [P, G] = \bigcup_{B \in \mathcal{M}} [B, G].$$

It follows that $P = C_P(G)[P, G]$.

If P is abelian, then $B = C_B(G) \times [B, G]$ (see [8, Theorem 5.2.3]). It follows that $C_P(G) \cap [P, G] = \langle 1 \rangle$, and therefore $P = C_P(G) \times [P, G]$. ∎

Lemma 2.13 *Let G be an AP–group and let g be an element of G. Suppose that H is an ascendant $\langle g \rangle$–invariant subgroup of G such that $\langle g \rangle \cap H = \langle 1 \rangle$. If H is a Gruenberg group, then $\langle h \rangle^{\langle g \rangle} = \langle h \rangle$ for each element $h \in H$.*

Proof. Put $L = \langle h \rangle^{\langle g \rangle}$. Then $\langle g, h \rangle = L \rtimes \langle g \rangle$ is a semidirect product. Since $\langle h \rangle$ is ascendant in H (and hence in G), $\langle h \rangle$ is permutable in G. Thus $\langle g, h \rangle = \langle g \rangle \langle h \rangle$ and

$$L = L \cap (\langle h \rangle \langle g \rangle) = \langle h \rangle (L \cap \langle g \rangle) = \langle h \rangle.$$ ∎

Lemma 2.14 *Let G be a periodic AP–group and let g be an element of G. Suppose that H is an ascendant $\langle g \rangle$–invariant p–subgroup of G, where p is a prime. If H is a Gruenberg group and $g \notin C_G(H)$ but $g^n \in C_G(H)$ for some p'–number n, then H is abelian and $[H, g] = H$.*

Proof. We have $g = bx$, where b is a p–element and x is a p'–element. Clearly $b \in C_G(H)$, so $h^g = h^x$ for each element $h \in H$. Let L be an arbitrary subgroup of H and let $K = L^{\langle x \rangle}$. Being a Gruenberg group, H is locally nilpotent. In particular, L is a finite nilpotent group. Then K is finite and $\langle x \rangle$-invariant. By Lemma 2.13, x induces on K a non-identity power automorphism. It follows from [9, Hilfssatz 5] that K is abelian. Hence every finite subgroup of H is abelian and, therefore, H is abelian. Applying Proposition 2.12, we obtain the direct decomposition

$$H = C_H(g) \times [H, \langle g \rangle] = C_H(g) \times [H, g].$$

By the choice of g, we have $H \neq C_H(g)$, and thus $[H, g] \neq \langle 1 \rangle$. Suppose that $C_H(g) \neq \langle 1 \rangle$ and choose in $C_H(g)$ an element c of order p. Let a be an

element of $[H, g]$ having order p. Since $a \notin C_H(G)$, $a^g = a^d$, where d is a p'–number, and moreover $d \not\equiv 1 \pmod{p}$. We have

$$(ac)^g = a^g c^g = a^d c,$$

and, since $ac \notin C_H(G)$, we have that

$$(ac)^g = (ac)^t,$$

where t is also a p'–number such that $t \not\equiv 1 \pmod{p}$. Hence

$$a^d c = (ac)^t = a^t c^t,$$

and therefore $d \equiv t \pmod{p}$ and $t \equiv 1 \pmod{p}$. This contradiction proves that $C_H(g) = \langle 1 \rangle$ and hence $[H, g] = H$. ∎

Lemma 2.15 *Let G be a periodic AP–group and let H be a normal p–subgroup of G, where p is a prime. Suppose that H is a Gruenberg group. If $QC_G(H)/C_G(H)$ is a non-identity p'–subgroup, then H is abelian and $[H, Q] = H$. In particular, $[H, G] = H$. Moreover, every subgroup L of H is Q–invariant and $[L, Q] = L$.*

We are now in a position to show the mentioned hypercyclicity of a radical hyperfinite AP–group.

