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Singular integrals in nonhomogeneous spaces: L^2 and L^p continuity from Hölder estimates

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Abstract

We present a result of L^p continuity of singular integrals of Calderón-Zygmund type in the context of bounded nonhomogeneous spaces, well suited to be applied to problems of a priori estimates for partial differential equations. First, an easy and selfcontained proof of L^2 continuity is got by means of C^α continuity, thanks to an abstract theorem of Krein. Then L^p continuity is derived adapting known results by Nazarov-Treil-Volberg about singular integrals in nonhomogeneous spaces.

1. Introduction

The theory of singular integrals of Calderón-Zygmund type has been successfully employed, since its very beginning in the mid 1950's, to the study of a-priori estimates for linear PDE's (see for instance Calderón-Zygmund [4]). Starting with the 1970's, the generalization of the theory to the context of spaces of homogeneous type, in the sense of Coifman-Weiss [7], has made possible the proof of a-priori estimates also for degenerate operators of Hörmander's type (see Folland [12], Rothschild-Stein [21]). In that context, the proof of L^2 continuity for the relevant singular integral operator was provided by means of the so-called *almost orthogonality principle*, a far-reaching idea, originally due to Cotlar [8] and then adapted by Knapp-Stein [14], suitable to a much broader application than the original Fourier transform technique, which worked so well in the Euclidean context, but not in more general settings. On the other hand, the part of the theory which deduces L^p continuity

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from L^2 continuity (via weak $(1,1)$ estimate and interpolation), was accomplished in [6], [7] by a suitable adaptation of the original Calderón-Zygmund construction, which is possible as soon as we have a *quasidistance* and a *doubling measure*. Later, in the mid 1980's, new deep advances were made with the "$T(1)$ theorem", by David-Journé [9], and the "$T(b)$ theorem", by David-Journé-Semmes [10]. These results made possible to prove the L^2 continuity of singular integral operators under considerably weaker assumptions on the kernel. More precisely, it was the *cancellation property* of the kernel which was expressed in a weaker sense, asking for instance -in the first of the two quoted results- $T(1)$ to belong to the space BMO, instead of being *bounded*, as traditionally required. The theorems $T(1)$ and $T(b)$ were also shown to hold in the general context of spaces of homogeneous type (see Christ [5]). Once the L^2 continuity was established, the existing machinery provided the extension to L^p continuity. In the late 1990's still another great step forward was done, with the works of Nazarov-Treil-Volberg (see e.g. [18, 19, 20, 25]), Tolsa ([23, 24]) and other authors, who showed that the *doubling condition*, which was considered the cornerstone of any extension to abstract frameworks of the theory of singular integrals, was not really necessary, but could be replaced by a less demanding condition. One of the main motivations of the new theory was the solution of several questions related to analytic capacity (see e.g. [25] for a discussion of these issues) and, as far as we know, the results which hold in *nonhomogeneous spaces* have not been applied yet to problems in PDE's. However, *there are* situations related to a-priori estimates for PDE's, where removing the doubling condition from the assumptions is useful. This paper is mainly motivated by this idea.

For instance, in the study of degenerate operators built with Hörmander's vector fields, the following ingredients are usually present: a bounded domain $\Omega \subset \mathbb{R}^n$; a distance or quasidistance d, defined in Ω, adapted to the differential operator; the Lebesgue measure. Usually, we know that the Lebesgue measure is *locally doubling* with respect to the metric balls; this means that, if we define

$$B_r(x) = \{y \in \Omega : d(x,y) < r\},$$

then for any $\Omega' \Subset \Omega$ there exist positive constants c, r_0 such that

$$|B_{2r}(x)| \leq c\,|B_r(x)| \text{ for any } x \in \Omega', r \leq r_0 \tag{1.1}$$

(a famous result of this kind is due to Nagel-Stein-Wainger [17]).

Clearly, this is not enough to say that (Ω, d, dx) or (Ω', d, dx) are spaces of homogeneous type: namely, in order to say that the doubling condition holds in (Ω, d, dx) one should know that (1.1) holds for any $x \in \Omega$ (and not just in Ω'), while to say that it holds in (Ω', d, dx) one should know that

$$|B_{2r}(x) \cap \Omega'| \leq c\,|B_r(x) \cap \Omega'| \text{ for any } x \in \Omega', r > 0.$$

But this condition requires some sort of regularity of the boundary of Ω', which can be hard to be proved also for domains which from the Euclidean point of view are smooth. For instance, it has been proved in [2] that this kind of regularity holds if d is the subelliptic distance induced by a system of Hörmander's vector fields and Ω' is a metric ball, and in [1] the analogous "parabolic version" of this result has been proved. However, there are still more general situations where a geometric regularity property of this kind seems difficult to be proved; hence in this context it is quite natural to apply the ideas of the theory of "Calderón-Zygmund operators in nonhomogeneous spaces". Namely, the results proved in this paper have been recently applied in [3] to prove some new local and global L^p estimates in $\mathbb{R}^n$ for a class of Ornstein-Uhlenbeck degenerate operators for which the natural quasidistance does not allow to check the doubling condition on bounded domains.

Like in the classical case, also in the nondoubling context the theory of Calderón-Zygmund operators proceeds in two steps:

1) the proof of L^2 continuity for an operator with kernel satisfying standard estimates plus some kind of cancellation property;

2) the proof of weak $(1,1)$ continuity for an operator which is continuous on L^2, with kernel satisfying standard estimates.

