Riesz transforms on forms and L^p-Hodge decomposition on complete Riemannian manifolds

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Abstract

In this paper we prove the Strong L^p-stability of the heat semigroup generated by the Hodge Laplacian on complete Riemannian manifolds with non-negative Weitzenböck curvature. Based on a probabilistic representation formula, we obtain an explicit upper bound of the L^p-norm of the Riesz transforms on forms on complete Riemannian manifolds with suitable curvature conditions. Moreover, we establish the Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvature.

1. Introduction

1.1. Background

It is well-known that the Riesz transforms R_j on $\mathbb{R}^n$, defined by the principal value of the singular integrals

$$R_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} \, dy, \quad j = 1, \ldots, n,$$

are weak $(1,1)$ and are bounded in $L^p(\mathbb{R}^n, dx)$ for all $p > 1$, see e.g. E. M. Stein [64]. In recent years, there has been considerable interest in finding the exact value or obtaining a good estimate of the L^p-norm of the Riesz

transforms. In 1972, Pichorides [57] proved that the L^p-norm of the Hilbert transform on the real line

$$Hf(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy$$

is given by

$$\|H\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall\, p > 1.$$

Here and throughout of this paper, we denote

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In 1996, Iwaniec and Martin [34] proved that the L^p-norm of the Riesz transforms R_j is also given by

$$\tag{1.1} \|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad j = 1, \ldots, n.$$

In [10], Bañuelos and Wang gave an alternative proof of (1.1) and proved that for all $p > 1$ the L^p-norm of the vector Riesz transform $\nabla(-\Delta)^{-1/2} = (R_1, \ldots, R_n)$ has an explicit and dimension-free upper bound

$$\tag{1.2} \left\|\nabla(-\Delta)^{-1/2}\right\|_{p,p} \leq 2(p^* - 1).$$

One of the motivations of the above study can be seen in Donaldson and Sullivan [20] and in Iwaniec and Martin [33, 34], where it has been pointed out that the knowledge of the exact value or a good estimate of the L^p-norm of the Riesz transforms can lead important applications in the study of quasi-conformal mappings and related nonlinear geometric PDEs as well as in the L^p-Hodge decomposition theory. In their 1993 Acta Math paper [33], using the L^p-boundedness of the Riesz transforms, Iwaniec and Martin proved the following version of the L^p-Hodge decomposition theorem on $\mathbb{R}^n$.

Theorem 1.1 *Let* $\omega \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n)$, $1 < p < \infty$, *and* $k = 1, \ldots, n-1$. *Then there is a* $(k-1)$-*form* α *and* $(k+1)$-*form* β *such that*

$$\omega = d\alpha + d^*\beta$$

and

$$d\alpha,\ d^*\beta \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n),$$

where d^* *denotes the* L^2-*adjoint of* d *with respect to the Lebesgue measure on* $\mathbb{R}^n$.

Moreover, $d\alpha$ and $d^*\beta$ are unique,

$$\alpha \in \operatorname{Ker} d^* \cap L_1^p(\mathbb{R}^n, \Lambda^{k-1}\mathbb{R}^n), \quad \beta \in \operatorname{Ker} d \cap L_1^p(\mathbb{R}^n, \Lambda^{k+1}\mathbb{R}^n),$$

and there exists a constant $C_p(k,n) > 0$ such that

$$\|\alpha\|_{L_1^p(\mathbb{R}^n)} + \|\beta\|_{L_1^p(\mathbb{R}^n)} \leq C_p(k,n)\|\omega\|_p,$$

where $L_1^p(\mathbb{R}^n, \Lambda^k\mathbb{R}^n)$ denotes the homogeneous Sobolev space of k-forms on $\mathbb{R}^n$ whose first order derivatives are L^p-integrable with respect to the Lebesgue measure on $\mathbb{R}^n$, on which $\|\omega\|_{L_1^p(\mathbb{R}^n)} := \|d\omega\|_p + \|d^*\omega\|_p$.

It might be interesting to point out that, in his 1858 *Crelle*'s paper [32], in order to solve boundary value problems arising from the study of hydrodynamic systems, Helmholtz formulated a result on the splitting of vector fields into vortices and gradients, which can be understood in a suitable form of what is now called the "Hodge decomposition". In his fundamental paper [38], Leray used the L^2-orthogonal projection $P = \operatorname{Id} + \nabla(-\Delta)^{-1}\operatorname{div}$ to study the Navier-Stokes equations. So far, it has been well-known that the L^p-boundedness of the Riesz transforms and the L^p-Hodge decomposition theorem have important applications in the study of elliptic and parabolic PDEs [26], the Navier-Stokes equations [47], boundary valued problems [55], quasi-conformal mappings and related nonlinear geometric PDEs [20, 33, 34, 58] as well as in stochastic differential equations [67].

1.2. Motivation

To what extent can the classical analysis on Euclidean space be extended to complete Riemannian manifolds? This is an important issue in analysis on complete non-compact Riemannian manifolds. Let (M, g) be a complete non-compact Riemannian manifold, $n = \dim M$, v the Riemannian volume measure, i.e., $dv(x) = \sqrt{\det g(x)}dx$. For any $k = 1, \ldots, n$, we use $C_0^\infty(\Lambda^k T^*M)$ to denote $C_0^\infty(M, \Lambda^k T^*M)$. Let d_k be the exterior differential on k-forms, d_k^* the formal L^2-adjoint of d_k with respect to ν. The Hodge Laplacian on k-forms is defined by

$$\square_k = d_{k-1}d_{k-1}^* + d_k^*d_k.$$

Inspired by the Witten deformation of the Morse theory [69], we consider more general weighted Hodge Laplacians on complete Riemannian manifolds. More precisely, let M be a complete Riemannian manifold equipped with a weighted volume measure

$$d\mu = e^{-\phi}dv,$$

where $\phi \in C^2(M)$.

Let $d^*_{\phi,k}$ the formal L^2-adjoint of d_k with respect to μ, i.e., for all $\alpha \in C^\infty_0(\Lambda^k T^*M)$ and $\beta \in C^\infty_0(\Lambda^{k+1} T^*M)$, we have

$$\int_M <d\alpha, \beta> d\mu = \int_M <\alpha, d^*_\phi \beta> d\mu.$$

The weighted Hodge Laplacian on k-forms with respect to the weighted volume measure μ, also called the Witten Laplacian on k-forms with respect to μ, is defined by

$$\square_{\phi,k} = d_{k-1} d^*_{\phi,k-1} + d^*_{\phi,k} d_k.$$

When $\phi \equiv 0$, we have $d^*_{0,k} = d^*_k$ and $\square_{0,k} = \square_k$, $k = 0, 1, \ldots, n$. For all $p > 1$, let $L^p(\Lambda^k T^*M, \mu)$ be the completion of $C^\infty_0(\Lambda^k T^*M)$ with respect to the L^p-norm $\|\cdot\|_p$ defined by

$$\|\omega\|^p_p := \int_M |\omega(x)|^p d\mu(x).$$

By [15, 65], it is well-known that $\square_k$ is essentially self-adjoint on $L^2(\Lambda^k T^*M, v)$. Similarly, we can prove that $\square_{\phi,k}$ is essentially self-adjoint on $L^2(\Lambda^k T^*M, \mu)$.

The purpose of this paper is to study the following fundamental problems.

Problem 1.2 *Under which conditions on a complete non-compact Riemannian manifold M and $\phi \in C^2(M)$, the Riesz transforms $d\square^{-1/2}_{\phi,k}$ and $d^*_\phi \square^{-1/2}_{\phi,k}$ are bounded in L^p with respect to the weighted measure $d\mu = e^{-\phi} dv$ for some or all $p > 1$?*

Problem 1.3 *Under which conditions on a complete non-compact Riemannian manifold M and $\phi \in C^2(M)$, the Weak L^p-Hodge decomposition theorem holds for some or all $p > 1$?*

When $\phi \equiv 0$, Problem 1.2 was originally raised by Strichartz in his 1983 celebrated paper [65] and sometimes is called the Strichartz problem by people working in harmonic analysis on complete non-compact manifolds. In [65], Strichartz has implicitly pointed out the deep relationship between the above two problems. When $p = 2$, using Gaffney's integration by parts formula, it is well-known that the Riesz transforms $d\square^{-1/2}_{\phi,k}$ and $d^*_\phi \square^{-1/2}_{\phi,k}$ are always bounded in L^2, and the Weak L^2-Hodge decomposition theorem is always true on all complete Riemannian manifolds with C^2-weighted volume measures, see e.g. [19, 65, 12]. However, for $p \neq 2$, the situation is very complicated. Since 1983, many people have studied the above problems on complete non-compact Riemannian manifolds with various geometric or analytic conditions, see e.g. [65, 48, 5, 62, 71, 72, 16, 17, 2, 14, 49, 41, 42] and reference therein.

1.3. Previous results

We now describe some results of Bakry [5]. Using a martingale approach to the Littlewood-Paley-Stein inequalities, Bakry [5] proved that, for any diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ on a complete Riemannian manifold M, where $\phi \in C^2(M)$, if the Ricci curvature associated with L is bounded below by $-a$, i.e., $Ric(L) = Ric + \nabla^2\phi \geq -a$, where $a \geq 0$ is a non-negative constant, then for all $p > 1$ the Riesz transform $\nabla(a - L)^{-1/2}$ is bounded in L^p with respect to the weighted volume measure $d\mu = e^{-\phi}dv$ and its L^p-norm is bounded above by a universal constant depending only on p. In particular, if $Ric(L) \geq 0$, then the Riesz transform $\nabla(-L)^{-1/2}$ is bounded in L^p for all $p > 1$ and its L^p-norm is bounded above by a dimensional free constant. This recaptures an earlier famous result due to P. A. Meyer [53] on the L^p-boundedness of the Riesz transforms associated with the Ornstein-Uhlenbeck operator on finite or infinite dimensional Gaussian spaces. In [5], Bakry also proved that, if M is a complete Riemannian manifold with $W_k \geq -a$ and $W_{k+1} \geq -a$ for some positive constant $a > 0$, where W_k denotes the k-th Weitzenböck curvature on (M, g), then the Riesz transforms $d(a+\square_k)^{-1/2}$ associated with the Hodge Laplacian $\square_k$ is bounded in L^p for all $p > 1$, and there exists a universal constant $C_{p,k} > 0$ which is independent of $n = \dim M$ and a, such that for all $\omega \in C_0^\infty(\Lambda^k T^*M)$,

$$(1.3) \qquad \left\|d(a + \square_k)^{-1/2}\omega\right\|_p \leq C_{p,k}\|\omega\|_{p,}.$$

Under the same conditions, the Riesz transform $d^*(a+\square_{k+1})^{-1/2}$ is bounded in L^p for all $p > 1$. Moreover,

$$(1.4) \qquad \left\|d^*(a + \square_{k+1})^{-1/2}\omega\right\|_p \leq C_{p,k}\|\omega\|_p.$$

It is very natural to ask whether the Riesz transforms $d\square_k^{-1/2}$ and $d^*\square_{k+1}^{-1/2}$ are bounded in L^p for all $1 < p < \infty$ if M is a complete Riemannian manifold with non-negative Weitezenböck curvatures $W_i \geq 0$, $i = k, k+1$. Whether or not this result is true is very important to obtain an affirmative answer to the Strichartz problem and to prove the Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weizenböck curvatures. When $k \geq 1$, it seems that one cannot find an explicit statement of this result in [5] even though it can be derived from Bakry's L^p-estimates (1.3) and (1.4) with universal constant. Indeed, (1.3) is equivalent to

$$(1.5) \quad \left\|d\omega\right\|_p \leq C_{p,k}\left\|(a + \square_k)^{1/2}\omega\right\|_p, \qquad \forall\omega \in C_0^\infty(\Lambda^k T^*M).$$

Using Lemma 5.2 in [5], there exists a constant $A > 0$ independent of a and p such that

$$(1.6) \quad \left\|(a + \Box_k)^{1/2}\omega\right\|_p \leq A\big(\sqrt{a}\|\omega\|_p + \|\Box_k^{1/2}\omega\|p\big), \quad \forall \omega \in C_0^\infty(\Lambda^k T^* M).$$

Since the universal constants $C_{p,k}$ and A are independent of $a > 0$, one can take the limit $a \to 0$ in (1.5) and (1.6), and deduce that

$$\|d\omega\|_p \leq A C_{p,k}\|\Box_k^{1/2}\omega\|_p, \qquad \forall \omega \in C_0^\infty(\Lambda^k T^* M).$$

Thus, the Riesz transform $d\Box_k^{-1/2}$ is bounded in L^p for all $p > 1$ on complete Riemannian manifolds with non-negative Weitezenböck curvatures $W_i \geq 0$, $i = k, k + 1$. By duality argument, under the same conditions, the Riesz transform $d^*\Box_{k+1}^{-1/2}$ is bounded in L^p for all $p > 1$, Moreover, for all $p > 1$ and $k = 1, \ldots, n$, we have

$$\|d\Box_k^{-1/2}\|_{p,p} \leq A C_{p,k},$$
$$\|d^*\Box_{k+1}^{-1/2}\|_{p,p} \leq A C_{p,k}.$$

Inspired by the above mentioned results due to of Pichorides [57], Iwaniec-Martin [34] and Bañuelos-Wang [10] on the L^p-norm estimates of the Riesz transforms on Euclidean space, it is very natural to ask what is the asymptotic behavior of the constant $C_{p,k}$ when $p \to 1$ and $p \to \infty$ for all $k = 0, 1, \ldots, n$. In our previous paper [42], we developed a new probabilistic approach in the study of the Riesz transforms on complete Riemannian manifolds and proved that, if the Bakry-Emery Ricci curvature associated with $L = \Delta - \nabla\phi \cdot \nabla$ is non-negative, i.e., $Ric(L) = Ric + \nabla^2\phi \geq 0$, then the L^p-norm of the Riesz transform $\nabla(-L)^{-1/2}$ with respect to the weighted volume measure μ satisfies

$$(1.7) \qquad \|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1.$$

In particular, on all complete Riemannian manifolds with non-negative Ricci curvature, we proved in [42] that

$$(1.8) \qquad \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1.$$

These can be considered as a refinement of the above mentioned result of Bakry [5] and a natural generalization of the above mentioned results due to Pichorides [57], Iwaniec-Martin [34] and Bañuelos-Wang [10]. Moreover, using the results of Iwaniec-Martin [34], Bañuelos-Wang [10], Arcozzi [1] and Larsson-Cohn [37], we pointed out that the above estimates (1.7) and (1.8) are asymptotic sharp when $p \to 1$ and when $p \to \infty$. Now, it is very natural to ask whether one can extend the above estimates (1.7) and (1.8) to the Riesz transforms associated with the Witten Laplacians on complete

Riemannian manifolds with non-negative Weitzenböck curvature. In this paper, we study this problem and establish the Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvature.

1.4. Notation and assumption

Let (M, g) be a complete non-compact Riemannian manifold, v the Riemannian volume measure, $n = \dim M$. Let ∇ be the Levi-Civita connection on TM or the Levi-Civita covariant derivative operator on ΛT^*M. Let $\phi \in C^2(M)$, and $d\mu = e^{-\phi} dv$. The Hodge Laplacian and the Witten Laplacian on k-forms are denoted by $\square_k$ and $\square_{\phi,k}$ respectively. When acting on k-forms on M, $k = 0, 1, \ldots, n$, we denote

$$\Delta_\phi = \Delta - \nabla_{\nabla\phi},$$

where $\Delta = \mathrm{tr}\nabla^2$ is the covariant Laplace-Beltrami operator on (M, g). In particular, when $k = 0$, we use L to denote $\Delta_\phi|_{C_0^\infty(M)}$. More precisely,

$$L = \Delta - \nabla\phi \cdot \nabla.$$

Note that, for all $f, g \in C_0^\infty(M)$, we have the integration by parts formula

$$\int_M < \nabla f, \nabla g > d\mu = -\int_M (Lf)g \, d\mu = -\int_M f(Lg) \, d\mu.$$

In other words, $L = -\square_{\phi,0}$. Similarly to the case $\phi = 0$ as in [15, 65], L and $\square_{\phi,k}$ are essentially self-adjoint on $L^2(M, \mu)$ and $L^2(\Lambda^k T^*M, \mu)$, $k = 1, \ldots, n$.

