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Bi-Lipschitz decomposition of Lipschitz functions into a metric space

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Abstract

We prove a quantitative version of the following statement. Given a Lipschitz function f from the k-dimensional unit cube into a general metric space, one can be decomposed f into a finite number of BiLipschitz functions $f|_{F_i}$ so that the k-Hausdorff content of $f([0,1]^k \setminus \cup F_i)$ is small. We thus generalize a theorem of P. Jones [7] from the setting of $\mathbb{R}^d$ to the setting of a general metric space. This positively answers problem 11.13 in "Fractured Fractals and Broken Dreams" by G. David and S. Semmes, or equivalently, question 9 from "Thirty-three yes or no questions about mappings, measures, and metrics" by J. Heinonen and S. Semmes. Our statements extend to the case of *coarse* Lipschitz functions.

1. Introduction

We prove the following theorem.

Theorem 1.1. *Let $\epsilon \geq 0$, $0 < \alpha < 1$ and $k \geq 1$ be given. There are universal constants $M = M(\alpha, k)$, $c_1 = c_1(k)$ and c_2 such that the following statements hold. Let $\mathcal{M}$ be any metric space. Let $f : [0,1]^k \to \mathcal{M}$ be an ϵ-coarse 1-Lipschitz function, i.e. such that*

$$\mathrm{dist}(f(x), f(y)) \leq |x - y| + \epsilon\,.$$

Then there are sets $F_1, ..., F_M \subset [0,1]^k$ so that for $1 \leq i \leq M$, $x, y \in F_i$ we have

$$\alpha|x - y| - c_2\epsilon \leq \mathrm{dist}(f(x), f(y)) \leq |x - y| + \epsilon\,,$$

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and

$$h^k(f([0,1]^k \setminus (F_1 \cup ... \cup F_M))) \leq c_1\alpha\,. \tag{1.1}$$

(h^k is the one-dimensional Hausdorff content, defined below).

As a corollary of the theorem (with $\epsilon = 0$) we get

Corollary 1.2. *Let $k > 0$. Let $\mathcal{M}$ be a k-Ahlfors-David regular metric space which has Big Pieces of Lipschitz images of $\mathbb{R}^k$. Then $\mathcal{M}$ has Big Pieces of BiLipshcitz images of $\mathbb{R}^k$.*

Thus we positively answer problem 11.13 in [4] by G. David and S. Semmes, or equivalently, question 9 in [6] by J. Heinonen and S. Semmes.

The statement of Corollary 1.2 for the case where $\mathcal{M} = \mathbb{R}^d$ was proved by Guy David [1]. In fact, David assumes less about the domain of the functions in question. In particular, the domains need not be Euclidean, however they are required to satisfy some geometric conditions. Shortly after, Peter Jones gave a proof for Theorem 1.1 for the case where $\{\epsilon = 0$ and $\mathcal{M} = \mathbb{R}^d\}$ [7]. (Jones' and David's results appeared in the same issue of Rev. Mat. Iberoamericana.) Their work was motivated by the study of singular integrals.

Theorem 1.1, with $\epsilon = 0$, can be thought of as a quantitative version of Sard's Theorem, where we think of the non-quantitative version as: a Lipschitz map can be written as a countable union of invertible maps, and a map whose range is a Lebesgue null set. We note that a non-quantitative variant of our theorem had already appeared in [8].

The proof we give follows the outline of [7]. An important point is that the $\mathbb{R}^d$ result relies on a sum of squares of wavelet coefficients (see the exposition in [2]) or their upper half space analogue (which was the way the proof went in [7]). We replace this by a statement about a metric space analogue of the Jones β numbers, or equivalently, a statement about certain Menger curvature averages (Lemma 2.1). It is the authors feeling that the lack of such a statement was the only thing that prevented this theorem from appearing 10-20 years ago. This theorem is another building block in the process of transferring (parts of) the Euclidean theory of quantitative rectifiability to the setting of general metric spaces.

