

Sharp linear and bilinear restriction estimates for paraboloids in the cylindrically symmetric case

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Abstract

For cylindrically symmetric functions dyadically supported on the paraboloid, we obtain a family of sharp linear and bilinear adjoint restriction estimates. As corollaries, we first extend the ranges of exponents for the classical *linear or bilinear adjoint restriction conjectures* for such functions and verify the *linear adjoint restriction conjecture* for the paraboloid. We also interpret the restriction estimates in terms of solutions to the Schrödinger equation and establish the analogous results when the paraboloid is replaced by the lower third of the sphere.

1. Introduction

Let $n \geq 3$ be a fixed integer and S be a smooth compact non-empty subset of the paraboloid $\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1} : \tau = |\xi|^2\}$. If $0 < p, q \leq \infty$, the classical *linear adjoint restriction estimate*¹ for the paraboloid is the *a priori* estimate

$$(1.1) \quad \|(gd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^{n-1})} \leq C_{p,q,n,S} \|g\|_{L^p(S,d\sigma)}$$

for all Schwartz functions g on S , where

$$(gd\sigma)^\vee(t, x) = \int_S g(\tau, \xi) e^{i(x \cdot \xi + t\tau)} d\sigma(\xi) = \int_{\mathbb{R}^{n-1}} g(|\xi|^2, \xi) e^{i(x \cdot \xi + t|\xi|^2)} d\xi$$

denotes the inverse space-time Fourier transform of the measure $gd\sigma$, and $d\sigma$ is the canonical measure of the paraboloid defined in Section 2.

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¹In the notation of [17], the estimate (1.1) is denoted by $R_S^*(p \rightarrow q)$ and the estimate (1.2) is denoted by $R_{S_1, S_2}^*(p \times p \rightarrow q)$.

By duality, the estimate (1.1) is equivalent to the following estimate

$$\|\hat{f}\|_{L^{p'}(S, d\sigma)} \leq C_{p,q,n,S} \|f\|_{L^q(\mathbb{R} \times \mathbb{R}^{n-1})}$$

for all Schwartz functions f , which roughly says that the Fourier transform of an $L^q(\mathbb{R} \times \mathbb{R}^{n-1})$ function can be “meaningfully” restricted to the paraboloid S . This leads to the *restriction problem*, one of the central problems in harmonic analysis, which concerns the optimal range of exponents p and q for which the estimate (1.1) should hold. It was originally proposed by Stein for the sphere [13] and then extended to smooth sub-manifolds of $\mathbb{R} \times \mathbb{R}^{n-1}$ with appropriate curvature [14] such as the paraboloid and the cone. The restriction problem is intricately related to other outstanding problems in analysis such as the Bochner-Riesz conjecture, the local smoothing conjecture, the Kakeya set conjecture and the Kakeya maximal function conjecture, see e.g., [17], [18].

In this paper, we will mainly focus on the restriction estimates for the paraboloid. The corresponding *linear adjoint restriction conjecture* for the paraboloid asserts that

Conjecture 1.1. *The inequality (1.1) holds with constants depending on S , n and p, q if and only if $q > \frac{2n}{n-1}$ and $\frac{n+1}{q} \leq \frac{n-1}{p'}$.*

The conditions on p and q are known to be best possible by the decay estimates of $(d\sigma)^\vee$ and the standard Knapp example, see e.g., [14], [17]. When $n = 2$, the non-endpoint case was first proven to be true by Fefferman and Stein [7] (and generalized to other oscillatory integrals by Carleson and Sjölin [5]), and the endpoint case was proven to be true by Zygmund [29]. When $n > 2$, it was proven with the additional condition $q > \frac{2(n+1)}{n-1}$ by Tomas [25] using real interpolation, and $q = \frac{2(n+1)}{n-1}$ by Stein [14] using complex interpolation. In 1977, Córdoba [6] gave an alternate proof for $n = 2$ by largely relying on the successful resolution of the Kakeya conjecture in two dimensions. In 1991, Bourgain [1] generalized Córdoba’s arguments to higher dimensions, so that nontrivial progress on the Kakeya problem might imply some nontrivial progress on the restriction result; using this technique, he proved estimates for some $q < \frac{2(n+1)}{n-1}$; in particular, $q > 4 - \frac{2}{15}$ when $n = 3$. Further improvements along this line were made by Moyua, Vargas, Vega and Wolff, see e.g., [11], [27]. The current best result $q > \frac{2(n+2)}{n}$ in higher dimensions $n \geq 3$ is due to Tao [20], based on the techniques in Wolff’s breakthrough paper on the cone restriction estimates [28].

Among various techniques developed to attack this problem, the bilinear method proves to be very powerful. Variants of this idea also have applications to the nonlinear dispersive equations, see e.g., [2], [3], [9], etc. More

precisely, we assume S_1 and S_2 to be two smooth compact non-empty subsets of the paraboloid in $\mathbb{R} \times \mathbb{R}^{n-1}$, which are transverse in the sense that the unit normals of S_1 and of S_2 are separated by at least some fixed angle $c > 0$. Then the classical *bilinear adjoint restriction conjecture* concerns the optimal range of exponents p and q for which the bilinear operator, $(f, g) \rightarrow (fd\sigma_1)^\vee(gd\sigma_2)^\vee$, should bound from $L^p \times L^p$ to L^q , where $d\sigma_1, d\sigma_2$ are the canonical Lebesgue measures of S_1, S_2 , respectively. The following formulation of this conjecture is taken from [23].

Conjecture 1.2. *Let S_1, S_2 be defined as above and $q \geq \frac{n}{n-1}, \frac{n+2}{2q} + \frac{n}{p} \leq n$ and $\frac{n+2}{2q} + \frac{n-2}{p} \leq n-1$. Then there exists a constant $0 < C < \infty$ depending on S_1, S_2, n and p, q such that*

$$(1.2) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^{n-1})} \leq C \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}$$

for all $f \in L^p(S_1)$ and $g \in L^p(S_2)$.

It is known that the bilinear restriction conjecture 1.2 is stronger than the linear restriction conjecture 1.1, see [23]. For a discussion of recent progress made on this problem, see [17]. We remark that the conditions on p and q in this conjecture are also known to be best possible by the decay estimates of $(d\sigma)^\vee$ and the variants of the standard Knapp examples such as the squashed caps and the stretched caps, see e.g., [23], [17].

However, none of these Knapp-type examples are cylindrically symmetric functions, i.e., functions on $\mathbb{R} \times \mathbb{R}^{n-1}$ invariant under spatial rotations. Hence we expect that further estimates are available if we assume that functions are cylindrically symmetric and supported on a dyadic subset of the paraboloid in the form of $\{(\tau, \xi) : M \leq |\xi| \leq 2M, \tau = |\xi|^2\}$ with $M \in 2^{\mathbb{Z}}$. We denote by \mathcal{L}_M this class of functions. Indeed, it is the case: when $n = 3$, the Tomas-Stein restriction estimate $L^2 \rightarrow L^4$ is known to be best possible; but for functions in \mathcal{L}_M , the estimate $L^2 \rightarrow L^q$ is true for any $q > 10/3$ by Corollary 2.3 in Section 3.

Our main theorems, Theorem 2.1 and 2.5, of this paper are to present a family of sharp linear adjoint restriction estimates for $f \in \mathcal{L}_1$, and bilinear ones for $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$ on the dyadic space-time slab $\mathbb{R} \times \{R/2 \leq |x| \leq R\}$ with $R \in 2^{\mathbb{Z}}$. The proofs essentially combine the two classical and elementary methods, the Carleson-Sjölin argument [14] and the bilinear method via the Whitney decomposition, which effectively solved the two dimensional *restriction conjecture*. In the arguments, we heavily exploit the rotational symmetry via the ‘‘Fourier-Bessel’’ formula, Lemma 3.2, for cylindrically symmetric functions to reduce matters to main term estimates by encoding the error term into certain integrals. A lot of effort is devoted to inventing counterexamples to show that the restriction

estimates are best possible by relying on the idea coming from the standard Knapp examples, the principles of both stationary phase and non-stationary phase [14] and the Khintchine inequality [17]. We remark that some of them are quite challenging, see e.g., Example 4.5.

As corollaries of the main theorems, we can verify the inequality (1.1) for \mathcal{L}_M when the exponents p and q are in a larger region (see Figure 2) and show that it is nearly sharp except for certain endpoints. Furthermore, we show that the *linear adjoint restriction conjecture* 1.1 holds for all cylindrically symmetric functions when p and q are restricted to the classical region. By similar arguments, one can also establish the analogous sharp restriction estimates when the paraboloid is replaced by the lower third of the sphere \mathcal{S}^{n-1} or more general cylindrically symmetric and compact hypersurfaces of elliptic type as defined in [12], [23]. As applications of the restriction estimates, we will interpret them in terms of the solutions to the Schrödinger equations and present another proof of the weighted Strichartz estimates in [26] for Schrödinger equations.

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2. Notations and Main Theorems

Let $n \geq 3$ be the fixed space-time dimension. In this paper, we interpret $\mathbb{R} \times \mathbb{R}^{n-1}$ as the space-time frequency space.

We will use the notations $X \lesssim Y$, $Y \gtrsim X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some constant $0 < C < \infty$, which may depend on p, q, n and S_1 or S_2 , but not on the functions. If $X \lesssim Y$ and $Y \lesssim X$ we will write $X \sim Y$. If the constant C depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For example, C_ε should be understood as a positive constant not only depending on p, q, n and S_1 or S_2 , but also on ε .

We denote by $d\sigma$ the canonical Lebesgue measure of the standard paraboloid $S = \{(\tau, \xi) : \tau = |\xi|^2\}$ in $\mathbb{R} \times \mathbb{R}^{n-1}$, which is the pullback of $n - 1$ -dimensional Lebesgue measure $d\xi$ under the projection map $(\tau, \xi) \mapsto \xi$; thus,

$$\int_S f(\tau, \xi) d\sigma = \int_{\mathbb{R}^{n-1}} f(|\xi|^2, \xi) d\xi.$$

By \mathcal{S}^{n-2} we denote the $n - 1$ dimensional unit sphere canonically embedded in \mathbb{R}^{n-1} , and by $d\mu$ its surface measure.

We define a dyadic number to be any number $R \in 2^{\mathbb{Z}}$ of the form $R = 2^j$ where j is an integer. For each dyadic number $R > 0$, we define the dyadic annulus in \mathbb{R}^{n-1} ,

$$A_R := \{x \in \mathbb{R}^{n-1} : R/2 \leq |x| \leq R\}.$$

We define the space-time norm $L_t^q L_x^r$ of f on $\mathbb{R} \times \mathbb{R}^{n-1}$ by

$$\|f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^{n-1})} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(t, x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}^{n-1}$ is replaced by a small region of space-time such as $\mathbb{R} \times A_R$. When $q = r$, we abbreviate it by $L_{t,x}^q$. Unless specified in this paper, all the space-time integrals are taken over $\mathbb{R} \times A_R$ with dyadic $R > 0$, which will be clear from the context.

We define the spatial Fourier transform of f on \mathbb{R}^{n-1} by

$$\hat{f}(\xi) = \int_{\mathbb{R}^{n-1}} f(x) e^{-ix \cdot \xi} dx.$$

We use 1_U to denote the indicator function of the set U , i.e.,

$$1_U(x) = \begin{cases} 1, & \text{for } x \in U, \\ 0, & \text{for } x \notin U. \end{cases}$$

For $1 \leq p \leq \infty$, we denote the conjugate exponent of p by p' , i.e., $1/p + 1/p' = 1$.

We start with stating the main theorem concerning the linear restriction estimates, which is proven in Section 3.

Theorem 2.1. *Suppose $f \in \mathcal{L}_1$ and $R > 0$ is a dyadic number. Then the following sharp restriction estimates hold:*

- for $q = 2$ and $2 \leq p \leq \infty$,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^2} \lesssim \min \{R^{\frac{1}{2}}, R^{\frac{n-1}{2}}\} \|f\|_{L^p(S)}.$$

- for $q = 4$ and $4 \leq p \leq \infty, \forall \varepsilon > 0$,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^4} \lesssim_\varepsilon \min \{R^{-\frac{n-2}{4}+\varepsilon}, R^{\frac{n-1}{4}}\} \|f\|_{L^p(S)}.$$

- for $q = 3p'$ and $1 \leq p < 4$,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^q} \lesssim \min \{R^{(n-2)(\frac{1}{q}-\frac{1}{2})}, R^{\frac{n-1}{q}}\} \|f\|_{L^p(S)}.$$

- for $q = \infty$ and $1 \leq p \leq \infty$,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^\infty} \lesssim \min \{R^{-\frac{n-2}{2}}, 1\} \|f\|_{L^p(S)}.$$

where A in $\{A, B\}$ is given by the case where $R \geq 2$ and B by $R \leq 1$. Furthermore, by interpolation we obtain the sharp restriction estimates $L^p \rightarrow L^q$ when p, q are in the region determined by these lines (Figure 1).

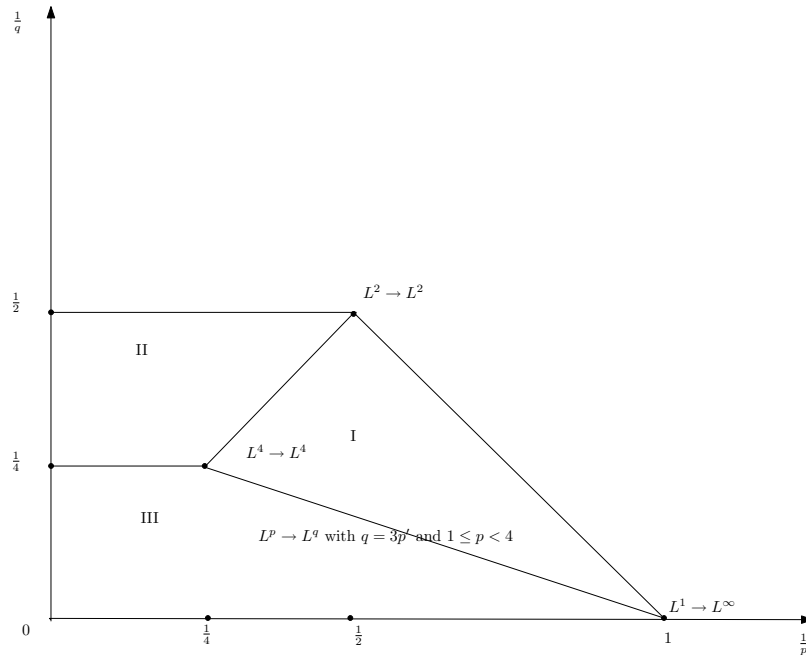


Figure 1: Linear restriction estimates on $\mathbb{R} \times A_R$.