Fix $x \in M$. Let $e_1, \ldots, e_n$ be a normal orthonormal basis of T_yM neat x such that $\nabla_{e_i} e_j(x) = 0$ for all $i, j = 1, \ldots, n$, and let $e_1^*, \ldots, e_n^*$ be its dual basis. By definition, the k-th Weitzenböck curvature associated with the Witten Laplacian $\square_\phi$ on k-forms, is defined as a symmetric endomorphism on $\Lambda^k T^*M$, and is given by

$$W_{\phi,k}(x) := \sum_{i,j} e_i^* \wedge i_{e_j} R(e_i, e_j) + d\Lambda^k \nabla^2\phi,$$

where R the Riemannian curvature of the Levi-Civita connection ∇ on (M, g), i_{e_j} denotes the interior multiplication induced by the contraction of the vector field e_j on $\Lambda^k(T^*M)$, $\nabla^2\phi$ denotes the Hessian of ϕ with respect to the Levi-Civita connection ∇, and $d\Lambda^k\nabla^2\phi$ is defined by

$$d\Lambda^k\nabla^2\phi(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^k v_1 \ldots \wedge \nabla^2\phi(v_i) \ldots \wedge v_k, \quad v_1, \ldots, v_k \in TM.$$

For simplicity, we make use the convention that $W_{\phi,k} \equiv 0$ for $k = 0, n+1$. When $\phi = 0$, we denote $W_k = W_{0,k}$, $k = 0, \ldots, n+1$.

Throughout this paper, we make the following basic assumption:

(A) The heat semigroup e^{tL} is a Markovian semigroup in the sense that $e^{tL}1(x) = 1$ for all $x \in M$. In other words, the heat semigroup e^{tL} is stochastically complete.

In the case $\phi \equiv 0$, Yau [70] proved that the heat semigroup $e^{t\Delta}$ is Markovian if M is a complete Riemannian manifold with Ricci curvature bounded from below. More general criterion for the stochastical completeness of the heat semigroup $e^{t\Delta}$ were given by Kanp-Li [36], Li-Schoen [39], and Grigor'yan [27, 28]. In the case $\phi \neq 0$, Bakry [4] proved that, if the Bakry-Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ is bounded from below by a negative constant, then e^{tL} is Markovian. More general criterion for the stochastical completeness of the heat semigroup e^{tL} are due to Sturm [68] and the author [40, p. 1306, Theorem 1.4].

1.5. Main results

To state the main results of this paper, we first recall the rigorous definitions of the Riesz transforms and the Riesz potentials associated with the Hodge Laplacian on complete Riemannian manifolds.

Definition 1.4 ([65]) Let M be a complete Riemannian manifold, $\square_{\phi,k}$ be the weighted Hodge Laplacian with respect to the weighted volume measure $d\mu = e^{-\phi}dv$. For $a \geq 0$, the Riesz transforms $d(a + \square_{\phi,k})^{-1/2}$ and $d_\phi^*(a + \square_{\phi,k})^{-1/2}$ as well as the Riesz potential $(a + \square_{\phi,k})^{-1/2}$ are defined as follows:

(i) A k-form $\omega \in L^p(\Lambda^k T^*M, \mu)$ is in the L^p domain of $d(a + \square_{\phi,k})^{-1/2}$ if

$$d(a + \square_{\phi,k})^{-1/2}\omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N de^{-t(a+\square_{\phi,k})}\omega \frac{dt}{\sqrt{t}}$$

exists in $L^p(\Lambda^{k+1}T^*M, \mu)$.

(ii) A k-form $\omega \in L^p(\Lambda^k T^*M, \mu)$ is in the L^p domain of $d_\phi^*(a + \square_{\phi,k})^{-1/2}$ if

$$d_\phi^*(a + \square_{\phi,k})^{-1/2}\omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N d_\phi^* e^{-t(a+\square_{\phi,k})}\omega \frac{dt}{\sqrt{t}},$$

exist in $L^p(\Lambda^{k-1}T^*M, \mu)$.

(iii) A k-form $\omega \in L^p(\Lambda^k T^*M, \mu)$ is in the L^p domain of $(a + \square_{\phi,k})^{-1/2}$ if

$$(a + \square_{\phi,k})_{\phi,k}^{-1/2}\omega := \frac{1}{\Gamma(1/2)} \lim_{N \to \infty} \int_0^N e^{-t(a+\square_{\phi,k})}\omega \frac{dt}{\sqrt{t}}$$

exists in $L^p(\Lambda^k T^*M, \mu)$.

Now we state the main results of this paper. The first result of this paper is the Strong L^p-stability of the Hodge Laplacian on complete Riemannian manifolds with non-negative Weitzenböck curvature operator. It seems that neither an explicit statement nor a proof of it can be found in the literature. It plays a crucial role in this paper.

Theorem 1.5 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that*

$$W_{\phi,k} \geq 0.$$

Then the heat semigroup $e^{-t\square_{\phi,k}}$ and the Poisson semigroup $e^{-t\sqrt{\square_{\phi,k}}}$ are L^p-strong stable in $(\mathrm{Ker}\square_{\phi,k})^{\perp} \cap L^p(\Lambda^k T^*M, \mu)$, i.e.,

$$\lim_{t\to\infty} \|e^{-t\square_{\phi,k}}\omega - \mathrm{H}_p\omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu),$$

$$\lim_{t\to\infty} \|e^{-t\sqrt{\square_{\phi,k}}}\omega - \mathrm{H}_p\omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu),$$

where

$$\mathrm{H}_p : L^p(\Lambda^k T^*M, \mu) \to (\mathrm{Ker}\square_{\phi,k}) \cap L^p(\Lambda^k T^*M, \mu)$$

denotes the L^p-Hodge harmonic projection.

The second result is a refinement of the above mentioned result due to Bakry [5].

Theorem 1.6 *Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that the heat semigroup e^{tL} is conservative, and $W_{\phi,i} \geq -a$, $i = k$, $k + 1$, where a is a non-negative constant. Then, there exists a constant $C_k > 0$ such that, for all $1 < p < \infty$, we have*

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2},$$
$$\|d_\phi^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}.$$

In particular, if the k-th and $(k + 1)$-th Weitzenböck curvatures are non-negative, i.e.,

$$W_{\phi,i} \geq 0, \quad i = k, k + 1,$$

then the Riesz transforms $d\square_{\phi,k}^{-1/2}$ and $d_\phi^*\square_{\phi,k+1}^{-1/2}$ are bounded in L^p for all $p > 1$. Moreover, for all $p > 1$,

$$\|d\square_{\phi,k}^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2},$$
$$\|d_\phi^*\square_{\phi,k+1}^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}.$$

The third result provides us with a reasonable condition on the Weitzenböck curvatures under which the asymptotically sharp L^p-norm estimates (1.2), (1.7) and (1.8) extend to the Riesz transforms associated with the Witten Laplacians on k-forms. See also Remark 6.5.

Theorem 1.7 *Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that e^{tL} is conservative, $W_{\phi,k} = -a$ and $W_{\phi,k+1} \geq -a$, where $a \geq 0$ is a non-negative constant. Then, there exists a constant C_k, such that for all $1 < p < \infty$, we have*

$$\|d(a + \Box_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1),$$

$$\|d_\phi^*(a + \Box_{\phi,k+1})^{-1/2}\|_{p,p} \leq C_k(p^* - 1),$$

where C_k is a positive constant depending only on $k = 0, 1, \ldots, n$.

Moreover, at least on the Euclidean spaces, an upper bound of the order $O(p^* - 1)$ for the L^p-norm of the Riesz transforms associated with the Hodge Laplacian is asymptotically sharp when $p \to 1$ and when $p \to \infty$.

By Gallot-Meyer [25], on the hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^n(-1)$ of constant sectional curvature -1, where Γ is any torsion-free discrete subgroup of $G = SO^+(n, 1)$ (the group of isometries on $\mathbb{H}^n(-1)$), we have

$$W_k = -k(n - k)\mathrm{Id}.$$

From this and Theorem 1.7 we have the following result which is also new in the literature.

Theorem 1.8 *Let $M = \Gamma\backslash\mathbb{H}^n(-1)$ be the hyperbolic manifold of constant sectional curvature -1, where Γ is any torsion-free discrete subgroup of $G = SO^+(n, 1)$. Then, for $k = \frac{n-1}{2}, \ldots, n$ if n is odd, $k = \frac{n}{2}, \ldots, n$ if n is even, and for all $p > 1$, we have*

$$\|d (k(n - k) + \Box_k)^{-1/2}\|_{p,p} \leq C_k(p^* - 1),$$

$$\|d^* (k(n - k) + \Box_{k+1})^{-1/2}\|_{p,p} \leq C_k(p^* - 1),$$

where C_k is a universal constant depending only on k.

Finally we establish the following Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvatures, which is a natural extension of Theorem 1.1.

Theorem 1.9 *Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that e^{tL} is conservative, and the Weitzenböck curvatures are non-negative:*

$$W_{\phi,i} \geq 0, \qquad i = k-1, k, k+1.$$

Then, for all $1 < p < \infty$, every $\omega \in L^p(\Lambda^k T^* M, \mu)$ has a unique decomposition in L^p:

$$\omega = \mathrm{H}_p \omega + d d_\phi^* \square_\phi^{-1} \omega + d_\phi^* d \square_\phi^{-1} \omega,$$

where $\mathrm{H}_p : L^p(\Lambda^k T^* M, \mu) \to \mathrm{Ker} \square_{\phi,k} \cap L^p(\Lambda^k T^* M, \mu)$ denotes Hodge harmonic projection.

1.6. Remarks

Remark 1.10 As far as we know, we cannot find an explicit statement of Theorem 1.5 and a proof for it in the literature. The L^p-contractivity of the heat semigroup $e^{-t\square_\phi}$ on differential forms plays an important role in the above proof. In [66], Strichartz studied the problem of the L^p-contractivity of the heat semigroup $e^{-t\square}$ on differential forms and the L^p-contractivity of the Hodge-Kodaira projection on complete non-compact Riemannian manifolds. He pointed out that the heat semigroup on k-forms is always L^2-contractive but one cannot expect that the L^p-contractivity of the heat semigroup is always "yes". In [66, p. 353], he wrote: "If $e^{t\Delta}$ were L^p contractive then by taking the limit as $t \to \infty$ we would obtain that the Kodaira projection operator T onto the harmonic k-forms is L^p-contractive." Here, according to the notation in [66], Δ denotes the negative Hodge-de Rham Laplacian on k-form.

Remark 1.11 By [57, 33, 10],

$$\|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right) \quad \text{and} \quad \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1.$$

From these one can derive that, for all $k = 0, 1, \ldots, n$, and for all $p > 1$,

$$\frac{2}{\pi}(p^* - 1)(1 + o(1)) \leq \|d\square_k^{-1/2}\|_{p,p} \leq C_k(p^* - 1),$$

$$\frac{2}{\pi}(p^* - 1)(1 + o(1)) \leq \|d^*\square_k^{-1/2}\|_{p,p} \leq C_{k-1}(p^* - 1),$$

where the left hand sides make sense when $p \to 1$ or $p \to \infty$. This can be viewed as an particular example of Theorem 1.7. It also indicates that at least in the case of Euclidean spaces, an upper bound of the order $O(p^* - 1)$ for the L^p-norm of the Riesz transforms $d\square_k^{-1/2}$ and $d^*\square_k^{-1/2}$ is asymptotically sharp when $p \to 1$ or $p \to \infty$. For details, see Example 6.6. However,

it is still an open problem to determine the exact value of the L^p-norm of the Riesz transforms $\nabla(-\Delta)^{-1/2}$ as well as $d\square_k^{-1/2}$ and $d^*\square_k^{-1/2}$ for $n \geq 2$ and $p \neq 2$.

Remark 1.12 To prove Theorem 1.6 and Theorem 1.7, we will first prove a probabilistic representation formula of the Riesz transforms on differential forms, cf. Theorem 5.3 below. In the proof of Theorem 5.3, we use the Littlewood-Paley identity. However, the proof of Theorem 1.6 and Theorem 1.7 does not need to use the Littlewood-Paley-Stein inequalities. The proof of Theorem 1.7 is inspired by the argument used in Bañuelos-Wang [10] for the estimates (1.1) and (1.2). Se also Remark 6.5.

Remark 1.13 In [44], using a probabilistic approach which is different from the present one, we proved that, under the condition $W_k \geq 0$, the singular integral operators $dd^*\square_k^{-1}$ and $d^*d\square_k^{-1}$ are bounded in L^p for all $p > 1$. This further implies that the Weak L^p-Hodge decomposition theorem holds on $L^p(\Lambda^k T^*M)$ for all $p > 1$ providing that $W_k \geq 0$.

1.7. Applications

The method and the main results of this paper will be used in three forthcoming papers [45, 43, 46]. In [45], we use the main results of this paper to prove the Strong L^p-Hodge decomposition theorem and to prove some vanishing theorems of the L^p-cohomology on complete Riemannian manifolds with suitable geometric conditions. In [43], we use the main results of this paper to prove some Sobolev inequalities on differential forms and to prove some vanishing theorems of the $L^{p,q}$-cohomology on complete Riemannian manifolds with suitable geometric conditions. In [46], we use the method of this paper to prove the L^p-boundedness of the Riesz transforms associated with the complex Kodaira-Hodge Laplacian on complete Kähler manifolds with non-negative curvature operator, and use this result to prove the L^p-estimates and existence theorems of the $\bar\partial$-operator on complete Kähler manifold with suitable curvature conditions. The main result in [46] can be viewed as a non-trivial extension of the famous Hörmander-Andreotti-Vesentini L^2-estimate and existence theorem of the $\bar\partial$-operator on complete Kähler manifolds with semi-positive curvature.

The rest of this paper is organized as follows. In Section 2, we recall the well-known probabilistic representation formulas of the heat semigroup and the Poisson semigroup generated by the Hodge Laplacian or the Witten Laplacians on forms on complete Riemannian manifolds, as well as the well-known semigroup domination inequalities. In Section 3, we prove the strong L^p-stability of the heat semigroup and the Poisson semigroup on forms on

complete Riemannian manifolds with non-negative Weitzenböck curvature. In Section 4, we formulate the Burkholder sharp L^p-inequality for martingale transforms on complete Riemannian manifolds. In Section 5, we prove the probabilistic representation formulas for the Riesz transforms and the Riesz potentials on forms on complete Riemannian manifolds. In Section 6, we prove Theorem 1.6, Theorem 1.7 and Theorem 1.8. In Section 7, we prove the Weak L^p-Hodge decomposition theorem (i.e., Theorem 1.9) on complete Riemannian manifolds with non-negative Weitzenböck curvatures.

2. Heat semigroup and Poisson semigroup on forms

2.1. The weighted Bochner-Weitzenböck formula

We first recall the weighted Bochner-Weitzenböck formula which has been well-known to experts. To sake the completeness of the paper, we give a proof for it.