Lemma 2.16 *Let G be a radical hyperfinite group. If G is an AP–group, then G is hypercyclic.*

Proof. Since G is hyperfinite and radical, G has a normal finite abelian p–subgroup for some prime p. It follows that G contains a minimal normal abelian p–subgroup A. The finiteness of A yields that $G/C_G(A)$ is finite. Being soluble, by Hall's theorem,

$$G/C_G(A) = (Q/C_G(A))(P/C_G(A)),$$

where $Q/C_G(A)$ is the Hall p'–subgroup of $G/C_G(A)$ and $P/C_G(A)$ is the Sylow p–subgroup of $G/C_G(A)$. Since $P/C_G(A)$ is a finite p–group, $C_A(P) = B \neq \langle 1 \rangle$. Moreover, B is a P–invariant subgroup of A. By Lemma 2.13, B is also Q–invariant. The equation $G = PQ$ yields that B is G–invariant. However, in this case $B = A$. This shows that $G/C_G(A)$ is a p'–group. Application of Lemma 2.13 again gives that A is cyclic. Hence G has a non-identity normal cyclic subgroup. From this statement and making use of transfinite induction, we obtain that G is hypercyclic, as required. ∎

3. Structure of hyperfinite AP–groups

As the title indicates, the first objective of this section is to determine the structure of a (radical) hyperfinite AP–group. We split the proof of our main theorem in several results.

Lemma 3.1 *Let G be a radical hyperfinite group and suppose that G has a normal Sylow p–subgroup P such that G/P is locally nilpotent, where p is a prime. If G is an AP–group and G is not locally nilpotent, then P is abelian. Moreover, if B is a subgroup of P, then B is G–invariant and $B = [B, G]$.*

Proof. Being hyperfinite, P is hypercentral. Then every subgroup of P is ascendant (see, for example [12, §63]). By Corollary 2.4, every subgroup of P is permutable. Suppose that P is non-abelian and $G = PC_G(P)$. Then $G = P \times Q$, where Q is the Sylow p'–subgroup of G ([5, Theorem 7]). It follows that G is locally nilpotent This is a contradiction that proves that $G \neq PC_G(P)$. By Lemma 2.15, P must be abelian.

Reasoning as above we can see that $G \neq PC_G(P)$. Let B be an arbitrary subgroup of P. Applying again Lemma 2.15, we obtain that B is G–invariant and $[B, G] = B$. ∎

Corollary 3.2 *Let G be a radical hyperfinite AP–group, and suppose that G is not locally nilpotent. If L is the locally nilpotent residual of G, then every Sylow p–subgroup of G is abelian for each $p \in \Pi(L)$. In particular, L is abelian. Moreover, every Sylow p–subgroup of L is the Sylow p–subgroup of G for each $p \in \Pi(L)$.*

Proof. By Lemma 2.16, G is hypercyclic. Therefore $[G, G]$ is hypercentral (see [12, §64]). The obvious inclusion $L \leq [G, G]$ implies that L is likewise hypercentral. Then

$$L = \mathrm{Dr}_{p \in \Pi(L)} L_p,$$

where L_p is the Sylow p–subgroup of L. Pick $p \in \Pi(L)$ and put $R = \mathrm{Dr}_{r \neq p} L_r$. Since G is locally finite, G/L is locally nilpotent. Thus G/L has an unique Sylow p–subgroup, say P/L. It follows that P/R is the unique Sylow p–subgroup of G/R. Since $R \neq L$, G/R is not locally nilpotent. By Lemma 3.1, P/R is abelian. Since P/R is the unique Sylow p–subgroup of G/R, $L_p R/R \leq P/R$, and thus

$$L_p \cong L_p/(L_p \cap R) \cong L_p R/R$$

and L_p is abelian. Since this holds for each $p \in \Pi(L)$, L is abelian too.

Suppose that $L_p R/R = L/R \neq P/R$. Being hyperfinite and locally nilpotent, G/L is hypercentral. It follows that $P/L \cap \zeta(G/L) = Z/L \neq \langle 1 \rangle$.