Remark 2.2. We observe that the estimates above in each case are “continuous” in the sense that they match when $R \sim 1$.

One can easily obtain the following corollary regarding the linear adjoint restriction conjecture.

Corollary 2.3. *Suppose f are cylindrically functions supported on the paraboloid.*

- If $f \in \mathcal{L}_M$, the linear adjoint restriction conjecture 1.1 holds with constants depending on p, q, n and M whenever $q > \frac{2n}{n-1}, \frac{1}{p} + \frac{1}{q} \leq 1$ and $\frac{1}{p} + \frac{2n-1}{q} < n - 1$ (Figure 2).
- The linear adjoint restriction conjecture 1.1 is true for all f (Figure 2).

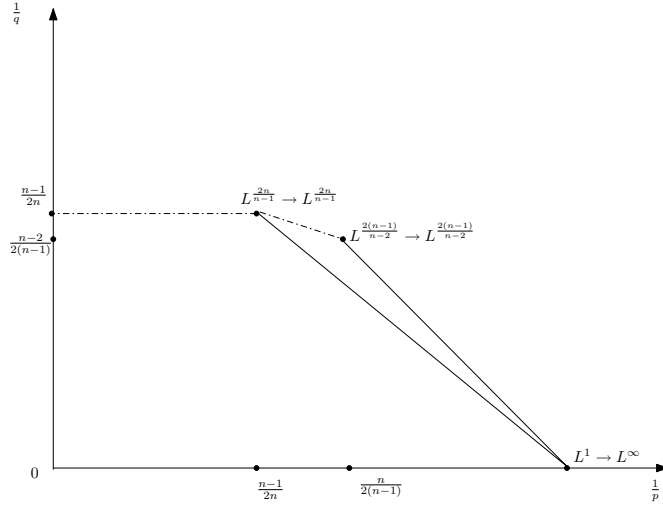


Figure 2: Linear restriction estimates for \mathcal{L}_M (cf. the inside trapezoid, the classical range of p and q for Conjecture 1.1).

Proof. By Theorem 2.1, the first assertion follows from scaling to $f \in \mathcal{L}_1$ and then summing all dyadic R and using interpolation.

To prove the second assertion, it is sufficient to obtain it under the boundary conditions $q > \frac{2n}{n-1}$ and $\frac{n+1}{q} = \frac{n-1}{p'}$ since other estimates are easily obtained by a standard argument of using the Hölder inequality. By interpolating between the $L^2 \rightarrow L^2$ estimate and the $L^p \rightarrow L^q$ estimates on the line segment $q = 3p'$ and $1 \leq p < 4$, we obtain that, for $q > \frac{2n}{n-1}$, $\frac{n+1}{q} = \frac{n-1}{p'}$ and $f \in \mathcal{L}_1$,

$$\|(fd\sigma)^\vee\|_{L^q_{t,x}} \lesssim R^{\alpha(R)} \|f\|_{L^p(S)},$$

where α is the step function

$$\alpha(R) = \begin{cases} -\frac{n-2}{2} [1 - \frac{2n}{q(n-1)}], & \text{for } R \geq 2, \\ \frac{n-1}{q}, & \text{for } 0 < R \leq 1. \end{cases}$$

We remark that the constant above $CR^{\alpha(R)}$ does not depend on ε . By scaling, when $f \in \mathcal{L}_M$, under the same conditions on p and q ,

$$\|(fd\sigma)^\vee\|_{L^q_{t,x}} \lesssim (RM)^{\alpha(RM)} \|f\|_{L^p(S)},$$

where α is defined as above. Then for all cylindrically symmetric f supported on the paraboloid, we decompose it into a sum of dyadically supported functions,

$$f = \sum_{M: \text{dyadic}} f1_{\{(\tau,\xi): \tau=|\xi|^2, M \leq |\xi| \leq 2M\}} = \sum_M f_M,$$

where $f_M := f1_{\{(\tau,\xi):\tau=|\xi|^2, M\leq|\xi|\leq 2M\}}$. Hence

$$\begin{aligned} \|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^{n-1})} &= \left(\sum_R \|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\right)^{1/q} \\ &= \left(\sum_R \left\|\sum_M (f_M d\sigma)^\vee\right\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\right)^{1/q} \\ &\leq \left(\sum_R \left(\sum_M \|(f_M d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\right)^q\right)^{1/q} \\ &\lesssim \left(\sum_R \left(\sum_M (RM)^{\alpha(RM)} \|f_M\|_{L^p(S)}\right)^q\right)^{1/q} \\ &\lesssim \left(\sum_M \|f_M\|_{L^p(S)}^p\right)^{1/p} = \|f\|_{L^p(S)}, \end{aligned}$$

where $R > 0$ and $M > 0$ are both dyadic numbers; in the last inequality, since

$$\begin{aligned} \forall R > 0, \sum_M (RM)^{\alpha(RM)} &< \infty, \\ \forall M > 0, \sum_R (RM)^{\alpha(RM)} &< \infty, \end{aligned}$$

we have used the Schur’s test for exponents p and q satisfying the condition $q > \frac{2n}{n-1} > p \geq 1$. ■

Remark 2.4. In the cylindrically symmetric case, we remark that $q > \frac{2n}{n-1}$ is still sharp since it is given by the decay estimate $|(d\sigma)^\vee(t, \xi)| \leq C_n(1 + |t| + |\xi|)^{(1-n)/2}$, see e.g., [14, Chapter 8, Theorem 3.1].

Next we state the theorem regarding the bilinear restriction estimates in the cylindrically symmetric case, which is proven in Section 4.

Theorem 2.5. *Suppose $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$. $R > 0$ is a dyadic number. Then the following sharp bilinear restriction estimates hold:*

- for $q = 1$ and $2 \leq p \leq \infty$,

$$\begin{aligned} &\|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \\ &\lesssim \min \left\{ RM^{\frac{n-2}{2}-\frac{n-1}{p}}, R^{\frac{n}{2}} M^{-1+\frac{n-1}{p'}}, R^{n-1} M^{-1+\frac{n-1}{p'}} \right\} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

- for $q = 2$ and $2 \leq p \leq \infty$,

$$\begin{aligned} & \| (fd\sigma_1)^\vee (gd\sigma_2)^\vee \|_{L^2_{t,x}} \\ & \lesssim \min \{ R^{-\frac{n-2}{2}} M^{\frac{n-1}{2} - \frac{n-1}{p}}, R^{\frac{1}{2}} M^{\frac{n-1}{p'}}, R^{\frac{n-1}{2}} M^{\frac{n-1}{p'}} \} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

- for $q = 4$ and $4 \leq p \leq \infty, \forall \varepsilon > 0$,

$$\begin{aligned} & \| (fd\sigma_1)^\vee (gd\sigma_2)^\vee \|_{L^4_{t,x}} \\ & \lesssim_\varepsilon \min \{ R^{-\frac{3(n-2)}{4} + \varepsilon} M^{\frac{n}{2} - \frac{n-1}{p}}, R^{-\frac{n-2}{4} + \varepsilon} M^{\frac{n-1}{p'}}, R^{\frac{n-1}{4}} M^{\frac{n-1}{p'}} \} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

- for $q = 3p'$ and $1 \leq p < 4$,

$$\begin{aligned} & \| (fd\sigma_1)^\vee (gd\sigma_2)^\vee \|_{L^q_{t,x}} \\ & \lesssim \min \{ R^{-\frac{n-2}{q}} M^{\frac{n}{2} - \frac{n-1}{p}}, R^{(n-2)(\frac{1}{q} - \frac{1}{2})} M^{\frac{n-1}{p'}}, R^{\frac{n-1}{q}} M^{\frac{n-1}{p'}} \} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

- for $q = \infty$ and $1 \leq p \leq \infty$,

$$\begin{aligned} & \| (fd\sigma_1)^\vee (gd\sigma_2)^\vee \|_{L^\infty_{t,x}} \\ & \lesssim \min \{ R^{-(n-2)} M^{\frac{n}{2} - \frac{n-1}{p}}, R^{-\frac{n-2}{2}} M^{\frac{n-1}{p'}}, M^{\frac{n-1}{p'}} \} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

where A in $\{A, B, C\}$ is given by the case where $R \geq 1/M$, B by $2 \leq R \leq 1/M$ and C by $R \leq 1$. Furthermore, by interpolation we obtain the sharp restriction estimates $L^p \times L^p \rightarrow L^q$ when p and q are in the region determined by these lines (Figure 3).

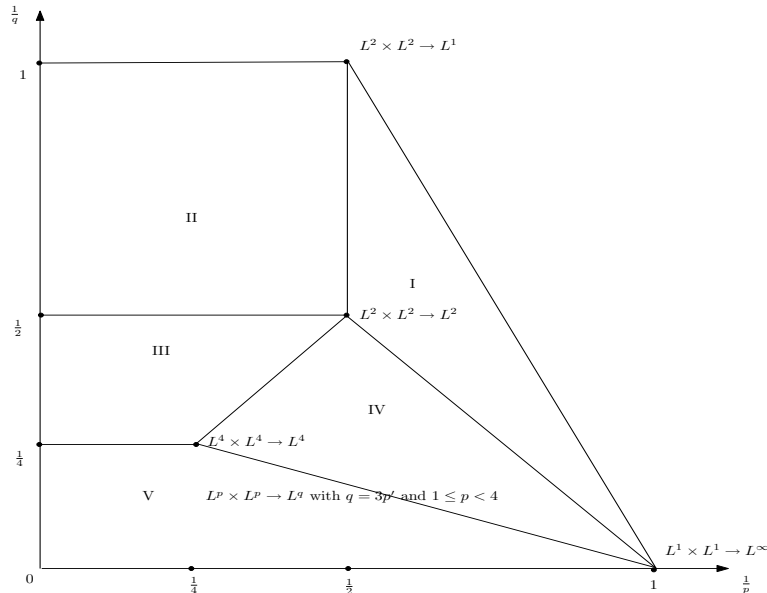


Figure 3: Bilinear restriction estimates on $\mathbb{R} \times A_R$.

Remark 2.6. We observe that $R \geq \frac{1}{M}$ if and only if $A \lesssim B$ and $R \geq 1$ if and only if $B \lesssim C$. In other words, the estimates are “continuous” in the sense that $A \sim B$ when $R \sim 1/M$ and $B \sim C$ when $R \sim 1$.

As a corollary of Theorem 2.5, we see that the *bilinear adjoint restriction conjecture* holds for exponents p and q in a larger region.

Corollary 2.7. *Suppose f and g are defined as Theorem 2.5. Then the bilinear adjoint restriction conjecture 1.2 holds with constants depending on p, q, n, S_1, S_2 and M , whenever $q > \frac{n}{n-1}$, $\frac{2}{p} + \frac{n}{q} < n$ and $\frac{2}{p} + \frac{1}{q} \leq 2$. These estimates are sharp except for certain endpoints (Figure 4).*

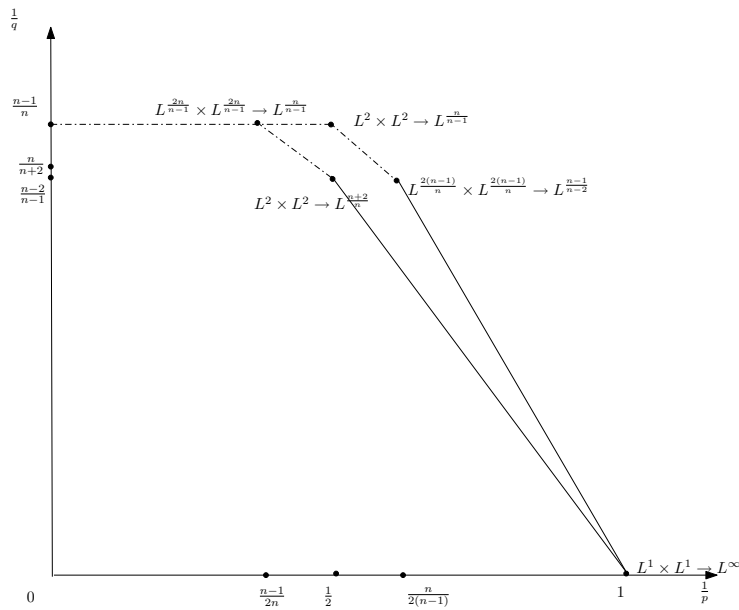


Figure 4: Bilinear restriction estimates for functions in \mathcal{L}_1 and \mathcal{L}_M with $0 < M \leq 1/4$ (cf. the inside pentagram, the classical range of p and q for Conjecture 1.2).

Remark 2.8. When S is replaced by the lower third of the sphere, the analogous results to Theorems 2.1 and 2.5 hold, which will be accomplished in Section 5. This is essentially due to the common geometric property of non-vanishing Gaussian curvature shared by the sphere and the paraboloid and the fact that the sphere locally resembles the paraboloid, which can be seen from the Taylor expansion $\sqrt{1 - |\xi|^2} \sim 1 - c(|\xi|)|\xi|^2$ when $|\xi|$ is small.

Remark 2.9. It is well known that the adjoint restriction estimates are closely related to the Strichartz estimates for the nonlinear dispersive equations such as the Schrödinger equation and the wave equation, see e.g., [16] and [14]. We will establish this connection in our case in Section 6.

This paper is organized as follows: Section 3 is devoted proving Theorem 2.1 and constructing counterexamples to show the linear estimates are sharp. Section 4 is devoted proving Theorem 2.5 and constructing counterexamples showing the bilinear estimates are sharp. In Section 5 we will establish analogous results for the cylindrically symmetric functions supported on the lower third of the sphere or the cylindrically symmetric and compact hypersurfaces of elliptic type. In Section 6 we will interpret the restriction estimates in terms of solutions to the Schrödinger equation to establish the Strichartz estimates.

3. Proof of Theorem 2.1: linear estimates and examples

For any cylindrically symmetric function f on the paraboloid, we set $F(|\xi|) = f(|\xi|^2, \xi)$. We observe that $(fd\sigma)^\vee(t, x)$ is also a cylindrically symmetric function. To begin the proof of Theorem 2.1, we first investigate the behavior of $(fd\sigma)^\vee$ on $\{|x| \leq 1\}$ via the following proposition.