Theorem 2.1 *Let ∇ be the Levi-Civita connection on (M, g), $\phi \in C^2(M)$. Let*

$$\Delta_\phi = \Delta - \nabla_{\nabla\phi}$$

be the covariant weighted Laplace-Beltrami operator, where $\Delta = \mathrm{Tr}\nabla^2$ is the covariant Laplace-Beltrami operator on k-forms. Let

$$W_{\phi,k} = \sum_{i,j} e_i^* \wedge i_{e_j} R(e_i, e_j) + d\Lambda^k \nabla^2 \phi$$

be the weighted Weitzenbock curvature, where $e_1, \ldots, e_n$ is a normal orthonormal basis at $T_y M$ for $y \in M$ near $x \in M$ such that $\nabla_{e_i} e_j(x) = 0$, $\nabla^2 \phi$ denotes the Hessian of ϕ with respect to the Levi-Civita connection ∇. Then

(2.1) $$\square_{\phi,k} = -\Delta_\phi + W_{k,\phi}.$$

Proof. By definition, the L^2-adjoint of d_k with respect to the measure μ, denoted by $d_{\phi,k}^* : C_0^\infty(\Lambda^{k+1}T^*M) \to L^2(\Lambda^k T^*M, \mu)$, satisfies

$$\int_M < d_k\omega, \eta > d\mu = \int_M < \omega, d_{\phi,k}^*\eta > d\mu.$$

Standard argument based on integration by parts formula shows that

$$d_{\phi,k}^* = d^* + i_{\nabla\phi},$$

where $i_{\nabla\phi}$ denotes the interior multiplication induced by the contraction of the vector field $\nabla\phi$ on $\Lambda^{k+1}T^*M$. This yields

$$\Box_{\phi,k} = d(d^* + i_{\nabla\phi}) + (d^* + i_{\nabla\phi})d = (dd^* + d^*d) + (di_{\nabla\phi} + i_{\nabla\phi}d).$$

Using the Cartan identity

$$di_{\nabla\phi} + i_{\nabla\phi}d = L_{\nabla\phi},$$

we obtain

(2.2) $$\Box_{\phi,k} = \Box + L_{\nabla\phi}.$$

By the standard Bochner-Weitzenböck formula, we have

(2.3) $$\Box = -\Delta + \sum_{i<j} e_i^* \wedge i_{e_j} R(e_i, e_j).$$

We now prove the following identity

(2.4) $$L_{\nabla\phi}\omega = \nabla_{\nabla\phi}\omega + d\Lambda^k \nabla^2\phi\omega.$$

To prove (2.4), notice that $\nabla_{e_j} i_{e_i} = 0$ holds at the point x. Hence

$$
\begin{aligned}
di_{\nabla\phi}\omega &= \sum_{i,j} e_j^* \wedge \nabla_{e_j}(< \nabla\phi, e_i > i_{e_i}\omega) \\
&= \sum_{i,j} \nabla_{e_j} < \nabla\phi, e_i > e_j^* \wedge i_{e_i}\omega + \sum_{i,j} < \nabla\phi, e_i > e_j^* \wedge i_{e_i}\nabla_{e_j}\omega \\
&= \sum_{i,j} \nabla^2\phi(e_i, e_j)e_i^* \wedge i_{e_j}\omega + \sum_{i,j} < \nabla\phi, e_i > e_j^* \wedge i_{e_i}\nabla_{e_j}\omega,
\end{aligned}
$$

and

$$
\begin{aligned}
i_{\nabla\phi}d\omega &= \sum_{i,j} < \nabla\phi, e_i > i_{e_i}e_j^* \wedge \nabla_{e_j}\omega \\
&= \sum_{i,j} < \nabla\phi, e_i > \nabla_{e_i}\omega - \sum_{i<j} < \nabla\phi, e_i > e_j^* \wedge i_{e_i}\nabla_{e_j}\omega \\
&= \nabla_{\nabla\phi}\omega - \sum_{i<j} < \nabla\phi, e_i > e_j^* \wedge i_{e_i}\nabla_{e_j}\omega.
\end{aligned}
$$

Combining the above identities, we obtain (2.4). From (2.2)-(2.4), we obtain (2.1). ∎

2.2. Heat semigroup on forms

In this subsection we recall the well-known probabilistic representation formula of the heat semigroup on forms and the semigroup domination inequalities.

Let X_t^x be a diffusion process on M with infinitesimal generator $L = \Delta - \nabla_{\nabla \phi}$ and with $X_0 = x$. By Itô's theory of diffusion process on Riemannian manifolds, there exists a Brownian motion W_t on $\mathbb{R}^n$, such that

$$dX_t^x = U_t \circ dW_t - \nabla\phi(X_t^x)dt,$$
$$X_0 = x,$$

where $U_t \in \mathrm{End}(T_x M, T_{X_t^x} M)$ denotes the stochastic parallel transport along the path $\{X_s^x, s \in [0, t]\}$ with respect to the Levi-Civita connection, and is the unique solution to the following covariant SDE along $\{X_s^x, s \in [0, t]\}$:

$$\nabla_{\circ dX_t^x} U_t = 0, \quad U_0 = Id_{T_x M},$$

where $\circ d$ denotes the Stratonovich stochastic differentiation.

By Itô's formula and the weighted Bochner–Lichnerowicz-Weitzenböck formula, and using the same argument as in the proof of the Feynman-Kac formula for vector valued function on $\mathbb{R}^n$, we have the following well-known probabilistic representation formula of the heat semigroup generated by the Witten Laplacian $\square_{\phi, k}$ on k-forms:

$$(2.5) \qquad e^{-t\square_{\phi, k}}\omega(x) = E\left[M_{t,k}^* \omega(X_t^x)\right],$$

where $M_{t,k} \in \mathrm{End}(\Lambda^k T_x^* M, \Lambda^k T_{X_t^x}^* M)$ is the solution of the covariant differential equation

$$(2.6) \qquad \frac{\nabla}{\partial t} M_{t,k} = -W_{\phi, k}(X_t^x) M_{t,k}, \quad M_{0,k} = Id_{\Lambda^k T_x^* M},$$

where $\frac{\nabla}{\partial t} := U_t \frac{\partial}{\partial t} U_t^{-1}$ denotes the covariant derivative operator with respect to the Levi-Civita connection along the path of $\{X_s^x, s \in [0, t]\}$. For a proof of (2.5), we refer the reader to Elworthy-Le Jan-X.-M. Li [21].
We have the following well-known semigroup domination inequality, cf. [21].

Theorem 2.2 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $W_{\phi, k} \geq a$, where $a \in \mathbb{R}$. Then, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$,*

$$(2.7) \qquad \left|e^{-t\square_{\phi, k}}\omega(x)\right| \leq e^{-at}e^{tL}|\omega|(x), \quad \forall x \in M, t > 0.$$

Remark 2.3 In the case $k = 0$ and $\phi \equiv 0$, the probabilistic representation formula (2.5) of the heat semigroup $e^{-t\square}$ and the semigroup domination inequality (2.7) can be traced back to Malliavin [50].

2.3. Poisson semigroup on k-forms

Using the Bochner subordination formula, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$, and for all $(x, t) \in M \times \mathbb{R}^+$, the Poisson semigroup $e^{-t\sqrt{a+\Box_{\phi,k}}}$ on one-forms can be defined as follows

$$e^{-t\sqrt{a+\Box_{\phi,k}}}\omega(x) \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}(a+\Box_{k,\phi})}\omega(x)e^{-u}u^{-1/2}du.$$

In the case $k = 0$, for all $f \in C_0^\infty(M)$, the Poisson semigroup $e^{-t\sqrt{a-L}}$ on functions can be given by

$$e^{-t\sqrt{a-L}}f(x) \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}(a-L)}f(x)e^{-u}u^{-1/2}du.$$

We can also give the probabilistic representation formulas for the Poisson semigroup $e^{-t\sqrt{a-L}}$ and the Poisson semigroup $e^{-t\sqrt{a+\Box_\phi}}$. To do so, let B_t be the Brownian motion with infinitesimal generator $\frac{d^2}{dy^2}$ instead of $\frac{1}{2}\frac{d^2}{dy^2}$ and starting from $B_0 = y > 0$, which is independent of X_t^x, and let

$$\tau_y = \inf\{t > 0 : B_t = 0\}.$$

Then it is well-known that for all $\lambda > 0$, we have

$$E_y\left[e^{-\lambda\tau_y}\right] = e^{-y\sqrt{\lambda}}.$$

This leads us to the following probabilistic representation formula for the Poisson semigroup on functions

$$e^{-y\sqrt{a-L}}f(x) = E_y\left[e^{-(a-L)\tau_y}f(x)\right] = E_y\left[e^{-a\tau_y}E[f(X_{\tau_y}^x)]\right].$$

That is

$$e^{-y\sqrt{a-L}}f(x) = E_y\left[e^{-a\tau_y}f(X_{\tau_y}^x)\right].$$

Similarly, for any $\omega \in C_0^\infty(\Lambda^k T^* M)$, the Poisson semigroup $e^{-y\sqrt{a+\Box_{\phi,k}}}\omega(x)$ can be represented by

$$e^{-y\sqrt{a+\Box_{\phi,k}}}\omega(x) = E_y\left[e^{-(a+\Box_{k,\phi})\tau_y}\omega(x)\right] = E_y\left[e^{-a\tau_y}E[M_{\tau_y,k}^*\omega(X_{\tau_y}^x)]\right].$$

Let $E_{(x,y)}$ denote the expectation with respect to the law of (X_t^x, B_t). Then

$$e^{-y\sqrt{a+\Box_{\phi,k}}}\omega(x) = E_{(x,y)}\left[e^{-a\tau_y}M_{\tau_y,k}^*\omega(X_{\tau_y}^x)\right].$$

3. Strong L^p-stability of the heat semigroup and the Poisson semigroup

In this section we prove the Strong L^p-stability of the heat semigroup and the Poisson semigroup on complete Riemannian manifolds with non-negative Weitzenböck curvatures. This result will play a crucial role in the proof of Theorem 1.6.

We first prove the following result which is essentially due to Bakry [5].

Theorem 3.1 *Let M be a complete Riemannian manifold. Suppose that $W_{\phi,k} \geq -a$, where $a > 0$ is a constant. Then, for all $p \in [1, \infty]$ and $t > 0$, we have*

$$(3.1) \qquad \left\| e^{-t(a + \square_{\phi,k})} \right\|_{p,p} \leq e^{-2\min\{\frac{1}{p}, 1 - \frac{1}{p}\}at},$$

$$(3.2) \qquad \left\| e^{-t\sqrt{a + \square_{\phi,k}}} \right\|_{p,p} \leq e^{-2\min\{\frac{1}{p}, 1 - \frac{1}{p}\}\sqrt{a}t}.$$

Proof. Using the semigroup domination inequality (2.7) and the fact that e^{tL} is L^p-contractive, under the curvature condition $W_{\phi,k} \geq -a$, we have

$$\|e^{-t(a + \square_{\phi,k})}\omega\|_p \leq \|e^{tL}|\omega|\|_p \leq \|\omega\|_p, \quad \forall p \in [1, \infty], t > 0.$$

Therefore

$$\|e^{-t(a + \square_{\phi,k})}\|_{p,p} \leq 1, \quad \forall p \in [1, \infty], t > 0.$$

This proves (3.1) for $p = 1, \infty$. On the other hand, for all $\omega \in L^2(\Lambda^k T^*M, \mu)$,

$$\frac{\partial}{\partial t} \left\| e^{-t\square_{\phi,k}}\omega \right\|_2^2 = -2 \int_M <\square_{\phi,k} e^{-t\square_{\phi,k}}\omega, e^{-t\square_{\phi,k}}\omega> d\mu \leq 0.$$

Hence

$$\left\| e^{-t\square_{\phi,k}}\omega \right\|_2 \leq \|\omega\|_2.$$

This yields

$$\|e^{-t(a + \square_{\phi,k})}\|_{2,2} \leq e^{-at}, \quad \forall t > 0.$$

By the Riesz convexity interpolation, for all $p \in (1, 2)$, letting $\theta \in (0, 1)$ be such that $1/p = \theta/1 + 1 - \theta/2$, i.e., $\theta = 2/p - 1$, we have

$$\left\| e^{-t(a + \square_{\phi,k})} \right\|_{p,p} \leq \left\| e^{-t(a + \square_{\phi,k})} \right\|_{1,1}^{\theta} \left\| e^{-t(a + \square_{\phi,k})} \right\|_{2,2}^{1-\theta}$$

$$\leq e^{-(1-\theta)at} = e^{-\frac{2(p-1)at}{p}}, \quad \forall t > 0.$$

Similarly, for all $p \in (2, \infty)$, letting $\theta \in (0, 1)$ be such that $1/p = 1 - \theta/2 + \theta/\infty$, i.e., $\theta = 1 - 2/p$, we have

$$\left\| e^{-t(a+\Box_{\phi,k})} \right\|_{p,p} \leq \left\| e^{-t(a+\Box_{\phi,k})} \right\|_{\infty,\infty}^{\theta} \left\| e^{-t(a+\Box_{\phi,k})} \right\|_{2,2}^{1-\theta}$$

$$\leq e^{-(1-\theta)at} = e^{-\frac{2at}{p}}, \quad \forall t > 0.$$

These finish the proof of (3.1) for all $p \in [1, \infty]$ and $t > 0$. Similarly, for all $p \in [1, \infty]$ and $t > 0$, we can prove (3.2). ∎

We would like to point out that Theorem 3.1 plays a crucial role in Bakry's proof of the L^p-boundedness of the Riesz transforms $d(a + \Box_{\phi,k})^{-1/2}$ on complete Riemannian manifolds with $W_{\phi,k} \geq -a$ and $W_{\phi,k+1} \geq -a$ for some positive constant $a > 0$. According to [22], Theorem 3.1 says that, under the condition $W_{\phi,k} \geq -a$ with $a > 0$, the heat semigroup $e^{-t(a+\Box_{\phi,k})}$ and the Poisson semigroup $e^{-t\sqrt{a+\Box_{\phi,k}}}$ are exponentially stable when $t \to \infty$. In particular, we obtain the following

Corollary 3.2 *Under the same conditions as in Theorem* 3.1*, for all* $p > 1$*, we have*

$$\lim_{t \to \infty} \left\| e^{-t(a+\Box_{\phi,k})} \right\|_{p,p} = 0,$$

$$\lim_{t \to \infty} \left\| e^{-t\sqrt{a+\Box_{\phi,k}}} \right\|_{p,p} = 0.$$

Remark 3.3 In the case where M is a complete non-compact Riemannian manifold with non-negative Weitzenböck curvature $W_{\phi,k} \geq 0$, if we do not assume some additional condition on M, we cannot prove that the heat semigroup $e^{-t\Box_{\phi,k}}$ and the Poisson semigroup $e^{-t\sqrt{\Box_{\phi,k}}}$ are exponentially stable. This might be the most important reason why we cannot find an explicit statement in [5] saying that the Riesz transform $d\Box_{\phi,k}^{-1/2}$ is bounded in L^p for all $p > 1$ if M is a complete Riemannian manifold with $W_{\phi,k} \geq 0$ and $W_{\phi,k+1} \geq 0$.

An important observation of this paper is that, even though the heat semigroup $e^{-t\Box_{\phi,k}}$ and the Poisson semigroup $e^{-t\sqrt{\Box_{\phi,k}}}$ are usually not exponentially stable on complete Riemannian manifolds with non-negative Weitzenböck curvatures $W_{\phi,k} \geq 0$, they are strong stable in L^p for all $p > 1$ in the sense of Theorem 1.5. It will play a crucial role in the proof of Theorems 1.6, 1.7 and Theorem 1.9.