In the already quoted papers by Nazarov-Treil-Volberg and by Tolsa, the two steps have been performed at different levels of generality: in step (1) the space is usually $\mathbb{R}^n$ (or just $\mathbb{R}^2$) with the Euclidean distance, while for step (2) they consider a separable metric space (X, d). In both cases, the measure usually satisfies the dimensional bound

$$\mu(B_r(x)) \leq cr^n$$

for some positive constants c, n, but can be nondoubling. The cancellation property considered in step (1) is usually very weak, inspired to the theorems $T(1)$, $T(b)$ or variants of them.

In contrast with this setting, in the applications to PDE's that we have in mind it is essential to prove L^2 (and therefore L^p) estimates in a bounded space endowed with a *general quasidistance* and a possibly nondoubling measure (satisfying the above "dimensional bound"); on the other hand, one can usually rely on some strong (and more classical) kind of cancellation property, which should make some arguments much simpler. As we shall see, the cancellation property we assume to prove L^2 continuity amounts to the *boundedness* of $T(1)$ and $T^*(1)$, plus the requirement that $T(1)-T^*(1)$ be Hölder continuous.

Therefore the existing theories, especially in the L^2 case, are not well suited to this situation. Our idea is to get a new proof of L^2 (and L^p) continuity for a Calderón-Zygmund operator, with the aforementioned features (see Theorem 3 in section 2 for the exact statement of our main result). To accomplish this goal, our strategy is the following:

i) thanks to the cancellation property we assume, it is not difficult to prove, also in the nondoubling context and for any quasidistance, that the singular integral operator (or a suitable variant of this) is continuous on Hölder spaces $C^\alpha(X)$, where X is our nonhomogeneous space (section 3);

ii) an abstract argument originally due to Krein [15] then allows to deduce the continuity of the singular integral operator on $L^2(X)$ (section 4); this idea has been applied, in the doubling context, in [26] and [11] but we think that this approach is not widely known.

iii) Once the L^2 continuity is proved, the weak $(1,1)$ continuity result proved by Nazarov-Treil-Volberg can be applied, with some minor adaptation (section 5): one has to check that their arguments actually work for any quasidistance, and not necessarily in a metric space. This immediately implies the desired L^p estimate.

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2. Basic definitions and statement of the main result

Let X be a set. A function $d : X\times X \to \mathbb{R}$ is called a *quasidistance* on X if there exists a constant $c_d \geqslant 1$ such that for any $x,y,z\in X$:

$$d(x,y)\geqslant 0 \text{ and } d(x,y)=0 \Leftrightarrow x=y;$$

$$d(x,y)=d(y,x); \tag{2.1}$$

$$d(x,y)\leqslant c_d\left(d(x,z)+d(z,y)\right). \tag{2.2}$$

We will say that two functions $d,d' : X\times X\to\mathbb{R}$ are *equivalent*, and we will write $d\simeq d'$, if there exist two positive constants c_1,c_2 such that $c_1 d'(x,y)\leqslant d(x,y)\leqslant c_2 d'(x,y)$ for any $x,y\in X$.

We will say that d is a *quasisymmetric quasidistance* if axiom (2.1) is replaced by the weaker

$$d(x,y)\leq c_d d(y,x).$$

If d is a quasisymmetric quasidistance, then

$$d^*(x,y)=d(x,y)+d(y,x)$$

is a quasidistance, equivalent to d; d^* will be called *the symmetrized quasidistance* of d.

For $r > 0$, let $B_r(x) = \{y \in X : d(x, y) < r\}$. These "balls" induce a topology. With respect to this topology, the balls $B_r(x)$ need not be open. Also, note that the property of openness of d-balls may not be conserved replacing d with an equivalent quasisymmetric quasidistance, while the topology itself remains the same.

Definition 1 *We will say that* (X, d, μ, k) *is a* nonhomogeneous space with Calderón-Zygmund kernel k *if:*

1. (X, d) *is a set endowed with a quasisymmetric quasidistance* d*, such that the* d*-balls are open with respect to the topology induced by* d*;*
2. μ *is a positive regular Borel measure on* X *(with respect to the topology induced by d), and there exist two positive constants* A, n *such that:*

$$\mu(B(x, r)) \leq Ar^n \text{ for any } x \in X; \tag{2.3}$$

3. $k(x, y)$ *is a real valued measurable kernel defined in* $X \times X$*, and there exists a positive constant* β *such that:*

$$|k(x, y)| \leq \frac{A}{d(x, y)^n} \quad \text{for any } x, y \in X; \tag{2.4}$$

$$|k(x, y) - k(x_0, y)| \leq A\frac{d(x_0, x)^\beta}{d(x_0, y)^{n+\beta}} \tag{2.5}$$

for any $x_0, x, y \in X$ *with* $d(x_0, y) \geq Ad(x_0, x)$*, where* n, A *are as in* (2.3).

Properties (2.4)–(2.5) are called "standard estimates" for k. We stress the fact that (2.3) weakens the standard doubling condition, required in the definition of space of homogeneous type, given by Coifman-Weiss in [7].

Remark 2 (i) If assumptions (2.3), (2.4), (2.5) hold for d, they still hold with respect to any equivalent d' (with the constant A possibly replaced by a different A', and the same n, β).

(ii) If assumptions (2.3), (2.4), (2.5) hold for some $A > 1$, then they hold for any $A' > A$. We can then assume A large enough, so that the condition $d(x_0, y) \geqslant Ad(x_0, x)$ appearing in (2.5) implies (by the quasitriangle inequality) that $d(x_0, y) \simeq d(x, y)$. We will use systematically this equivalence.