Proposition 3.1. *Suppose $f \in \mathcal{L}_1$. Then for any $1 \leq p \leq \infty$, $q \geq \max\{2, p'\}$ and $R \leq 1$, we have a sharp estimate*

$$(3.1) \quad \|(fd\sigma)^\vee\|_{L^q_{t,x}} \lesssim R^{\frac{n-1}{q}} \|f\|_{L^p(S)}.$$

Proof. If we change to polar coordinates, the left-hand side of (3.1) is

$$\begin{aligned} \|(fd\sigma)^\vee\|_{L^q_{t,x}} &= \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{1 \leq |\xi| \leq 2} f(\xi) e^{i(x \cdot \xi - t \cdot |\xi|^2)} d\xi \right|^q dt dx \right)^{1/q} \\ &= \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{n-2} e^{-its^2} \int_{S^{n-2}} e^{irs\omega} d\mu(\omega) ds \right|^q dt r^{n-2} dr \right)^{1/q}, \end{aligned}$$

where $I = [1, 2]$. Then we change variables back $s \rightarrow a^{1/2}$ to majorize it by

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I'} a^{(n-3)/2} F(a^{1/2}) (d\mu)^\vee(ra^{1/2}e_1) e^{-ita} da \right|^q dt r^{n-2} dr \right)^{1/q},$$

where $I' = [1, \sqrt{2}]$ and $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$.

Then by the Hausdorff-Young inequality when $q > 2$ or the Plancherel theorem when $q = 2$, changing $s \rightarrow a = s^2$ and the fact that $\|(d\mu)^\vee\|_{L^\infty} \lesssim 1$, the left-hand side of (3.1) is further bounded by

$$R^{\frac{n-2}{q}} \left(\int_{R/2}^R \|F\|_{L^{q'}(I)}^q dr \right)^{1/q} \sim R^{\frac{n-1}{q}} \|F\|_{L^{q'}(I)}.$$

Then by the Hölder inequality and the fact $\|F\|_{L^p(I)} \sim \|f\|_{L^p(S)}$, (3.1) follows.

Now we will construct a counterexample to show the estimate (3.1) is sharp when $1 \leq p \leq \infty$ and $q \geq \max\{2, p'\}$. We take

$$f(|\xi|^2, \xi) = F(|\xi|) = |\xi|^{-(n-2)} 1_{\{1 \leq |\xi| \leq 2\}} e^{it_0|\xi|^2},$$

where $t_0 \in \mathbb{R}$. Then the left-hand side of (3.1) reduces to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I e^{-i(t-t_0)s^2} \int_{\mathcal{S}^{n-2}} e^{irs\omega} d\mu(\omega) ds \right|^q dt r^{n-2} dr \right)^{1/q}.$$

We choose r and t satisfying that

$$r \in [R/100, R/50], |t - t_0| \leq 1/100.$$

Then the left-hand side of (3.1) $\gtrsim R^{\frac{n-1}{q}}$ and its right-hand side $\lesssim R^{\frac{n-1}{q}}$. Hence (3.1) is easily seen to be sharp. ■

Before investigating the behavior of $(fd\sigma)^\vee$ on $|x| \geq 1$, we shall exploit the spatial rotation-invariant symmetry of f in the following proposition.

Lemma 3.2 (Fourier-Bessel formula). *Suppose f is a cylindrically symmetric function supported on the paraboloid. Then*

$$\begin{aligned} & (fd\sigma)^\vee(t, x) \\ &= c_n r^{-\frac{n-2}{2}} \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds + c_n r^{-\frac{n-2}{2}} \int_I F(s) s^{\frac{n-2}{2}} e^{-i(rs+ts^2)} ds \\ & \quad + c_n \int_I F(s) s^{n-2} e^{-its^2-irs} \int_0^\infty e^{-rsy} y^{\frac{n-4}{2}} [(y+2i)^{\frac{n-4}{2}} - (2i)^{\frac{n-4}{2}}] dy ds \\ (3.2) \quad & \quad - c_n \int_I F(s) s^{n-2} e^{-its^2+irs} \int_0^\infty e^{-rsy} y^{\frac{n-4}{2}} [(y-2i)^{\frac{n-4}{2}} - (-2i)^{\frac{n-4}{2}}] dy ds, \end{aligned}$$

where $r = |x|$ and I is the projection interval in the radial direction.

Proof. We first expand $(fd\sigma)^\vee$ in the polar coordinates,

$$(fd\sigma)^\vee(t, x) = \int_{\{|\xi| \in I\}} f(|\xi|^2, \xi) e^{i(x\xi-t|\xi|^2)} d\xi = \int_I F(s) e^{-its^2} s^{n-2} (d\mu)^\vee(rse_1) ds,$$

where $d\mu$ is the surface measure of the sphere \mathcal{S}^{n-2} .

On the one hand, the inverse Fourier transform of $d\mu$ is given by

$$(d\mu)^\vee(\xi) = c_n |\xi|^{\frac{3-n}{2}} J_{\frac{n-3}{2}}(|\xi|),$$

where $J_{\frac{n-3}{2}}$ is the Bessel function of order $\frac{n-3}{2}$, see e.g. [14] or [15].

On the other hand, by using the same argument as proving [15, Lemma 3.11], we obtain that, for fixed $m \geq 0$,

$$J_m(r) = c_m \frac{e^{ir} - e^{-ir}}{r^{1/2}} + c_m r^m e^{-ir} \int_0^\infty e^{-ry} y^{\frac{2m-1}{2}} \left[(y+2i)^{\frac{2m-1}{2}} - (2i)^{\frac{2m-1}{2}} \right] dy \\ - c_m r^m e^{ir} \int_0^\infty e^{-ry} y^{\frac{2m-1}{2}} \left[(y-2i)^{\frac{2m-1}{2}} - (-2i)^{\frac{2m-1}{2}} \right] dy.$$

Hence (3.2) holds after we combine these two estimates and set $m = \frac{n-3}{2}$. ■

Therefore we define the error term of $(fd\sigma)^\vee$ by

$$\mathcal{E}f(t, x) := \pm c_n \int_I F(s) s^{n-2} e^{-its^2 \mp irs} \times \\ \times \int_0^\infty e^{-rsy} y^{\frac{n-4}{2}} \left[(y \pm 2i)^{\frac{n-4}{2}} - (\pm 2i)^{\frac{n-4}{2}} \right] dy ds,$$

where by \pm we denote a sum of two terms where $+$ and $-$ appear alternatively. Heuristically, one should think of $\mathcal{E}f$ as a term comparable to $r^{-n/2} \int_I F(s) s^{\frac{n-4}{2}} e^{-its^2} ds$, which comes from estimating the error term of Bessel function $J_m(r)$ by $r^{-3/2}$. At the first approximation, this simplification is easy to deal with and intuitively provides what the bound involving the error term should be. However in this paper we will establish it rigorously in the following proposition, which shows that the information about its contribution to the linear estimates when $|x| \geq 1$. It is acceptable compared to the main term estimates established in the next propositions.

Proposition 3.3. *Suppose $f \in \mathcal{L}_1$. Then for any $q \geq \max\{2, p'\}$ and any dyadic number $R \geq 2$ and $f \in L^p(S)$,*

$$(3.3) \quad \|\mathcal{E}f\|_{L^q_{t,x}} \lesssim R^{-\frac{n}{2} + \frac{n-1}{q}} \|f\|_{L^p(S)}.$$

Proof. We set

$$E(r) := \int_0^\infty e^{-ry} y^{\frac{n-4}{2}} \left[(y \pm 2i)^{\frac{n-4}{2}} - (\pm 2i)^{\frac{n-4}{2}} \right] dy.$$

For $r \geq 1$, we first estimate $E(r)$ by repeating the proof of [15, Lemma 3.11] for readers' convenience.

$$|E(r)| \leq \int_0^1 e^{-ry} y^{\frac{n-4}{2}} \left| (y \pm 2i)^{\frac{n-4}{2}} - (\pm 2i)^{\frac{n-4}{2}} \right| dy + \\ + \int_1^\infty e^{-ry} y^{\frac{n-4}{2}} \left| (y+2i)^{\frac{n-4}{2}} - (2i)^{\frac{n-4}{2}} \right| dy \\ =: I + II$$

For I , where $0 \leq y \leq 1$, by the mean value theorem we have $|(y \pm 2i)^{\frac{n-4}{2}} - (\pm 2i)^{\frac{n-4}{2}}| \lesssim y$. For II , where $y \geq 1$, we take y out and then use the mean value theorem to obtain $|(y \pm 2i)^{\frac{n-4}{2}} - (\pm 2i)^{\frac{n-4}{2}}| \lesssim y^{\frac{n-4}{2}}$. Then combining these estimates above,

$$\begin{aligned} |E(r)| &\leq \int_0^1 e^{-ry} y^{\frac{n-2}{2}} dy + \int_1^\infty e^{-ry} y^{n-4} dy \\ &\lesssim r^{-n/2} \int_0^r e^{-y} y^{\frac{n}{2}-1} dy + r^{-(n-3)} \int_r^\infty e^{-y} y^{n-4} dy. \end{aligned}$$

By the definition of the Gamma function Γ ,

$$r^{-n/2} \int_0^r e^{-y} y^{\frac{n}{2}-1} dy \lesssim \Gamma(n/2) r^{-n/2}.$$

Then using integration by parts $n - 4$ times when $n \geq 4$ and $y^{-1} \leq r^{-1}$ when $n = 3$, we obtain $r^{-(n-3)} \int_r^\infty e^{-y} y^{n-4} dy \lesssim e^{-r} r^{-1}$.

Since $e^{-r} r^{-1+\frac{n}{2}}$ is continuous on $[1, \infty)$ and decays to 0 as $r \rightarrow \infty$, $e^{-r} r^{-1} \lesssim r^{-\frac{n}{2}}$ holds for $r \geq 1$. Therefore

$$(3.4) \quad |E(r)| \lesssim r^{-n/2}.$$

Next let us turn to the estimate (3.3). By changing to polar coordinates, the left-hand side of (3.3) is comparable to

$$(3.5) \quad \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{n-2} e^{-its^2 \mp irs} E(rs) ds \right|^q dt r^{n-2} dr \right)^{1/q},$$

where $I = [1, 2]$. After changing variables $s = a^{1/2}$, (3.5) is comparable to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I'} F(a^{1/2}) a^{\frac{n-3}{2}} E(ra^{1/2}) e^{\mp ira^{1/2}} e^{-ita} da \right|^q dt r^{n-2} dr \right)^{1/q},$$

where $I' = [1, \sqrt{2}]$. Then by the Hausdorff-Young inequality when $q > 2$ or the Plancherel theorem when $q = 2$, changing variables back $a = s^2$ and $s \sim 1$, the left-hand side of (3.5) is further majorized by

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I |F(s) E(rs)|^{q'} ds \right|^{q/q'} dt r^{n-2} dr \right)^{1/q}.$$

Since $rs \geq 1$ for any $r \in [R/2, R]$ and $s \in I$, (3.4) and the Hölder inequality give $R^{-\frac{n}{2} + \frac{n-1}{q}} \|F\|_{L^p(I)}$. Since $\|F\|_{L^p(I)} \sim \|f\|_{L^p(S)}$, (3.3) follows. ■

When $|x| \geq 1$, we are left with estimating the main term of $(fd\sigma)^\vee$,

$$(3.6) \quad \mathcal{M}f(t, x) := c_n r^{-\frac{n-2}{2}} \int_I F(s) s^{\frac{n-2}{2}} e^{i(\pm rs - ts^2)} ds,$$

where by \pm we denote the summation of two terms. We call it *the heuristic approximation of $(fd\sigma)^\vee$* . We are going to prove the positive part “estimates” of Theorem 2.1 in the following four propositions. In the remainder of this section, we will prove its negative part “sharpness” by certain counterexamples.

Proposition 3.4 ($q = 2$ line). *Suppose $f \in \mathcal{L}_1$. Then for $q = 2, 2 \leq p \leq \infty$ and $R \geq 2$, we have a sharp estimate*

$$(3.7) \quad \|(fd\sigma)^\vee\|_{L^2_{t,x}} \lesssim R^{1/2} \|f\|_{L^p(S)}.$$

Proof. We observe that it is sufficient to estimate the term of $\mathcal{M}f$ with + sign. In the propositions followed, we will make the same reduction unless specified. Hence by the heuristic approximation (3.6) of $(fd\sigma)^\vee$ with + sign, changing variables and then the Plancherel theorem in t followed by the Hölder inequality,

$$\begin{aligned} \|(fd\sigma)^\vee\|_{L^2_{t,x}} &= \left(\int_{R/2 \leq |x| \leq R} \int_{\mathbb{R}} |(fd\sigma)^\vee(t, x)|^2 dt dx \right)^{1/2} \\ &\sim \left(\int_{R/2}^R \int_{\mathbb{R}} \left| r^{-\frac{n-2}{2}} \int_I F(s) s^{\frac{n-2}{2}} e^{i[rs - ts^2]} ds \right|^2 dt r^{n-2} dr \right)^{1/2} \\ &= \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{\frac{n-2}{2}} e^{irs} e^{-ts^2} ds \right|^2 dt dr \right)^{1/2} \\ &= \left(\int_{R/2}^R \|F\|_{L^2_s(I)}^2 dr \right)^{1/2} \lesssim R^{1/2} \|f\|_{L^2(S)} \lesssim R^{1/2} \|f\|_{L^p(S)}, \end{aligned}$$

where $I = [1, 2], 2 \leq p \leq \infty$. Hence (3.7) follows. ■

Now let us deal with the estimates on the line $q = 4$. The estimate $L^4 \rightarrow L^4$ is the endpoint of two dimensional ($n = 2$) linear adjoint restriction conjecture and hence the classical TT^* approach, namely the Carleson-Sjölin argument used in Proposition 3.6, unfortunately fails because we can not apply the Hardy-Littlewood-Sobolev inequality at one step. Instead, we can perform a Whitney-type decomposition to I to create some frequency separation. Similar arguments can be found in [20], [22], [23].