Proof of Theorem 1.5. Using the semigroup domination inequality (2.7), as $W_{\phi,k} \geq 0$, we have

$$\left| e^{-t\Box_{\phi,k}} \omega(x) \right| \leq e^{tL} |\omega|(x), \quad \forall \omega \in L^p(\Lambda^k T^* M, \mu), x \in M, t > 0.$$

Since e^{tL} is a Markovian semigroup, it is L^p-contractive. Therefore

$$\|e^{-t\Box_{\phi,k}}\omega\|_p \leq \|\omega\|_p, \quad \forall t > 0.$$

By Bochner subordination, we have

$$\|e^{-t\sqrt{\Box_{\phi,k}}}\omega\|_p \leq \|\omega\|_p, \quad \forall t > 0.$$

Slightly modifying the argument used in the proof of Theorem 1 in E. M. Stein [63, p. 67] or Theorem 4.2.12 in Jacob [35, p. 293], using the fact that $\Box_{\phi,k}$ is a self-adjoint operator on $L^2(\Lambda^k T^*M, \mu)$, and that $e^{-t\Box_{\phi,k}}$ is L^p-contractive for all $p \in [1, \infty]$, we can prove that $e^{-t\Box_{\phi,k}}$ has an analytic continuation from the sector $\Sigma_{\theta(p)}$ to $L^p(\Lambda^k T^*M, \mu)$, where

$$\Sigma_{\theta(p)} = \left\{ z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2}\left(1 - \left| \frac{2}{p} - 1 \right| \right) \right\}, \quad p > 1.$$

By the theory of analytic semigroups, see e.g. Theorem 4.2.7 in Jacob [35], there exists two constants $\delta \in (0, \frac{\pi}{2})$ and $M > 0$ such that

$$\Sigma_{\frac{\pi}{2}+\delta} = \left\{ \lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta \right\} \cup \{0\} \subset \rho_p(\Box_{\phi,k}),$$

and

$$\|(\lambda - \Box_{\phi,k})^{-1}\|_{p,p} \leq \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_{\frac{\pi}{2}+\delta} \setminus \{0\},$$

where $\rho_p(\Box_{\phi,k})$ denotes the resolvent set of $\Box_{\phi,k}$ in $L^p(\Lambda^k T^*M, \mu)$. Equivalently, the L^p-spectra of $\Box_{\phi,k}$ satisfies

$$\sigma_p(\Box_{\phi,k}) \cap i\mathbb{R} \subset \{0\}.$$

By a theorem of Arendt-Batty-Lyubich-Vũ, see [22] (Ch. VI Theorem 2.21, see also Example 2.23(i) of Ch. VI), we obtain

$$\lim_{t \to \infty} \|e^{-t\Box_{\phi,k}}\omega - H_p\omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu).$$

This proves that $e^{-t\Box_{\phi,k}}$ is strong stable in $(\text{Ker}\Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^*M, \mu)$. Similarly, we can prove that the Poisson semigroup $e^{-t\sqrt{\Box_{\phi,k}}}$ is strong stable in $(\text{Ker}\Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^*M, \mu)$. The proof of Theorem 1.5 is completed. ∎

4. Martingale transforms on complete Riemannian manifolds

In this section, we formulate the Burkholder L^p-sharp inequality for martingale transforms on complete Riemannian manifolds. For this, we borrow some ideas from Bañuelos-Wang [10] and Arcozzi [1].

Let $\mathcal{F}_t$ be the filtration generated by the n-dimensional Brownian motion W_t. Let Y_t be a real-valued L^2-martingale on the Brownian filtration $\mathcal{F}_t$. Then

$$Y_t = E[Y_t] + \int_0^t H_s \cdot dW_s,$$

where H_t is a predictable process with value in $\mathbb{R}^n$. Let $A \in M(n, \mathbb{R})$ be an $n \times n$ real matrix-valued predictable process. Define the martingale transformation

$$(A * Y)_t = \int_0^t A(s)H_s \cdot dW_s.$$

Let

$$\||A\|| = \sup_{s \geq 0} \|A(s)\|, \quad \text{where } \|A(s)\| = \sup \left\{ \|A(s)v\|_{\mathbb{R}^n} : v \in \mathbb{R}^n, \|v\|_{\mathbb{R}^n} \leq 1 \right\}.$$

Then

$$< A * Y >_t - < A * Y >_s \leq \||A\||^2 < Y >_t - \||A\||^2 < Y >_s .$$

According to [13, 10, 1], this means that $(A*Y)_t$ is differentially subordinate to $\||A\||Y_t$.

The following theorem is essentially due to Burkholder [13]. It gives the best constant in the L^p-inequality for the differential subordinate martingales.

Theorem 4.1 ([13, 10, 1]) *Let Y and $A*Y$ be as above. Then, for any $p > 1$, we have*

$$\|A * Y\|_p \leq (p^* - 1)\||A\|| \, \|Y\|_p$$

and the constant $(p^* - 1)\||A\||$ is the best possible here.

Let M be a complete Riemannian manifold, $n = \dim M$, and $\phi \in C^2(M)$. Suppose that the L-diffusion process X_t is stochastically complete, i.e., the lifetime of the L-diffusion process is infinite. By Itô's SDE theory, we have

$$dX_t = U_t \circ dW_t - \nabla\phi(X_t)dt,$$

where W_t is the standard Brownian motion on $\mathbb{R}^n$, U_t denotes the stochastic parallel transport along the trajectory of $\{X_s : s \in [0,t]\}$, i.e., U_t satisfies the following covariant SDE on the orthonormal frame bundle $O(M)$ over M:

$$\nabla_{\circ dX_t} U_t = 0.$$

Note that the Brownian filtration $\mathcal{F}_t$ coincides with the filtration generated by X_t, since we can reconstruct the $\mathbb{R}^n$-valued Brownian motion W_t from the M-valued L-diffusion process X_t in the following way:

$$W_t = \int_0^t U_s^{-1} \circ dX_s + \int_0^t U_s^{-1} \nabla \phi(X_s) ds,$$

where $\circ dX_s$ denotes the Stratonovich differential along the trajectory of $\{X_s, s \in [0,t]\}$.

Definition 4.2 Let F be a vector bundle over M, $< \cdot, \cdot >_F$ be an inner product over F. Let $E = \Lambda^\cdot T^* M \otimes F$. Let $U_t^F : F_{X_0} \to F_{X_t}$ be the stochastic parallel transport along the trajectory of $\{X_s : s \in [0,t]\}$. An $\mathrm{End}(E)$-valued martingale transformer over the L-diffusion process X_t on M is a bounded and continuous process A_t such that

$$A_t(\omega) \in \mathrm{End}\big(\Lambda^\cdot T^*_{X_t(\omega)} M \otimes F_{X_t(\omega)}\big).$$

Let Ψ_t be a continuous, bounded process with values in $E = \Lambda^\cdot T^* M \otimes F$ over X. The martingale transform of the F_{X_0}-valued Itô stochastic integral

$$(I_\Psi)_t = \int_0^t U_s^{F,-1} \Psi_s U_s dW_s$$

by the martingale transformer $A \in \mathrm{End}(E)$, denoted by $(A * I_\Psi)_t$, is the F_{X_0}-valued martingale defined by

$$(A * I_\Psi)_t = \int_0^t U_s^{F,-1} A_s \Psi_s U_s dW_s.$$

The following theorem is a straightforward extension of a result due to Arcozzi [1] where F is a trivial vector bundle over M, i.e., $F = \mathbb{R}^l$.

Theorem 4.3 *Let X_t be a stochastically complete L-diffusion process on M. Let A_t be a martingale transformer over X_t. Suppose that*

$$\||A|\| = \sup_{s>0} \sup_{\omega \in \Omega} \|A_s(\omega)\|_{\mathrm{op}} < \infty,$$

where $\|A_s(\omega)\|_{\mathrm{op}}$ denotes the operator norm of $A_s(\omega) \in \mathrm{End}(\Lambda^\cdot T^*_{X_s} M \otimes F_{X_s})$. Then

$$\|A * I_\Psi\|_p \leq (p^* - 1)\||A|\| \, \|I_\Psi\|_p.$$

Proof. By Theorem 4.1, we need only to prove that $A * I_\Psi$ is differentially subordinate to I_Ψ. The covariance process of the martingale transformation $A * I_\Psi$ is

$$< A * I_\Psi >_t = \int_0^t \text{Tr}(A_s \Psi_s \otimes A_s \Psi_s) ds$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_x M$, and let $f_1, \ldots, f_l$ be an orthonormal basis of $F_{X_s(\omega)}$. Then $\{e_i \otimes f_j, 1 \leq i \leq n, 1 \leq j \leq l\}$ is an ONB of $E_{X_s(\omega)} = \Lambda \cdot T^*_{X_s(\omega)} M \otimes F_{X_s(\omega)}$. By definition, we have

$$\text{Tr}(A_s \Psi_s \otimes A_s \Psi_s)(X_s(\omega)) = \sum_{i=1}^n \sum_{j=1}^l < A_s \Psi_s(e_i \otimes f_j), A_s \Psi_s(e_i \otimes f_j) >_{E_{X_s(\omega)}}$$

$$= \sum_{i=1}^n \sum_{j=1}^l | < A_s \Psi_s, e_i \otimes f_j >_{E_{X_s(\omega)}} |^2$$

$$= \|A_s \Psi_s\|^2_{E_{X_s(\omega)}} \leq \|A_s\|^2_{\text{op}} \|\Psi_s\|^2_{E_{X_s(\omega)}} \leq \||A\||^2 \|\Psi_s\|^2_{E_{X_s(\omega)}}.$$

Hence

$$< A * I_\Psi >_t \leq \||A\||^2 \int_0^t \|\Psi_s\|^2_{E_{X_s}} ds = \||A\||^2 < I_\Psi >_t .$$

This yields that $A * Y$ is a differential subordination to Y, i.e., we have

$$< A * I_\Psi >_t - < A * I_\Psi >_s \leq \||A\|| < I_\Psi >_t - \||A\|| < I_\Psi >_s, \ \forall \ 0 \leq s < t.$$

The proof of Theorem 4.3 is completed. ∎

5. Martingale representation of Riesz transforms and Riesz potentials

5.1. Background radiation process

From this section, let X_t^μ be the diffusion process on M whose infinitesimal generator is L and whose initial measure is μ, and let B_t be a one-dimensional Brownian motion starting from $B_0 = y$ with infinitesimal generator $\frac{d^2}{dy^2}$ (instead of $\frac{1}{2}\frac{d^2}{dy^2}$) and independent of the horizontal diffusion X_t^μ. Note that $dB_t \cdot dB_t = 2dt$ (instead of $dB_t \cdot dB_t = dt$). Following P. A. Meyer [52] and Gundy [30], we introduce the so-called *background radiation process* on $M \times \mathbb{R}^+$ as follows

$$Z_t^\mu := (X_t^\mu, B_t).$$

In fact, $\{Z_t^\mu, t \in [0, \tau]\}$ is a diffusion process on $M \times \mathbb{R}^+$ whose infinitesimal generator is $L + \frac{d^2}{dy^2}$ and whose initial distribution is $\mu \otimes \delta_y$ supported on the hypersurface $M \times \{y\}$ at time $t = 0$. The process $\{Z_t^\mu, t \in [0, \tau_y]\}$ terminates at time $t = \tau_y$ upon hitting the boundary $M \times \{0\}$.

Let $P_{(x,y)}$ be the probability law of the process $Z_t^x = (X_t^x, B_t)$ starting at $(x, y) \in M \times \mathbb{R}^+$. We define the measures $\{P_y, y > 0\}$ on the path space $C([0, \infty), M \times \mathbb{R})$ as

$$P_y(Z_t^\mu \in \mathcal{B}) = \int_M P_{(x,y)}(Z_t^x \in \mathcal{B}) d\mu(x),$$

for all Borel sets $\mathcal{B} \subset M \times \mathbb{R}^+$. Let E_y be the expectation corresponding to P_y.

In the sequel, to simplify the notation, we use τ to denote τ_y and use X_t (respectively, Z_t) to denote X_t^μ (respectively, Z_t^μ).

5.2. Covariant Itô's calculus

The following proposition will be used in the proof of the results of the next sections.

Proposition 5.1 *For all $\omega \in C_0^\infty(\Lambda^k T^* M)$ and all $a \geq 0$, we have*

(5.1)
$$e^{-at}\omega(X_\tau) = e^{a\tau}M_\tau^{*,-1}\omega_a(Z_0) + \int_0^\tau e^{a(\tau-s)}M_\tau^{*,-1}M_s^*\left(\nabla, \frac{\partial}{\partial y}\right)\omega_a(Z_s) \cdot (U_s dW_s, dB_s).$$

where
$$\omega_a(x, y) := e^{-y\sqrt{a+\square_\phi}}\omega(x), \quad \forall (x, y) \in M \times \mathbb{R}^+.$$

Proof. By (2.6), we have

$$\frac{\nabla}{\partial t}(M_t^*) = -M_t^* W_{\phi,k}(X_t).$$

Using the covariant version of the Itô formula acting on differential forms, cf. Elworthy-Le Jan-X. M. Li [21] and Norris [56], we have

$$\begin{aligned}
\nabla(e^{-at}M_t^*\omega(X_t)) = &-ae^{-at}M_t^*\omega(X_t)dt + e^{-at}\nabla M_t^*\omega(X_t)dt \\
&+ e^{-at}M_t^*(\nabla\omega)(X_t) \circ dX_t \\
= &-e^{-at}M_t^*(a + W_{\phi,k}(X_t))\omega(X_t)dt + e^{-at}M_t^*(\nabla\omega)(X_t)dX_t \\
&+ e^{-at}M_t^*\nabla^2\omega(X_t)(dX_t, dX_t).
\end{aligned}$$

Note that
$$dX_t = U_t \circ dW_t - \nabla\phi(X_t)dt.$$

Hence

$$\nabla^2\omega(X_t)(dX_t, dX_t) = \sum_{i,j} \nabla^2\omega(X_t)(U_te_i, U_te_j)dW_t^i dW_t^j$$
$$= \sum_{i,j} \nabla^2\omega(X_t)(Ue_i, Ue_j)\delta_{ij}dt$$
$$= \mathrm{Tr}\nabla^2\omega(X_t)dt$$
$$= \Delta\omega(X_t)dt.$$

This yields that

$$\nabla(e^{-at}M_t^*\omega(X_t)) = e^{-at}M_t^*(\nabla\omega)(X_t)dX_t$$
$$+ e^{-at}M_t^*(\Delta - W_{\phi,k}(X_t) - a)\omega(X_t)dt$$
$$= e^{-at}M_t^*(\nabla\omega)(X_t)U_tdW_t$$
$$+ e^{-at}M_t^*(\Delta - \nabla_{\nabla_\phi} - W_{\phi,k}(X_t) - a)\omega(X_t)dt$$
$$= e^{-at}M_t^*(\nabla\omega)(X_t)U_tdW_t$$
$$- e^{-at}M_t^*(-\Delta_\phi + W_{\phi,k}(X_t) + a)\omega(X_t)dt.$$

By the weighted Bochner-Weitzenbock formula (2.1), we obtain

$$\nabla(e^{-at}M_t^*\omega(X_t)) = e^{-at}M_t^*(\nabla\omega)(X_t)U_tdW_t - e^{-at}M_t^*(a + \Box_{\phi,k})\omega(X_t)dt.$$

Therefore, for all $\omega \in \mathrm{Ker}(a + \Box_\phi)$, we have

$$\nabla(e^{-at}M_t^*\omega(X_t)) = e^{-at}M_t^*(\nabla\omega)(X_t)U_tdW_t.$$

Integrating from s to t along the trajectory of the diffusion process X we get

$$e^{-at}M_t^*\omega(X_t) = e^{-as}M_s^*\omega(X_s) + \int_s^t e^{-as}M_r^*(\nabla\omega)(X_r)U_rdW_r.$$

Replacing X_t by the background radiation process $Z_t = (X_t, B_t)$, and replacing the $(a + \Box_\phi)$-harmonic form $\omega \in \mathrm{Ker}(a + \Box_\phi)$ on M by the Poisson semigroup $\omega_a(x,y) = e^{-y\sqrt{a + \Box_{\phi,k}}}\omega(x)$ on $M \times \mathbb{R}^+$, we get

$$e^{-at}M_t^*\omega_a(Z_t) = e^{-as}M_s^*\omega_a(Z_s) + \int_s^t e^{-ar}M_r^*\left(\nabla, \frac{\partial}{\partial y}\right)\omega_a(Z_r)\cdot(U_rdW_r, dB_r).$$

In particular, at $t = \tau$ and $s = 0$, we get

$$\omega(X_\tau) = e^{a\tau}M_\tau^{*,-1}\omega_a(Z_0) + \int_0^\tau e^{a(\tau-s)}M_\tau^{*,-1}M_s^*\left(\nabla, \frac{\partial}{\partial y}\right)\omega_a(Z_s)\cdot(U_sdW_s, dB_s).$$

The proof of Proposition 5.1 is completed. ∎

5.3. A probabilistic representation formula of k-forms

The formula (5.1) in Proposition 5.1 gives a probabilistic representation of k-form in terms of the time derivative and the covariant derivative of its Poisson semigroup composed with the background radiation process on $M \times \mathbb{R}^+$. In this subsection we prove a probabilistic representation formula of k-forms which uses only the time derivative of the Poisson semigroup.

