Rev. Mat. Iberoamericana **26** (2010), no. 3, 965–974

Overdetermined problems in unbounded domains with Lipschitz singularities

Alberto Farina and Enrico Valdinoci

Abstract

We study the overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma, \end{cases}$$

where Ω is a locally Lipschitz epigraph, that is C^3 on $\Gamma \subseteq \partial\Omega$, with $\partial\Omega \setminus \Gamma$ consisting in nonaccumulating, countably many points.

We provide a geometric inequality that allows us to deduce geometric properties of the sets Ω for which monotone solutions exist.

In particular, if $\mathscr{C} \in \mathbb{R}^n$ is a cone and either $n = 2$ or $n = 3$ and $f \geqslant 0$, then there exists no solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathscr{C}, \\ u > 0 & \text{in } \mathscr{C}, \\ u = 0 & \text{on } \partial\mathscr{C}, \\ \partial_\nu u = c & \text{on } \partial\mathscr{C} \setminus \{0\}. \end{cases}$$

This answers a question raised by Juan Luis Vázquez.

Introduction

Let $n \geqslant 2$. We consider an epigraph in $\mathbb{R}^n$, that is

$$\Omega := \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \Phi(x')\big\}.$$

We suppose that Ω is locally Lipschitz and that it is C^3 except, at most, at a countable family of points that do not accumulate.

2000 Mathematics Subject Classification: 35J25, 35J20, 35B65.
Keywords: Elliptic partial differential equations, rigidity results, nonexistence of solutions.

Explicitly, we suppose that there exists $\mathscr{J} \subseteq \mathbb{N}$ and a family

$$\mathscr{F} := \{p'_j \in \mathbb{R}^{n-1},\ j \in \mathscr{J}\}$$

such that

$$\inf_{j,k \in \mathscr{J}} |p'_j - p'_k| > 0 \tag{0.1}$$

and

$$\Phi \in C^3\Big(\mathbb{R}^{n-1} \setminus \Big(\bigcup_{j \in \mathscr{J}} p'_j\Big)\Big).$$

Notice that $\nabla\Phi$ exists a.e.: we suppose that

$$\|\nabla\Phi\|_{L^\infty(K)} < +\infty,$$

for any bounded set K in $\mathbb{R}^{n-1}$.

We denote $p_j := \big(p'_j, \Phi(p'_j)\big)$ and

$$\Gamma := \partial\Omega \setminus \Big(\bigcup_{j \in \mathscr{J}} p_j\Big).$$

We remark that, by construction, the exterior derivative ν is always well defined at points of Γ.

Given $c \in \mathbb{R}$, we will study the following overdetermined elliptic problem:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma. \end{cases} \tag{0.2}$$

We will prove a geometric inequality for solutions of (0.2) and some rigidity results in low dimension.

For this, we introduce some notation.

Given a smooth function v, one may consider the level sets of v: in the vicinity of $\{\nabla v \neq 0\}$, these level sets are smooth manifolds, so one can consider the principal curvatures

$$\kappa_1, \dots, \kappa_{n-1}$$

at any point of such manifolds.

We set

$$\mathscr{K} := \sqrt{\kappa_1^2 + \cdots + \kappa_{n-1}^2}.$$

Also, it is customary to consider the tangential gradient along level sets of v at these points, that is

$$\nabla_T g := \nabla g - \left(\nabla g \cdot \frac{\nabla v}{|\nabla v|}\right) \frac{\nabla v}{|\nabla v|}.$$

Thus, we may state the main results of this paper as follows:

Theorem 1. *Let $u \in C^2(\Omega \cup \Gamma) \cap C(\overline{\Omega}) \cap W^{1,\infty}_{\rm loc}(\Omega)$ be a solution of* (0.2), *with $\partial_n u > 0$ in Ω.*

Then, for any $\xi \in C^\infty_0(\mathbb{R}^n)$,

$$\int_\Omega \Big(|\nabla u|^2 \mathscr{K}^2 + |\nabla_T |\nabla u||^2 \Big) \xi^2 \leqslant \int_\Omega |\nabla u|^2 |\nabla \xi|^2. \tag{0.3}$$

Theorem 2. *Let $u \in C^2(\Omega \cup \Gamma) \cap W^{1,\infty}(\Omega)$ be a solution of* (0.2), *with $u > 0$ in Ω and Ω globally Lipschitz.*