Proposition 3.5 (*q = 4 line*). *Suppose $f \in \mathcal{L}_1$. Then for $q = 4$, $4 \leq p \leq \infty$ and $\varepsilon > 0$ and $R \geq 2$, we have a sharp estimate up to R^ε ,*

$$(3.8) \quad \|(fd\sigma)^\vee\|_{L^4_{t,x}} \lesssim_\varepsilon R^{-(n-2)/4+\varepsilon} \|f\|_{L^p(S)}.$$

Proof. By the same reasoning as that used in the proof of Proposition 3.4, we will only prove the estimate when $p = 4$. By the heuristic approximation (3.6) of $(fd\sigma)^\vee$ with + sign,

$$\|(fd\sigma)^\vee\|_{L^4_{t,x}}^2 \sim R^{-(n-2)/2} \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds \right|^4 dt dr \right)^{1/2},$$

where $I = [1, 2]$.

We set $(Fd\sigma)^\vee(t, r) := \int_I F(s) s^{\frac{n-2}{2}} e^{i[rs-ts^2]} ds$, which can be regarded as the inverse space-time Fourier transform of $f(\tau, \xi) |\xi|^{\frac{n-2}{2}}$ restricted to the parabola. To prove (3.8), it suffices to prove that, for any $\varepsilon > 0$,

$$(3.9) \quad \left(\int_{R/2}^R \int_{\mathbb{R}} |(Fd\sigma)^\vee(Fd\sigma)^\vee|^2 dt dr \right)^{1/2} \lesssim_\varepsilon R^\varepsilon \|F\|_{L^4}^2.$$

Next we will perform a Whitney-type decomposition to $I = [1, 2]$. For each $j \geq 0$ we break up I into $O(2^j)$ dyadic intervals τ_k^j of length 2^{-j} , and write $\tau_k^j \simeq \tau_{k'}^j$ if τ_k^j and $\tau_{k'}^j$ are not adjacent but have adjacent parents. For each $j \geq 0$, let $F = \sum F_k^j$ where $F_k^j = F1_{\tau_k^j}$. Then

$$(Fd\sigma)^\vee(Fd\sigma)^\vee = \sum_j \sum_{k, k': \tau_k^j \simeq \tau_{k'}^j} (F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee.$$

From the triangle inequality, the left-hand side of (3.9) is bounded by

$$\begin{aligned} & \sum_{2^j \leq R} \left\| \sum_{k, k': \tau_k^j \simeq \tau_{k'}^j} (F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee \right\|_{L^2_{t,r}(\mathbb{R} \times \mathbb{R})} + \\ & + \sum_{2^j \geq R} \sum_{k, k': \tau_k^j \simeq \tau_{k'}^j} \left\| (F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee \right\|_{L^2_{t,r}(\mathbb{R} \times A_R)} =: A + B. \end{aligned}$$

We will first estimate A . By the quasi-orthogonality property of functions among $(F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee$ [23, Lemma 6.1],

$$A \lesssim \sum_{2^j \leq R} \left(\sum_{k, k': \tau_k^j \simeq \tau_{k'}^j} \|(F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee\|_{L^2_{t,r}(\mathbb{R} \times \mathbb{R})}^2 \right)^{1/2}.$$

By using the Plancherel theorem and the Cauchy-Schwarz inequality and $s_i \sim 1$ for $i = 1, 2$,

$$\|(F_k^j d\sigma)^\vee (F_{k'}^j d\sigma)^\vee\|_{L^2_{t,r}(\mathbb{R} \times \mathbb{R})}^2 \lesssim \|F_k^j\|_{L^2(I_k^j)}^2 \|F_{k'}^j\|_{L^2(I_{k'}^j)}^2 \|d\sigma_k^j * d\sigma_{k'}^j\|_{L^\infty},$$

where $d\sigma_k^j$ and $d\sigma_{k'}^j$ are two arc measures of the parabola $\{\tau = |\xi|^2\}$ in $\mathbb{R} \times \mathbb{R}$ supported on τ_k^j and $\tau_{k'}^j$, respectively.

On the one hand, from the geometric properties of the paraboloid,

$$\|d\sigma_k^j * d\sigma_{k'}^j\|_{L^\infty} \lesssim 2^j.$$

On the other hand, by the Hölder inequality, $\|F_k^j\|_{L^2(I_k^j)} \leq 2^{-j/4} \|F_k^j\|_{L^4(I_k^j)}$. Thus after combining these estimates,

$$A \lesssim \sum_{2^j \leq R} \left(\sum_{k,k':\tau_k^j \simeq \tau_{k'}^j} \|F_k^j\|_{L^4(I_k^j)}^2 \|F_{k'}^j\|_{L^4(I_{k'}^j)}^2 \right)^{1/2}.$$

We also observe that for each k , there are $O(1)$ k' such that $\tau_k^j \simeq \tau_{k'}^j$. Hence by the Cauchy-Schwarz inequality, for any $\varepsilon > 0$,

$$A \lesssim (\log R) \|F\|_{L^4}^2 \lesssim_\varepsilon R^\varepsilon \|F\|_{L^4}^2.$$

Next let us estimate B . On the one hand, by the Cauchy-Schwarz inequality,

$$\|(F_{k'}^j d\sigma)^\vee\|_{L^\infty_{t,x}} \lesssim 2^{-j/2} \|F_{k'}^j\|_{L^2(\tau_{k'}^j)}.$$

On the other hand, by the Plancherel theorem in t ,

$$\|(F_k^j d\sigma)^\vee\|_{L^2_{t,r}(\mathbb{R} \times A_R)} = \left(\int_{R/2}^R \int_{\mathbb{R}} |(F_k^j d\sigma)^\vee|^2 dt dr \right)^{1/2} \lesssim R^{1/2} \|F_k^j\|_{L^2(\tau_k^j)}.$$

Since there are $O(1)$ k' such that $\tau_k^j \simeq \tau_{k'}^j$ for each k , by using the Cauchy-Schwarz inequality,

$$B \lesssim R^{1/2} \sum_{2^j \geq R} \sum_{k,k':\tau_k^j \simeq \tau_{k'}^j} 2^{-j/2} \|F_{k'}^j\|_{L^2(\tau_{k'}^j)} \|F_k^j\|_{L^2(\tau_k^j)} \lesssim R^{1/2} \sum_{2^j \geq R} 2^{-j/2} \|F\|_{L^2}^2.$$

Thus summing in j and using the Hölder inequality, (3.9) follows. ■

In contrast to the proof of the estimate $L^4 \rightarrow L^4$ in Proposition 3.5, the estimates $L^p \rightarrow L^q$ when $q = 3p'$ and $1 \leq p < 4$ can be proven by the Carleson-Sjölin argument or equivalently the TT^* method. Such arguments can also be used to prove the non-endpoint Strichartz estimates as in [8].

Proposition 3.6 ($q = 3p'$ line). *Suppose $f \in \mathcal{L}_1$. Then for $1 \leq p < 4$, $q = 3p'$ and $R \geq 2$, we have a sharp estimate*

$$(3.10) \quad \|(fd\sigma)^\vee\|_{L^q_{t,x}} \lesssim R^{(n-2)(1/q-1/2)} \|f\|_{L^p(S)}.$$

Proof. By the heuristic approximation (3.6) with + sign,

$$\|(fd\sigma)^\vee\|_{L^q_{t,x}} \sim R^{(n-2)(\frac{1}{q}-\frac{1}{2})} \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds \right|^q dt dr \right)^{1/q},$$

where $I = [1, 2]$.

Setting $(Fd\sigma)^\vee(t, r) := \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds$, we see that it suffices to prove

$$(3.11) \quad \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds \right|^q dt dr \right)^{1/q} \lesssim \|F\|_{L^p(S)}.$$

Squaring $(Fd\sigma)^\vee$, we obtain

$$\{(Fd\sigma)^\vee(t, r)\}^2 = \int_{I \times I} F(s_1)F(s_2)(s_1s_2)^{\frac{n-2}{2}} e^{i(r \cdot (s_1+s_2) - t(s_1^2+s_2^2))} ds_1 ds_2,$$

which is an oscillatory integral with a phase function $r(s_1+s_2) - t(s_1^2+s_2^2)$. Its Hessian is $2(s_2-s_1)$ which vanishes when $s_1 = s_2$. But we can make a change of variables $(s_1, s_2) \rightarrow (a, b)$ with $a = s_1 + s_2$, $b = s_1^2 + s_2^2$. It is easy to see that the Jacobian is $2(s_2 - s_1)$. Let Ω be the image in $\mathbb{R} \times \mathbb{R}$ of $I \times I$ under such change of variables. Then $\{(Fd\sigma)^\vee(t, r)\}^2 = \int_{\Omega} \tilde{F}(a, b) e^{i(ra-tb)} da db$, where $\tilde{F}(a, b) = F(s_1)F(s_2)(s_1s_2)^{\frac{n-2}{2}}/|s_1 - s_2|$ is a function of s_1 and s_2 .

Setting $q = 2r'$. Since $r' > 2$, by the Hausdorff-Young inequality and $s_i \sim 1$ for $i = 1, 2$,

$$\begin{aligned} \|(Fd\sigma)^\vee(t, r)\|_{L^q(\mathbb{R} \times A_R)}^2 &\leq \|\{(Fd\sigma)^\vee(t, r)\}^2\|_{L^{r'}(\mathbb{R} \times \mathbb{R})} \\ &\leq \left(\int_{\Omega} |\tilde{F}(s_1, s_2)|^r ds_1 ds_2 \right)^{1/r} \\ &\sim \left(\int_{I^2} |F(s_1)|^r |F(s_2)|^r \frac{1}{|s_1 - s_2|^{r-1}} ds_1 ds_2 \right)^{1/r} \\ &= \left(\int_I |F(s_1)|^r \int_I |F(s_2)|^r \frac{1}{|s_1 - s_2|^{1-(2-r)}} ds_2 ds_1 \right)^{1/r} \\ &\sim \left(\int_I |F(s_1)|^r I_{2-r}(|F|^r)(s_1) ds_1 \right)^{1/r}, \end{aligned}$$

where I_{2-r} is the Riesz potential of order $2-r$ defined via the spatial Fourier transform by $\widehat{I_s f}(\xi) = |\xi|^{-s} \hat{f}(\xi)$. Since $\|f\|_{L^p} \sim \|F\|_{L^p(I)}$, it then suffices to prove that

$$\left(\int_I |F|^r I_{2-r}(|F|^r) ds_1 \right)^{1/r} \lesssim \|F\|_{L^p(I)}^2.$$

By the Hölder inequality, we obtain

$$\int_I |F|^r I_{2-r}(|F|^r) ds_1 \leq \| |F|^r \|_{L^{p/r}(I)} \| I_{2-r}(|F|^r) \|_{L^{1/(1-r/p)}(I)}.$$

Since $\| |F|^r \|_{L^{p/r}(I)}^{1/r} = \|F\|_{L^p(I)}$, it suffices to show $\| I_{2-r}(|F|^r) \|_{L^{1/(1-r/p)}} \lesssim \| |F|^r \|_{L^{p/r}(I)}$, which will follow from the Hardy-Littlewood-Sobolev inequality. Hence the inequality (3.11) follows. ■

Proposition 3.7 ($q = \infty$ line). *Suppose $f \in \mathcal{L}_1$. Then for $q = \infty, 1 \leq p \leq \infty$ and $R \geq 2$, we have a sharp estimate*

$$(3.12) \quad \|(fd\sigma)^\vee\|_{L_{t,x}^\infty} \lesssim R^{-(n-2)/2} \|f\|_{L^p(S)}.$$

Proof. By the heuristic approximation (3.6) of $(fd\sigma)^\vee$ with + sign and the Hölder inequality, for any $p \geq 1$,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^\infty} \sim R^{-(n-2)/2} \left\| \int_I F(s) s^{\frac{n-2}{2}} e^{i(rs-ts^2)} ds \right\|_{L_{t,r}^\infty} \lesssim R^{-(n-2)/2} \|f\|_{L^p(S)}.$$
■

Now we see that the restriction estimates in Theorem 2.1 follow from Propositions 3.3, 3.4, 3.5, 3.6 and 3.7.

The remainder of this section is devoted constructing counterexamples. In view of the propositions above, we will use the main term of $(fd\sigma)^\vee$,

$$\mathcal{M}f(t, x) = c_n r^{-\frac{n-2}{2}} \int_I F(s) s^{\frac{n-2}{2}} e^{i(\pm rs-ts^2)} ds,$$

since the bound B given by the error terms are much smaller than that by the main terms when R is sufficiently large .

Our first counterexample is of Knapp-type, which is designed to show the estimates in the region I in Figure 1 determined by the estimates $L^2 \rightarrow L^2$, $L^4 \rightarrow L^4$ and $L^1 \rightarrow L^\infty$ are sharp. The strength of the standard Knapp example or its variants lie in the idea of using both spatial localization and frequency localization. In this example, we will only show that the estimate $L^2 \rightarrow L^2$ is sharp since the computations for others are similar.

Example 3.8 (I). If $R \geq 2$, the $L^2 \rightarrow L^2$ estimate goes back to (3.7) in Proposition 3.4. We take

$$f(|\xi|^2, \xi) = F(|\xi|) = |\xi|^{-(n-2)/2} 1_{\{1 \leq |\xi| \leq 1+R^{-1/2}\}} e^{-ir_0|\xi|+it_0|\xi|^2},$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. Thus the left-hand side of (3.7) is comparable to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+R^{-1/2}} e^{[-i(t-t_0)s^2+i(\pm r-r_0)s]} ds \right|^2 dt dr \right)^{1/2},$$

where by \pm it denotes a summation of two terms. We observe that

$$\begin{aligned} & \left| \int_1^{1+R^{-1/2}} e^{[-i(t-t_0)s^2+i(\pm r-r_0)s]} ds \right| \\ &= \left| \int_1^{1+R^{-1/2}} e^{-i\{(t-t_0)(s-1)^2-[(\pm r-r_0)-2(t-t_0)](s-1)\}} ds \right|, \end{aligned}$$

and hence choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that

$$\begin{aligned} R/100 &\leq t - t_0 \leq R/50, \\ |(r - r_0) - 2(t - t_0)](s - 1)| &\leq R^{1/2}/100. \end{aligned}$$

Thus r and t are in the intersection region of two tubes whose size is of $R^{1/2} \times R$.