Theorem 5.2 *Suppose that $W_{\phi,k} \geq -a$, where a is a non-negative constant. Then, for all $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$(5.2) \quad \frac{1}{2}\omega(x) = \lim_{y \to \infty} E_y\left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s^\mu, B_s) dB_s \mid X_\tau^\mu = x\right],$$

where

$$\omega_a(x,y) = e^{-y\sqrt{a+\square_\phi}}\omega(x), \quad \forall\, (x,y) \in M \times \mathbb{R}^+.$$

Proof. Let $\eta \in C_0^\infty(\Lambda^k T^*M)$. By Proposition 5.1, we have

$$\eta(X_\tau) = e^{a\tau} M_\tau^{*,-1} \eta_a(X_0, B_0) + \int_0^\tau e^{a(\tau-s)} M_\tau^{*,-1} M_s^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

Hence

$$\int_M \left\langle E_y\left[\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s \bigg| X_\tau = x\right], \eta(x)\right\rangle d\mu(x)$$

$$= E_y\left[\left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, \eta(X_\tau)\right\rangle\right]$$

$$= I_1 + I_2,$$

where

$$I_1 = E_y\left[\left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, e^{a\tau} M_\tau^{*,-1} \eta_a(X_0, B_0)\right\rangle\right],$$

$$I_2 = E_y\left[\left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y}\omega_a(Z_s) dB_s, \right.\right.$$

$$\left.\left. \int_0^\tau e^{a(\tau-s)} M_\tau^{*,-1} M_s^*(\nabla, \partial_y)\eta_a(Z_s) \cdot (U_s dW_s, dB_s)\right\rangle\right].$$

Using the martingale property of the Itô stochastic integral, we have

$$I_1 = E_y\left[\left\langle \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s, \eta_a(X_0, B_0)\right\rangle\right]$$

$$= E_y\left[\left\langle E\left[\int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y}\omega_a(X_s, B_s) dB_s \bigg| (X_0, B_0)\right], \eta_a(X_0, B_0)\right\rangle\right]$$

$$= 0.$$

By the L^2-isometry of the Itô stochastic integral, we have

$$
\begin{aligned}
I_2 &= E_y\left[\int_0^\tau \left\langle M_\tau M_s^{-1}\frac{\partial}{\partial y}\omega_a(Z_s),\, M_\tau^{*,-1}M_s^*\frac{\partial}{\partial y}\eta_a(Z_s)\right\rangle ds\right]\\
&= E_y\left[\int_0^\tau \left\langle \frac{\partial}{\partial y}\omega_a(Z_s),\, M_s^{-1,*}M_\tau^* M_\tau^{*,-1}M_s^*\frac{\partial}{\partial y}\eta_a(Z_s)\right\rangle ds\right]\\
&= E_y\left[\int_0^\tau \left\langle \frac{\partial}{\partial y}\omega_a(Z_s),\, \frac{\partial}{\partial y}\eta_a(Z_s)\right\rangle ds\right].
\end{aligned}
$$

The Green function of the background radiation process is given by $2(y\wedge z)$. Thus

$$
E_y\left[\int_0^\tau \left\langle \frac{\partial}{\partial y}\omega_a(X_s,B_s),\, \frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right] =
$$
$$
= 2\int_M\int_0^\infty (y\wedge z)\left\langle \frac{\partial}{\partial z}\omega_a(x,z),\, \frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dz d\mu(x).
$$

Using the spectral decomposition, we have the Littlewood-Paley identity

$$
\lim_{y\to\infty}\int_M\int_0^\infty (y\wedge z)\left\langle \frac{\partial}{\partial z}\omega_a(x,z),\, \frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dz d\mu(x) = \int_M \langle \omega(x),\eta(x)\rangle d\mu(x).
$$

Therefore

$$
\langle \omega,\eta\rangle_{L^2(\mu)} = 2\lim_{y\to\infty}\int_M\left\langle E_y\left[\int_0^\tau e^{a(s-\tau)}M_\tau M_s^{-1}\right.\right.
$$
$$
\left.\left.\frac{\partial}{\partial y}\omega_a(X_s,B_s)\cdot dB_s\Big|X_\tau = x\right],\eta(x)\right\rangle d\mu(x).
$$

Since the above identity holds for all $\eta\in C_c^\infty(M,\Lambda^k T^*M)$, we get

$$
\omega(x) = 2\lim_{y\to\infty}E_y\left[\int_0^\tau M_\tau M_s^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)\cdot dB_s\mid X_\tau = x\right].
$$

The proof of Theorem 5.2 is completed.   ∎

5.4. Representation of Riesz transforms on k-forms

Following [65, 5], we consider the Riesz transforms $d(a+\Box_{\phi,k})^{-1/2}$ and $d_\phi^*(a+\Box_{\phi,k})^{-1/2}$ associated with the Witten Laplacian on k-forms. To simplify the notations, let

$$
\begin{aligned}
R_a^1(\Box_{\phi,k}) &:= d(a+\Box_{\phi,k})^{-1/2}\in \mathrm{End}(\Lambda^k T^*M, \Lambda^{k+1}T^*M),\\
R_a^2(\Box_{\phi,k}) &:= d_\phi^*(a+\Box_{\phi,k})^{-1/2}\in \mathrm{End}(\Lambda^k T^*M, \Lambda^{k-1}T^*M).
\end{aligned}
$$

We have the following probabilistic representation formulas of the Riesz transforms on k-forms.

Theorem 5.3 *For all $\omega \in C_0^\infty(\Lambda^k T^* M)$, $a \geq 0$, let*

$$Q_{k,a}\omega(x,y) = e^{-y\sqrt{a+\square_{\phi,k}}}\omega(x), \quad \forall \ x \in M, \ y \geq 0.$$

Then, for all $x \in M$, we have

$$(5.3) \qquad -\frac{1}{2}R_a^1(\square_{\phi,k})\omega(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} \right.$$
$$\left. d_k Q_{k,a}(\omega)(Z_s) dB_s \Big| X_\tau = x \right],$$

$$(5.4) \qquad -\frac{1}{2}R_a^2(\square_{\phi,k})\omega(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k-1} M_{s,k-1}^{-1} \right.$$
$$\left. d_{\phi,k}^* Q_{k,a}(\omega)(Z_s) dB_s \Big| X_\tau = x \right].$$

Proof. By Theorem 5.2, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$, we have

$$\frac{1}{2}\omega(x) = \lim_{y\to+\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k} M_{s,k}^{-1} \frac{\partial}{\partial y} Q_{k,a}\omega(Z_s) dB_s \mid X_\tau = x \right].$$

Replacing ω by $d_k(a + \square_{\phi,k})^{-1/2}\omega$, we obtain

$$-\frac{1}{2}R_a^1(\square_{\phi,k})\omega(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} \sqrt{a + \square_{\phi,k+1}} Q_{k+1,a} \right.$$
$$\left. d_k(a + \square_{\phi,k})^{-1/2}\omega(Z_s) dB_s \Big| X_\tau = x \right].$$

Using the commutative formula

$$d_k \sqrt{a + \square_{\phi,k}}\omega = \sqrt{a + \square_{\phi,k+1}} d_k \omega,$$

we obtain

$$\frac{1}{2}R_a^1(\square_{\phi,k})\omega(x) = -\lim_{y\to\infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} Q_{k+1,a} \right.$$
$$\left. d_k\omega(Z_s) dB_s \Big| X_\tau = x \right].$$

Using again $d_k Q_{k,a} = Q_{k+1,a} d_k$, we prove (5.3). Similarly, we can prove (5.4). ∎

Remark 5.4 When $k = 0$, the formula (5.3) was proved in [42]. It is a natural extension of the well-known Gundy-Varopoulos probabilistic representation formula of the Riesz transforms on $\mathbb{R}^n$ [31] and Gundy's representation formula of the P. A. Meyer Riesz transforms on the Weiner space [30].

5.5. Representation of Riesz potential on k-forms

In this subsection we give a probabilistic representation of the Riesz potential $\Box_{\phi,k}^{-1/2}$ and the Bessel potential $(a + \Box_{\phi,k})^{-1/2}$ on k-forms, even though we do not need it in the study of the Riesz potentials on Riemannian manifolds.

Theorem 5.5 *Under the same notations as in Theorem 5.3, for all $a \geq 0$, we have*

$$(5.5) \quad \frac{1}{2}(a + \Box_{\phi,k})^{-1/2}\omega(x) = -\lim_{y \to \infty} E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k} M_{s,k}^{-1} e^{-B_s \sqrt{a + \Box_{\phi,k}}} \right.$$

$$\left. \omega(X_s) dB_s \middle| X_\tau = x \right].$$

In particular, for $a = 0$,

$$(5.6) \quad \frac{1}{2}\Box_{\phi,k}^{-1/2}\omega(x) = -\lim_{y \to \infty} E_y \left[\int_0^\tau M_{\tau,k} M_{s,k}^{-1} e^{-B_s \sqrt{\Box_{\phi,k}}} \omega(X_s) dB_s \middle| X_\tau = x \right].$$

Proof. Applying the general representation formula (5.2) to $(a + \Box)_{\phi,k}^{-1/2}\omega$, the formula (5.5) follows. Taking $a = 0$, we get (5.6). ∎

6. Proof of Theorem 1.6 and Theorem 1.7

In this section we prove Theorem 1.6 and Theorem 1.7. It would be interesting to ask whether one can give an analytic proof of Theorem 1.6 without using "the magic world of Brownian motion".

6.1. Proof of Theorem 1.6

In this subsection, we prove Theorem 1.6. More precisely, we prove the following

Theorem 6.1 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that e^{tL} is conservative, $W_{\phi,k} \geq -a$ and $W_{\phi,k+1} \geq -a$, where a is a non-negative constant. Then, there exists a constant $C_k > 0$ such that, for all $p > 1$,*

$$(6.1) \qquad \|d(a + \Box_{\phi,k})^{-1/2}\|_{p,p} \leq CA_k(p^* - 1)^{3/2},$$
$$(6.2) \qquad \|d_\phi^*(a + \Box_{\phi,k+1})^{-1/2}\|_{p,p} \leq CA_k(p^* - 1)^{3/2},$$

where $C > 0$ is a constant independent of p and k, A_k denotes the uniform norm in the following inequality

$$\|d\omega\|_\infty \leq A_k \|\nabla \omega\|_\infty, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M).$$

Proof. In the case $p = 2$, it is well-known that, using the Gaffney integration by parts formula [23], the Riesz transform $d(a + \Box_{\phi,k})^{-1/2}$ is bounded in L^2 on all complete Riemannian manifolds and its L^2-norm is less than 1. Below, we consider the case $p \neq 2$.

For all $p > 1$, since conditional expectation $E[\cdot \mid X_\tau = x]$ is contractive in L^p, we have

$$\|d(a + \Box_{\phi,k})^{-1/2}\omega\|_p^p$$
$$= 2^p \int_M \lim_{y \to \infty} \left| E_y \left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \mid X_\tau = x \right] \right|^p d\mu(x)$$
$$\leq 2^p \liminf_{y \to \infty} \int_M E_y \left[\left| \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \right|^p \Big| X_\tau = x \right] d\mu(x)$$
$$\leq 2^p \liminf_{y \to \infty} E_y \left[\left| \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \right|^p \right],$$

where in the second step we have used Fatou's lemma. Let

$$I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s.$$

Then

(6.3) $$\|d(a + \Box_{\phi,k})^{-1/2}\omega\|_p \leq 2 \liminf_{y \to \infty} \|I_y\|_p.$$

We now estimate $\|I_y\|p$. Notice that, at any fixed point $x \in M$, for all $\omega \in \Gamma(\Lambda^k T^*M \otimes \mathbb{R})$,

$$d\omega(x, \cdot) = \sum_{i=1}^n e_i^* \wedge \nabla_{e_i}\omega(x, \cdot),$$

where $e_1, \ldots, e_n$ is a normal orthonormal basis at $T_x M$ such that $\nabla_{e_i} e_j(x) = 0$ for all $i, j = 1, \ldots, n$, and $e_1^*, \ldots, e_n^*$ is its dual basis. Let $\overline{\nabla} = \nabla^{TM \otimes \mathbb{R}}$. Then, for all $\omega \in \Gamma(\Lambda^k T^*M \otimes \mathbb{R})$, we have

$$d\omega(x,y) = \sum_{j=1}^n e_j^* \wedge \nabla_{e_j}\omega(x,y) = \begin{pmatrix} 0 & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ e_1^* \wedge & \ldots & e_j^* \wedge & \ldots & e_n^* \wedge & 0 \end{pmatrix} \begin{pmatrix} \nabla_{e_1}\omega \\ \ldots \\ \nabla_{e_j}\omega \\ \ldots \\ \nabla_{e_n}\omega \\ \partial_y\omega \end{pmatrix}(x).$$

Let A denote the $(n+1) \times (n+1)$ operator-valued matrix before $(\nabla\omega, \partial_y\omega)$. Then

$$d\omega(x,y) = A\overline{\nabla}\omega(x,y).$$

Hence, the Itô integral I_y can be reformulated as a martingale transform given by

$$(6.4) \qquad I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A \overline{\nabla} Q_{a,k} \omega(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

By the Burkholder-Davies-Gundy inequality, we have

$$\|I_y\|_p \le C_p \left\| \left\{ \int_0^\tau |e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A|^2 |\overline{\nabla} Q_{a,k} \omega(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p,$$

where (cf. Shigekawa [61, p. 50])

$$C_p = \left\{ \frac{1}{2} p(p-1) \left(\frac{p}{p-1} \right)^p \right\}^{1/2}.$$

Note that

$$A_k = \|A\| := \sup_{\omega \in \Gamma(\Lambda^k T^* M), \nabla\omega \ne 0} \frac{\|d\omega\|_\infty}{\|\nabla\omega\|_\infty} < +\infty$$

is a positive constant depending only on $k = 0, 1, \ldots, n$. Moreover, under the curvature assumptions $W_{\phi,k+1} \ge -a$, we have

$$\sup_{s \in [0,\tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1}\| \le 1.$$

Therefore

$$(6.5) \qquad \|I_y\|_p \le A_k C_p \|J_y\|_p$$

where

$$J_y = \left\{ \int_0^\tau |\overline{\nabla} Q_{a,k} \omega(X_s, B_s)|^2 ds \right\}^{1/2}.$$

Proposition 6.2 *Let $p > 1$. Then, for all $a \ge 0$, $\omega \in L^p(\Lambda^k T^* M, \mu)$ (with additional assumption $\mathrm{H}_p \omega = 0$ if $a = 0$), we have*

$$(6.6) \qquad \liminf_{y \to \infty} \|J_y\|_p \le B_p \|\omega\|_p,$$

where

$$B_p = \begin{cases} \dfrac{(2p)^{1/2}}{(p-1)^{3/2}}, & p \in (1, 2), \\[2mm] 1, & p = 2, \\[2mm] \dfrac{p}{\sqrt{2(p-2)}}, & p > 2. \end{cases}$$

Proof. Let $\omega_a(x,y) = Q_{a,k}\omega(x,y)$. By the Bochner-Weitzenböck formula, as $W_{\phi,k} \geq -a$, we have

$$
\left(\frac{\partial^2}{\partial y^2} + L\right)|\omega_a(x,y)|^2
$$

$$
= 2|\overline{\nabla}\omega_a(x,y)|^2 + 2\left\langle\left(\frac{\partial^2}{\partial y^2} - \Box_{\phi,k} + W_{\phi,k}\right)\omega_a(x,y), \omega_a(x,y)\right\rangle
$$

$$
= 2|\overline{\nabla}\omega_a(x,y)|^2 + 2 < (a + W_{\phi,k})\omega_a(x,y), \omega_a(x,y) >
$$

$$
\geq 2|\overline{\nabla}\omega_a(x,y)|^2.
$$

Therefore

$$
\sqrt{2}J_y \leq \left\|\left\{\int_0^\tau \left(\frac{\partial^2}{\partial y^2} + L\right)|\omega_a(X_s, B_s)|^2 ds\right\}^{1/2}\right\|_p,
$$

Let $N_t = |\omega_a(X_{\tau \wedge t}, B_{\tau \wedge t})|^2 - |\omega_a(X_0, B_0)|^2$. Then N_t is a continuous submartingale with the Doob-Meyer decomposition $N_t = M_t + A_t$, where M_t is the continuous martingale

$$
M_t = N_t - \int_0^{\tau \wedge t} \left(\frac{\partial^2}{\partial y^2} + L\right)|\omega_a(X_s, B_s)|^2 ds,
$$

and A_t is the bounded variation part of N_t given by

$$
A_t = \int_0^{\tau \wedge t} \left(\frac{\partial^2}{\partial y^2} + L\right)|\omega_a(X_s, B_s)|^2 ds.
$$

(i) Let $p = 2$. By Doob's stopping time theorem, we have

$$
E[A_\infty] = E[N_\infty] = \|\omega_a(X_\tau, B_\tau)\|_2^2 - \|\omega_a(X_0, B_0)\|_2^2 \leq \|\omega\|_2.
$$

This proves (6.6) with $B_2 = 1$.