(iii) Conditions (2.3) and (2.4) immediately imply that for any fixed $c_1, c_2 > 0$,

$$\int_{c_1 r < d(x,y) < c_2 r} |k(x,y)|\, d\mu(y) \leqslant A^2 \left(\frac{c_2}{c_1}\right)^n \tag{2.6}$$

for any $r > 0, x \in X$.

(iv) Condition (2.3) implies that the measure μ is nonatomic ($\mu(\{x\}) = 0$ for any $x \in X$); moreover, if X is bounded, then μ is finite.

(v) If X is bounded, as soon as we know that (2.3) holds for any $r \le r_0$, and some fixed r_0, we can conclude it holds for any $r > 0$.

(vi) Since the d-balls are open, the family of balls $\{B(x,r), r > 0, r \in \mathbb{Q}\}$ is a neighborhood basis of x; therefore, X is first countable.

We now state our main result:

Theorem 3 *Let (X, d, μ, k) be a bounded and separable nonhomogeneous space with Calderón-Zygmund kernel k. Also, assume that*

(i) $k^*(x,y) \equiv k(y,x)$ satisfies (2.5);

(ii) there exists a constant $B > 0$ such that

$$\left|\int_{d'(x,y)>r} k(x,y)\, d\mu(y)\right| + \left|\int_{d'(x,y)>r} k^*(x,y)\, d\mu(y)\right| \leqslant B \tag{2.7}$$

for any $r > 0, x \in X$, where d' is any quasisymmetric quasidistance on X, equivalent to d, and fixed once and for all.

(iii) for a.e. $x \in X$, the limits

$$h(x) \equiv \lim_{r\to 0} \int_{d'(x,y)>r} k(x,y)\, d\mu(y); \quad h^*(x) \equiv \lim_{r\to 0} \int_{d'(x,y)>r} k^*(x,y)\, d\mu(y)$$

exist. Moreover

$$h - h^* \in C^\gamma(X)$$

for some $\gamma > 0$ (see Definition 4 below).

Then the operator

$$Tf(x) \equiv \lim_{\varepsilon\to 0} T_\varepsilon f(x) \equiv \lim_{\varepsilon\to 0} \int_{d'(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y)$$

is well defined for any $f \in L^1(X)$, and

$$\|Tf\|_{L^p(X)} \le c_p \|f\|_{L^p(X)} \text{ for any } p \in (1,\infty);$$

moreover, T is weakly $(1,1)$ continuous. The constant c_p only depends on all the constants implicitly involved in the assumptions: $p, c_d, A, B, n, \beta, \operatorname{diam}(X)$ and $|h - h^*|_{C^\gamma(X)}$.

3. Singular integrals on Hölder spaces

In this section we will always assume that (X, d, μ, k) is a nonhomogeneous space with Calderón-Zygmund kernel k (see Definition 1). Any other assumption will be explicitly required when needed.

Definition 4 (Hölder spaces) *For any $\alpha > 0$, $u : X \to \mathbb{R}$, let:*

$$\begin{aligned} |u|_{C^\alpha(X)} &= \sup\left\{\frac{|u(x) - u(y)|}{d(x,y)^\alpha} : x, y \in X, x \neq y\right\} \\ \|u\|_{C^\alpha(X)} &= |u|_{C^\alpha(X)} + \sup_X |u| \\ C^\alpha(X) &= \left\{u : X \to \mathbb{R} : \|u\|_{C^\alpha(X)} < \infty\right\}. \end{aligned}$$

If X is unbounded, we will also define $C_0^\alpha(X)$ as the space of $C^\alpha(X)$ functions with bounded support.

A basic result proved by Macías-Segovia (see [16, Theor. 2]) states that:

Proposition 5 *Let d be any quasidistance on a set X. Then there exists another quasidistance $\widetilde{d}$ on X, equivalent to d, a constant $c > 0$ and an exponent $\alpha_0 \in (0, 1]$ such that for every $r > 0, x, y, z \in X$ with $\widetilde{d}(x, z) < r$, $\widetilde{d}(y, z) < r$,*

$$\left|\widetilde{d}(x,z) - \widetilde{d}(y,z)\right| \leqslant c\widetilde{d}(x,y)^{\alpha_0} r^{1-\alpha_0} \tag{3.1}$$

Remark 6 If d is a quasisymmetric quasidistance, this proposition can be applied to d^*, and says that the function $x \longmapsto \widetilde{d}(x, z)$ (for z fixed) is locally Hölder continuous (with respect to $\widetilde{d}$ and therefore also to d^* and d). This allows to prove that if μ is any positive regular Borel measure defined on X, then

$$C_0^\alpha(X) \text{ (or } C^\alpha(X) \text{ if } X \text{ is bounded) is dense in } L^p(X, \mu)$$

for any $p \in [1, \infty)$ and any $\alpha \leqslant \alpha_0$ (with α_0 as in (3.1)). A detailed proof of this fact can be found in [26, Lemma 2.3].

Let us turn now to the study of the action of singular integrals on Hölder spaces $C^\alpha(X)$. This study is performed both for its own interest, and for its application to the proof of L^2 continuity (which will be performed in the next section).

Theorem 7 *Let (X, d, μ, k) be a nonhomogeneous space with Calderón-Zygmund kernel k. Moreover, assume that either for some $R > 0$ the kernel $k(x, y)$ vanishes for $d(x, y) > R$, or X is bounded with* $\operatorname{diam}(X) = R$.