Suppose

- *either that $n = 2$*
- *or that $n = 3$ and $f \geqslant 0$.*

Then, Ω cannot be coercive, that is, it cannot be that

$$\lim_{|x'| \to +\infty} \Phi(x') = +\infty.$$

The result in Theorem 1 may be seen as a weighted Poincaré inequality. Similar inequalities have been used first in [5, 6], where no boundary term was present, and in [2, 3] to deduce symmetry results for PDEs. In [4] related inequalities have been used for problems like (0.2) in smooth domains. Differently than [4], in this paper we take into account also domains with Lipschitz singularities: indeed, when the domains are smooth, Theorems 1 and 2 here boil down to Theorems 1.1 and 1.6 in [4].

As a side remark, we also notice that the left hand side of (0.3) is well-defined, since $\nabla u \neq 0$ in Ω.

We now use Theorem 2 in order to answer a question posed to us by Juan Luis Vázquez [7]. For this, let $\mathscr{C}$ be a cone.

More precisely, if $n = 2$, we write

$$t^+ := \begin{cases} |t| & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases} \quad \text{and} \quad t^- := \begin{cases} |t| & \text{if } t < 0, \\ 0 & \text{if } t \geqslant 0, \end{cases}$$

and, given α^+, $\alpha^- \in (0, +\infty)$, we define the cone

$$\mathscr{C} := \big\{ (x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } x_2 > \alpha^+ x_1^+ + \alpha^- x_1^- \big\}.$$

When $n \geqslant 3$, given $\alpha \in (0, +\infty)$, we write the cone as

$$\mathscr{C} := \big\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \alpha |x| \big\}.$$

With this notation, we obtain the following result:

Corollary 3. *If*

- *either* $n = 2$
- *or* $n = 3$ *and* $f \geqslant 0$,

then there exists no solution $u \in C^2(\overline{\mathscr{C}} \setminus \{0\}) \cap W^{1,\infty}(\mathscr{C})$ *of*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathscr{C}, \\ u > 0 & \text{in } \mathscr{C}, \\ u = 0 & \text{on } \partial\mathscr{C}, \\ \partial_\nu u = c & \text{on } \partial\mathscr{C} \setminus \{0\}. \end{cases} \tag{0.4}$$

Corollary 3 is a simple consequence of Theorem 2. We also recall that solutions of (0.4) satisfy

$$\partial_n u(x) > 0 \text{ for any } x \in \Omega,$$

thanks to Theorem 1.3 in [1].

We prove Theorems 1 and 2 in the forthcoming Sections 1 and 2, respectively.

1. Proof of Theorem 1

Let now $\xi \in C_0^\infty(\mathbb{R}^n)$.

We define

$$\rho := \inf_{j,k \in \mathscr{J}} |p'_j - p'_k|.$$

We recall that $\rho > 0$ because of (0.1).

We fix $K > 2$ and $\eta > 0$ such that

$$\eta \leqslant \min \left\{ 1, \frac{\rho}{2}, \frac{1}{\log K} \right\}. \tag{1.1}$$

We define

$$\tau_{\eta,K}(x) := \begin{cases} 0 & \text{if } |x| \leqslant \eta/K, \\ \dfrac{\log|x| - \log(\eta/K)}{\log K} & \text{if } \eta/K < |x| \leqslant \eta, \\ 1 & \text{if } |x| > \eta. \end{cases}$$

Notice that $\tau_{\eta,K}$ is Lipschitz continuous and

$$|\nabla \tau_{\eta,K}(x)| \leqslant \frac{\chi_{(B_\eta \setminus B_{\eta/K})}(x)}{|x|\, \log K}. \tag{1.2}$$

Here above, as customary, we have denoted by χ_A the characteristic function of the set A.

We also set

$$\xi_{\eta,K}(x) := \xi(x) \cdot \prod_{j\in\mathscr{J}} \tau_{\eta,K}(x-p_j).$$

This function is well-defined, since $\tau_{\eta,K}(x-p_j) = 1$ for $x \notin B_\eta(p_j)$ and these balls are disjoint.