Let us quickly define the relevant notions. A metric space $\mathcal{M}$ is said to be a k-Ahlfors-David regular (with constant c_1) if for any $x \in \mathcal{M}$, $0 < r < \operatorname{diam}(\mathcal{M})$ we have $c_1^{-1}r^k \leq \mathcal{H}^k(\operatorname{Ball}(x,r)) \leq c_1 r^k$. A k-Ahlfors-David regular metric space $\mathcal{M}$ is said to have Big Pieces of Lipschitz images (with constants L_1 and c_1) if for any $x \in \mathcal{M}$, $0 < r < \operatorname{diam}(\mathcal{M})$ we have an L_1 Lipschitz function $f : A \to \mathcal{M}$, where $A \subset \mathrm{Ball}_{\mathbb{R}^k}(0,r)$ such that $\mathcal{H}^k(f(A) \cap \mathrm{Ball}(x,r)) \geq c_1 r^k$. A k-Ahlfors-David regular metric space $\mathcal{M}$ is said to have Big Pieces of BiLipschitz images (with constants L_2 and c_2) if for any $x \in \mathcal{M}$, $0 < r < \mathrm{diam}(\mathcal{M})$ we have an L_2 BiLipschitz function $f : A \to \mathcal{M}$, where $A \subset \mathrm{Ball}_{\mathbb{R}^k}(0,r)$ such that $\mathcal{H}^k(f(A) \cap \mathrm{Ball}(x,r)) \geq c_2 r^k$.

See [3] or [4] for more details. We note that in question 9 of [6] there is an error in the definition of Big Pieces of Lipschitz images.

Proof of Corollary 1.2. Let c_1, x, r, L_1 f, A as in the definition of Big Pieces of Lipschitz images be given. Let $e : \mathcal{M} \to L^\infty(\mathcal{M})$ be the Kuratowski embedding. Using the McShane-Whitney extension lemma for each coordinate, we extend $e \circ f$ to a L_1-Lipschitz function $\tilde{f} : [-r,r]^k \to L^\infty(M)$. See page 10 in [5] for more details. We now apply Theorem 1.1 (rescaled) to $\tilde{f} : [-r,r]^k \to L^\infty(M)$ with sufficiently small α (depending on the k-Ahlfors-David-regularity constant of $\mathcal{M}$, as well as L_1, and k) and $\epsilon = 0$ to get $\mathcal{H}^k(f(E \setminus (\cup F_i))) \leq \frac{1}{2} c_1 r^k$. Hence one of the sets $E \cap F_i$ must satisfy $\mathcal{H}^k(f(E \cap F_i) \cap \mathrm{Ball}(x,r)) \geq \frac{c_1}{2M} r^k$, as desired. ∎

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2. Proof of Theorem 1.1

2.1. Definitions

For a set E, define the one-dimensional Hausdorff content of a set K as

$$h^k(E) = \inf\{\sum \mathrm{diam}(U_i)^k : \cup U_i \supset E\}\,.$$

Let p be a function with range contained in $\mathcal{M}$. Define

$$\partial_1(x,y,z) = \partial_1^{(p)}(x,y,z)$$

by

$$\partial_1(x,y,z) = \mathrm{dist}(p(x),p(y)) + \mathrm{dist}(p(y),p(z)) - \mathrm{dist}(p(x),p(z))\,.$$

Define for an interval $I = [a,b] \subset \mathbb{R}$ the quantity $\tilde{\beta}(I) = \tilde{\beta}_{(p)}(I)$ by

$$\tilde{\beta}(I)^2 \mathrm{diam}(I) = \mathrm{diam}(I)^{-3} \int_{x=a}^{x=b} \int_{y=x}^{y=b} \int_{z=y}^{z=b} \partial_1(x,y,z)\, dz\, dy\, dx\,.$$

We extend this definition to higher dimensional cubes by rotations.

Let $k > 1$. Define for a cube $Q \in \mathbb{R}^k$ the quantity $\tilde{\beta}^{(k)}(Q) = \tilde{\beta}^{(k)}_{(p)}(Q)$ by

$$\tilde{\beta}^{(k)}(Q)^2 \mathrm{side}(Q)^{k-1} = \\ = \int_{g\in G_k}\int_{x\in\mathbb{R}^k\ominus g\mathbb{R}} \chi_{\{|(x+g\mathbb{R})\cap 7Q|\geq \mathrm{side}(Q)\}}\tilde{\beta}((x+g\mathbb{R})\cap 7Q)^2 dx d\mu(g)$$