(a) For any $f \in C^{\alpha}(X)$, let

$$\widehat{T}f(x) = \int k(x,y)\,[f(y) - f(x)]\,d\mu(y)$$

$$T_{\varepsilon}f(x) = \int_{d'(x,y)>\varepsilon} k(x,y)\,f(y)\,d\mu(y) \tag{3.2}$$

where d' is any quasisymmetric quasidistance on X, equivalent to d, and fixed once and for all. Then the integrals defining $\widehat{T}f(x)$, $T_{\varepsilon}f(x)$ are absolutely convergent for any $f \in C^{\alpha}(X), \alpha > 0, x \in X, \varepsilon > 0$.

(b) Assume there exists a constant $B > 0$ such that

$$\left| \int_{d'(x,y)>r} k(x,y)\,d\mu(y) \right| \leqslant B \tag{3.3}$$

for any $r > 0, x \in X$. Then the operator $\widehat{T}$ is continuous on $C^{\alpha}(X)$ for any $\alpha < \beta$ (β being the exponent in (2.5)) more precisely:

$$\left|\widehat{T}f\right|_{C^{\alpha}(X)} \leqslant c\,|f|_{C^{\alpha}(X)}\,, \qquad \left\|\widehat{T}f\right\|_{\infty} \leqslant cR^{\alpha}\,|f|_{C^{\alpha}(X)}$$

where R is the number appearing in the assumption, and c depends on $A, B, c_d, n, \alpha, \beta$.

(c) Assume now that X is bounded and "$T(1) \in C^{\gamma}(X)$", that is: for every $x \in X$ there exists

$$h(x) \equiv \lim_{\varepsilon \to 0} \int_{d'(x,y)>\varepsilon} k(x,y)\,d\mu(y)$$

where d' is the same quasidistance appearing in (3.2), and

$$h \in C^{\gamma}(X) \text{ for some } \gamma > 0. \tag{3.4}$$

Then for every α such that $\alpha < \beta, \alpha \leq \gamma$, every $f \in C^{\alpha}(X)$ and $x \in X$ the following limit exists:

$$Tf(x) \equiv P.V. \int_X k(x,y)\,f(y)\,d\mu(y) = \lim_{\varepsilon \to 0} T_{\varepsilon}f(x)$$

and the operator T is continuous on $C^{\alpha}(X)$; more precisely:

$$\|Tf\|_{C^{\alpha}(X)} \leqslant c\,\|f\|_{C^{\alpha}(X)} \tag{3.5}$$

where c depends on A, B, c_d, n, α, β, γ, $\|h\|_{C^{\gamma}(X)}$ and $R = \mathrm{diam}(X)$. If X is unbounded (but $k(x,y)$ vanishes for $d(x,y) > R$), the conclusion remains true provided $\gamma = \alpha$.

Remark 8 (i) Note that (3.4) is obviously necessary, in order for T to be continuous on $C^{\alpha}(X)$, since the constant 1 belongs to $C^{\alpha}(X)$. However, (3.4) is not required to prove the $C^{\alpha}(X)$ continuity of $\widehat{T}$. Point (c) in the above theorem is stated for its own interest, but will not be used to prove the L^2 continuity of T.

(ii) In some applications of the abstract theory of singular integrals it is useful to switch from one quasidistance to another one, having different good properties. In particular, in the definition of principal value of a singular integral, the small region around the pole which is removed and shrunk needs not to be a ball with respect to the original quasidistance. Note that, unlike all the other assumptions on the kernel, condition (3.4) is not generally preserved replacing the quasidistance d' with an equivalent d''. Hence, once we have checked all the other assumptions, we can look for a good equivalent quasidistance which satisfies also (3.4). An instance where this fact is useful occurs in [1] (in a doubling context).

A result similar to Theorem 7 has been proved in [26] (see also [1]) assuming the doubling condition, the symmetry of the quasidistance, and standard estimates on the kernel involving the function $|B(x, d(x,y))|$ instead of $d(x,y)^n$ (moreover, in [26] a slightly different definition of C^{α} is given).

To prove the theorem, let us start noting that assumption (2.3) implies, by a standard computation, the following:

Lemma 9 *For any $\beta > 0$ there exists a constant $c = \frac{A2^n}{1-2^{-\beta}}$ such that*

$$\int_{d(x,y)<r} \frac{d(x,y)^{\beta}}{d(x,y)^n} d\mu(y) \leqslant cr^{\beta} \ ;$$

$$\int_{d(x,y)>r} \frac{d(x,y)^{-\beta}}{d(x,y)^n} d\mu(y) \leqslant cr^{-\beta}.$$

Proof.

$$\int_{d(x,y)<r} \frac{d(x,y)^{\beta}}{d(x,y)^n} d\mu(y) = \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+1}} \leq d(x,y) < \frac{r}{2^k}} \frac{d(x,y)^{\beta}}{d(x,y)^n} d\mu(y)$$

$$\leq \sum_{k=0}^{\infty} \left(\frac{r}{2^k}\right)^{\beta} \cdot \frac{\mu\left(B\left(x, \frac{r}{2^k}\right)\right)}{\left(\frac{r}{2^{k+1}}\right)^n}$$

$$\overset{\text{by (2.3)}}{\leq} \sum_{k=0}^{\infty} \left(\frac{r}{2^k}\right)^{\beta} \cdot \frac{A\left(\frac{r}{2^k}\right)^n}{\left(\frac{r}{2^{k+1}}\right)^n} = 2^n A r^{\beta} \sum_{k=0}^{\infty} \frac{1}{2^{k\beta}} = \frac{A2^n}{1-2^{-\beta}} \cdot r^{\beta}.$$