We now take $\Omega_{\eta,K}$ to be an open set with C^3 boundary such that

$$\Omega \setminus \Big(\bigcup_{j\in\mathscr{J}} B_{\eta/(2K)}(p_j)\Big) \subset \Omega_{\eta,K} \subset \Omega \setminus \Big(\bigcup_{j\in\mathscr{J}} B_{\eta/(4K)}(p_j)\Big).$$

We make use of (1.4) of [4] to obtain that

$$\begin{aligned} &\int_{\Omega_{\eta,K}} \Big(|\nabla u|^2 \mathscr{K}^2 + |\nabla_T |\nabla u||^2\Big)\varphi^2 \\ &\qquad + \limsup_{\epsilon\to 0^+} \int_{\partial\Omega_{\eta,K}} \frac{\varphi^2}{\epsilon + \partial_n u}\Big(|\nabla u|^2 \partial^2_{n,\nu} u - \partial^2_{i,\nu} u \partial_i u \partial_n u\Big) \leqslant \\ &\qquad \leqslant \int_{\Omega_{\eta,K}} |\nabla u|^2 |\nabla \varphi|^2 \end{aligned} \tag{1.3}$$

for any $\varphi \in W^{1,\infty}_0(\mathbb{R}^n)$.

On the other hand, from (3.15) in [4] and (0.2) here, we know that

$$|\nabla u|^2 \partial^2_{n,\nu} u - \partial^2_{i,\nu} u \partial_i u \partial_n u = 0 \qquad \text{on } \Gamma. \tag{1.4}$$

Also, by construction,

$$\xi_{\eta,K} = 0 \qquad \text{in } \bigcup_{j\in\mathscr{J}} B_{\eta/(2K)}(p_j). \tag{1.5}$$

From (1.4) and (1.5) we thus obtain

$$\begin{aligned} &\int_{\partial\Omega_{\eta,K}} \frac{\xi^2_{\eta,K}}{\epsilon + \partial_n u}\Big(|\nabla u|^2 \partial^2_{n,\nu} u - \partial^2_{i,\nu} u \partial_i u \partial_n u\Big) = \\ &\quad = \sum_{j\in\mathscr{J}} \int_{\partial\Omega_{\eta,K} \cap B_{\eta/(2K)}(p_j)} \frac{\xi^2_{\eta,K}}{\epsilon + \partial_n u}\Big(|\nabla u|^2 \partial^2_{n,\nu} u - \partial^2_{i,\nu} u \partial_i u \partial_n u\Big) \;=\; 0. \end{aligned}$$

Consequently, by taking $\varphi := \xi_{\eta,K}$ in (1.3), we obtain

$$\int_{\Omega_{\eta,K}} \Big(|\nabla u|^2 \mathscr{K}^2 + |\nabla_T |\nabla u||^2\Big) \xi_{\eta,K}^2 \leqslant \int_{\Omega_{\eta,K}} |\nabla u|^2 |\nabla \xi_{\eta,K}|^2. \tag{1.6}$$

Also, recalling (1.1) and (1.2), a straightforward computation gives that

$$\begin{aligned} |\nabla \xi_{\eta,K}(x)| - |\nabla \xi(x)| &\leqslant |\nabla \xi_{\eta,K}(x) - \nabla \xi(x)| \leqslant \\ &\leqslant \|\xi\|_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j\in\mathscr{J}} \left(1 + \frac{1}{|x-p_j|\,\log K}\right) \chi_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))}(x) \\ &\leqslant 2\|\xi\|_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j\in\mathscr{J}} \frac{\chi_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))}(x)}{|x-p_j|\,\log K}. \end{aligned} \tag{1.7}$$

We now fix an auxiliary parameter $\delta > 0$ and we use a scaled Cauchy Inequality to deduce from (1.7) that

$$\begin{aligned} |\nabla \xi_{\eta,K}(x)|^2 &\leqslant (1+\delta)|\nabla \xi(x)|^2 \\ &\quad + C_\delta \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \left[\sum_{j\in\mathscr{J}} \frac{\chi_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))}(x)}{|x-p_j|\,\log K}\right]^2. \end{aligned}$$

Since the balls $B_\eta(p_j)$ are disjoint, we can write the above inequality as

$$\begin{aligned} |\nabla \xi_{\eta,K}(x)|^2 &\leqslant (1+\delta)|\nabla \xi(x)|^2 \\ &\quad + C_\delta \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j\in\mathscr{J}} \frac{\chi_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))}(x)}{|x-p_j|^2\,(\log K)^2}. \end{aligned} \tag{1.8}$$

Now, we denote by $\mathscr{S} \subset \mathbb{R}^n$ the support of ξ, and we define

$$\mathscr{J}_{\mathscr{S}} := \big\{ j \in \mathscr{J} \text{ s.t. } B_\eta(p_j) \cap \mathscr{S} \neq \emptyset \big\}.$$

We remark that $\mathscr{J}_{\mathscr{S}}$ is a finite set, so we denote by $C_{\mathscr{S}} \in \mathbb{N}$ its cardinality.