where $\mathbb{R}$ is identified with $\{\mathbb{R}, 0, ..., 0\} \subset \mathbb{R}^k$, G_k is the group of all rotations of $\mathbb{R}$ in $\mathbb{R}^k$ equipped with the its Haar measure $d\mu$, and dx is the $k-1$ dimensional Lebesgue measure on $\mathbb{R}^k \ominus g\mathbb{R}$, the orthogonal complement of $g\mathbb{R}$ in $\mathbb{R}^k$. We write $\tilde{\beta}^{(1)} = \tilde{\beta}$. and note that any $k \geq 1$, we have that $\tilde{\beta}^{(k)}$ is scale invariant. This type of quantity is connected to Menger curvature. See [10] for more details.

Define $\mathcal{D}_0$ the standard dyadic partition of $\mathbb{R}$, i.e.

$$\mathcal{D}_0 := \left\{ \left[\frac{j_1}{2^{j_2}}, \frac{j_1+1}{2^{j_2}}\right] : j_1, j_2 \in \mathbb{Z}\right\}.$$

Define $\mathcal{D}_1$ a dyadic partition of $\mathbb{R}$ given by shifting the standard dyadic partition by $\frac{1}{3}$, i.e. $\mathcal{D}_0 + \frac{1}{3}$. For $i = (i_1, ..., i_k) \in \{0,1\}^k$ we define

$$\mathcal{D}^k_i := \mathcal{D}_{i_1} \oplus \cdots \oplus \mathcal{D}_{i_k}.$$

The fact that now a ball $\mathrm{Ball}(x,r) \subset \mathbb{R}^k$ with $r < \frac{1}{6}$ is contained in a cube $Q \in \cup_{i\in\{0,1\}^k}\mathcal{D}^k_i$ with $\mathrm{side}(Q) \sim r$ earns this setup the (now standard) name *the one third trick*.

For simplicity of notation in the proof, we extend f to be 1-Lipschitz with domain $\mathbb{R}^k$ (by say fixing f on rays emanating from $(\frac{1}{2}, \ldots, \frac{1}{2})$ and outside $(0,1)^k$.

We call two dyadic cubes Q_1 and Q_2 **semi-adjacent** if

$$0 < \mathrm{dist}(Q_1, Q_2) \leq 2\,\mathrm{diam}(Q_1) = 2\,\mathrm{diam}(Q_2).$$

Hence every cube Q has at most $C(k)$ semi-adjacent cubes.

2.2. The proof

We start by defining a Lipschitz function p.

Let $X = X_\epsilon \subset [0,1]^k$ be an ϵ-net for $[0,1]^k$ and $Z = Z_\epsilon \subset \mathcal{M}$ be an ϵ-net for $\mathcal{M}$. Consider a function $f' : X \to \mathcal{M}$ defined as follows. for $z \in \mathcal{M}$, let $z_\epsilon \in Z$ be such that $\mathrm{dist}(z, z_\epsilon) \leq \epsilon$ (chosen arbitrarily if there is more then one such z_ϵ). Define $f'(x) = f(x)_\epsilon$. We have for any $x, y \in X$

$$\mathrm{dist}(f'(x), f(x)) \leq \epsilon, \quad \mathrm{dist}(f'(x), f'(y)) \leq |x-y| + \epsilon + 2\epsilon,$$

We get that for any $x, y \in X$

$$\mathrm{dist}(f'(x), f'(y)) \leq 4|x-y|.$$

Now extend f' to a 4-Lipschitz function $p : [0,1]^k \to L^\infty(\mathcal{M})$ using the Kuratowski embedding and the McShane-Whitney extension (as in Corollary 1.2). Denote by $\tilde{f} : [0,1]^k \to L^\infty(\mathcal{M})$, the map given by using the Kuratowski embedding of $\mathcal{M}$ in $L^\infty(\mathcal{M})$. We have for $x \in [0,1]^k$ and $x_\epsilon \in X$ such that $\mathrm{dist}(x, x_\epsilon) \le \epsilon$

$$\begin{aligned} \mathrm{dist}(\tilde{f}(x), p(x)) &\le \mathrm{dist}(\tilde{f}(x), \tilde{f}(x_\epsilon)) + \mathrm{dist}(\tilde{f}(x_\epsilon), p(x_\epsilon)) + \mathrm{dist}(p(x_\epsilon), p(x)) \\ &\le 2\epsilon + \epsilon + 4\epsilon\,. \end{aligned} \tag{2.1}$$

This p is the one we use in the above definitions of ∂_1 and β.