Analogously one proves the other inequality. ■

Let us now come to the

Proof of Theorem 7. (a) By (2.4), for any $f \in C^{\alpha}(X)$ we have, for the number $R > 0$ appearing in the assumption,

$$\int_X \left|k(x,y)[f(y)-f(x)]\right| d\mu(y) \leq A\,|f|_{\alpha} \int_{d(x,y)<R} \frac{d(x,y)^{\alpha}}{d(x,y)^{n}}\, d\mu(y) \leq A\,|f|_{\alpha}\, cR^{\alpha}$$

by Lemma 9. Analogously, by (2.4) and (2.3),

$$\int_{d'(x,y)>\varepsilon} \left|k(x,y)f(y)\right| d\mu(y) \leq A\|f\|_{\infty} \int_{\varepsilon<d'(x,y)<cR} \frac{d\mu(y)}{d(x,y)^{n}} \leq A^{2}\|f\|_{\infty}\, \frac{c}{\varepsilon^{n}}\, R^{n}.$$

(b) Let us write:

$$\begin{aligned}
&\widehat{T}f(x) - \widehat{T}f(x_0) = \\
&= \int_X k(x,y)\,[f(y)-f(x)]\,d\mu(y) - \int_X k(x_0,y)\,[f(y)-f(x_0)]\,d\mu(y) \\
&= \int_{d(x_0,y)\geqslant Ad(x_0,x)} \left\{k(x,y)\,[f(y)-f(x)] - k(x_0,y)\,[f(y)-f(x_0)]\right\} d\mu(y) \\
&\quad + \int_{d(x_0,y)<Ad(x_0,x)} \left\{k(x,y)\,[f(y)-f(x)] - k(x_0,y)\,[f(y)-f(x_0)]\right\} d\mu(y) \\
&\equiv A_1 + A_2.
\end{aligned}$$

$$\begin{aligned}
A_1 &= \int_{d(x_0,y)\geqslant Ad(x_0,x)} \left\{[k(x,y)-k(x_0,y)]\,[f(y)-f(x_0)]\right\} d\mu(y) \\
&\quad + [f(x_0)-f(x)] \int_{d(x_0,y)\geqslant Ad(x_0,x)} k(x,y)\,d\mu(y) \\
&\equiv A_{11} + A_{12}.
\end{aligned}$$

$$\begin{aligned}
|A_{11}| &\leqslant \int_{d(x_0,y)\geqslant Ad(x_0,x)} A\frac{d(x_0,x)^{\beta}}{d(x_0,y)^{n+\beta}}\,|f|_{\alpha}\, d(x_0,y)^{\alpha}\, d\mu(y) \\
&= c\,|f|_{\alpha}\, d(x_0,x)^{\beta} \int_{d(x_0,y)\geqslant Ad(x_0,x)} \frac{1}{d(x_0,y)^{n+\beta-\alpha}}\, d\mu(y) \\
&\overset{\text{since } \alpha<\beta, \text{ by Lemma 9}}{\leqslant} c\,|f|_{\alpha}\, d(x_0,x)^{\beta}\, d(x_0,x)^{\alpha-\beta} = c\,|f|_{\alpha}\, d(x_0,x)^{\alpha}.
\end{aligned}$$

As to the second term,

$$|A_{12}| \leqslant |f|_{\alpha}\, d(x_0,x)^{\alpha} \left| \int_{d(x_0,y)\geqslant Ad(x_0,x)} k(x,y)\,d\mu(y) \right|.$$

By Remark 2, $d(x_0,y) \geqslant Ad(x_0,x) \Rightarrow d(x,y) \geqslant cd(x_0,x)$ for some $c>0$. Then

$$\int_{d(x_0,y)\geqslant Ad(x_0,x)} k(x,y)\,d\mu(y) =$$
$$= \int_{d(x,y)\geqslant cd(x_0,x)} k(x,y)\,d\mu(y) - \int_{d(x_0,y)<Ad(x_0,x),\ d(x,y)\geqslant cd(x_0,x)} k(x,y)\,d\mu(y)$$

and, by (3.3) and (2.6),

$$\begin{aligned}|A_{12}| &\leqslant |f|_\alpha\, d(x_0,x)^\alpha \left\{\left|\int_{d(x,y)\geqslant cd(x_0,x)} k(x,y)\,d\mu(y)\right| + \right.\\ &\qquad \left. + \int_{d(x_0,y)<Ad(x_0,x),\ d(x,y)\geqslant cd(x_0,x)} |k(x,y)|\,d\mu(y)\right\}\\ &\leqslant |f|_\alpha\, d(x_0,x)^\alpha \left\{c + \int_{cd(x_0,x)\leqslant d(x,y)\leqslant c_1 d(x_0,x)} |k(x,y)|\,d\mu(y)\right\}\\ &\leqslant c\,|f|_\alpha\, d(x_0,x)^\alpha .\end{aligned}$$

$$\begin{aligned}|A_2| \leqslant &\int_{d(x_0,y)<Ad(x_0,x)} |k(x,y)|\,|f(y)-f(x)|\,d\mu(y) + \\ &+ \int_{d(x_0,y)<Ad(x_0,x)} |k(x_0,y)|\,|f(y)-f(x_0)|\,d\mu(y) \equiv A_{21}+A_{22}.\end{aligned}$$

$$|A_{22}| \leqslant A\,|f|_\alpha \int_{d(x_0,y)<cd(x_0,x)} \frac{d(x_0,y)^\alpha}{d(x_0,y)^n}\,d\mu(y) \overset{\text{by Lemma 9}}{\leqslant} c\,|f|_\alpha\, d(x_0,x)^\alpha .$$