Then $[Z/R, G/R] \leq L/R$. We have already noted that G/R is not locally nilpotent. Since Z/R is a G–invariant subgroup of P/R, Lemma 3.1 yields that $[Z/R, G/R] = Z/R$. This contradiction proves the equation $L_pR/R = L/R = P/R$. It follows that L_p is the Sylow p–subgroup of G. ∎

Lemma 3.3 *A radical hyperfinite AP–group G is metabelian.*

Proof. Let L be the locally nilpotent residual of G. Since $L \leq [G, G]$,

$$G/[G, G] \cong (G/L)/([G, G]/L).$$

Since G/L is locally nilpotent,

$$G/L = \mathrm{Dr}_{p \in \Pi(G/L)} S_p/L,$$

where S_p/L is the Sylow p–subgroup of G/L. Then

$$[G, G]/L = [G/L, G/L] = \mathrm{Dr}_{p \in \Pi(G/L)} [S_p/L, S_p/L].$$

Put $D_p/L = [S_p/L, S_p/L]$. Being hyperfinite, G/L is hypercentral. In particular, every subgroup of G/L is ascendant in G/L (see [12, §63]). Corollary 2.5 yields that every Sylow p–subgroup S_p/L of G/L is metabelian, so that D_p/L is abelian for each $p \in \Pi(G/L)$.

On the other hand, by Lemma 2.16, G is hypercyclic. Therefore $[G, G]$ is hypercentral (see[12, §63]). Pick $p \in \Pi([G, G]) \setminus \Pi(L)$ and choose the Sylow p–subgroup P of $[G, G]$. By Corollary 3.2, $P \cap L = \langle 1 \rangle$. Therefore

$$P \cong P(P \cap L) \cong PL/L \leq D_p/L,$$

and then P is abelian. By Corollary 3.2, the Sylow p–subgroup of L is abelian for each $p \in \Pi(L)$. It follows that $[G, G]$ is abelian. ∎

Lemma 3.4 *Let G be a radical hyperfinite group and let R be the locally nilpotent radical of G. If G is an AP–group, then $R = L \times Z$, where L is the locally nilpotent residual of G and Z is the upper hypercenter of G.*

Proof. By Corollary 3.2, L is an abelian normal subgroup of G, so that $L \leq R$. Being locally nilpotent, R is the product of its Sylow subgroups. Let $\pi = \Pi(L)$. By Corollary 3.2, L is the Hall π–subgroup of R. Hence $R = L \times Z$, where Z is the Hall π'–subgroup of R. Since the factor group G/L is locally nilpotent and hyperfinite, it is hypercentral. Let

$$\langle 1 \rangle = C_0/L \leq C_1/L \leq \ \cdots \ \leq C_\alpha/L \leq C_{\alpha+1}/L \leq \ \cdots \ C_\gamma/L = G/L$$

be the upper central series of G/L. Let Z_α be the subgroup of Z such that $Z_\alpha L/L = C_\alpha/L \cap ZL/L$, for every $\alpha < \gamma$. We have

$$[Z_{\alpha+1}, G]L = [Z_{\alpha+1}L/L, G/L] \leq C_\alpha/L \cap ZL/L = Z_\alpha L/L,$$

so that

$$[Z_{\alpha+1}, G] \leq Z_\alpha L \cap Z = Z_\alpha(L \cap Z) = Z_\alpha$$

for all $\alpha < \gamma$. In other words, the series

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq Z_{\alpha+1} \leq \cdots Z_\gamma = Z$$

is G–central. This shows that the upper hypercenter $\zeta_\infty(G)$ of G contains Z. On the other hand, $\zeta_\infty(G) \leq R$, and then