With this choice of r and t , $|(t - t_0)(s - 1)^2 - [(r - r_0) - 2(t - t_0)](s - 1)|$ is less than a small number, say $\pi/6$. Then by direct computations, the term with $+$ sign will be bounded below by $R^{1/4}$. However for the term with $-$ sign, given this choice of r and t , we see that the roots of the quadratic polynomial, $(t - t_0)(s - 1)^2 - [(r + r_0) + 2(t - t_0)](s - 1)$, will be strictly less than -1 , which consequently are not located in the interval $[0, R^{1/2}]$. Thus by the principle of non-stationary phase, we see that the term with $-$ sign will be bounded above by $O_N(R^{-N})$ for any $N > 0$. Then by choosing N sufficiently large, from the triangle inequality the left-hand side of (3.7) $\gtrsim R^{1/4}$. Also it is easy to see that its right-hand side $\lesssim R^{1/4}$. Thus we see that the estimate $L^2 \rightarrow L^2$ when $R \geq 2$ is sharp.

Our second counterexample is to show that the estimates in the region *II* in Figure 1 determined by the lines $q = 2$ and $q = 4$ are sharp. In this example we will only show the estimates on the line $q = 2$ in Proposition 3.4 are sharp by using the principle of stationary phase.

Example 3.9 (II). If $R \geq 2$, the estimate $L^p \rightarrow L^2$ when $2 \leq p \leq \infty$ goes back to Proposition 3.4. We take the example

$$f(|\xi|^2, \xi) = F(|\xi|) = |\xi|^{-\frac{n-2}{2}} 1_{\{1 \leq |\xi| \leq 2\}} e^{-ir_0|\xi|+it_0|\xi|^2},$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. Then the left-hand side of (3.7) is comparable to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I e^{-i\{(t-t_0)(s-1)^2 - [(\pm r-r_0)-2(t-t_0)](s-1)\}} ds \right|^2 dt dr \right)^{1/2}.$$

We choose $r \in [R/100, R/50]$ and $t \in \mathbb{R}$ such that $(r - r_0)/2(t - t_0) \in [1, 2]$. Then

$$-\frac{-(r - r_0) - 2(t - t_0)}{2(t - t_0)} \in [0, 1], \quad -\frac{(r + r_0) + 2(t - t_0)}{2(t - t_0)} < -1.$$

Then from the principles of stationary phase and non-stationary phase,

$$\begin{aligned} \left| \int_I e^{-i\{(t-t_0)(s-1)^2 - [(r-r_0)-2(t-t_0)](s-1)\}} ds \right| &\gtrsim R^{-1/2}, \\ \left| \int_I e^{-i\{(t-t_0)(s-1)^2 + [(r+r_0)+2(t-t_0)](s-1)\}} ds \right| &\lesssim_N R^{-N}, \end{aligned}$$

for any $N \geq 0$. Then if choosing N sufficiently large, the triangle inequality gives

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_I e^{-i\{(t-t_0)(s-1)^2 - [(\pm r-r_0)-2(t-t_0)](s-1)\}} ds \right|^2 dt dr \right)^{1/2} \gtrsim R^{1/2}.$$

Its right-hand side $\lesssim R^{1/2}$ for $2 \leq p \leq \infty$. Thus we see that the estimates on the line $q = 2$ when $R \geq 2$ are sharp.

The third counterexample shows that the estimates inside the region *III* determined by lines $q = 4$, $q = 3p'$ and $q = \infty$ in Figure 1 are sharp. In this example, we will only carry out the computations for the estimates $L^p \rightarrow L^q$ on the line $q = \infty$ in Proposition 3.7.

Example 3.10 (III). If $R \geq 2$, the estimate $L^p \rightarrow L^\infty$ when $1 \leq p \leq \infty$ goes back to Proposition 3.7. We take

$$f(|\xi|^2, \xi) = F(|\xi|) = |\xi|^{-\frac{n-2}{2}} 1_{\{1 \leq |\xi| \leq 2\}} e^{-ir_0|\xi|+it_0|\xi|^2},$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. They are chosen such that $\|(fd\sigma)^\vee\|_{L_{t,x}^\infty}$ can be realized at (t_0, x_0) with $r_0 = |x_0|$.

Hence the left-hand side of the inequality (3.12) is comparable to

$$R^{-\frac{n-2}{2}} \left| \int_I e^{i(\pm r_0 - r_0)s} ds \right|.$$

Since $r_0 \in [R/2, R]$, $|\int_I e^{-i2r_0s} ds| \lesssim R^{-N}$ holds for any $N \geq 0$. Then the triangle inequality yields $R^{-\frac{n-2}{2}} |\int_I e^{i(\pm r_0 - r_0)s} ds| \gtrsim R^{-\frac{n-2}{2}}$. On the other hand, the right-hand side of (3.12) $\lesssim R^{-(n-2)/2}$ for $1 \leq p \leq \infty$. Hence the estimates on the line $q = \infty$ when $R \geq 2$ are sharp.

For lines $q = 4$ and $q = 3p'$, the estimates go back to (3.8) and (3.10). We choose $r \in [R/2, R]$ and t such that $2 \leq r - r_0 \leq 4$, $2 \leq t - t_0 \leq 4$. Then by the same reasoning as Example 3.8, these estimates are sharp.

Thus the proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.5: bilinear estimates and examples

For $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$, we set $I_1 = [1, 2]$, $I_M = [M, 2M]$ and $F(|\xi|) = f(|\xi|^2, \xi)$, $G(|\xi|) = g(|\xi|^2, \xi)$. In the bilinear case, 1 and $1/M$ will bring in two natural separation scales in the physical space. In light of the proof of Theorem 2.1, we will have the following permutations of the product $(fd\sigma_1)^\vee (gd\sigma_2)^\vee$.

- when $|x| = r \geq 1/M$,

$$|(fd\sigma_1)^\vee (gd\sigma_2)^\vee| = |\mathcal{M}f\mathcal{M}g + \mathcal{M}f\mathcal{E}g + \mathcal{M}g\mathcal{E}f + \mathcal{E}f\mathcal{E}g|.$$
- when $1 \leq |x| = r \leq 1/M$,

$$|(fd\sigma_1)^\vee (gd\sigma_2)^\vee| = |\mathcal{M}f(gd\sigma_2)^\vee + \mathcal{E}f(gd\sigma_2)^\vee|.$$
- when $|x| \leq 1$, $(fd\sigma_1)^\vee (gd\sigma_2)^\vee$ remains unchanged.

We are going to prove the “estimates” part of Theorem 2.5 via the following three propositions and its “sharpness” part by building counterexamples in three cases followed.

Proposition 4.1. *Suppose $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$, and $R \leq 1$ is a dyadic number. Then we have sharp estimates*

- for $q = 1$ and $2 \leq p \leq \infty$,

$$(4.1) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim R^{n-1} M^{-1+\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

- for $q \geq \max\{2, p'\}$,

$$(4.2) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^q_{t,x}} \lesssim R^{\frac{n-1}{q}} M^{\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

Proof. If we change to the polar coordinates, the left-hand side of (4.1) reduces to

$$(4.3) \quad \int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I_1} F(s_1) e^{-its_1^2} (d\mu)^\vee (rs_1 e_1) s_1^{n-2} ds_1 \times \right. \\ \left. \times \int_{I_M} G(s_2) e^{-its_2^2} (d\mu)^\vee (rs_2 e_1) s_2^{n-2} ds_2 \right| dt r^{n-2} dr.$$

We use the Cauchy-Schwarz inequality and the Plancherel theorem in t to bound (4.3) by

$$(4.4) \quad R^{n-1} M^{n-2} \|F\|_{L^2(I_1)} M^{-1/2} \|G\|_{L^2(I_M)}.$$

Then by the Hölder inequality, (4.4) is bounded by

$$R^{n-1} M^{n-2-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

Hence the inequality (4.1) follows.

To prove (4.2), by the Hölder inequality,

$$\|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^q_{t,x}} \lesssim \|(fd\sigma_1)^\vee\|_{L^q_{t,x}} \|(gd\sigma_2)^\vee\|_{L^\infty_{t,x}}.$$

On the one hand, by Proposition 3.1,

$$\|(fd\sigma_1)^\vee\|_{L^q_{t,x}} \lesssim R^{\frac{n-1}{q}} \|f\|_{L^p(S_1)}.$$

On the other hand, by the Hölder inequality,

$$\|(gd\sigma_2)^\vee\|_{L^\infty_{t,x}} \lesssim M^{\frac{n-1}{p'}} \|g\|_{L^p(S_2)}.$$

Hence the inequality (4.2) follows. ■

The following proposition concerns the case where $1 \leq |x| \leq 1/M$.

Proposition 4.2. *Suppose $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$, and $2 \leq R \leq 1/M$. Then*

- for $q = 1$ and $2 \leq p \leq \infty$,

$$(4.5) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim R^{\frac{n}{2}} M^{-1+\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

- for $q \geq \max\{2, p'\}$,

$$(4.6) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^q_{t,x}} \lesssim \|R^*\|_{L^p \rightarrow L^q} M^{\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $\|R^*\|_{L^p \rightarrow L^q}$ denotes the operator norm of $f \rightarrow (fd\sigma)^\vee$ from $L^p(S_1)$ to $L^q_{t,x}$ given by Theorem 2.1.

Proof. To prove (4.5), it suffices to prove the following inequalities by Lemma 3.2,

$$\int_{R/2}^R \int_{\mathbb{R}} |\mathcal{M}f(t, re_1)| |(gd\sigma_2)^\vee(t, re_1)| dt r^{n-2} dr \lesssim R^{\frac{n}{2}} M^{-1+\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

and

$$\int_{R/2}^R \int_{\mathbb{R}} |\mathcal{E}f(t, re_1)| |(gd\sigma_2)^\vee(t, re_1)| dt r^{n-2} dr \lesssim R^{\frac{n}{2}-1} M^{-1+\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

These two estimates above can be proven along similar lines as proving (4.1). We choose to prove the second. The Hölder inequality yields,

$$\|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^q_{t,x}} \lesssim \|(fd\sigma_1)^\vee\|_{L^q_{t,x}} \|(gd\sigma_2)^\vee\|_{L^\infty_{t,x}}.$$

Then by using the same reasoning as in the proof of (4.2), the inequality (4.6) follows. We note that the error term $\mathcal{E}f$ gives a better decay estimate as expected. ■

Next let us concentrate on the case where $|x| \geq 1/M$. As indicated at the beginning of this section, we will have to deal with estimates involving $|\mathcal{M}f\mathcal{M}g|$, $|\mathcal{M}f\mathcal{E}g|$, $|\mathcal{E}f\mathcal{M}g|$ and $|\mathcal{E}f\mathcal{E}g|$.

Proposition 4.3 (Bilinear main term estimates). *Suppose $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_M$ with $0 < M \leq 1/4$, and $R \geq 1/M$. Then*

- for $q = 1$ and $2 \leq p \leq \infty$,

$$(4.7) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim RM^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

- for $q = 2$ and $2 \leq p \leq \infty$,

$$(4.8) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^2_{t,x}} \lesssim R^{-\frac{n-2}{2}} M^{\frac{n-1}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

- for $q \geq \max\{4, 3p'\}$ and $1 \leq p \leq \infty$,

$$(4.9) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^q_{t,x}} \lesssim \|R^*\|_{L^p \rightarrow L^q} R^{-\frac{n-2}{2}} M^{\frac{n}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}.$$

Proof. To prove (4.7), it suffices to prove the following inequalities

$$(4.10) \quad \begin{aligned} \|\mathcal{M}f\mathcal{M}g\|_{L^1_{t,x}} &\lesssim RM^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{E}f\mathcal{E}g\|_{L^1_{t,x}} &\lesssim R^{-1} M^{\frac{n-4}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{M}f\mathcal{E}g\|_{L^1_{t,x}} &\lesssim M^{\frac{n-4}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{E}f\mathcal{M}g\|_{L^1_{t,x}} &\lesssim M^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

In what follows we will only prove (4.10) since other estimates involving error terms will follow similarly. In fact these inequalities give better decay estimates than those given by (4.10). By the heuristic approximation (3.6) with + sign, the left-hand side of (4.10) reduces to

$$(4.11) \quad \int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I_1} F(s_1) s_1^{\frac{n-2}{2}} e^{i(rs_1-ts_1^2)} ds_1 \int_{I_M} G(s_2) s_2^{\frac{n-2}{2}} e^{i(rs_2-ts_2^2)} \right| dt dr.$$

After changing variables and using the Cauchy-Schwarz inequality and the Plancherel theorem in t , we see that (4.11) is bounded by

$$RM^{\frac{n-3}{2}} \|F\|_{L^2(I_1)} \|G\|_{L^2(I_M)} \sim RM^{-\frac{1}{2}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}.$$

Then from the Hölder inequality, the inequality (4.10) follows. Similarly, to prove (4.8), it suffices to prove the following inequalities

$$(4.12) \quad \begin{aligned} \|\mathcal{M}f\mathcal{M}g\|_{L^2_{t,x}} &\lesssim R^{-\frac{n-2}{2}} M^{\frac{n-1}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{E}f\mathcal{E}g\|_{L^2_{t,x}} &\lesssim R^{-\frac{n+1}{2}} M^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{M}f\mathcal{E}g\|_{L^2_{t,x}} &\lesssim R^{-\frac{n}{2}+\frac{1}{2}} M^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|\mathcal{E}f\mathcal{M}g\|_{L^2_{t,x}} &\lesssim R^{-\frac{n}{2}+\frac{1}{2}} M^{\frac{n}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

Since the estimates above involving error terms give better decay estimates than (4.12), we will also only prove (4.12). We rewrite its left-hand side as

$$R^{-\frac{n-2}{2}} \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I_1 \times I_M} F(s_1)G(s_2) \times \right. \right. \\ \left. \left. \times (s_1 s_2)^{\frac{n-2}{2}} e^{i(r(s_1+s_2)-t(s_1^2+s_2^2))} ds_1 ds_2 \right|^2 dt dr \right)^{1/2}.$$

Setting $x := s_1 + s_2$ and $y := s_1^2 + s_2^2$, we observe that the Jacobian $\sim |1 - M| \sim 1$ provided $M \leq 1/4$. From the Plancherel theorem both in t and r , the left-hand side of (4.12) is further majorized by

$$R^{-\frac{n-2}{2}} M^{\frac{n-2}{2}} \|F\|_{L^2(I_1)} \|G\|_{L^2(I_M)} \sim R^{-\frac{n-2}{2}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}.$$

By using the Hölder inequality again, we see that (4.12) follows. Finally we prove (4.9). In fact, it suffices to prove the following two inequalities

$$\begin{aligned} \|(fd\sigma_1)^\vee \mathcal{M}g\|_{L^q_{t,x}} &\lesssim \|R^*\|_{L^p \rightarrow L^q} R^{-\frac{n-2}{2}} M^{\frac{n}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \\ \|(fd\sigma_1)^\vee \mathcal{E}g\|_{L^q_{t,x}} &\lesssim \|R^*\|_{L^p \rightarrow L^q} R^{-\frac{n}{2}} M^{\frac{n}{2}-1-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)}. \end{aligned}$$

The first follows from the Hölder inequality and the linear estimate in Theorem 2.1, and the second follows along similar lines. ■

Therefore the restriction estimates in Theorem 2.5 are obtained from Propositions 4.1, 4.2 and 4.3.