Analogously, since $d(x_0,y) < Ad(x_0,x) \Longrightarrow d(x,y) < cd(x_0,x)$

$$|A_{21}| \leqslant \int_{d(x,y)<cd(x_0,x)} |k(x,y)|\,|f(y)-f(x)|\,d\mu(y) \leqslant c\,|f|_\alpha\, d(x,x_0)^\alpha .$$

We have therefore proved that

$$\left|\widehat{T}f(x) - \widehat{T}f(x_0)\right| \leqslant c\,|f|_\alpha\, d(x,x_0)^\alpha .$$

On the other hand, we have already proved in point (a) that

$$\left|\widehat{T}f(x)\right| \leqslant c\,|f|_\alpha\, R^\alpha$$

for some fixed $R>0$, and this concludes the proof of (b).

(c) Since

$$\begin{aligned} T_\varepsilon f(x) &= \int_{d'(x,y)>\varepsilon} k(x,y) f(y) \, d\mu(y) = \\ &= \int_{d'(x,y)>\varepsilon} k(x,y) [f(y) - f(x)] \, d\mu(y) + f(x) \int_{d'(x,y)>\varepsilon} k(x,y) \, d\mu(y) \end{aligned}$$

we have, under the assumptions of (c), that

$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x) = \widehat{T} f(x) + h(x) f(x).$$

Then, by what we have already proved in (b), if we show that

$$\|hf\|_\alpha \le c \|f\|_\alpha \tag{3.6}$$

then we will conclude

$$\|Tf\|_\alpha \le c \|f\|_\alpha. \tag{3.7}$$

To show (3.6): if X is bounded, for any $\alpha \le \gamma$ (recalling that $h \in C^\gamma(X)$)

$$\begin{aligned} |hf|_\alpha &\le |h|_\alpha \|f\|_\infty + \|h\|_\infty |f|_\alpha \le |h|_\gamma R^{\gamma-\alpha} \|f\|_\infty + \|h\|_\infty |f|_\alpha \\ &\le \left(|h|_\gamma R^{\gamma-\alpha} + \|h\|_\infty \right) \|f\|_\alpha \end{aligned}$$

hence

$$\|hf\|_\alpha \le \left(|h|_\gamma R^{\gamma-\alpha} + 2 \|h\|_\infty \right) \|f\|_\alpha.$$

If X is unbounded but $h \in C^\alpha(X)$,

$$|hf|_\alpha \le |h|_\alpha \|f\|_\infty + \|h\|_\infty |f|_\alpha \le 2 \|h\|_\alpha \|f\|_\alpha$$

and the same conclusion holds. Note that (3.7) holds with c depending on $A, B, n, c_d, \alpha, \gamma, R, \|h\|_\gamma$. ■

4. L^2 continuity via continuity on C^α

The link between Theorem 7 and L^2 continuity relies in the following abstract theorem due to Krein [15]. Since its proof is really short and elegant, we will reproduce it here below, for convenience of the reader, in a form similar to that which can be found in [11].

Theorem 10 *Let H be a (real, for simplicity) Hilbert space and Y a linear normed space for which the inclusion $i : Y \to H$ is well defined, continuous and with dense range. Let $T, T^* : Y \to Y$ be two linear continuous operators on Y such that*

$$(Tx, y) = (x, T^*y) \text{ for any } x, y \in Y,$$

where $(,)$ denotes the scalar product in H. Then T and T^* extend to linear continuous operators on H, with

$$\|T\|_{H\to H}\ , \|T^*\|_{H\to H} \le \|T\|_{Y\to Y}^{1/2} \cdot \|T^*\|_{Y\to Y}^{1/2} .$$

Proof. For any $x \in Y$

$$\|Tx\|_H^2 = (Tx, Tx) = (x, T^*Tx) \le \|x\|_H \|T^*Tx\|_H$$

hence

$$\|Tx\|_H \le \|x\|_H^{1/2} \|T^*Tx\|_H^{1/2} . \tag{4.1}$$

We now apply (4.1) with T replaced by T^*T. Since $(T^*T)^* = T^*T$ we get

$$\|T^*Tx\|_H \le \|x\|_H^{1/2} \left\|(T^*T)^2 x\right\|_H^{1/2}$$

which combined with (4.1) gives

$$\|Tx\|_H \le \|x\|_H^{1/2+1/4} \left\|(T^*T)^2 x\right\|_H^{1/4} . \tag{4.2}$$

We then apply (4.2) with T replaced by $(T^*T)^2$, and so on; iteration yields

$$\begin{aligned}
\|Tx\|_H &\le \|x\|_H^{1/2+1/4+\cdots+1/2^j} \left\|(T^*T)^{2^{j-1}} x\right\|_H^{1/2^j} \\
&\le \|x\|_H^{1/2+1/4+\cdots+1/2^j} \|i\|_{Y\to H}^{1/2^j} \left\|(T^*T)^{2^{j-1}} x\right\|_Y^{1/2^j} \\
&\le \|x\|_H^{1/2+1/4+\cdots+1/2^j} \|i\|_{Y\to H}^{1/2^j} \left\|(T^*T)^{2^{j-1}}\right\|_{Y\to Y}^{1/2^j} \|x\|_Y^{1/2^j} \\
&\le \|x\|_H^{1/2+1/4+\cdots+1/2^j} \|i\|_{Y\to H}^{1/2^j} \|T^*T\|_{Y\to Y}^{2^{j-1}/2^j} \|x\|_Y^{1/2^j} .
\end{aligned}$$