Lemma 2.1. *For an L-Lipschitz function p,*

$$\sum_{\substack{Q\in\mathcal{D}_i^k,\ Q\subset[0,1]^k \\ i\in\{0,1\}^k}} \tilde{\beta}^{(k)}(Q)^2 side(Q)^k \lesssim L\,.$$

We postpone the proof of this lemma to Section 3.

Let $\alpha' = 10\alpha$. For $x_1, x_2 \in \mathbb{R}^k$, let $[x_1, x_2]$ be the straight segment connecting x_1 and x_2. Let

$$\begin{aligned} \mathcal{E}_1 \ := \Big\{Q_1 \in \mathcal{D}_0^k :\ & Q_1 \subset [0,1]^k, \quad \exists x_1 \in Q_1,\ x_2 \in Q_2, \\ & Q_1, Q_2 \text{ semi-adjacent}, \quad \mathrm{diam}(p([x_1,x_2])) \ge \alpha'|x_1 - x_2|, \\ & \mathrm{dist}(p(x_1), p(x_2)) \le \frac{\alpha'}{10}|x_1 - x_2|, \quad \alpha'|x_1 - x_2| \ge 10\epsilon\Big\}, \\ \mathcal{E}_2 \ := \Big\{[x_1, x_2] :\ & x_i \in Q_i, \quad Q_1 \subset [0,1]^k, \\ & Q_1, Q_2 \text{ semi-adjacent}, \quad \mathrm{diam}(p([x_1,x_2])) \le \alpha'|x_1 - x_2|\,, \\ & \alpha'|x_1 - x_2| \ge 10\epsilon\Big\}. \\ B \ := \Big\{Q_0 \in \mathcal{E}_1 :\ & \exists Q_1, ..., Q_N \in \mathcal{E}_1, \text{ such that } Q_0 \subsetneq Q_1 \subsetneq ... \subsetneq Q_N\Big\}. \end{aligned}$$

The constant N will be chosen later, and will depend only on α' and k. (Note that in the definition of B the cubes $Q_1, ..., Q_N$ may be of wildly different scales.)

Lemma 2.2.

$$h^k(f(\cup\mathcal{E}_2)) \lesssim \alpha'\,.$$

Proof. Assume $[x_1, x_2] \in \mathcal{E}_2$, and let Q_1, Q_2 be the corresponding semi-adjacent cubes. Recall that $|x_1 - x_2| \sim \mathrm{diam}(Q_1)$. Define for $c = 3, 30$

$$U^c_{x_1,x_2} = f^{-1}\mathrm{Ball}(f(x_1), c\alpha'|x_1 - x_2|)\,.$$

Then

$$U^3_{x_1,x_2} \supset \{x \in \mathbb{R}^k : \mathrm{dist}(x,[x_1,x_2]) < \alpha'|x_1 - x_2|\}\,. \tag{2.2}$$

Note that

$$\mathrm{diam}(fU^{30}_{x_1,x_2}) \lesssim \alpha'|x_2 - x_1|$$

implying that (together with (2.2))

$$\mathrm{diam}(fU^{30}_{x_1,x_2})^k \lesssim \alpha'\mathcal{H}^k(U^3_{x_1,x_2})\,. \tag{2.3}$$

Consider the set

$$\mathcal{U} = \{U^3_{x_1,x_2} : [x_1,x_2] \in \mathcal{E}_2\}\,.$$

We will show $h^k(\cup f\mathcal{U}) \lesssim \alpha'$, which will give the lemma as $\cup\mathcal{E}_2 \subset \cup\mathcal{U}$.