In the remainder of this section we will construct counterexamples to show these estimates are sharp or nearly sharp up to R^ε . Since the error terms give much better decay estimates, we will use the heuristic approximations (3.6) of $(fd\sigma_1)^\vee$ and $(gd\sigma_2)^\vee$ when computing these examples. We will distinguish them into three cases as follows.

Case 1: $R \geq 1/M$.

We start with a common example to show the estimates in the region I in Figure 3 determined by $L^2 \times L^2 \rightarrow L^1$, $L^2 \times L^2 \rightarrow L^2$ and $L^1 \times L^1 \rightarrow L^\infty$ are sharp by using the idea coming from standard Knapp example. In the following example, we will only do the computations when $p = 2$ and $q = 1$.

Example 4.4 (I). If $R \geq 1/M$, $RM^{-\frac{1}{2}}$ is best possible in the following inequality

$$(4.13) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim RM^{-\frac{1}{2}}\|f\|_{L^2(S_1)}\|g\|_{L^2(S_2)}.$$

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= F(|\xi|) = |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} 1_{\{1 \leq |\xi| \leq 1+R^{-1}M\}}, \\ g(|\eta|^2, \eta) &= G(|\eta|) = |\eta|^{-\frac{n-2}{2}} e^{i(-r_0|\eta|+t_0|\eta|^2)} 1_{\{M \leq |\eta| \leq M+R^{-1}\}}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. By the heuristic approximation (3.6), the left-hand side of (4.13) is comparable to

$$\int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+R^{-1}M} e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \int_M^{M+R^{-1}} e^{i[(\pm r-r_0)s_2-(t-t_0)s_2^2]} ds_2 \right| dt dr,$$

which we understood is a summation of four terms. For the integral on $[1, 1 + R^{-1}M]$, we will choose r and t such that $R/100 \leq r - r_0 \leq R/50$ and $RM^{-1}/100 \leq t - t_0 \leq RM^{-1}/50$. Then we have

$$\begin{aligned} &\left| [(r - r_0) - 2(t - t_0)](s_1 - 1) - (t - t_0)(s_1 - 1)^2 \right| \leq 1, \\ &-\frac{(r + r_0) + 2(t - t_0)}{2(t - t_0)} = -\left(1 + \frac{r - r_0}{2(t - t_0)} \right) \notin [0, R^{-1}M]. \end{aligned}$$

Hence the integral on $[1, 1 + R^{-1}M]$ with $+$ sign is $\gtrsim R^{-1}M$ while the one with $-$ sign $\lesssim_N R^{-N}$ for any $N \geq 0$. Similarly for the integral on

$[M, M + R^{-1}]$ with this choice of r and t . Then if N is sufficiently large, the triangle inequality gives

$$\int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+R^{-1}M} e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \right. \\ \left. \times \int_{M-R^{-1}}^M e^{i[(\pm r-r_0)s_2-(t-t_0)s_2^2]} ds_2 \right| dt dr \gtrsim 1.$$

Then by direct computations, the right-hand side of (4.13) $\lesssim 1$. Thus we see that the estimate $L^2 \times L^2 \rightarrow L^1$ is sharp when $R \geq 1/M$.

By modifying the above “narrow” Example 4.4, namely taking a linear combination to create a “spreading-out” example, we will show that the estimates in the region II in Figure 3 determined by the lines $q = 1$ and $q = 2$ are sharp by using the Khintchine inequality. A similar construction by Lee and Vargas can be found in [10] to show the sharp null form estimates for the wave equation. In the following example we will only do computations for the estimates on the line $q = 1$.

Example 4.5 (II). If $R \geq 1/M$, $RM^{(n-2)/2-(n-1)/p}$ is best possible in the following inequality

$$(4.14) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim RM^{\frac{n-2}{2}-\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $2 \leq p \leq \infty$.

We define two index sets $J := \{j \in \mathbb{Z} : 1 \leq j \leq [RM^{-1}]\}$ and $K := \{k \in \mathbb{Z} : 1 \leq k \leq [RM]\}$, where $[x]$ denotes the biggest integer which is less than or equal to $x \in \mathbb{R}$. For each $j \in J, k \in K$ we define

$$f_j(|\xi|^2, \xi) = F_j(|\xi|) = |\xi|^{\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} \mathbf{1}_{\{1+(j-1)R^{-1}M \leq |\xi| \leq 1+jR^{-1}M\}}, \\ g_k(|\eta|^2, \eta) = G_k(|\eta|) = |\eta|^{-\frac{n-2}{2}} e^{i(-r_0|\eta|+t_0|\eta|^2)} \mathbf{1}_{\{M+(k-1)R^{-1} \leq |\eta| \leq M+kR^{-1}\}}.$$

Also we set $f = \sum_{j \in J} \varepsilon_j f_j$ and $g = \sum_{k \in K} \tilde{\varepsilon}_k g_k$, where $\{\varepsilon_j : j \in J\}$ and $\{\tilde{\varepsilon}_k : k \in K\}$ are sets of i.i.d. (independent identically distributed) random variables taking ± 1 with an equal probability $1/2$. Note that f_j and g_k are “narrow”, disjoint and in the form of Example 4.4; but f and g “spread out” and support on the whole set S_1 and S_2 . By the Khintchine inequality, we estimate the left-hand side of (4.14) by

$$(4.15) \quad \mathbb{E} \left(\|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \right) \sim \left\| \left(\sum_{j,k} |(f_j d\sigma_1)^\vee (g_k d\sigma_2)^\vee|^2 \right)^{1/2} \right\|_{L^1_{t,x}},$$

where $\mathbb{E}(X)$ denotes the expectation of the random variable X . By the heuristic approximation (3.6), the right-hand side of (4.15) is comparable to

$$(4.16) \quad \int_{R/2}^R \int_{\mathbb{R}} \left(\sum_{j,k} \left| \int_{a_j-R^{-1}M}^{a_j} F_j(s_1) e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \times \right. \right. \\ \left. \left. \times \int_{b_k-R^{-1}}^{b_k} G_k(s_2) e^{i[(\pm r-r_0)s_2-(t-t_0)s_2^2]} ds_2 \right|^2 \right)^{1/2} dt dr,$$

where $a_j = 1+jR^{-1}M$ and $b_k = M+kR^{-1}$, and by \pm we denote a summation of four terms. We choose r and t such that $R/100 \leq r - r_0 \leq R/50$ and $RM^{-1} \leq t - t_0 \leq RM^{-1}/50$. By this choice of r and t and similar discussions as in Example 4.4, the triangle inequality gives,

$$\left| \int_{a_j-R^{-1}M}^{a_j} F_j(s_1) e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \times \int_{b_k-R^{-1}}^{b_k} G_k(s_2) e^{i[(\pm r-r_0)s_2-(t-t_0)s_2^2]} ds_2 \right| \gtrsim R^{-2}M.$$

Then (4.16) is bounded below by $R^2M^{-1}(|J||K|)^{1/2}R^{-2}M$, i.e., $(|J||K|)^{1/2}$. Here $|J| \sim RM^{-1}$ denotes the cardinality of the index set J , similarly for $|K| \sim MR$. Hence we obtain that the left-hand side of (4.14) is $\gtrsim R$. On the other hand, the right-hand side of (4.14) $\lesssim R$ for $2 \leq p \leq \infty$. Hence the estimates on the line $q = 1$ when $R \geq 1/M$ are sharp.

The following example shows that the estimates in the region III in Figure 3 determined by the lines $q = 2$ and $q = 4$ are sharp by the principle of stationary phase. We do computations when $q = 2$.

Example 4.6 (III). If $R \geq 1/M$, $R^{-\frac{n-2}{2}}M^{\frac{n-1}{2}-\frac{n-1}{p}}$ is best possible in the following inequality.

$$(4.17) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^2_{t,x}} \lesssim R^{-\frac{n-2}{2}}M^{\frac{n-1}{2}-\frac{n-1}{p}}\|f\|_{L^p(S_1)}\|g\|_{L^p(S_2)},$$

where $2 \leq p \leq \infty$.

We take

$$f(|\xi|^2, \xi) = |\xi|^{-\frac{n-2}{2}}e^{i(-r_0|\xi|+t_0|\xi|^2)}\mathbf{1}_{\{1 \leq |\xi| \leq 2\}}, \\ g(|\eta|^2, \eta) = |\eta|^{-\frac{n-2}{2}}e^{i(-r_0|\eta|+t_0|\eta|^2)}\mathbf{1}_{\{M \leq |\eta| \leq 2M\}},$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. By the heuristic approximation (3.6), the left-hand side of (4.17) is comparable to

$$R^{-\frac{n-2}{2}} \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I_1} e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \int_{I_M} e^{i[(\pm r-r_0)s_2-(t-t_0)s_2^2]} ds_2 \right|^2 dt dr \right)^{1/2}.$$

We choose $r \in [R/2, R]$ and t such that $M^{-1}/100 \leq r - r_0 \leq M^{-1}/50$ and $(r - r_0)/2(t - t_0) \in I_1$. Then from the principles of stationary phase and non-stationary phase, for any $N \geq 0$,

$$\begin{aligned} \left| \int_{I_1} e^{i[(r-r_0)s_1 - (t-t_0)s_1^2]} ds_1 \right| &\gtrsim M^{1/2}, \\ \left| \int_{I_1} e^{-i[(r+r_0)s_1 + (t-t_0)s_1^2]} ds_1 \right| &\lesssim_N M^N, \\ \left| \int_{s_2 \sim M} e^{i[(r-r_0)s_2 - (t-t_0)s_2^2]} ds_2 \right| &\gtrsim M, \\ \left| \int_{s_2 \sim M} e^{-i[(r+r_0)s_2 + (t-t_0)s_2^2]} ds_2 \right| &\lesssim_N M^N. \end{aligned}$$

Then from the triangle inequality, the left-hand side of (4.17) $\gtrsim R^{-\frac{(n-2)}{2}} M^{1/2}$. By direct computations, the right-hand side of (4.17) $\lesssim R^{-\frac{(n-2)}{2}} M^{1/2}$. Thus the estimates on the line $q = 2$ when $R \geq 1/M$ are sharp.

The next example will show that the estimates in the region IV in Figure 3 determined by $L^2 \times L^2 \rightarrow L^2$, $L^4 \times L^4 \rightarrow L^4$ and $L^1 \times L^1 \rightarrow L^\infty$ are sharp up to R^ε by using the idea of Knapp example. We will only do computations for the estimate when $p = q = 2$.

Example 4.7 (IV). If $R \geq 1/M$, $R^{-\frac{n-2}{2}}$ is best possible in the following inequality.

$$(4.18) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^2_{t,x}} \lesssim R^{-\frac{n-2}{2}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}.$$

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi| + t_0|\xi|^2)} 1_{\{1 \leq |\xi| \leq 1 + M^{1/2}\}}, \\ g(|\eta|^2, \eta) &= |\eta|^{-\frac{n-2}{2}} e^{i(-r_0|\eta| + t_0|\eta|^2)} 1_{\{M \leq |\eta| \leq 2M\}}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. By the heuristic approximation (3.6), the left-hand side of (4.18) is comparable to

$$R^{-\frac{n-2}{2}} \left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+M^{\frac{1}{2}}} e^{i[(\pm r - r_0)s_1 - (t - t_0)s_1^2]} ds_1 \int_{I_M} e^{i[(\pm r - r_0)s_2 - (t - t_0)s_2^2]} ds_2 \right|^2 dt dr \right)^{1/2}.$$

We choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that

$$\begin{aligned} |(r - r_0) - 2(t - t_0)| &\leq M^{-1/2}, \\ |(r - r_0) - 2M(t - t_0)| &\leq M^{-1}, \\ M^{-1}/100 &\leq t - t_0 \leq M^{-1}/50, \end{aligned}$$

i.e., r and t are located in the intersection area of two tubes which has size $M^{-1} \times M^{-1/2}$. Then by similar discussions as Example 4.4, the left-hand side of (4.18) $\gtrsim R^{-(n-2)/2}M^{3/4}$; by direct computations the right-hand side of (4.18) $\lesssim R^{-(n-2)/2}M^{3/4}$. Hence we see that the estimate $L^2 \times L^2 \rightarrow L^2$ when $R \geq 1/M$ is sharp.

The next example shows that the estimates in the region V in Figure 3 determined by the lines $q = 4$, $q = \infty$ and $q = 3p'$ are sharp. In this example, we will do the computations for the estimates on the line $q = \infty$.

Example 4.8 (V). If $R \geq 1/M$, $R^{-(n-2)} \times M^{\frac{n}{2} - \frac{n-1}{p}}$ is best possible in the following inequality

$$(4.19) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L_{t,x}^\infty} \lesssim R^{-(n-2)} M^{\frac{n}{2} - \frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $1 \leq p \leq \infty$.

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi| + t_0|\xi|^2)} \mathbf{1}_{\{1 \leq |\xi| \leq 2\}}, \\ g(|\eta|^2, \eta) &= |\eta|^{-\frac{n-2}{2}} e^{i(-r_0|\eta| + t_0|\eta|^2)} \mathbf{1}_{\{M \leq |\eta| \leq 2M\}}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$ satisfying the L^∞ norms $\|(fd\sigma)^\vee\|_{L_{t,x}^\infty(\mathbb{R} \times A_R)}$ and $\|(gd\sigma)^\vee\|_{L_{t,x}^\infty(\mathbb{R} \times A_R)}$ can be realized at (t_0, x_0) with $|x_0| = r_0$. By the heuristic approximation (3.6), the left-hand side of (4.19) is comparable to

$$R^{-(n-2)} \left| \int_{I_1} e^{i(\pm r_0 - r_0)s_1} ds_1 \int_{I_M} e^{i(\pm r_0 - r_0)s_2} ds_2 \right|.$$

Then by the same reasoning as Example 3.10, the above $\gtrsim R^{-(n-2)}M$. On the other hand, the right-hand side of (4.19) $\lesssim R^{-(n-2)}M$. Hence the estimates on the line $q = \infty$ when $R \geq 1/M$ are sharp.