For fixed x and $j \to +\infty$ we get

$$\|Tx\|_H \le \|x\|_H \|T^*T\|_{Y\to Y}^{1/2} \le \|x\|_H \|T^*\|_{Y\to Y}^{1/2} \|T\|_{Y\to Y}^{1/2}$$

which shows that T can be extended continuously to H, with

$$\|T\|_{H\to H} \le \|T\|_{Y\to Y}^{1/2} \cdot \|T^*\|_{Y\to Y}^{1/2} .$$

An analogous reasoning shows the same holds for T^*. ∎

We want to apply the above theorem to:

$$H = L^2(X); Y = C^\alpha(X)$$

with X bounded and T our Calderón-Zygmund operator:

Theorem 11 *Let (X, d, μ, k) be a bounded nonhomogeneous space with Calderón-Zygmund kernel k. Also, assume that*

(i) $k^*(x,y) \equiv k(y,x)$ satisfies (2.5);

(ii) condition (2.7) holds (with respect to possibly different, but equivalent, quasisymmetric quasidistances);

(iii) for a.e. $x \in X$, the limits

$$h(x) \equiv \lim_{r\to 0} \int_{d'(x,y)>r} k(x,y)\, d\mu(y); \quad h^*(x) \equiv \lim_{r\to 0} \int_{d'(x,y)>r} k^*(x,y)\, d\mu(y)$$

exist. Moreover

$$h - h^* \in C^\gamma(X)$$

for some $\gamma > 0$.

Then the operators

$$Tf(x) \equiv \lim_{\varepsilon\to 0} T_\varepsilon f(x) \equiv \lim_{\varepsilon\to 0} \int_{d'(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y)$$

and T^* (analogously defined by k^*) can be extended to continuous operators on $L^2(X)$:

$$\|Tf\|_{L^2(X)} + \|T^* f\|_{L^2(X)} \le c\, \|f\|_{L^2(X)}$$

with c depending on the constants A, B, c_d, n, α, β, $|h - h^*|_{C^\gamma(X)}$ and $R = \operatorname{diam} X$, involved in our assumptions.

Remark 12 Even in some *doubling* contexts, the present proof of this theorem can be seen as an easier way to get L^2 continuity of singular integrals. Compare, for instance, with the proof of L^2 continuity given in [14] or [22] for singular integrals in homogeneous groups, using the almost orthogonality principle.

Proof. Let us write, for any $f \in C^\alpha$,

$$\begin{aligned} T_\varepsilon f(x) &= \int_{d'(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y) \\ &= \int_{d'(x,y)>\varepsilon} k(x,y)\,[f(y) - f(x)]\, d\mu(y) + f(x) \int_{d'(x,y)>\varepsilon} k(x,y)\, d\mu(y); \end{aligned}$$

hence, by assumption (iii), there exists

$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x) = \widehat{T} f(x) + f(x)\, h(x)$$

with $\widehat{T}$ as in Theorem 7 and $h \in L^\infty(X)$ by assumption (ii). Hence

$$\|Tf\|_{L^2(X)} \leq \left\|\widehat{T} f\right\|_{L^2(X)} + \|h\|_{L^\infty(X)} \|f\|_{L^2(X)} . \tag{4.3}$$

On the other hand, under our assumptions we can apply Theorem 7 to the operator $\widehat{T}$, concluding its continuity on $C^\alpha(X)$ for α small enough. In order to apply Theorem 10, we also need to check $C^\alpha(X)$ continuity of $\widehat{T}^*$. An easy computation shows that

$$\widehat{T}^* f(x) = \widehat{T^*} f(x) + f(x)\left[h^*(x) - h(x)\right].$$

Since k^* satisfies the same assumptions as k, again Theorem 7 implies that $\widehat{T^*}$ is $C^\alpha(X)$ continuous for α small enough and, since $h - h^* \in C^\gamma(X)$ for some positive γ, the same is true for $\widehat{T}^*$. Moreover, for α small enough the inclusion of $C^\alpha(X)$ in $L^2(X)$ is continuous and with dense range (see Remark 6), hence by Theorem 10 the operators $\widehat{T}$ and $\widehat{T}^*$ are continuous on $L^2(X)$, and by (4.3) the same is true for T, and therefore for T^*. ∎

5. L^p continuity

Once we have established the L^2 continuity of T and T^*, we can apply a general result proved by Nazarov-Treil-Volberg in the context of nonhomogeneous space, and deduce the weak $(1,1)$ continuity of T, and therefore, via interpolation and duality, the continuity on L^p for $1 < p < \infty$, that is Theorem 3.

Let us first state the main result proved in [18]:

Theorem 13 *Let (X, d) be a separable metric space, let μ be a nonnegative Borel measure on X, and $k : X \times X \to \mathbb{C}$ be a kernel such that* (2.3), (2.4) *and* (2.5) *hold, and* (2.5) *is satisfied also by k^*. Assume that $T : L^2(X) \to L^2(X)$ is a linear continuous operator such that*

$$Tf(x) = \int_X k(x, y)\, f(y)\, d\mu(y) \;\; \textit{for any } f \in L^2(X), x \notin \textit{sprtf}.$$

Then for any $p \in (1, \infty)$, the operator T is bounded on $L^p(X)$, in the sense that

$$\|Tf\|_{L^p} \leq C \|f\|_{L^p} \;\; \textit{for any } f \in L^2(X) \cap L^p(X).$$

Moreover, T is weakly $(1,1)$ continuous.