We use a Vitali covering type argument. We find a disjoint sub-collection $\mathcal{U}' \subset \mathcal{U}$ so that if $U \in \mathcal{U}$ then $U \cap U' \neq \emptyset$ for some $U' \in \mathcal{U}'$ with $2\mathrm{diam}(fU') \geq \mathrm{diam}(fU)$ as follows. Write $\mathcal{U} = \cup\mathcal{U}_j$ where $U \in \mathcal{U}_j$ implies

$$2^{-j-1} < \mathrm{diam}(fU) \leq 2^{-j}\,.$$

We greedily construct $\mathcal{U}'$ by adding sets to it from $\mathcal{U}_j$, inducting on j. We start with $\mathcal{U}' = \emptyset$. Place a maximal (with respect to inclusion) disjoint subset of $\mathcal{U}_0$ in $\mathcal{U}'$. At stage $j > 0$, consider all sets $S_j \subset \mathcal{U}_j$ which have disjoint elements and have elements disjoint from all current $\mathcal{U}'$ elements. Take a maximal (with respect to inclusion) such S_j, and add S_j to $\mathcal{U}'$. This defines $\mathcal{U}'$ as desired.

Now, we note that if $U \in \mathcal{U}$ and $U \cap U' \neq \emptyset$, $U' \in \mathcal{U}'$ with $2\mathrm{diam}(fU') \geq \mathrm{diam}(fU)$, and $U' = U^3_{x_1,x_2}$, then (by looking at the push-forward by f) $U \subset U^{30}_{x_1,x_2}$. Hence

$$\cup\{U : U \in \mathcal{U}\} \subset \cup\{U^{30}_{x_1,x_2} : U^3_{x_1,x_2} \in \mathcal{U}'\}\,.$$

Using the disjointness of elements in $\mathcal{U}'$ and inequality (2.3) we get the desired result. ∎

Lemma 2.3. *There is an $\epsilon_0 = \epsilon_0(\alpha', k) > 0$ such that for any $Q_1 \in \mathcal{E}_1$ we have $\tilde{\beta}^{(k)}(7Q_1) \geq \epsilon_0$.*

Proof. Let Q_2, x_1, x_2 be as in the definition of $\mathcal{E}_1$. Set

$$D = \mathrm{side}(Q_1) \geq |x_2 - x_1|.$$

Let $x_3 \in [x_1,x_2]$ be a point such that $\mathrm{dist}(p(x_3), p(x_1)) > \frac{\alpha'}{2}D$. Then for $x_i' \in \mathrm{Ball}(x_i, \frac{\alpha'}{100}D)$ we have $\partial_1(x_1', x_3', x_2') \geq \frac{\alpha'}{10}D$. Now use definition of $\tilde{\beta}^{(k)}$. ∎

Let

$$B_1 = \bigcup_{\substack{Q \in B \\ Q \subset [0,1]^k}} 7Q.$$

From Lemma 2.1, Lemma 2.3, and the one-third trick, we have:

Lemma 2.4. *By taking $N = N(\alpha', k)$ large enough (universally determined)*

$$\mathcal{H}^k(B_1) \lesssim \alpha' .$$

and

$$h^k(f(B_1)) \lesssim \alpha' .$$

Proof. The first inequality follows from

$$\int \chi_{B_1} \le \int \frac{1}{N+1} \sum_{Q \in \mathcal{E}_1} \chi_{7Q} \le \frac{C}{N+1} .$$

If $\epsilon = 0$ this is more then enough for the second inequality as well. If $\epsilon > 0$, to see the second inequality we note that

$$\begin{aligned} h^k(f([0,1]^k \cap B_1)) &\le \frac{1}{N+1} \sum_{Q \in \mathcal{E}_1} \text{side}(7Q) + \epsilon \\ &\le \frac{1}{N+1} \sum_{Q \in \mathcal{E}_1} 2\,\text{side}(7Q) \le \frac{C}{N+1} . \end{aligned}$$

■

Now denote by $G = [0,1]^k \setminus (B_1 \cup (\cup \mathcal{E}_2))$. We would like to split G into $M(\alpha', k)$ sets as desired. We split according to the behavior of the function p using $\mathcal{E}_1$ as our guide. One goes through the dyadic tree (large scale to fine scale) and makes sure that if $Q_1, Q_2 \in \mathcal{E}_1$ are semi-adjacent, then they are not in the same F_i. Since we excise the intervals in B (which gave us B_1), we can do this with only a finite number of sets F_i (namely $M = 2^{C(k)N}$). For more details see pages 81-82 of [2] (starting at the bottom of page 81, with the same notation). This gives $F_1, ..., F_M \subset [0,1]^k$ so that for $1 \le i \le M$, and $x, y \in F_i$ such that $\alpha' |x_1 - x_2| \ge 10\epsilon$ we have

$$\frac{1}{10} \alpha' |x - y| \le \text{dist}(p(x), p(y)) ,$$

and

$$h^k(f([0,1]^k \setminus (F_1 \cup ... \cup F_M))) \le c_1 \alpha' .$$

Using equation (2.1) we get that if $x, y \in F_i$ we have

$$\frac{1}{10} \alpha' |x - y| - 14\epsilon \le \text{dist}(f(x), f(y)) ,$$

This concludes the proof of Theorem 1.1.