$$\zeta_\infty(G) = Z \times (\zeta_\infty(G) \cap L).$$

We claim that $\zeta_\infty(G) \cap L = \langle 1 \rangle$. For, otherwise $\zeta(G) \cap L = Y \neq \langle 1 \rangle$. Pick $p \in \Pi(Y)$, and let P be the Sylow p–subgroup of L and let Q be the Hall p'–subgroup of L. By Corollary 3.2, P is abelian. If we suppose that $C_G(P) = G$, then $C_{G/Q}(PQ/Q) = G/Q$. It follows that $G/Q = PQ/Q \times S/Q$, where S/Q is the Hall p'–subgroup of G/Q ([5, Theorem 7]). This means that the factor-group G/Q is hypercentral, which contradicts the choice of L. This contradiction shows that $C_G(P) \neq G$, so there is a p'–element g such that $g \notin C_G(P)$. By Proposition 2.12, $P = C_P(g) \times [P, g]$. By the choice of P, we have that $C_P(g) \neq \langle 1 \rangle$. Therefore $[P, g] \neq P$. However Lemma 2.15 yields that $[P, g] = P$. This contradiction shows our claim, that is $\zeta_\infty(G) \cap L = \langle 1 \rangle$. Therefore $\zeta_\infty(G) = Z$, as required. ∎

We are now in a position to prove our result on the description of the structure of radical hyperfinite AP–groups.

Theorem 3.5 *Let G be a radical hyperfinite AP–group. Then the following assertions hold:*

(i) *G is metabelian;*

(ii) *if R is the locally nilpotent radical of G, then $R = L \times Z$, where L is the locally nilpotent residual of G and Z is the upper hypercenter of G;*

(iii) *The Sylow p–subgroup of L is the Sylow p–subgroup of G for every $p \in \Pi(L)$;*

(iv) *every subgroup of L is G–invariant; and*

(v) *G/L is hypercentral and every subgroup of G/L is permutable.*

Moreover, if the factor group G/L is countable, then G splits over L. In particular, if G is countable, then G splits over L.

On the other hand, if G is a periodic group having a normal abelian subgroup L that satisfies the conditions (iii)–(v), then G is an AP–group.

Proof. The statement (i) follows from Lemma 3.3, the statement (ii) follows from Lemma 3.4 and the statement (iii) follows from Corollary 3.2. As we see above, $C_G(L) \neq G$, so that $G/C_G(L)$ is a non-identity group such that $\Pi(L) \cap \Pi(G/L) = \emptyset$. Therefore, Lemma 2.15 yields (iv).

Finally G/L is locally nilpotent and, being hyperfinite, G/L is hypercentral. In particular, every subgroup of G/L is ascendant in G/L (see [12, §63]). Hence every subgroup of G/L is permutable, that is (v) holds. To obtain the mentioned splitting of G over L it suffices to apply [6, Theorem 2.4.5], for example.

The last assertion is an immediate consequence of Lemma 2.11.   ∎

We remark that the description of hypercentral periodic groups, every subgroup of which is permutable has been obtained above in Corollary 2.5. Since a radical hyperfinite AP–group G is hypercyclic by Lemma 2.16, it is not hard to see that G has a descending Sylow series, that is G has a series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n \geq G_{n+1} \geq \cdots \bigcap_{n \in \mathbb{N}} G_n = \langle 1 \rangle$$

of normal subgroups satisfying the following conditions:

  (i) for every $n \in \mathbb{N}$, G_n/G_{n+1} is a p_n–group, for some prime p_n; and

  (ii) $p_0 < p_1 < \cdots < p_n < p_{n+1} < \cdots$.

As a consequence, we can see that $2 \notin \Pi(L)$. Another consequence is the following important result.

Corollary 3.6 *Let G be a radical hyperfinite AP–group. Then every subgroup of G is an AP–group.*

The corresponding result for PT–groups is not true as the locally dihedral 2–group shows. In this sense, we say that a group G is a $\overline{PT}$–group if every subgroup of G is a PT–group, that is, if for subgroups H, K and L such that H is permutable in K and K is permutable in L, it is always true that H is permutable in L. It is easy to prove, using transfinite induction, that every ascendant subgroup of a $\overline{PT}$–group is permutable. Since every AP–group is a PT–group, we also deduce from Corollary 3.6 that $\overline{PT}$–groups are exactly AP–groups in the universe of all radical hyperfinite groups.

We recall that a group G is said to be a T–*group* if normality is a transitive relation in G, that is if K is a normal subgroup of H and H is a normal subgroup of G implies that K is normal in G. The study of infinite T–groups has began in a paper of D. J. S. Robinson [13]. This study has been continued in many papers, where infinite T–groups have been studied from different points of view. The next results show connections between the classes of AP–groups and T–groups.