We are going to prove that the same conclusions of the above theorem hold under the assumptions of our Theorem 3 which, in turn, are exactly the same of Theorem 11, plus the separability of (X, d), a technical assumption which we will need.

Proof of Theorem 3. Revising carefully the proof of Theorem 13 given in [18, pp. 469-479], one can check that it actually holds just assuming d a quasisymmetric quasidistance. In the following we shall point out some remarks which should convince of this fact anybody who has read the aforementioned proof.

1. First of all, it is not restrictive to assume that d is symmetric. Otherwise, we can consider the symmetrized quasidistance d^* and note that both the assumptions and the conclusion of the theorem are preserved replacing d with an equivalent d^*.

2. For any quasidistance d, the following "engulfing property" holds: there exists $C > 1$ such that

$$B(x, r) \cap B(y, r) \neq \emptyset \Longrightarrow B(y, r) \subset B(x, Cr)$$

for any $x, y \in X, r > 0$. The constant C depends on the constant c_d in the quasitriangle inequality of d. If d is a distance, then $C = 3$; if d is a quasidistance, all the arguments in [18, pp. 469-479] (Vitali covering lemma, the definition of the maximal function $\widetilde{M}$, and so on) must (and actually can) be rewritten with the constant 3 replaced by this C.

3. Whitney's decomposition is a key point in the proof in [18], where the assumption that (X, d) is a separable *metric* space is explicitly used. So, let us show how the argument in [18, p. 478] can be modified in our case. We shall make use of the Withney decomposition proved for a quasidistance in [22, Lemma 2, pp. 15-16].

Lemma 14 *Let (X, d) be as in Definition 1, and let X be separable. Given an open set $G \subset X$ with nonvoid complement G^c, there exists a sequence of balls $\{B_i\}$ such that*

(a) the B_i's are pairwise disjoint;

(b) $\cup_i B_i^* = G$, where $B_i^* = B(x_i, c^* r_i)$ if $B_i = B(x_i, r_i)$;

(c) $B_i^{**} \cap G^c \neq \emptyset$ for each i, where $B_i^* = B(x_i, c^{**} r_i)$ if $B_i = B(x_i, r_i)$. Here c^*, c^{**} are constants > 1 depending on the quasidistance d. More precisely,

$$B_i^* = B\left(x_i, \frac{\delta(x_i)}{2}\right)$$

where $\delta(x_i) =$dist(x_i, G^c).

Moreover, there exists a collection $\{Q_i\}$ *of Borel sets ("cubes") such that:*

(a') the Q_i*'s are pairwise disjoint;*

(b') $B_i \subset Q_i \subset B_i^*$;

(c') $\cup_i Q_i = G$.

By the way, we note that the separability of X is not explicitly asked in [22] because there the context is that of subsets of $\mathbb{R}^n$. However all the arguments hold in an abstract context; the assumption of separability implies that (X, d) has a countable basis (since (X, d) is first-countable, see Remark 2, (vi)), and therefore, by Lindelöf's Theorem, allows to extract a countable covering from any open covering of G, a fact which is implicitly used in the proof of this Lemma in [22].

With the notation of [18, p. 14], we now apply this Lemma to the open set

$$G = \{x \in X : |f(x)| > t\}$$

and set:

$$\begin{aligned} f_i &= f \cdot \chi_{Q_i}; \\ \alpha_i &= \int_X f_i d\mu = \int_{Q_i} f d\mu; \\ x_i &= \text{the center of } B_i; \\ \nu_N &= \sum_{i=1}^N \alpha_i \delta_{x_i}; \\ f^N &= \sum_{i=1}^N f_i \end{aligned}$$

(where δ_{x_i} stands for the Dirac measure concentrated at x_i). Then we can bound:

$$\int_{X \setminus G} \left| T f^N - T \nu_N \right| d\mu \leq \sum_{i=1}^N \int_{X \setminus G} \left| T \left[f_i d\mu - \alpha_i \delta_{x_i} \right] \right| d\mu.$$

Next, we note that the measure $\mu_i = f_i d\mu - \alpha_i \delta_{x_i}$ is supported in $Q_i \subset B_i^* = B\big(x_i, \frac{\delta(x_i)}{2}\big)$, $\mu_i(X) = 0$, and

$$\sum_{i=1}^N \int_{X \setminus G} |T\mu_i| \, d\mu \leq \sum_{i=1}^N \int_{X \setminus B(x_i, \delta(x_i))} |T\mu_i| \, d\mu.$$

Then we can apply Lemma 3.4 in [18, p. 8] to assure that

$$\int_{X\setminus B(x_i,\delta(x_i))} |T\mu_i|\, d\mu \le c\,\|\mu_i\| = c\,|\alpha_i|\,,$$

so that

$$\int_{X\setminus G} \left|Tf^N - T\nu_N\right| d\mu \le c\sum_{i=1}^{N} |\alpha_i| \le c\,\|f\|_1\,.$$

This is the same conclusion proved in [18] under the assumption that (X, d) is a separable metric space. Hence the reasoning in [18] can be repeated for any quasidistance, and the proof of Theorem 3 is completed.

Note added in proof. After the acceptance of this paper, Eduardo Gatto has pointed out to my attention his recent paper [13], which contains results partially overlapping with those in the present paper. I thank him for this communication.

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