3. Curvature estimates

In this section we prove Lemma 2.1. We first consider the case $k = 1$, and then use it to prove the lemma for $k > 1$.

Lemma 3.1.

$$\sum_{\substack{\mathcal{D}_0 \cup \mathcal{D}_1 \\ I \subset [0,1]}} \tilde{\beta}(I)^2 \mathrm{diam}(I) \lesssim L\,.$$

This lemma is stated and proved in [9]. The setting we were interested in there was that of *Ahlfors-regular curves*, however the proof given there for this lemma is correct for the setting we have here. We give most of the proof's details in the appendix of this paper.

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. We use Lemma 3.1 and the definition of $\tilde{\beta}^{(k)}$. Fix $i \in \{0,1\}^k$ and write

$$\mathcal{D}^k = \mathcal{D}^k_i.$$

We have

$$\begin{aligned}
&\sum_{\substack{Q \in \mathcal{D}^k \\ Q \subset [0,1]^k}} \tilde{\beta}^{(k)}(Q)^2 \mathrm{sd}(Q)^k \\
&= \sum_{\substack{Q \in \mathcal{D}^k \\ Q \subset [0,1]^k}} \int\limits_{g \in G_k} \int\limits_{x \in \mathbb{R}^k \ominus g\mathbb{R}} \chi_{\{|(x+g\mathbb{R}) \cap 7Q| \geq \mathrm{sd}(Q)\}} \tilde{\beta}((x+g\mathbb{R}) \cap 7Q)^2 \mathrm{sd}(Q) dx d\mu(g) \\
&= \int\limits_{g \in G_k} \int\limits_{x \in \mathbb{R}^k \ominus g\mathbb{R}} \sum_{\substack{Q \in \mathcal{D}^k \\ Q \subset [0,1]^k}} \chi_{\{|(x+g\mathbb{R}) \cap 7Q| \geq \mathrm{sd}(Q)\}} \tilde{\beta}((x+g\mathbb{R}) \cap 7Q)^2 \mathrm{sd}(Q) dx d\mu(g) \\
&\lesssim L + \int\limits_{g \in G_k} \int\limits_{x \in \mathbb{R}^k \ominus g\mathbb{R}} \chi_{\{x \in C[0,1]^k\}} \sum_{\substack{I \in \mathcal{D}_0 \cup \mathcal{D}_1 \\ I \subset [0,1]}} \tilde{\beta}(x+gI)^2 \mathrm{diam}(I) dx d\mu(g) \\
&\lesssim L + L \int\limits_{g \in G_k} \int_{x \in \mathbb{R}^k \ominus g\mathbb{R}} \chi_{\{x \in C[0,1]^k\}} dx d\mu(g) \lesssim L + L \int\limits_{g \in G_k} d\mu(g) \\
&\lesssim L\,.
\end{aligned}$$

where above, the notation $\mathrm{sd}(Q)$ is short for $\mathrm{side}(Q)$, the side length of the cube Q. ■

4. Appendix

We review the proof of Lemma 3.1, taken from [9]. For a little more details see the original.

For numbers $r, v \in [0,1]$ we will look at the mapping $\psi^{v,r} : [0,1] \to [0,1]$ given by $\psi^{v,r}(t) = v + rt \mod 1$.

For an interval $I \subset [0,1]$ write $I = [a(I), b(I)]$.