(ii) Let $p > 2$. By the Lenglart-Lépingle-Pratelli inequality, we have

$$
E\left[A_\infty^{\frac{p}{2}}\right] \leq p^{\frac{p}{2}} E\left[\sup_{t \geq 0} |N_t|^{\frac{p}{2}}\right]
$$

Moreover, using Doob's martingale inequality, we have

$$
E\left[\sup_{t \geq 0} |N_t|^{\frac{p}{2}}\right] \leq \left(\frac{p/2}{p/2 - 1}\right)^{\frac{p}{2}} E\left[|N_\infty|^{\frac{p}{2}}\right].
$$

Hence

$$
\begin{aligned}
\|A_\infty^{1/2}\|_p^2 &\le \frac{p^2}{p-2}\|N_\infty\|_{\frac{p}{2}} \\
&= \frac{p^2}{p-2}\left\| |\omega(X_\tau)|^2 - |\omega_a(X_0, B_0)|^2 \right\|_{\frac{p}{2}} \\
&\le \frac{p^2}{p-2}\left(\left\| |\omega(X_\tau)|^2 \right\|_{\frac{p}{2}} + \left\| |\omega_a(X_0, B_0)|^2 \right\|_{\frac{p}{2}} \right) \\
&= \frac{p^2}{p-2}\left(\|\omega(X_\tau)\|_p^2 + \|\omega_a(X_0, B_0)\|_p^2 \right) \\
&= \frac{p^2}{p-2}\left(\|\omega\|_p^2 + \left\| e^{-y\sqrt{a+\Box_\phi}}\omega \right\|_p^2 \right).
\end{aligned}
$$

By Theorem 3.1 and Theorem 1.5, we can therefore deduce that

$$
(6.7) \quad \liminf_{y\to\infty} J_y \le \frac{1}{\sqrt{2}}\liminf_{y\to\infty}\left\| \left\{ \int_0^\tau \left(\frac{\partial^2}{\partial y^2} + L \right)|\omega_a(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p
$$
$$
\le \frac{p}{\sqrt{2(p-2)}}\|\omega\|_p.
$$

This proves (6.6) with $B_p = \frac{p}{\sqrt{2(p-2)}}$ for all $p > 2$.

  (iii) Let $1 < p < 2$. By [71, p. 641, Lemma 6.1] or [41, p. 622, Lemma 4.5], we have

$$
|\nabla\omega_a(x, y)|^2 \le \frac{1}{p(p-1)}|\omega_a(x, y)|^{2-p}\liminf_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial y^2} + L \right)(|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}}.
$$

This implies

$$
\begin{aligned}
J_y &\le \frac{1}{\sqrt{p(p-1)}}\left\| \left\{ \int_0^\tau |\omega_a(X_s, B_s)|^{2-p}\liminf_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial y^2} + L \right) \right.\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.\left. (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}}ds \right\}^{1/2} \right\|_p \\
&\le \frac{1}{\sqrt{p(p-1)}}\left\| \sup_{s\ge 0}|\omega_a(X_s, B_s)|^{\frac{2-p}{2}}\left\{ \int_0^\tau \liminf_{\varepsilon\to 0}\left(\frac{\partial^2}{\partial y^2} + L \right) \right.\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.\left. (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}}ds \right\}^{1/2} \right\|_p.
\end{aligned}
$$

Using the Hölder inequality to $2/(2 - p)$ and $2/p$, we get

(6.8) $\qquad J_y \leq \dfrac{1}{\sqrt{p(p-1)}} \|\sup_{s \geq 0} |\omega_a(X_s, B_s)|\|_p^{\frac{2-p}{2}}$

$$\left\| \int_0^\tau \liminf_{\varepsilon \to 0} \left(\frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\|_1^{1/2}.$$

We need to prove two preliminary results.

Proposition 6.3 *For all $1 < p < 2$, we have*

(6.9) $\qquad \liminf_{y \to \infty} \left\| \int_0^\tau \liminf_{\varepsilon \to 0} \left(\frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\|_1$

$$\leq 2 \left(\frac{p}{p-1} \right)^p \|\omega\|_p^p.$$

Proof. Similarly to Yoshida [71, p. 644], we set $f(x, y) = (|\omega_a|^2(x, y) + \varepsilon^2)^{p/2}$, and $N_t = f(X_t, B_t)$. Then $N_t = M_t + A_t$ is a $P_{(x,y)}$-submartingale, where M_t is the martingale part given by

$$M_t = N_t - N_0 - \int_0^t \left(\frac{\partial^2}{\partial y^2} + L \right) f(X_s, B_s) ds,$$

and A_t is the bounded variation part given by

$$A_t = \int_0^{t \wedge \tau} \left(\frac{\partial^2}{\partial y^2} + L \right) f(X_s, B_s) ds.$$

By the Lenglart-Lépingle-Pratelli inequality, we have

$$E[A_\tau] \leq 2E \left[\sup_{t > 0} |N_t - N_0| \right]$$

$$= 2E \left[\sup_{t > 0} (|\omega_a|^2(X_t, B_t) + \varepsilon^2)^{p/2} \right] + 2E \left[(|\omega_a|^2(X_0, B_0) + \varepsilon^2)^{p/2} \right].$$

Using the elementary inequality $(a + b)^{p/2} \leq a^{p/2} + b^{p/2}$ for $p \in (1, 2)$, and $a, b \geq 0$, we have

$$E[A_\tau] \leq 2E \left[\sup_{t > 0} |\omega_a|^p(X_t, B_t) \right] + 2E[|\omega_a|^p(X_0, B_0)] + 4\varepsilon^p.$$

By Doob's martingale inequality, we have

$$E \left[\sup_{t > 0} |\omega_a|^p(X_t, B_t) \right] \leq \left(\frac{p}{p-1} \right)^p E[|\omega_a|^p(X_\tau, B_\tau)].$$

Therefore

$$E\left[A_\tau\right] \leq 2\left(\frac{p}{p-1}\right)^p E\left[|\omega_a|^p(X_\tau, B_\tau)\right] + 2E\left[|\omega_a|^p(X_0, B_0)\right] + 4\varepsilon^p.$$

Taking $\varepsilon \to 0$ and using Fato's lemma, we have

$$E\left[\liminf_{\varepsilon \to 0}\int_0^\tau \left(\frac{\partial^2}{\partial y^2} + L\right)\left(|\omega_a(X_s, B_s)|^2 + \varepsilon^2\right)^{p/2}ds\right]$$

$$\leq \liminf_{\varepsilon \to 0} E\left[\int_0^\tau \left(\frac{\partial^2}{\partial y^2} + L\right)\left(|\omega_a(X_s, B_s)|^2 + \varepsilon^2\right)^{p/2}ds\right]$$

$$\leq 2\left(\frac{p}{p-1}\right)^p E\left[|\omega_a|^p(X_\tau, B_\tau)\right] + 2E\left[|\omega_a|^p(X_0, B_0)\right]$$

$$= 2\left(\frac{p}{p-1}\right)^p \|\omega(X_\tau)\|_p^p + 2E\left[|\omega_a|^p(X_0, B_0)\right]$$

$$= 2\left(\frac{p}{p-1}\right)^p \|\omega\|_p^p + 2E\left[|\omega_a|^p(X_0, B_0)\right].$$

By Theorem 3.1 and Theorem 1.5, for all $\omega \in L^p(\Lambda^k T^*M, \mu)$ (with $H_p\omega = 0$ in the case $a = 0$), we have

$$\lim_{y \to \infty} E\left[|\omega_a|^p(X_0, B_0)\right] = \lim_{y \to \infty}\left\|e^{-y\sqrt{a + \Box_{\phi,k}}}\omega\right\|_p^p = 0.$$

Hence

$$\liminf_{y \to \infty} E\left[\liminf_{\varepsilon \to 0}\int_0^\tau \left(\frac{\partial^2}{\partial y^2} + L\right)\left(|\omega_a(X_s, B_s)|^2 + \varepsilon^2\right)^{p/2}ds\right] \leq 2\left(\frac{p}{p-1}\right)^p \|\omega\|_p^p.$$

This completes the proof of (6.9). ∎

Proposition 6.4 *For all* $1 < p < 2$, *we have*

(6.10) $$\|\sup_{s \geq 0}|\omega_a(X_s, B_s)|\|_p \leq \frac{p}{p-1}\|\omega\|_p.$$

Proof. Using the semigroup domination inequality (2.7), as $W_{\phi,k} \geq -a$, we have

$$|e^{-t\Box_{\phi,k}}\omega(x)| \leq e^{at}e^{tL}|\omega|(x), \quad \forall x \in M, t > 0.$$

From this and using the Bochner subordination, we get

$$|\omega_a(x, y)| = \left|e^{-y\sqrt{a + \Box_{\phi,k}}}\omega(x)\right| \leq e^{-y\sqrt{-L}}|\omega|(x) = |\omega|(x, y).$$

This yields that

$$\left\|\sup_{s \geq 0}|\omega_a(X_s, B_s)|\right\|_p \leq \left\|\sup_{s \geq 0}|\omega|(X_s, B_s)\right\|_p.$$

Since
$$|\omega|(x,y) = e^{-y\sqrt{-L}}|\omega|(x)$$
is a $\left(\frac{\partial^2}{\partial y^2} + L\right)$-harmonic function, $|\omega|(X_{\tau \wedge t}, B_{\tau \wedge t})$ is a martingale. By Doob's martingale inequality, we have

$$\Big\| \sup_{s \geq 0} |\omega|(X_{\tau \wedge t}, B_{\tau \wedge t}) \Big\|_p \leq \frac{p}{p-1} \big\| |\omega|(X_\tau, B_\tau) \big\|_p = \frac{p}{p-1} \|\omega\|_p.$$

The maximal inequality (6.10) is proved. ∎

End of Proof of Proposition 6.2. It remains to prove (6.6) for $1 < p < 2$. Combining (6.8), (6.9) and (6.10), for all $1 < p < 2$, we have

$$(6.11) \qquad J_y \leq \frac{1}{\sqrt{p(p-1)}} \sqrt{2\left(\frac{p}{p-1}\right)^p \left(\frac{p}{p-1}\right)^{\frac{2-p}{2}}} \|\omega\|_p.$$

This completes the proof of (6.6) for all $1 < p < 2$ with

$$B_p = \frac{(2p)^{1/2}}{(p-1)^{3/2}}.$$ ∎

End of Proof of Theorem 6.1. By (6.3), (6.5), (6.6) and Proposition 6.2, we have

$$\|d(a + \Box_{\phi,k})^{-1/2}\omega\|_p \quad \leq \quad \frac{2p\,C_p\,A_k}{\sqrt{2(p-2)}} \|\omega\|_p, \qquad \forall p > 2,$$

$$\|d(a + \Box_{\phi,k})^{-1/2}\omega\|_p \quad \leq \quad \frac{2C_p\,(2p)^{1/2}A_k}{(p-1)^{3/2}} \|\omega\|_p, \quad \forall 1 < p < 2.$$

Note that
$$C_p = \left\{ \frac{1}{2}p(p-1)\left(\frac{p}{p-1}\right)^p \right\}^{1/2}.$$

Therefore, for all $p > 2$,

$$\frac{2p\,C_p}{\sqrt{2(p-2)}} = \sqrt{\left(\frac{p}{p-1}\right)^p \frac{p^3(p-1)}{p-2}} \leq \sqrt{e}(p-1)^{3/2}(1 + o(1)),$$

and for all $1 < p < 2$,

$$\frac{2C_p\,(2p)^{1/2}}{(p-1)^{3/2}} = 2\left(\frac{p}{p-1}\right)^{1+p/2} \leq 2\sqrt{e}\left(\frac{1}{p-1}\right)^{3/2}(1 + o(1)).$$

This finishes the proof of (6.1) in Theorem 1.5.

It remains to prove (6.2). Let $(\ )^*_\phi$ denote the L^2-adjoint of the operator in $(\ )$ with respect to $d\mu = e^{-\phi}dv$. Then

$$d^*_{\phi,k-1}(a + \square_{\phi,k})^{-1/2} = ((a + \square_{\phi,k})^{-1/2}d_{k-1})^*_\phi = (d_{k-1}(a + \square_{\phi,k-1})^{-1/2})^*_\phi.$$

Using the duality argument, for all $p > 1$, we have

$$(6.12) \qquad \|d^*_{\phi,k-1}(a + \square_{\phi,k})^{-1/2}\|_{p,p} = \|(a + \square_{\phi,k})^{-1/2}d_{k-1}\|_{q,q}$$
$$= \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q}$$

Indeed, for all $\omega \in L^p(\Lambda^k T^*M, \mu)$ and all $\eta \in L^q(\Lambda^{k-1}T^*M, \mu)$, we have

$$\|d^*_{\phi,k-1}(a + \square_{k,\phi})^{-1/2}\omega\|_p = \sup_{\|\eta\|_q=1} \int_M <d^*_{\phi,k-1}(a + \square_{\phi,k})^{-1/2}\omega, \eta> d\mu$$
$$= \sup_{\|\eta\|_q=1} \int_M <\omega, (a + \square_{\phi,k})^{-1/2}d_{k-1}\eta> d\mu$$
$$\leq \sup_{\|\eta\|_q=1} \|\omega\|_p \|(a + \square_{k,\phi})^{-1/2}d_{k-1}\eta\|_q$$
$$\leq \|(a + \square_{\phi,k})^{-1/2}d_{k-1}\|_{q,q} \|\omega\|_p$$
$$= \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q} \|\omega\|_p.$$

Thus

$$\|d^*_{\phi,k-1}(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q}.$$

Similarly, we can prove that

$$\|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q} \leq \|d^*_{\phi,k-1}(a + \square_{\phi,k})^{-1/2}\|_{p,p}.$$

This proves (6.12). By (6.1) and (6.12), we finish the proof of (6.2) in Theorem 1.5. ∎

6.2. Proof of Theorem 1.7 and Theorem 1.8

By (6.4), the Itô integral I_y can be reformulated as a martingale transform on $M \times \mathbb{R}$:

$$I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M^{-1}_{s,k+1} A U_s U^{-1}_s \overline{\nabla}\omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

By the Burkholder sharp L^p-inequality for martingale transforms, cf. Theorem 4.3, we get

$$\|I_y\|_p \leq (p^* - 1) \sup_{s \in [0,\tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M^{-1}_{s,k+1} A U_s\|_{\text{op}}$$
$$\times \left\| \int_0^\tau U^{-1}_s \overline{\nabla}\omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p.$$

Under the curvature assumption $W_{\phi,k+1} \geq -a$, we have

$$\sup_{s \in [0,\tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A U_s\|_{\mathrm{op}} \leq A_k.$$

Therefore

$$(6.13) \qquad \|I_y\|_p \leq (p^* - 1) A_k \left\| \int_0^\tau U_s^{-1} \overline{\nabla} \omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p.$$

By Itô's formula, we can prove that

$$U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0) = \int_0^\tau U_s^{-1} \overline{\nabla} \omega_a(X_t, B_t)(U_t dW_t, dB_t)$$
$$- \int_0^\tau U_s^{-1} \left(\frac{\partial^2}{\partial y^2} + \Delta - \nabla_{\nabla\phi} \right) \omega_a(X_s, B_s) ds.$$