For lines $q = 4, 4 \leq p \leq \infty$ or $q = 3p', 1 \leq p < 4$, the estimates go back to (4.9). We will choose $r \in [R/2, R]$ and t such that $2 \leq r - r_0 \leq 4$ and $2 \leq t - t_0 \leq 4$. Then by similar reasoning, the estimates on these lines are sharp.

Case 2: $2 \leq R \leq 1/M$.

In this subcase, we will construct counterexamples to show the restriction estimates in Theorem 2.5 are sharp when $2 \leq R \leq 1/M$. As in the *Case 1*, we will start with a “narrow” example which shows that estimates in the region I in Figure 3 determined by $L^2 \times L^2 \rightarrow L^1$, $L^2 \times L^2 \rightarrow L^2$ and $L^1 \times L^1 \rightarrow L^\infty$ are sharp. In this example, we will do computations for the estimate $L^2 \times L^2 \rightarrow L^1$.

Example 4.9 (I). If $2 \leq R \leq 1/M$, $R^{\frac{n}{2}} M^{\frac{n-3}{2}}$ is best possible in

$$(4.20) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim R^{\frac{n}{2}} M^{\frac{n-3}{2}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}.$$

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} \mathbf{1}_{\{1 \leq |\xi| \leq 1+M^2\}}, \\ g(|\eta|^2, \eta) &= |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. By the heuristic approximation (3.6) only for $(fd\sigma)^\vee$, we see that the left-hand side of (4.20) is comparable to

$$R^{\frac{n-2}{2}} \int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+M^2} e^{i((\pm r-r_0)s_1-(t-t_0)s_1^2)} ds_1 \int_{I_M} e^{-i(t-t_0)s_2^2} (d\mu)^\vee(rse_1) ds_2 \right| dt dr.$$

We choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that

$$R/100 \leq r - r_0 \leq R/50, \quad M^{-2}/100 \leq t - t_0 \leq M^{-2}/50.$$

Then we have $|[(r - r_0) - 2(t - t_0)](s_1 - 1)| \leq c$ with a small $c > 0$ and $-\frac{(r+r_0)+2(t-t_0)}{2(t-t_0)} < -1$. From the principle of non-stationary phase and the triangle inequality, the left-hand side of (4.20) $\gtrsim R^{\frac{n}{2}} M$. On the other hand, the right-hand side of (4.20) $\lesssim R^{\frac{n}{2}} M$. Thus the estimate $L^2 \times L^2 \rightarrow L^1$ when $2 \leq R \leq 1/M$ is sharp.

In the next example, we are going to show the estimates in the region *II* in Figure 3 determined by the lines $q = 1$ and $q = 2$ are sharp. In this example, we will do computations for the estimates on the line $q = 1$.

Example 4.10 (II). If $2 \leq R \leq 1/M$, $R^{\frac{n}{2}} M^{-1+\frac{n-1}{p}}$ is best possible in the following inequality

$$(4.21) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^1_{t,x}} \lesssim R^{\frac{n}{2}} M^{-1+\frac{n-1}{p}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $2 \leq p \leq \infty$. We define an index set $J := \{j : 1 \leq j \leq [M^{-2}]\}$. For each $j \in J$, we set

$$f_j(|\xi|^2, \xi) = F_j(|\xi|) = |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} \mathbf{1}_{\{1+(j-1)M^2 \leq |\xi| \leq 1+jM^2\}}.$$

Then we define

$$f = \sum_j \varepsilon_j f_j, \quad g(|\eta|^2, \eta) = |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M},$$

where $\{\varepsilon_j : j \in J\}$ is a set of i.i.d. random variables taking ± 1 with an equal probability $1/2$.

By using the Khintchine inequality, we obtain

$$\mathbb{E} \left(\|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^1_{t,x}(\mathbb{R} \times A_R)} \right) \sim \left\| \left(\sum_j |(f_j d\sigma_1)^\vee (g d\sigma_2)^\vee|^2 \right)^{1/2} \right\|_{L^1_{t,x}(\mathbb{R} \times A_R)},$$

where $\mathbb{E}(X)$ denotes the expectation of the random variable X . By the heuristic approximation (3.6), the right-hand side of the above is comparable to

$$R^{\frac{n-2}{2}} \int_{R/2}^R \int_{\mathbb{R}} \left(\sum_j \left| \int_{c_j-M^2}^{c_j} e^{i[(\pm r-r_0)s_1 - (t-t_0)s_1^2]} ds_1 \times \right. \right. \\ \left. \left. \times \int_{I_M} e^{-i(t-t_0)s_2^2} (d\mu)^\vee (rs_2 e_1) ds_2 \right|^2 \right)^{1/2} dt dr,$$

where $c_j = 1 + jM^2$. We choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that

$$R/100 \leq r - r_0 \leq R/50, \quad M^{-2}/100 \leq t - t_0 \leq M^{-2}/50.$$

This gives $|(r - r_0)2(t - t_0)|(s_1 - c_j)| \leq c$ and $-\frac{(r+r_0)+2(t-t_0)}{2(t-t_0)} < -1$. Then by the principle of non-stationary phase and the triangle inequality, the above is bounded below by $RM^{-2}|J|^{1/2}M^3$, where $|J| \sim M^{-2}$ denotes the cardinality of the set J . Hence the left-hand side of (4.21) $\gtrsim R^{n/2}$ and the right-hand side of (4.21) $\lesssim R^{n/2}$ for $2 \leq p \leq \infty$. Thus the estimates on the line $q = 1$ when $2 \leq R \leq 1/M$ are sharp.

In the following example, we will see the estimates in the region *III* in Figure 3 determined by the lines $q = 2$ and $q = 4$ are sharp in the case $2 \leq R \leq 1/M$. We will do computations for the estimates on the line $q = 2$ below.

Example 4.11 (III). If $2 \leq R \leq 1/M$, $R^{1/2}M^{(n-1)/p'}$ is best possible in the following inequality.

$$(4.22) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L^2_{t,x}} \lesssim R^{1/2}M^{(n-1)/p'} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $2 \leq p \leq \infty$.

We take

$$f(|\xi|^2, \xi) = |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi| + t_0|\xi|^2)} \mathbf{1}_{\{1 \leq |\xi| \leq 2\}}, \\ g(|\eta|^2, \eta) = |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M},$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. Then by the heuristic approximation (3.6), the left-hand side of (4.22) is comparable to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_{I_1} e^{i[(\pm r-r_0)s_1 - (t-t_0)s_1^2]} ds_1 \int_{I_M} e^{-i(t-t_0)s_2^2} (d\mu)^\vee (rs_2 e_1) ds_2 \right|^2 dt dr \right)^{1/2}.$$

We choose $r \in [R/100, R/50]$ and $t \in \mathbb{R}$ such that $\frac{r-r_0}{2(t-t_0)} \in I_1$. Then r and t are in the region of size $\sim R^2$. The principles of stationary phase and non-stationary phase again give, for any $N \geq 0$,

$$\left| \int_{I_1} e^{i[(r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \right| \gtrsim R^{-\frac{1}{2}}, \quad \left| \int_{I_1} e^{-i[(r+r_0)s_1+(t-t_0)s_1^2]} ds_1 \right| \lesssim_N R^{-N}.$$

With this choice of r and t , we have $\left| \int_{I_M} e^{-i(t-t_0)s_2^2} (d\mu)^\vee(ir s_2 \omega) ds_2 \right| \gtrsim M$. Hence from the triangle inequality, the left-hand side of (4.22) $\gtrsim R^{1/2}M$. On the other hand, the right-hand side of (4.22) $\lesssim R^{1/2}M$. Thus we see that the estimates on the line $q = 2$ when $2 \leq R \leq 1/M$ are sharp.

The next example will show that the estimates in the region *IV* in Figure 3 determined by $L^2 \times L^2 \rightarrow L^2$, $L^4 \times L^4 \rightarrow L^4$ and $L^1 \times L^1 \rightarrow L^\infty$ are sharp. We will do the computations for the estimate $L^2 \times L^2 \rightarrow L^2$.

Example 4.12 (IV). If $2 \leq R \leq 1/M$, $R^{1/2}M^{(n-1)/2}$ is best possible in the following inequality.

$$(4.23) \quad \|(fd\sigma_1)^\vee(gd\sigma_2)^\vee\|_{L^2_{t,x}} \lesssim R^{\frac{1}{2}}M^{\frac{n-1}{2}}\|f\|_{L^2(S_1)}\|g\|_{L^2(S_2)}.$$

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} \mathbf{1}_{\{1 \leq |\xi| \leq 1+R^{-1/2}\}}, \\ g(|\eta|^2, \eta) &= |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. By the heuristic approximation for $(fd\sigma)^\vee$, the left-hand side of (4.23) is comparable to

$$\left(\int_{R/2}^R \int_{\mathbb{R}} \left| \int_1^{1+R^{-1/2}} e^{i[(\pm r-r_0)s_1-(t-t_0)s_1^2]} ds_1 \int_{I_M} e^{-i(t-t_0)s_2^2} (d\mu)^\vee(ir s_2 e_1) ds_2 \right|^2 dt dr \right)^{1/2}.$$

We choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that

$$|(r - r_0) - 2(t - t_0)| \leq R^{1/2}/100, \quad R^{1/2}/100 \leq t - t_0 \leq R^{1/2}/50.$$

Then r and t are located in the intersection area of two tubes which has size of $R \times R^{1/2}$. Hence the left-hand side of (4.23) $\gtrsim R^{1/4}M$. On the other hand, its right-hand side $\lesssim R^{1/4}M$. Thus we see that the estimate $L^2 \times L^2 \rightarrow L^2$ when $2 \leq R \leq 1/M$ is sharp.

The following example will show that the estimates in the region *V* in Figure 3 determined by the lines $q = 4$, $q = \infty$ and $q = 3p'$ are sharp.

Example 4.13 (V). If $2 \leq R \leq 1/M$, $R^{-(n-2)/2} M^{(n-1)/p'}$ is best possible in the following inequality

$$(4.24) \quad \|(fd\sigma_1)^\vee (gd\sigma_2)^\vee\|_{L_{t,x}^\infty} \lesssim R^{-\frac{n-2}{2}} M^{\frac{n-1}{p'}} \|f\|_{L^p(S_1)} \|g\|_{L^p(S_2)},$$

where $1 \leq p \leq \infty$.

We take

$$\begin{aligned} f(|\xi|^2, \xi) &= |\xi|^{-\frac{n-2}{2}} e^{i(-r_0|\xi|+t_0|\xi|^2)} \mathbf{1}_{\{1 \leq |\xi| \leq 2\}}, \\ g(|\eta|^2, \eta) &= |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M}, \end{aligned}$$

where $r_0 \in [R/2, R]$ and $t_0 \in \mathbb{R}$. They are chosen such that $\|(fd\sigma_1)^\vee\|_{L_{t,x}^\infty}$ and $\|(gd\sigma_2)^\vee\|_{L_{t,x}^\infty}$ can be realized at (t_0, x_0) with $|x_0| = r_0$. By the heuristic approximation (3.6),

$$R^{-\frac{n-2}{2}} \left| \int_{I_1} e^{i(\pm r_0 - r_0)s_1} ds_1 \int_{I_M} (d\mu)^\vee(rs_2e_1) ds_2 \right|.$$

Then from the triangle inequality, the left-hand side of (4.24) $\gtrsim R^{-(n-2)/2} M$. On the other hand, its right-hand side $\lesssim R^{-(n-2)/2} M$ for $1 \leq p \leq \infty$. Thus the estimates on the line $q = \infty$ when $2 \leq R \leq 1/M$ are sharp.

When $q = 4, 4 \leq p \leq \infty$, or $q = 3p', 1 \leq p < 4$, the estimates go back to (4.6). We choose $r \in [R/2, R]$ and $t \in \mathbb{R}$ such that $2 \leq r - r_0 \leq 4$ and $2 \leq t - t_0 \leq 4$.

Case 3: $R \leq 1$.

In this subcase, we will construct counterexamples to show the estimates (4.1) and (4.2) are sharp. We will omit the computations for simplicity.

The following example shows that the estimates in the region I determined by $L^2 \times L^2 \rightarrow L^1, L^2 \times L^2 \rightarrow L^2$ and $L^1 \times L^1 \rightarrow L^\infty$ are sharp.

Example 4.14 (I). We take

$$\begin{aligned} f(|\xi|^2, \xi) &= F(|\xi|) = |\xi|^{-(n-2)} e^{it_0|\xi|^2} \mathbf{1}_{\{1 \leq |\xi| \leq 1+M^2\}}, \\ g(|\eta|^2, \eta) &= G(|\eta|) = |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M}, \end{aligned}$$

where $t_0 \in \mathbb{R}$. The r and t are chosen such that $\frac{R}{2} \leq r \leq R$ and $\frac{1}{100M^2} \leq t - t_0 \leq \frac{1}{50M^2}$.

The next example shows that the estimates in the region II determined by the lines $q = 1$ and $q = 2$ are sharp.

Example 4.15 (II). We define an index set $J := \{j : 1 \leq j \leq [M^{-2}]\}$. For each $j \in J$, we set

$$f_j(|\xi|^2, \xi) = F_j(|\xi|) = |\xi|^{-(n-2)} e^{it_0|\xi|^2} \mathbf{1}_{\{1+(j-1)M^2 \leq |\xi| \leq 1+jM^2\}}.$$

Then we define

$$f = \sum_j \varepsilon_j f_j, \quad g(|\eta|^2, \eta) = |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M},$$

where $\{\varepsilon_j : j \in J\}$ is a set of i.i.d. random variables taking ± 1 with an equal probability $1/2$, and the r and t are chosen such that $R/2 \leq r \leq R$ and $1/2 \leq t - t_0 \leq 1$.

The third example shows that the estimate (4.2) is sharp. Hence the estimates in the regions III, IV and V when $R \leq 1$ are sharp.