Then, by (1.8),

$$
\begin{aligned}
&\int_{\Omega_{\eta,K}} |\nabla u(x)|^2 |\nabla \xi_{\eta,K}(x)|^2 \, dx \leqslant \\
&\leqslant \int_{\mathscr{S}} |\nabla u(x)|^2 \Big[(1+\delta)|\nabla \xi(x)|^2 + C_\delta \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \sum_{j\in\mathscr{J}} \frac{\chi_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))}(x)}{|x-p_j|^2 \, (\log K)^2} \Big] \, dx \\
&\leqslant (1+\delta) \int_\Omega |\nabla u(x)|^2 |\nabla \xi(x)|^2 \, dx \\
&\qquad + C_\delta \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \|u\|^2_{W^{1,\infty}(\mathscr{S})} \sum_{j\in\mathscr{J}_{\mathscr{S}}} \int_{(B_\eta(p_j)\setminus B_{\eta/K}(p_j))} \frac{1}{|x-p_j|^2 \, (\log K)^2} \, dx \\
&= (1+\delta) \int_\Omega |\nabla u(x)|^2 |\nabla \xi(x)|^2 \, dx \\
&\qquad + \frac{C_\delta \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \|u\|^2_{W^{1,\infty}(\mathscr{S})}}{(\log K)^2} \sum_{j\in\mathscr{J}_{\mathscr{S}}} \int_{\eta/K}^{\eta} \frac{r^{n-1}}{r^2} \, dr \\
&\leqslant (1+\delta) \int_\Omega |\nabla u(x)|^2 |\nabla \xi(x)|^2 \, dx \\
&\qquad + \frac{C_\delta C_{\mathscr{S}} \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \|u\|^2_{W^{1,\infty}(\mathscr{S})}}{(\log K)^2} \int_{\eta/K}^{\eta} \frac{1}{r} \, dr \\
&= (1+\delta) \int_\Omega |\nabla u(x)|^2 |\nabla \xi(x)|^2 \, dx + \frac{C_\delta C_{\mathscr{S}} \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \|u\|^2_{W^{1,\infty}(\mathscr{S})}}{\log K}.
\end{aligned}
$$

This and (1.6) give that

$$
\begin{aligned}
&\int_{\Omega_{\eta,K}} \Big(|\nabla u|^2 \mathscr{K}^2 + |\nabla_T |\nabla u||^2 \Big) \xi^2_{\eta,K} \leqslant \\
&\qquad \leqslant (1+\delta) \int_\Omega |\nabla u(x)|^2 |\nabla \xi(x)|^2 \, dx + \frac{C_\delta C_{\mathscr{S}} \|\xi\|^2_{W^{1,\infty}(\mathbb{R}^n)} \|u\|^2_{W^{1,\infty}(\mathscr{S})}}{\log K}.
\end{aligned}
$$

We now take $\eta = 1/\log K$ and we send $K \to +\infty$ (notice that (1.1) allows us to do so), so that we obtain

$$
\int_\Omega \Big(|\nabla u|^2 \mathscr{K}^2 + |\nabla_T |\nabla u||^2 \Big) \xi^2 \leqslant (1+\delta) \int_\Omega |\nabla u|^2 |\nabla \xi|^2.
$$

By taking δ as small as we wish, we obtain (0.3), thus completing the proof of Theorem 1.

2. Proof of Theorem 2

We observe that, under the assumptions of Theorem 2,

$$u \in W^{1,\infty}(\Omega) \subset C(\overline{\Omega}).$$

We suppose, by contradiction, that Ω is coercive.

Then, $\partial_n u > 0$, thanks to Theorem 1.3 in [1].

Thus, when $n = 2$, the claim of Theorem 2 follows from (0.3) here and Lemma 5.1 in [4].

Thus, we focus on the case in which $n = 3$ and $f \geqslant 0$.