By the Bochner-Weitzenböck formula $\Box_{\phi,k} = -\Delta_\phi + W_{\phi,k}$, we have

$$\left(\frac{\partial^2}{\partial y^2} + \Delta - \nabla_{\nabla\phi} \right) \omega_a(x, y) = (\Box_{\phi,k} + a + \Delta - \nabla_{\nabla\phi}) \omega_a(x, y)$$
$$= (-\Delta_\phi + W_{\phi,k} + a + \Delta - \nabla_{\nabla\phi}) \omega_a(x, y)$$
$$= (a + W_{\phi,k}) \omega_a(x, y).$$

Therefore, as $W_{\phi,k} = -a$, it holds that

$$(6.14) \quad \int_0^\tau U_s^{-1} \overline{\nabla} \omega_a(X_t, B_t)(U_t dW_t, dB_t) = U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0).$$

Combining (6.3), (6.13) with (6.14), we obtain

$$\|R_a^1(\Box_{\phi,k})\omega\|_p \leq 2(p^* - 1) A_k \liminf_{y \to \infty} \left\| U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0) \right\|_p$$
$$= 2(p^* - 1) A_k \liminf_{y \to \infty} \|U_\tau^{-1} \omega(X_\tau) - \omega(X_0, B_0)\|_p$$
$$\leq 2(p^* - 1) A_k \liminf_{y \to \infty} \left[\|\omega(X_\tau)\|_p + \|\omega_a(X_0, B_0)\|_p \right].$$

Since X_τ has the law μ, we have

$$\|\omega(X_\tau)\|_p = \|\omega\|_p.$$

On the other hand, by Theorem 1.5 and Theorem 3.1, for all $p > 1$, and $\omega \in L^p(\Lambda^k T^* M, \mu)$ (with the additional condition $H_p \omega = 0$ when $a = 0$), we have

$$(6.15) \qquad \lim_{y \to \infty} \|\omega_a(X_0, B_0)\|_p = \lim_{y \to \infty} \left\| e^{-y\sqrt{a + \Box_{\phi,k}}} \omega \right\|_p = 0.$$

Thus

$$\|R_a^1(\square_{\phi,k})\omega\|_p \le 2(p^* - 1)A_k\|\omega\|_p.$$

This implies that

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \le 2A_k(p^* - 1).$$

From this and using (6.12), we have

$$\|d_\phi^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \le 2A_k(p^* - 1).$$

The proof of Theorem 1.7 is completed. Note that when $k = 0$, we have $A_0 = 1$.

In the particular case where $M = \Gamma\backslash\mathbb{H}^n(-1)$, we have $W_k = -k(n-k)\mathrm{Id}$. From this and using Theorem 1.7, we can easily prove Theorem 1.8. ■

Remark 6.5 In general, using the weighted Bochner-Weitzenböck formula, we have

$$\int_0^\tau U_s^{-1}\left(\frac{\partial^2}{\partial y^2} + \Delta - \nabla_{\nabla\phi}\right)\omega_a(X_s, B_s)ds = \int_0^\tau U_s^{-1}\left(a + W_{\phi,k}\right)\omega_a(X_s, B_s)ds \ne 0$$

except that $W_{\phi,k} = -a$. Under the assumption of Theorem 1.6, one cannot prove that its L^p norm is dominated by $C_k(p^* - 1)\|\omega\|_p$. This explains why we claimed before the statement of Theorem 1.7 that $W_{\phi,k+1} \ge -a$ and $W_{\phi,k} = -a$ give us a reasonable condition to extend the asymptotically sharp bound of the form $O(p^*-1)$ to the Riesz transforms associated with the Hodge (or Witten) Laplacian on forms on complete Riemannian manifolds.

6.3. Examples

Example 6.6 Let $M = \mathbb{R}^n$, $\phi = 0$, and $d\mu(x) = dx$. Then $\|d\square_k^{-1/2}\|_{p,p} \le C_k(p^* - 1)$ for all $p > 1$ and $k = 0, 1 \ldots, n$. Let $\omega = \omega_I dx^I$, where $I = (i_1, \ldots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$. Then $\square_k\omega = -\Delta\omega_I dx^I$, and

$$d\square_k^{-1/2}\omega = d(-\Delta)^{-1/2}\omega_I dx^I = \sum_{j\notin I}\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}\omega_I dx^j \wedge dx^I.$$

By the estimate (1.1) of Pichorides [57], Iwaniec and Martin [34] and Bañuelos-Wang [10],

$$\|d\square_k^{-1/2}\|_{p,p} = \sup_{\omega\ne 0}\frac{\|d\square_k^{-1/2}\omega\|_p}{\|\omega\|_p} \ge \sup_{\omega_I\ne 0}\frac{\|R_j\omega_I\|_p}{\|\omega\|_p} = \|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right).$$

Thus, when $p \to 1$, we have

$$\|d\square_k^{-1/2}\|_{p,p} \geq \frac{2}{\pi} \frac{1}{p-1} (1 + o(1)).$$

Combining this with $\|d\square_k\|_{p,p} \leq C_k(p^* - 1)$, when $p \to 1$, we have

$$\frac{2}{\pi} \frac{1}{p-1} (1 + o(1)) \leq \|d\square_k^{-1/2}\|_{p,p} \leq \frac{C_k}{p-1}.$$

Similarly, when $p \to \infty$, we have

$$\|d\square_k^{-1/2}\|_{p,p} \geq \cot\left(\frac{\pi}{2p^*}\right) \geq \frac{2}{\pi}(p-1)(1 + o(1)).$$

Hence

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d\square_k^{-1/2}\|_{p,p} \leq C_k(p-1).$$

By (6.12), we have $\|d^*\square_k^{-1/2}\|_{p,p} = \|d\square_{k-1}^{-1/2}\|_{q,q}$. The above estimates for $d\square_k^{-1/2}$ lead to similar estimates for the L^p-norm of $d^*\square_k^{-/2}$. More precisely, when $p \to 1$,

$$\frac{2}{\pi}(p-1)^{-1}(1 + o(1)) \leq \|d^*\square_k^{-1/2}\|_{p,p} \leq C_{k-1}(p-1)^{-1},$$

and when $p \to \infty$,

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d^*\square_k^{-1/2}\|_{p,p} \leq C_{k-1}(p-1).$$

Thus, in the Euclidean case, an upper bound of the order $O(p^* - 1)$ of the L^p-norm of the Riesz transforms $d\square_k^{-1/2}$ and $d^*\square_k^{-1/2}$ are asymptotically sharp when $p \to 1$ and when $p \to \infty$.

Example 6.7 Let $M = \mathbb{R}^n$, $\phi(x) = \frac{\|x\|^2}{2}$, and μ be the standard Gaussian measure on $\mathbb{R}^n$. Then $L = \Delta - x \cdot \nabla$ is the Ornstein-Uhlenbeck operator. The k-th Witten Laplacian on the Gaussian space is given by

$$\square_{\phi,k} = k - L, \qquad k = 0, \ldots, n,$$

and the k-th Weitzenböck curvature is

$$W_{\phi,k} = k, \qquad k = 0, \ldots, n.$$

By Theorem 1.6, for all $p > 1$ and all $a \geq 0$, we have

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}.$$

On the other hand, taking again $\omega = \omega_I dx^I$ as in the above example, we get

$$d\square_{\phi,k}^{-1/2}\omega = d(k-L)^{-1/2}\omega_I dx^I = \sum_{j \notin I} \frac{\partial}{\partial x_j}(k-L)^{-1/2}\omega_I dx^j \wedge dx^I,$$

from which we get

$$\|d\square_{\phi,k}^{-1/2}\|_{p,p} = \sup_{\omega \neq 0} \frac{\|d\square_{\phi,k}^{-1/2}\omega\|_p}{\|\omega\|_p} \geq \sup_{\omega_I \neq 0} \frac{\|\frac{\partial}{\partial x_j}(k-L)^{-1/2}\omega_I\|_p}{\|\omega_I\|_p}$$
$$= \left\|\frac{\partial}{\partial x_j}(k-L)^{-1/2}\right\|_{p,p}.$$

For $n = 1$, let $D = d/dx$. Then $L = D^2 - xD$. By Larsson-Cohn [37], for any $a > 0$, taking $f_a(x) = x/a$ for $|x| \leq a$ and $f_a(x) = \text{sign}(x)$ for $|x| > a$, where $x \in \mathbb{R}$, we have

$$\limsup_{a \to 0} \frac{\|D^2(-L)^{-1/2}f_a\|_p}{\|f_a'\|_p} \geq \frac{2}{\pi}\frac{1}{p-1}(1 + o(1)).$$

Notice that $D^2(-L)^{-1/2}f_a = D(1-L)^{-1/2}f_a'$. Taking $\omega_I(x_1,\ldots,x_n) = f_a(x_j)$, we can prove that when $p \to 1$,

$$\|d\square_{\phi,1}^{-1/2}\|_{p,p} \geq \frac{2}{\pi}\frac{1}{p-1}(1 + o(1)).$$

Therefore, when $p \to 1$, we have

$$\frac{2}{\pi}\frac{1}{p-1}(1 + o(1)) \leq \|d\square_{\phi,1}^{-1/2}\|_{p,p} \leq \frac{C_1}{(p-1)^{3/2}}.$$

Similarly, we can prove that, when $p \to \infty$, we have $\|d\square_{\phi,1}^{-1/2}\|_{p,p} \geq \frac{2}{\pi}(p-1)(1 + o(1))$. Hence

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d\square_{\phi,1}^{-1/2}\|_{p,p} \leq C_1(p-1)^{3/2}.$$

By (6.12), we have $\|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} = \|d\square_{\phi,1}^{-1/2}\|_{q,q}$. The above estimates for $d\square_{\phi,1}^{-1/2}$ lead to similar estimates for the L^p-norm of $d_\phi^*\square_{\phi,2}^{-1/2}$. More precisely, when $p \to 1$,

$$\frac{2}{\pi}(p-1)^{-1}(1 + o(1)) \leq \|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} \leq C_1(p-1)^{-3/2},$$

and when $p \to \infty$,

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} \leq C_1(p-1)^{3/2}.$$

It would be very interesting to know whether we can replace $(p-1)^{3/2}$, $(p-1)^{-3/2}$ in the above estimates by $(p-1)$, $(p-1)^{-1}$ respectively.

7. L^p-Hodge decomposition

In this section we prove the Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvatures.

Let M be a complete Riemannian manifold. By spectral decomposition, cf. [44], we can prove that

$$dd_\phi^* \Box_\phi^{-1} \omega := \lim_{N \to \infty} \int_0^N dd_\phi^* e^{-s\Box_\phi} \omega ds$$

exists in L^2 and hence μ-a.s. Similarly,

$$d_\phi^* d \Box_\phi^{-1} \omega := \lim_{N \to \infty} \int_0^N d_\phi^* d e^{-s\Box_\phi} \omega ds$$

exists in L^2 and hence μ-a.s. This yields the Weak $L^2(\mu)$-Hodge orthogonal decomposition formula

$$\omega = \mathrm{H}\omega + dd_\phi^* \Box_\phi^{-1} \omega + d_\phi^* d \Box_\phi^{-1} \omega,$$

where $\mathrm{H}\omega$ denotes the harmonic projection of ω from $L^2(\Lambda^k T^* M, \mu)$ to $\mathrm{Ker}\Box_{\phi,k} \cap L^2(\Lambda^k T^* M, \mu)$.

By duality argument, we can prove that, if the Riesz transform $d\Box_{\phi,k}^{-1/2}$ is a bounded operator from $L^p(\Lambda^k T^* M, \mu)$ into $L^p(\Lambda^{k+1} T^* M, \mu)$ for a fixed $p > 1$, then $\Box_{\phi,k}^{-1/2} d_\phi^* = (d\Box_{\phi,k}^{-1/2})_\phi^*$ is bounded from $L^q(\Lambda^{k+1} T^* M, \mu)$ into $L^q(\Lambda^k T^* M, \mu)$, where $q = \frac{p}{p-1}$. Moreover,

$$\|(d\Box_{\phi,k}^{-1/2})_\phi^*\|_{p,p} = \|d\Box_{\phi,k}^{-1/2}\|_{q,q}.$$

Since $d_\phi^* \Box_{\phi,k+1}^{-1/2} = \Box_{\phi,k}^{-1/2} d_\phi^*$, we obtain

$$\|d_\phi^* \Box_{\phi,k+1}^{-1/2}\|_{p,p} = \|d\Box_{\phi,k}^{-1/2}\|_{q,q}.$$

Suppose that the Riesz transforms $d\Box_{\phi,k-1}^{-1/2}$ and $d\Box_{\phi,k}^{-1/2}$ are bounded in $L^p(\mu)$ and in $L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then, for all $\omega \in C_0^\infty(\Lambda^k T^* M)$, we have

$$\begin{aligned}
\|dd_\phi^* \omega\|_p &\leq \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \, \|\Box_{\phi,k-1}^{1/2} d_\phi^* \omega\|_p \\
&= \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \, \|d_\phi^* \Box_{\phi,k}^{1/2} \omega\|_p \\
&\leq \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \, \|d_\phi^* \Box_{\phi,k}^{-1/2}\|_{p,p} \, \|\Box_{\phi,k}^{1/2} \Box_{\phi,k}^{1/2} \omega\|_p \\
&= \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \, \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} \, \|\Box_{\phi,k} \omega\|_p.
\end{aligned}$$

Equivalently,

$$\|dd_\phi^* \Box_{\phi,k}^{-1} \omega\|_p \leq \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \, \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} \|\omega\|_p.$$

Similarly, we have

$$\|d_\phi^* d\omega\|_p \leq \|d_\phi^* \Box_{\phi,k+1}^{-1/2}\|_{p,p} \|\Box_{\phi,k+1}^{1/2} d\omega\|_p = \|d_\phi^* \Box_{\phi,k+1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{1/2}\omega\|_p$$
$$\leq \|d_\phi^* \Box_{\phi,k+1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|\Box_{\phi,k}^{1/2}\Box_{\phi,k}^{1/2}\omega\|_p$$
$$= \|d_\phi^* \Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q} \|\Box_{\phi,k}\omega\|_p.$$

Equivalently,

$$\|d_\phi^* d\Box_{\phi,k}^{-1}\omega\|_p \leq \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q}\|\omega\|_p.$$

Therefore

(7.1)
$$\|\omega - \mathrm{H}\omega\|_p \leq \left[\|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} + \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q} \right] \|\omega\|_p.$$

Now we are able to give the proof of Theorem 1.9.