Example 4.16 (III, IV, V). We take

$$\begin{aligned} f(|\xi|^2, \xi) &= F(|\xi|) = |\xi|^{-(n-2)} e^{it_0|\xi|^2} \mathbf{1}_{I_1}, \\ g(|\eta|^2, \eta) &= G(|\eta|) = |\eta|^{-(n-2)} e^{it_0|\eta|^2} \mathbf{1}_{I_M}, \end{aligned}$$

where $t_0 \in \mathbb{R}$. The r and t will be chosen such that $\frac{R}{2} \leq r \leq R$ and $\frac{1}{2} \leq t - t_0 \leq 1$.

Thus the proof of Theorem 2.5 is complete.

5. Connection with the restriction estimates for the sphere or the hypersurface of elliptic type

In this section we are concerned with whether the analogous results of Theorems 2.1 and 2.5 remain valid if S is replaced with the lower third of the sphere S^{n-1} or a cylindrically symmetric and compact hypersurface of elliptic type.

Let us first consider the case where the paraboloid is replaced by the sphere S^{n-1} in \mathbb{R}^n . Suppose f is a cylindrically symmetric function supported on a compact set of S^{n-1} , $S := \{(-\sqrt{1 - |\xi|^2}, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1} : M \leq |\xi| \leq 2M\}$, where $0 < M \leq 1/6$. Then

$$(5.1) \quad (fd\mu)^\vee(t, x) = \int_{M \leq |\xi| \leq 2M} e^{i(x \cdot \xi - t\sqrt{1 - |\xi|^2})} F(|\xi|) d\xi,$$

where $d\mu$ is the surface measure of the sphere and $F(|\xi|) = f(-\sqrt{1 - |\xi|^2}, \xi)$.

Since f is cylindrically symmetric, we see that $(fd\mu)^\vee$ is also cylindrically symmetric. Then if we change (5.1) to the polar coordinates to obtain

$$(5.2) \quad (fd\mu)^\vee(t, r) = \int_I F(s)e^{-it\sqrt{1-s^2}}(d\mu)^\vee(rse_1)s^{n-2}ds,$$

where $I = [M, 2M]$. By the Taylor expansion of $\sqrt{1-s^2}$ at $s = 0$,

$$(5.3) \quad -\sqrt{1-s^2} = -1 + \frac{1}{2}s^2 + C_1(s)s^4,$$

where $C_1(s) \sim 1$ for all $s \in I$. Then from (5.2),

$$(5.4) \quad \left| (fd\mu)^\vee(2t, r) \right| = \left| \int_I F(s)e^{it(s^2+C_2s^4)}(d\mu)^\vee(rse_1)s^{n-2}ds \right|,$$

where $C_2(s) \sim 1$ for all $s \in I$. The factor “2” in $(fd\mu)^\vee(2t, r)$ is artificial since we are going to integrate t in \mathbb{R} . We make two key observations similar to those we used in Theorem 2.1 and 2.5 as follows.

- Since $0 \leq M \leq 1/6$,

$$\frac{d(s^2 + C_2(s)s^4)}{ds} \sim s, \quad \frac{d^2(s^2 + C_2(s)s^4)}{d^2s} \sim 1.$$

Heuristically, this condition means that if we change variables $s^2 + C_2(s)s^4 \rightarrow a$, it is almost like changing $s^2 \rightarrow a$. Hence the analogous result to Proposition 3.6 will hold for the lower third of the sphere.

- Form the geometric properties of the sphere,

$$\|d\mu_1 * d\mu_2\|_{L_{i,r}^\infty} \lesssim O(1),$$

when $d\mu_1$ and $d\mu_2$ are the canonical Lebesgue measure of two arcs of size $O(1)$ supported on the sphere \mathcal{S}^1 but separated by a distance $O(1)$. Hence the analogous result to Proposition 3.5 will hold.

Those observations enable us to run all the arguments in Theorems 2.1 and 2.5.

We now turn to the case where S is replaced by a cylindrically symmetric and compact hypersurface S of elliptic type, i.e., S is of the form

$$(5.5) \quad S := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1} : \tau = |\xi|^2 + \varepsilon\phi(\xi)\}$$

where the error function $\phi(\xi)$ is radial and smooth, and ε is a sufficiently small parameter depending on the smooth norms of ϕ and on the size of S , or more generally on the separation of S_1 and S_2 . In other words, S is the small perturbation of the standard paraboloid. By similar observations we made on the sphere, we can establish the analogous results to Theorems 2.1 and 2.5 for the cylindrically symmetric functions compactly supported on S defined in (5.5).

6. Connection with Strichartz inequalities of the Schrödinger equation

6.1. Linear Strichartz estimates

The restriction problem is closely related to that of estimating solutions to linear PDE such as the wave equation and the Schrödinger equation. Strichartz first observed this connection in [16], which initiated the intensive study on various Strichartz estimates. In this section we will interpret our restriction estimates regarding $(fd\sigma)^\vee$ in terms of the solutions to the Schrödinger equations.

Suppose $f(\tau, \xi)$ is a function supported on the paraboloid S in $\mathbb{R} \times \mathbb{R}^{n-1}$. Functions of the form $u(t, x) := (fd\sigma)^\vee$, where $d\sigma$ is the canonical Lebesgue measure on S , can be easily seen to solve the free Schrödinger equation

$$(6.1) \quad iu_t + \Delta u = 0, \quad u(0, x) = u_0(x),$$

where the spatial Fourier transform $\hat{u}_0(\xi) = f(|\xi|^2, \xi)$. It is easy to deduce that f is cylindrically symmetric on $\mathbb{R} \times \mathbb{R}^{n-1}$ if u_0 is radial on \mathbb{R}^{n-1} . By this interpretation, the linear estimate $L^p \rightarrow L^q$ or the bilinear restriction estimate $L^p \times L^p \rightarrow L^q$ will correspond to certain Strichartz estimates. For instance, the Tomas-Stein restriction estimate $L^2 \rightarrow L^{2(n+1)/(n-1)}$ implies the Strichartz estimate

$$\|e^{it\Delta}u_0\|_{L_{t,x}^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times \mathbb{R}^{n-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^{n-1})},$$

where we have denoted u by $e^{it\Delta}u_0(x) = \int_{\mathbb{R}^{n-1}} e^{i(x\xi+t|\xi|^2)}\hat{u}_0(\xi)d\xi$. This is known to be best possible simply by the scaling property associated to the Schrödinger equation. In fact, we have the following optimal result called the linear Strichartz estimates [8],

$$(6.2) \quad \|e^{it\Delta}u_0\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^{n-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^{n-1})}$$

if and only if

$$(6.3) \quad \frac{2}{q} + \frac{n-1}{r} = \frac{n-1}{2}, \quad q \geq 2, r \geq 2, (q, r, n) \neq (2, \infty, 3).$$

A natural question arises: if we assume that \hat{u}_0 is radial and supported on a compact set $U := \{\xi \in \mathbb{R}^{n-1} : M \leq |\xi| \leq 2M\}$ with dyadic $M > 0$, do we have further estimates available? The answer is confirmed in Corollary 2.3. In particular, we have

Corollary 6.2. *Suppose u_0 is defined as above. Then for any $q > \frac{4n-2}{2n-3}$,*

$$(6.4) \quad \|e^{it\Delta}u_0\|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^{n-1})} \lesssim M^{\frac{n-1}{2} - \frac{n+1}{q}} \|u_0\|_{L^2(\mathbb{R}^{n-1})}.$$

Remark 6.3. For such functions, one can easily extend the current range (6.3) for the linear mixed norm Strichartz estimates (6.2) by interpolating them with the estimate (6.4).

In another direction, one can also obtain various weighted Strichartz estimates. This type of estimates for radial data has proven very useful in establishing the global well-posedness and scattering results for certain Schrödinger equations, see e.g., [24]. In [26], Vilela showed that, assuming $u_0 \in L^2(\mathbb{R}^{n-1})$ to be radial,

$$(6.5) \quad \|D_x^s e^{it\Delta} u_0\|_{L^2_{t,x}(|x|^{-\alpha})} \lesssim \|u_0\|_{L^2},$$

if and only if $\alpha = 2(1 - s)$, $1 < \alpha < n - 1$ and $n \geq 3$, where $D_x^s f$ is defined via the spatial Fourier transform by $\widehat{D_x^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$. The “only if” part is given in [26] by the decay estimate of $(d\sigma)^\vee$ and scaling. Here we will give another proof of the “if” part by using the linear dyadic restriction restriction estimates given by Theorem 2.1.

Proof. We first assume that u_0 has dyadically localized frequency, i.e., \hat{u}_0 supported on the set $\{\xi : M/2 \leq |\xi| \leq M\}$ with dyadic M . Then we set $f(|\xi|^2, \xi) = \hat{u}_0(M\xi)$, i.e., $f \in \mathcal{L}_1$. Then from the estimate $L^2 \rightarrow L^2$ in Theorem 2.1, we obtain, $\forall \varepsilon > 0$,

$$\| |x|^{-(1+\varepsilon)/2} (fd\sigma)^\vee \|_{L^2(\mathbb{R} \times \{|x| \geq 1\})} \lesssim_\varepsilon \|f\|_{L^2(\mathbb{R}^{n-1})}.$$

If we restrict $0 < \varepsilon < n - 2$, by the Plancherel theorem in t ,

$$\| |x|^{-(1+\varepsilon)/2} (fd\sigma)^\vee \|_{L^2(\mathbb{R} \times \{|x| \leq 1\})} \lesssim_\varepsilon \|f\|_{L^2(\mathbb{R}^{n-1})}.$$

Hence

$$\| |x|^{-(1+\varepsilon)/2} (fd\sigma)^\vee \|_{L^2(\mathbb{R} \times \mathbb{R}^{n-1})} \lesssim_\varepsilon \|f\|_{L^2(\mathbb{R}^{n-1})}.$$

By re-scaling by M ,

$$\| |x|^{-(1+\varepsilon)/2} M^{(1-\varepsilon)/2} e^{it\Delta} u_0 \|_{L^2(\mathbb{R} \times \mathbb{R}^{n-1})} \lesssim_\varepsilon \|u_0\|_{L^2(\mathbb{R}^{n-1})}.$$

By the weighted Hörmander-Mikhlin theorem [24, Lemma 2.2],

$$\| D^{(1-\varepsilon)/2} e^{it\Delta} u_0 \|_{L^2(|x|^{-(1+\varepsilon)})} \lesssim_\varepsilon \|u_0\|_{L^2(\mathbb{R}^{n-1})}.$$

Setting $s = (1 - \varepsilon)/2$ and $\alpha = 1 + \varepsilon$, we obtain (6.5) for frequency localized u_0 . Then we follow the approach of using the Khintchine inequality to prove the Littlewood-Paley inequality and use the weighted inequalities for singular integrals [14, Chapter 5, Corollary 4.2] ($|x|^{-\alpha}$ is a A_2 weight) to obtain (6.5). ■

6.4. Bilinear Strichartz estimates

From the linear Strichartz estimates (6.2), we see their bilinear analogues,

$$(6.6) \quad \|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})}$$

if and only if

$$(6.7) \quad \frac{2}{q} + \frac{n-1}{r} = n-1; \quad q, r \geq 1; (q, r, n) \neq (1, \infty, 3).$$

For the necessity of excluding the endpoint $(1, \infty, 3)$, see [21].

The estimate (6.6) becomes more interesting when we assume u_0 and v_0 are compactly supported and separated by a distance comparable to $O(1)$. In this case, we expect that there are more estimates available. For instance, when $q = r$, Klainerman and Machedon [9] conjectured that (6.6) holds if and only if $q = r \geq (n + 2)/n$. The exponent $(n + 2)/n$ is best possible, see e.g., [23], [20]. This conjecture has been verified by Tao in [20] up to the endpoint $(n + 2)/n$. The analogous results in the cone setting were established by Wolff in the non-endpoint case [28] and Tao in the endpoint case [19].

As shown in Corollary 2.7, we have further estimates available if we assume that \widehat{u}_0 and \widehat{v}_0 are radial functions and compactly supported on $U_1 = \{\xi \in \mathbb{R}^{n-1} : M_1/2 \leq |\xi| \leq M_1\}$ and $U_2 = \{(\xi \in \mathbb{R}^{n-1} : M_2/2 \leq |\xi| \leq M_2\}$, respectively. Here $M_1 > 0, M_2 > 0$ are dyadic numbers satisfying $M_2 \leq M_1/4$. For instance, as a corollary of Theorem 2.5, we have the following bilinear Strichartz estimates by interpolation and summing in dyadic R .

Corollary 6.5. *Suppose u_0, v_0 are defined as above. Then*

- for $\frac{n}{n-1} < q \leq 2$,

$$\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L_{t,x}^q(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim M_1^{-\frac{1}{2}}M_2^{\frac{2n-1}{2}-\frac{n+1}{q}}\|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})}.$$

- for $2 \leq q \leq \frac{2(2n-1)}{2n-3}$,

$$\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L_{t,x}^q(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim M_1^{-\frac{3}{2q}+\frac{1}{4}}M_2^{\frac{4n-5}{4}-\frac{2n-1}{2q}}\|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})}.$$

- for $q \geq \frac{2(2n-1)}{2n-3}$,

$$\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L_{t,x}^q(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim M_1^{\frac{n-1}{2}-\frac{n+1}{q}}M_2^{\frac{n-1}{2}}\|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})}.$$

Remark 6.6. It is clear that, $\forall q > \frac{n}{n-1}$, $n \geq 3$ and $M_1, M_2 \sim 1$,

$$\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})},$$

which improves $q > \frac{n+2}{n}$.

Remark 6.7. When $q = 2$ and $n \geq 3$, we have the following sharp estimates for u_0, v_0 defined as above,

$$\|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim M_1^{-\frac{1}{2}}M_2^{\frac{n-2}{2}}\|u_0\|_{L^2(\mathbb{R}^{n-1})}\|v_0\|_{L^2(\mathbb{R}^{n-1})},$$

which generalizes Bourgain's following estimates to all dimensions

$$\begin{aligned} \|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} &\lesssim M_1^{-\frac{1}{2}}M_2^{\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^2)}\|v_0\|_{L^2(\mathbb{R}^2)}, \\ \|e^{it\Delta}u_0e^{it\Delta}v_0\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^3)} &\lesssim M_1^{-\frac{1}{2}}M_2\|u_0\|_{L^2(\mathbb{R}^3)}\|v_0\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

But we remark that Bourgain's estimates are for general u_0 and v_0 without the radial assumption, see [2], [4].

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