Corollary 3.7 *Let G be a radical hyperfinite AP–group. If the Sylow 2–subgroups of G are Dedekind groups and the Sylow p–subgroups of G are abelian for $p \neq 2$, then G is a metabelian T–group.*

Proof. Let L be the locally nilpotent residual of G. If H is a given subnormal subgroup of G, we put $D = L \cap H$. By the condition (i) of Theorem 3.5, D is normal in G. Therefore it suffices to prove that H/D is normal in G/D. Without loss of generality, we can suppose that $D = \langle 1 \rangle$, that is $H \cap L = \langle 1 \rangle$. By Lemma 2.10, $H \le O_\pi(G)$, where $\pi = \Pi(H)$. Put $\sigma = \Pi(L)$. It follows that $[H, L] \le O_\pi(G) \cap O_\sigma(G) = \langle 1 \rangle$ since $\pi \cap \sigma = \emptyset$ by the condition (iii) of Theorem 3.5. Then $[H, L] = \langle 1 \rangle$. It suffices to show that $g^{-1}Hg = H$, where g is a p–element and p is an arbitrary prime. If $p \in \sigma$, then $g \in L$ and $\langle g \rangle H = H \langle g \rangle$. In particular, $g^{-1}Hg = H$. Suppose now that $p \notin \sigma$ and put $R = \langle H, g \rangle$. By Lemma 2.10, $H \le O_\pi(G)$. Let $\omega = \pi \cup \{p\}$ so that $\Pi(R) = \omega$ and, by the condition (iii) of Theorem 3.5, $R \cap L = \langle 1 \rangle$. Since G/L is Dedekind, HL/L is normal in G/L. Hence for each element $h \in H$ we have $[hL, gL] = [h, g]L \in HL/L$, that is $[h, g] \in HL$. On the other hand, $g, h \in R$ and so $[h, g] \in R$. Thus $[h, g] \in HL \cap R = H(L \cap R) = H$. This means that $g^{-1}Hg = H$, which proves the result.   ∎

Corollary 3.8 *Let G be a radical hyperfinite AP–group. If the Sylow p–subgroups of G are abelian for all primes $p \in \Pi(G)$, then G is a metabelian T–group.*

Corollary 3.9 *Let G be a radical hyperfinite AP–group. If the locally nilpotent residual of G coincides with the derived subgroup, then G is a metabelian T–group.*

Proof. Let p be a prime, and let P be a Sylow p–subgroup of G. Let L be the locally nilpotent residual of G. If $p \in \Pi(L)$, then by the condition (iii) of Theorem 2.7, $P \le L$, and the condition (iv) of Theorem 3.5, yields that P is abelian. If $p \notin \Pi(L)$, then, by the condition (iii) of Theorem 3.5, $P \le L = \langle 1 \rangle$. Since G/L is abelian, P is abelian too. It suffices to apply Corollary 3.8.   ∎

Corollary 3.10 *Let G be a radical hyperfinite AP–group. If $\zeta(G) \cap [G, G] = \langle 1 \rangle$, then G is a metabelian T–group.*

Proof. Let R be the locally nilpotent radical of G. By the condition (ii) of Theorem 3.5, $R = L \times Z$, where L is the locally nilpotent residual of G, and Z is the upper hypercenter of G. Lemma 2.16 yields that G is hypercyclic. It follows that $[G, G]$ is hypercentral (see [12, §64]). In particular, $[G, G] \le R$. On the other hand, $L \le [G, G]$, so that

$$[G, G] = L \times ([G, G] \cap Z) = L$$

and it suffices to apply Corollary 3.9.   ∎

Corollary 3.11 *Let G be a radical hyperfinite AP–group. If $\zeta(G) = \langle 1 \rangle$, then G is a metabelian T–group.*

4. The structure of AP–groups in some wider classes of groups

In this section we extend some of our previous results to certain extensions of the class of radical hyperfinite groups. They are the hyperabelian groups, hyper–N–groups, hyper–Gruenberg groups. In general, these classes of groups are much wider than the class of radical hyperfinite groups. Really the class of hyperfinite groups is "very narrow". In particular, there are hyperabelian (even soluble) groups, which are not hyperfinite. In fact, for every prime p and every periodic locally cyclic p'–group L there exists a simple $\mathbb{F}_p L$–module A, where $\mathbb{F}_p$ is the field of order p (see, for example, [11, Chapter 2]). The natural semidirect product of A by L gives an example of a metabelian group which is not hyperfinite. However, as we will see in this section, periodic AP–groups in the above classes are hyperfinite.