Proof of Theorem 1.9. By Theorem 1.6, for all $p > 1$ and $q = \frac{p}{p-1}$, we get

$$\|dd_\phi^* \Box_{\phi,k}^{-1}\omega\|_p \leq \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} \|\omega\|_p$$
$$\leq C_{k-1}^2 (p^* - 1)^{3/2}(q^* - 1)^{3/2}\|\omega\|_p = C_{k-1}^2 (p^* - 1)^3 \|\omega\|_p.$$

Similarly, we have

$$\|d_\phi^* d\Box_{\phi,k}^{-1}\omega\|_p \leq \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q} \|\omega\|_p \leq C_k^2 (p^* - 1)^3 \|\omega\|_p.$$

Therefore, the operators $dd_\phi^* \Box_{\phi,k}^{-1}$ and $d_\phi^* d\Box_{\phi,k}^{-1}$ are bounded in L^p for all $p > 1$. Note that

$$dd_\phi^* \Box_\phi^{-1}\omega - \int_0^N dd_\phi^* e^{-s\Box_\phi}\omega ds = \int_N^\infty dd_\phi^* e^{-s\Box_\phi}\omega ds = \int_0^\infty dd_\phi^* e^{-(N+t)\Box_\phi}\omega dt$$
$$= \int_0^\infty dd_\phi^* e^{-t\Box_\phi}\left(e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right) dt = dd_\phi^* \Box_\phi^{-1}\left(e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right),$$

which yields

$$\left\|\int_0^N dd_\phi^* e^{-t\Box_\phi}\omega dt - dd_\phi^* \Box_\phi^{-1}\omega\right\|_p = \left\|dd_\phi^* \Box_\phi^{-1}\left(e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right)\right\|_p$$
$$\leq \left\|dd_\phi^* \Box_\phi^{-1}\right\|_{p,p} \left\|e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right\|_p.$$

Similarly,

$$\left\|\int_0^N d_\phi^* d e^{-t\Box_\phi}\omega dt - d_\phi^* d\Box_\phi^{-1}\omega\right\|_p = \left\|d_\phi^* d\Box_\phi^{-1}\left(e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right)\right\|_p$$
$$\leq \left\|d_\phi^* d\Box_\phi^{-1}\right\|_{p,p} \left\|e^{-N\Box_\phi}\omega - \mathrm{H}\omega\right\|_p.$$

Since $W_k \geq 0$, by Theorem 1.5, $\lim_{N\to\infty} \|e^{-N\square_\phi}\omega - \mathrm{H}\omega\|_p = 0$. As $dd_\phi^*\square_{\phi,k}^{-1}$ and $dd_\phi^*\square_{\phi,k}^{-1}$ are bounded in L^p, we have

$$(7.2) \qquad \lim_{N\to\infty} \left\| \int_0^N dd_\phi^* e^{-s\square_\phi}\omega\, ds - dd_\phi^*\square_{\phi,k}^{-1}\omega \right\|_p = 0,$$

$$(7.3) \qquad \lim_{N\to\infty} \left\| \int_0^N d_\phi^* d\, e^{-s\square_\phi}\omega\, ds - d_\phi^* d\square_{\phi,k}^{-1}\omega \right\|_p = 0.$$

Now

$$e^{-t\square_\phi}\omega - \omega = \int_0^t \frac{\partial}{\partial s} e^{-s\square_\phi}\omega\, ds \quad \text{in} \quad L^p(\Lambda^k T^* M, \mu).$$

Taking $t \to \infty$ and using $\frac{\partial}{\partial s} e^{-s\square_\phi}\omega = -\square_\phi e^{-s\square_\phi}\omega$, we get

$$\omega - \mathrm{H}\omega = \int_0^\infty (dd_\phi^* + d_\phi^* d) e^{-t\square_\phi}\omega\, ds.$$

Combining this with (7.2) and (7.3), we obtain the Weak L^p-Hodge decomposition

$$\omega = \mathrm{H}\omega + dd_\phi^*\square_{\phi,k}^{-1}\omega + d_\phi^* d\square_{\phi,k}^{-1}\omega.$$

Finally, from (7.1) we can deduce that

$$\|(\mathrm{I} - \mathrm{H})\omega\|_p \leq (C_{k-1}^2 + C_k^2)(p^* - 1)^3 \|\omega\|_p.$$

Thus, the Hodge harmonic projection

$$\mathrm{H} : L^p(\Lambda^k T^* M, \mu) \to (\mathrm{Ker}\square_\phi)^\perp \cap L^p(\Lambda^k T^* M, \mu)$$

is bounded. The proof of Theorem 1.9 is completed. ∎

The argument used in the proof of Theorem 1.9 goes back to [65]. It can be considered as a natural extension of the heat equation approach initiated by Milgram-Rosenbloom [54] for the Hodge decomposition theorem on compact Riemannian manifolds and developed by Gaffney [24] for the L^2-Hodge decomposition theory on complete Riemannian manifolds.

References

[1] ARCOZZI, N.: Riesz transforms on compact Lie groups, spheres and Gauss space. *Ark. Mat.* **36** (1998), no. 2, 201–231.

[2] AUSCHER, P., COULHON, T., DUONG, X. T. AND HOFMANN, S.: Riesz transform on manifolds and heat kernel regularity. *Ann. Sci. École Norm. Sup. (4)* **37** (2004), no. 6, 911–957.

[3] BAKRY, D.: Transformations de Riesz pour les semi-groupes symétriques. II. Étude sous la condition $\Gamma_2 \geq 0$. In *Séminaire de Probabilités, XIX*, 145–174. Lecture Notes in Math. **1123**. Springer, Berlin, 1985.

[4] BAKRY, D.: Un critére de non-explosion pour certaines diffusions sur une variété Riemannienne compléte. *C.R. Acad. Sci. Paris Sér. I. Math.* **303** (1986), no. 1, 23–26.

[5] BAKRY, D.: Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée. In *Séminaire de Probabilités, XXI*, 137–172. Lecture Notes in Math. **1247**. Springer, Berlin, 1987.

[6] BAKRY, D.: The Riesz transforms associated with second order differential operators. In *Seminar on Stochastic Processes, 1988*, 1–43. Progr. Probab. **17**. Birkhäuser Boston, Boston, Ma, 1989.

[7] BERLINE, N., GETZLER, E. AND VERGNE, M.: *Heat kernels and Dirac operators.* Grundlehren der Mathematischen Wissenschaften **298**. Springer-Verlag, Berlin, 1992.

[8] BISMUT, J. M.: The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem. *J. Funct. Anal.* **57** (1984), no. 1, 56–99.

[9] BAÑUELOS, R. AND LINDEMAN, A. II.: A martingale study of the Beurling-Ahlfors transforms in $\mathbb{R}^n$. *J. Funct. Anal.* **145** (1997), no. 1, 224–265.

[10] BAÑUELOS, R. AND WANG, G.: Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms. *Duke Math. J.* **80** (1995), no. 3, 575–600.

[11] BAÑUELOS, R. AND MÉNDEZ-HERNÁNDEZ, P.: Space-time Brownian motion and the Beurling-Ahlfors transform. *Indiana Univ. Math J.* **52** (2003), no. 4, 981–990.

[12] BUELER, E. L.: The heat kernel weighted Hodge laplacian on noncompact manifolds. *Trans. Amer. Math. Soc.* **351** (1999), no. 2, 683–713.

[13] BURKHOLDER, D. L.: A sharp and strict L^p-inequality for stochastic integrals. *Ann. Probab.* **15** (1987), no. 1, 268–273.

[14] CARRON, G., COULHON, T. AND HASSELL, A.: Riesz transform and L^p cohomology for manifolds with Euclidean ends. *Duke Math. J.* **133** (2006), no. 1, 59–93.

[15] CHERNOFF, P. R.: Essential self-adjointness of powers of generators of hyperbolic equations. *J. Funct. Anal.* **12** (1973), 401–414.

[16] COULHON, T. AND DUONG, X. T.: Riesz transforms for $1 \leq p \leq 2$. *Trans. Amer. Math. Soc.* **351** (1999), no. 3, 1151–1169.

[17] COULHON, T. AND DUONG, X. T.: Riesz transform and related inequalities on noncompact Riemannian manifolds. *Comm. Pure Appl. Math.* **56** (2003), no. 12, 1728–1751.

[18] DELLACHERIE, C., MAISONNEUVE, B. AND MEYER, P. A.: *Probabilies et Potentiel, Chapitres XVII a XXIV, Processus de Markov (fin) Complements de Calcul Stochastiques.* Hermann, 1992.

[19] De Rham, G.: *Variétés Différentielles.* Hermain, Paris, 1960.

[20] Donaldson, S. and Sullivan, D.: Quasiconformal 4-manifolds. *Acta Math.* **163** (1989), no. 3-4, 181–252.

[21] Elworthy, D., LeJan, Y. and Li, X. M.: *On the geometry of diffusion operators and stochastic flows.* Lecture Notes in Mathematics **1720**. Springer-Verlag, Berlin, 1999.

[22] Engel, K. J. and Nagel, R.: *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics **194**. Springer-Verlag, New York, 2000.

[23] Gaffney, M.: A special Stokes's theorem for complete Riemannian manifolds. *Ann. of Math. (2)* **60** (1954), 140–145.

[24] Gaffney, M.: Hilbert spaces methodes in the theory of harmonic integrals. *Trans. Amer. Math. Soc.* **78** (1955), 426–444.

[25] Gallot, S. and Meyer, D.: Opérateur de courbure et laplacien des formes différentielles d'une variété riemannnienne. *J. Math. Pures Appl. (9)* **54** (1975), no. 3, 259–284.

[26] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order.* Classics in Math. Springer-Verlag, Berlin, 2001.

[27] Grigor'yan, A. A.: On stochastically complete manifolds. *DAN SSSR* **290** (1986), 534–537. Engl. transl. *Soviet Math. Dokl.* **34** (1987), 310–313.

[28] Grigor'yan, A. A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc. (N. S.)* **36** (1999), no. 2, 135–249.

[29] Gundy, R. F.: Sur les transformations de Riesz pour le semi-groupe d'Ornstein-Uhlenbeck. *C. R. Acad. Sci. Paris Sér. I. Math* **303** (1986), no. 19, 967–970.

[30] Gundy, R. F.: *Some Topics in Probability and Analysis.* CBMS Regional Conference Series in Math. **70**. Amer. Math. Soc., Providence, RI, 1989.

[31] Gundy, R. F. and Varopoulos, N. Th.: Les transformations de Riesz et les intégrales stochastiques. *C. R. Acad. Sci. Paris Sér. A-B.* **289** (1979), no. 1, A13–A16.

[32] Helmholtz, H.: Über Integrale der hydrodynamischen Gle ichungen, welche den Wirbelbewegunggen entsprechen. *J. Reine Angew. Math.* **55** (1858), 25–55.

[33] Iwaniec, T. and Martin, G.: Quasiregular mappings in even dimensions. *Acta Math.* **170** (1993), no. 1, 29–81.

[34] Iwaniec, T. and Martin, G.: Riesz transforms and related singular integrals. *J. Reine Angew. Math.* **473** (1996), 25–57.

[35] Jacob, N.: *Pseudo differential operators and Markov processes. Vol. I. Fourier analysis and semigroups.* Imperial College Press, London, 2002.

[36] Karp, L., and Li, P.: *The heat equation on complete Riemannian manifolds.* Unpublished manuscript, 1982.

[37] Larsso-Cohn, L.: On the constants in the Meyer inequality. *Monatsch. Math.* **137** (2002), no. 1, 51–56.

[38] Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63** (1934), no. 1, 193–248.

[39] Li, P. and Schoen, R.: L^p and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.* **153** (1984), no. 3–4, 279–301.

[40] Li, X. D.: Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds. *J. Math. Pures Appl. (9)* **84** (2005), no. 10, 1295–1361.

[41] Li, X. D.: Riesz transforms for symmetric diffusion operators on complete Riemannian manifolds. *Rev. Mat. Iberoamericana* **22** (2006), no. 2, 591–648.

[42] Li, X. D.: Martingale transforms and L^p-norm estimates of Riesz transforms on complete Riemannian manifolds. *Probab. Theory Related Fields* **141** (2008), no. 1-2, 247–281.

[43] Li, X. D.: Sobolev inequalities on differential forms and $L^{p,q}$-cohomology on complete Riemannian manifolds. *J. Geom. Anal.* **20** (2010), 354–387.

[44] Li, X. D.: On the weak L^p-Hodge decomposition over complete Riemannian manifolds. *Probab. Theory Related Fields*, `doi: 10.1007/s00440-010-0270-2`.

[45] Li, X. D.: On the strong L^p-Hodge decomposition over complete Riemannian manifolds. *J. Funct. Anal.* **257** (2009), no. 11, 3617–3646.

[46] Li, X. D.: L^p-estimates and existence theorems of the $\bar{\partial}$-operator on complete Kähler manifolds. *Adv. in Math.* **224** (2010), 620–647.

[47] Lions, P. L.: *Mathematical Topics in Fluid Mechanics, Vol 1, Incompressible Methods, and Vol 2 Compressible Models.* Oxford Lecture Series in Mathematics and its Applications **3** and **10**. The Clarendon Press, Oxford University Press, New York, 1966.

[48] Lohoué, N.: Comparaison des champs de vecteurs et des puissances du Laplacien sur une variété riemannienne à courbure non positive. *J. Funct. Anal.* **61** (1985), no. 2, 164–201.

[49] Lohoué, N.: Estimation des projecteurs de de Rham Hodge de certaines variétés Riemanniennes non-compactes. *Math. Nachr.* **279** (2006), no. 3, 272–298.

[50] Malliavin, P.: Formule de la moyenne, calcul des perturbations et théorèmes d'annulation pour les formes harmoniques. *J. Functional Analysis* **17** (1974), 274–291.

[51] Meyer, P. A.: Démonstrations probabilistes des inégalités de Littlewood-Paley. I. Les inégalités classiques. In *Séminaire de Probabilités, X*, 125–141. Lecture Notes in Math. **511**. Springer, Berlin, 1976.

[52] Meyer, P. A.: Le dual de $H^1(\mathbb{R}^v)$: démonstrations probabilistes. In *Séminaire de Probabilités, XI*, 132–195. Lecture Notes in Math. **581**. Springer, Berlin, 1977.

[53] Meyer, P. A.: Transformations de Riesz pour les lois gaussiennes. In *Seminar on Probability, XVIII*, 179–193. Lecture Notes in Math. **1059**. Springer, Berlin, 1984.

[54] Milgram, A. and Rosenbloom, P.: Harmonic forms and heat conduction, I: Closed Riemannian manifolds. *Proc. Nat. Acad. Sci.* **37** (1951), 180–184.

[55] Morrey, C. B. Jr.: *Multiple integrals in the calculus of variations.* Die Grundlehren der mathematischen Wissenschaften **130**. Springer-Verlag, New York, 1966.

[56] Norris, J. R.: A complete differential formalism for stochastic calculus in manifolds. In *Séminaire de Probabilités XXVI*, 189–209. Lecture Notes in Math. **1526**. Springer, Berlin, 1992.

[57] Pichorides, S. K.: On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. *Studia Math.* **44** (1972), 165–179.

[58] Robbin, J. W., Rogers, R. C. and Temple, B.: On weak continuity and the Hodge decomposition. *Trans. Amer. Math. Soc.* **303** (1987), no. 2, 609–618.

[59] Rogers, L. C. G. and Williams, D.: *Diffusions, Markov processes and martingales, Vol. 1. Foundations.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000.

[60] Scott, C.: L^p theory of differential forms on manifolds. *Trans. Amer. Math. Soc.* **347** (1995), no. 6, 2075–2096.

[61] Shigekawa, I.: *Stochastic Analysis.* Translations of Mathematical Monographs **224**. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2004.

[62] Shigekawa, I. and Yoshida, N.: Littlewood-Paley-Stein inequality for a symmetric diffusion. *J. Math. Soc. Japan* **44** (1992), no. 2, 251–280.

[63] Stein, E. M.: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series **30**. Princeton University Press, Princeton, N.J., 1970.

[64] Stein, E. M.: *Topics in harmonic analysis related to the Littlewood-Paley theory.* Annals of Mathematics Studies **63**. Princeton University Press, Princeton, N.J.; University Tokyo Press, Tokyo, 1970.

[65] Strichartz, R.: Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.* **52** (1983), no. 1, 48–79.

[66] Strichartz, R.: L^p-contractive projections and the heat semigroup for differential forms. *J. Funct. Anal.* **65** (1986), no. 3, 348–357.

[67] Stroock, D. W. and Varadhan, S. R. S.: *Multidimensional Diffusion Processes.* Grundlehren der Mathematischen Wissenschaften **233**. Springer-Verlag, Berlin-New York, 1979.

[68] Sturm, K. T.: Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.* **456** (1994), 173–196.

[69] Witten, E.: Supersymmetry and Morse theory. *J. Differential Geom.* **17** (1982), no. 4, 661–692.

[70] Yau, S. T.: On the heat kernel of a complete Riemannian manifold. *J. Math. Pures Appl. (9)* **57** (1978), no. 2, 191–201.

[71] Yoshida, N.: Sobolev spaces on a Riemannian manifold and their equivalence. *J. Math. Kyoto Univ.* **32** (1992), no. 3, 621–654.

[72] Yoshida, N.: The Littlewood-Paley-Stein inequality on an infinite dimensional manifold. *J. Funct. Anal.* **122** (1994), no. 2, 402–427.

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