Remark 4.1. *Let $|I|$ be the diameter of the interval I. When doing addition* mod 1*, we have (by change of variable) for any I' with $|I'| = 2^{-k}$*

$$\sum_{\substack{I\in\mathcal{D}_0\\|I|=2^{-k}}} |I|^{-3} \int_{a(I)}^{b(I)} \int_x^{b(I)} \int_y^{b(I)} \partial_1(x,y,z)dzdydx$$
$$\leq |I'|^{-3} \int_{v=0}^1 \int_{r=0}^1 \int_{y\in v+rI'} \partial_1(v+ra(I'), y, v+rb(I'))dy \cdot |I'|drdv$$

giving

$$\sum_{\substack{I\in\mathcal{D}_0\\|I|=2^{-k}}} |I|^{-3} \int_{a(I)}^{b(I)} \int_x^{b(I)} \int_y^{b(I)} \partial_1(x,y,z)dzdydx$$
$$\leq \sum_{\substack{I\in\mathcal{D}_0\\|I|=2^{-k}}} |I|^{-3} \int_{v=0}^1 \int_{r=0}^1 \int_{y\in v+rI} \partial_1(v+ra(I), y, v+rb(I))dy \cdot |I|dr|I|dv\,.$$

Let $I' = [a,b] \in \mathcal{D}_0$. Define

$$\partial_{dyadic}(\psi^{v,r}(I')) := \partial_1((v+ra), (v+r\frac{a+b}{2}), (v+rb)).$$

The triangle inequality gives the following lemma.

Lemma 4.2. *Let $I \in \mathcal{D}_0$. Let $v, r \in [0,1]$ be chosen such that $\psi^{v,r}(I) = [x,z] \ni y$. Then*

$$\partial_1(x,y,z) \leq \sum_{\substack{I'\in\mathcal{D}, I'\subset I\\ y\in\psi^{v,r}(I')}} \partial_{dyadic}(\psi^{v,r}(I')).$$

Via telescoping sums, one get the following lemma.

Lemma 4.3. *Let $r, v \in [0,1]$ be fixed. Then*

$$\sum_{I'\in\mathcal{D}_0} \partial_{dyadic}(\psi^{v,r}(I')) \lesssim L$$

Finally, putting the above together, one gets (with the diameter of I denoted by $|I|$)

$$\begin{aligned}
&\sum_{I\in\mathcal{D}_0}|I|^{-3}\int_{a(I)}^{b(I)}\int_{x}^{b(I)}\int_{y}^{b(I)}\partial_1(x,y,z)dzdydx\\
&\leq \sum_{I\in\mathcal{D}_0}|I|^{-3}\int_{v=0}^{1}\int_{r=0}^{1}\int_{y\in v+rI}\partial_1(v+ra(I),y,v+rb(I))dy\cdot|I|dr\cdot|I|dv\\
&\leq \sum_{I\in\mathcal{D}_0}|I|^{-3}\int_{v=0}^{1}\int_{r=0}^{1}\sum\nolimits_{\substack{I'\in\mathcal{D}_0\\ I'\subset I}}\int_{y\in v+rI'}\partial_{dyadic}(\psi^{v,r}(I'))\cdot dy\cdot|I|dr\cdot|I|dv\\
&= \sum_{I\in\mathcal{D}_0}|I|^{-3}\int_{v=0}^{1}\int_{r=0}^{1}\sum\nolimits_{\substack{I'\in\mathcal{D}_0\\ I'\subset I}}\partial_{dyadic}(\psi^{v,r}(I'))\cdot r\mathcal{H}^1(I')\cdot|I|dr\cdot|I|dv\\
&= \int_{v=0}^{1}\int_{r=0}^{1}\sum_{I\in\mathcal{D}_0}\frac{1}{|I|}\sum\nolimits_{\substack{I'\in\mathcal{D}_0\\ I'\subset I}}\partial_{dyadic}(\psi^{v,r}(I'))\cdot r\mathcal{H}^1(I')drdv\\
&= \int_{v=0}^{1}\int_{r=0}^{1}\sum_{I'\in\mathcal{D}_0}\sum\nolimits_{I\supset I'}\frac{\mathcal{H}^1(I')}{|I|}\partial_{dyadic}(\psi^{v,r}(I'))\cdot rdrdv\\
&\lesssim \int_{v=0}^{1}\int_{r=0}^{1}\sum_{I'\in\mathcal{D}_0}\partial_{dyadic}(\psi^{v,r}(I'))\cdot rdrdv\lesssim L.
\end{aligned}$$

which gives Lemma 3.1.

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