Lemma 4.1 *Let G be an AP–group and let P be a normal p–subgroup of G for some prime p. Suppose that P has an ascending series of normal subgroups*

$$\langle 1 \rangle = D_0 \leq D_1 \leq \ \cdots \ \leq D_\alpha \leq D_{\alpha+1} \leq \ \cdots \ D_\gamma = P$$

whose factors are N–groups. Then every subgroup of P is permutable in G.

Proof. We proceed by induction on γ. If $\gamma = 1$, then P itself is a N–group. In this case every subgroup of P is ascendant in P and hence in G. It follows that every subgroup of P is permutable in G.

Let now $\gamma > 1$ and suppose that we have already proved that every subgroup of D_α is permutable in G for all $\alpha < \gamma$. If γ is a limit ordinal, then the result follows from Corollary 2.8. Therefore we assume that $\gamma - 1$ exists. Put $D = D_{\gamma-1}$. By induction, every subgroup of D is permutable in G. Being a p–group, D is hypercentral (see [17, Theorem 2.4.14]). Since every subgroup of P/D is ascendant in P/D, Corollary 2.4 yields that every subgroup of P/D is permutable in G/D and so P/D is likewise hypercentral. Let $\mathcal{L}$ be the family of all finite subgroups of P/D and let U and V be P–invariant subgroups of D such that $U \leq V$ and the factor V/U is D–central. If $L/D \in \mathcal{L}$, then $L = KD$ for some finite subgroup K of P. Pick $v \in V$, and put $T = \langle v, K \rangle$. Being a finite p–subgroup, T is nilpotent and then TU/U is nilpotent too. Since V is normal in P,

$$Z/U = V/U \cap \zeta(T/U) \neq \langle 1 \rangle.$$

This shows that

$$C_{V/U}(L/D) \neq \langle 1 \rangle.$$

Applying transfinite induction, we obtain that V/U has an ascending L–central series. Since this holds for each factor of the upper central series of D, L is hypercentral. Since L is ascendant in G, every subgroup of L is ascendant in G. Applying Lemma 2.6, we see that every subgroup of P is permutable in G, as required. ∎

Lemma 4.2 *Let G be a locally finite group. Suppose that, for some prime p, G has a normal Sylow p–subgroup P such that P does not contain $C_G(P)$. Then $O_{p'}(G) \neq \langle 1 \rangle$. Moreover, if G/P is infinite and $G/C_G(P)$ is finite, then $O_{p'}(G)$ is infinite.*

Proof. Put $C = C_G(P)$ and $A = P \cap C$. The isomorphism

$$C/A = C/(C \cap P) \cong CP/P$$

shows that C/A is a p'–group. It follows that A is the Sylow p–subgroup of C and $C = A \times O_{p'}(C)$ ([5, Theorem 7]). The conditions of the Lemma imply that $C \neq A$, therefore $O_{p'}(C) \neq \langle 1 \rangle$. Since C is normal in G, $O_{p'}(C) \leq O_{p'}(G)$, and the first assertion follows. The second one is immediate. ∎

Lemma 4.3 *Let G be a periodic AP–group. Suppose that $O_{p'}(G) = \langle 1 \rangle$ for some prime p. If G is a hyper–N–group, then $G/O_p(G)$ is finite.*

Proof. Obviously we can suppose that G is infinite. Put $P = O_p(G)$. Suppose the contrary, that is G/P is infinite. We claim that P is infinite. For, otherwise, $O_{p'}(G)$ is infinite by Lemma 4.2, which is impossible. Hence P is infinite. Since G is a hyper–N–group, then we fall into one of the following cases.

(a) G has a finite series of normal subgroups

$$P = D_0 \leq D_1 \leq \cdots \leq D_n \leq D_{n+1},$$

where the factors D_j/D_{j-1} are finite r_j–groups for some prime r_j, $1 \leq j \leq n$, and D_{n+1}/D_n is an infinite q–group for some prime q.

(b) G has an infinite series of normal subgroups

$$P = D_0 \leq D_1 \leq \cdots \leq D_n \leq D_{n+1} \leq \cdots,$$

where the factors D_j/D_{j-1} are finite r_j–groups for some prime r_j, $1 \leq j$.

Consider first the case (a). Applying Lemma 4.2, after finitely many steps, we see that G has a normal subgroup Q such that $P \leq Q$ and Q/P is an infinite q–group. By definition of P, we have $p \neq q$. By Lemma 4.1, every subgroup of P is permutable in G. Then Corollary 2.5 yields that either P is abelian or $P = B\langle b \rangle$, for some element b and some abelian normal subgroup B of P such that every cyclic subgroup of B is P–invariant. In both cases, Lemma 2.13 proves that every subgroup of A (respectively of B) is Q–invariant, while, in particular, in the second case B is normal in Q.

If P is abelian, then $Q/C_Q(P)$ is finite (see [17, Theorem 1.5.6]). Applying Lemma 4.2 in this case, $O_{p'}(G) = \langle 1 \rangle$, which is impossible. Now we suppose that P is non-abelian. Since P/B is finite, by Lemma 4.2, $R/B = O_{p'}(Q/B)$ is infinite. Repeating now the same arguments that we used when supposed P abelian, we obtain again that $O_{p'}(G) \neq \langle 1 \rangle$. Therefore the case (a) leads to a contradiction.

Consider now the case (b). Put $D = \bigcup_{n \in \mathbb{N}} D_n$. Since D is normal in G, D is an AP–group by Lemma 2.2. Being hyperfinite, D/P is metabelian by Theorem 3.5. Let L/P be the locally nilpotent radical of D/P. If we assume that L/P is finite, then D/P is finite, because $C_{D/P}(L/P) \leq L/P$. However this contradicts our assumption. Hence L/P is infinite. By definition of P, L/P is a p–group. Reasoning as in case (a), we obtain again that $O_{p'}(G) \neq \langle 1 \rangle$. As in case (a), this is a contradiction. This contradiction shows that G/P is finite, as required. ∎

Theorem 4.4 *Let G be a periodic AP–group. If G is a hyper–N–group, then G is hyperfinite.*

Proof. Let

$$\langle 1 \rangle = D_0 \trianglelefteq D_1 \trianglelefteq \cdots \trianglelefteq D_\alpha \trianglelefteq D_{\alpha+1} \trianglelefteq \cdots D_\gamma = G$$

be an ascending series of normal subgroups of G whose factors are N–groups. Pick $p \in \Pi(D_1)$, and let S be the Sylow p–subgroup of D_1. Put $Q = O_{p'}(G)$ and $P/Q = O_p(G/Q)$. By Lemma 4.3, G/P is finite. Then Lemma 4.1 shows that every subgroup of P/Q is permutable in G/Q. Being a p–group, P/Q is hypercentral by Corollary 2.5. Since SQ/Q is normal in P/Q,

$$SQ/Q \cap \zeta(P/Q) \neq \langle 1 \rangle.$$

Pick $1 \neq z \in S$ such that $zQ \in SQ/Q \cap \zeta(P/Q)$. Then $C_{G/Q}(zQ)$ has finite index. Put $Z = \langle z \rangle^G$ so that

$$ZQ/Q \leq SQ/Q \cap \zeta(P/Q),$$

and hence ZQ/Q is finite. Since S is normal in G, $Z \leq S$, which implies the finiteness of Z. Consequently, G has a non-identity finite normal subgroup. Applying transfinite induction, we obtain that G is hyperfinite. ∎

It is worth noting that the groups in Theorem 4.4 are in fact hypercyclic by Lemma 2.16. Thus, we obtain the following consequences.

Corollary 4.5 *Let G be a periodic AP–group. If G is a hyper–N–group, then G is a hypercyclic metabelian AP–group.*

Corollary 4.6 *Let G be a periodic AP–group. If G is a hyper–Gruenberg group, then G is a hypercyclic metabelian AP–group. In particular, if G is a countable radical group, then G is a hypercyclic metabelian group.*

Proof. Let $U \leq V$ be two normal subgroups of G such that U/V is a Gruenberg group. By Corollary 2.9, every Sylow p-subgroup of U/V is nilpotent, for all primes p. It follows that G is hypernilpotent. In particular, it is a hyper–N–group and we can apply Corollary 4.5. ∎

Corollary 4.7 *Let G be a periodic AP–group. If G is hyperabelian, then G is a hypercyclic metabelian AP–group.*

Corollary 4.8 *Let G be a periodic AP–group. If G is soluble, then G is a hypercyclic metabelian AP–group.*

Corollary 4.9 *Let G be a periodic AP–group. If G is residually soluble, then G is a hypercyclic metabelian AP–group.*

Proof. By Lemma 2.1, every soluble factor-group of G is an AP–group, and, by Corollary 4.8, this factor-group is metabelian. If follows that G itself is metabelian. By Corollary 4.8, G is also hypercyclic. ∎

We recall that a group G is called *a Baer group*, if every finitely generated subgroup of G is subnormal in G. Since every Baer group is a Gruenberg group, we have

Corollary 4.10 *Let G be a periodic AP–group. If G is a hyper–Baer group, then G is a hypercyclic metabelian AP–group.*

5. The construction of periodic soluble AP–groups

The main object of this section is to show how our results allows us to construct all periodic soluble AP–groups.

Let G be a periodic soluble AP–group and let R be the locally nilpotent radical of G. In particular, it follows from Theorem 4.4 that G is hyperfinite. By virtue of Lemma 3.4, we have $R = L \times Z$, where L is the locally nilpotent residual of G and Z is the upper hypercenter of G. It is clear then that G is embedded into $G/L \times G/Z$. In addition, applying Theorem 3.5, G/L is a locally nilpotent group with all subgroups permutable. Consider $H = G/Z$ and let $K = LZ/Z$. Then $K = \mathrm{Dr}_{p \in \Pi(K)} K_p$ where K_p is the Sylow p–subgroup of K. Let U_p be the normal subgroup of G which is maximal with the property: $\mathrm{Dr}_{q \neq p} K_q$ is contained in U_p and $U_p \cap K_p = 1$. Now we consider the quotient H/U_p. Then $C_{H/U_p}(K_p U_p/U_p) = K_p U_p/U_p$. Furthermore, by Theorem 3.5, every Sylow subgroup of L is the Sylow subgroup of G. It follows that $(H/U_p)/(K_p U_p/U_p)$ is a p'–group. Since H/U_p induces power automorphisms on $K_p U_p/U_p$, then being periodic, it is finite of order at most $p - 1$. Hence H/U_p is the semidirect product of $K_p U_p/U_p$ and V_p/U_p where V_p/U_p is finite of order at most $p - 1$. Since the intersection of all U_p is trivial (it lies in the centralizer of K), then H is embedded in the Cartesian product of all the $(H/U_p)'s$.

Let π be a non-empty set of primes such that $2 \notin \pi$. Let A_p be an abelian p–group for every prime $p \in \pi$. We consider the natural semidirect product C_p of A_p and F_p where F_p is a finite group of order at most $p-1$ such that F_p acts as power automorphisms group on A_p. Let B be the Cartesian product of C_p, $p \in \pi$, and let A be the direct product of A_p, $p \in \pi$. Choose an arbitrary periodic subgroup P of B including A and consider G the direct product of P and Q where Q is a locally nilpotent π'–group such that every subgroup of Q is permutable. Then, by the above paragraph, every periodic soluble AP–group arises as a subgroup of such group.

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