For any $t \geqslant 0$ and any $(x', x_3) \in \Omega$, we define

$$u_t(x', x_3) := u(x', x_3 + t).$$

Due to standard elliptic regularity theory, we have that the following limit exists for any $x' \in \mathbb{R}^2$, with $(x', x_3) \in \Omega$, and it is attained in $C^2(\mathbb{R}^2)$:

$$u_\infty(x') := \lim_{t \to +\infty} u_t(x', x_3). \tag{2.1}$$

In particular,

$$\Delta u_\infty + f(u_\infty) = 0 \quad \text{in } \mathbb{R}^2. \tag{2.2}$$

We also set

$$F(r) := \int_0^r f(s)\, ds.$$

Note that $F' = f \geqslant 0$ and so F is nondecreasing. Accordingly,

$$F(u(x)) \leqslant F(u_t(x)) \qquad \text{for any } x \in \Omega$$

and so

$$F(u(x', x_3)) \leqslant F(u_\infty(x')) \qquad \text{for any } (x', x_3) \in \Omega. \tag{2.3}$$

Now, we take $\Omega_\epsilon \subseteq \Omega$ to be a C^3 coercive epigraph that approaches Ω when $\epsilon \to 0^+$.

We make use of Lemma 9.1 in [4] (applied here to u in the smooth domain Ω_ϵ): we obtain, for any $t \geqslant 0$,

$$\int_{B_R \cap \Omega_\epsilon} \frac{|\nabla u|^2}{2} - F(u)\, dx \leqslant CR^2 + \int_{B_R \cap \Omega_\epsilon} \frac{|\nabla u_t|^2}{2} - F(u_t)\, dx,$$

for a suitable constant $C \geqslant 0$.

Therefore, keeping t fixed and sending $\epsilon \to 0^+$,

$$\int_{B_R\cap\Omega} \frac{|\nabla u|^2}{2} - F(u)\,dx \leqslant CR^2 + \int_{B_R\cap\Omega} \frac{|\nabla u_t|^2}{2} - F(u_t)\,dx.$$

We now send $t \to +\infty$ and we conclude that

$$\int_{B_R\cap\Omega} \frac{|\nabla u(x)|^2}{2} - F(u(x))\,dx \leqslant CR^2 + \int_{B_R\cap\Omega} \frac{|\nabla u_\infty(x')|^2}{2} - F(u_\infty(x'))\,d(x',x_3).$$

Therefore, from (2.3),

$$\int_{B_R\cap\Omega} \frac{|\nabla u(x)|^2}{2}\,dx \leqslant CR^2 + \int_{B_R\cap\Omega} \frac{|\nabla u_\infty(x')|^2}{2}\,d(x',x_3). \tag{2.4}$$

We now observe that, by (2.2), it holds that $\Delta u_\infty \leqslant 0$ in $\mathbb{R}^2$ and therefore, by a classical Liouville Theorem, we have that u_∞ is constant.

Hence, (2.4) becomes

$$\int_{B_R\cap\Omega} \frac{|\nabla u(x)|^2}{2}\,dx \leqslant CR^2. \tag{2.5}$$

Thus, in the light of (0.3) and (2.5), we may now apply Corollary 9.4 of [4]: we obtain that $\partial\Omega$ is a hyperplane, in contradiction with the fact that Ω is coercive.

This completes the proof of Theorem 2.

References

[1] Berestycki, H., Caffarelli, L. A. and Nirenberg, L.: Monotonicity for elliptic equations in unbounded Lipschitz domains. *Comm. Pure Appl. Math.* **50** (1997), no. 11, 1089–1111.

[2] Farina, A.: *Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires.* Habilitation à diriger des recherches, Paris VI, 2002.

[3] Farina, A., Sciunzi, B. and Valdinoci, E.: Bernstein and De Giorgi type problems: new results via a geometric approach. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **7** (2008), no. 4, 741–791.

[4] Farina, A. and Valdinoci, E.: Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems. *Arch. Ration. Mech. Anal.* **195** (2010), no. 3, 1025–1058.

[5] Sternberg, P. and Zumbrun, K.: Connectivity of phase boundaries in strictly convex domains. *Arch. Rational Mech. Anal.* **141** (1998), no. 4, 375–400.

[6] Sternberg, P. and Zumbrun, K.: A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. *J. Reine Angew. Math.* **503** (1998), 63–85.

[7] Vázquez, J. L.: Personal communication, 2008.

Recibido: 23 de diciembre de 2008

Alberto Farina
LAMFA – CNRS UMR 6140
Université de Picardie Jules Verne
Faculté des Sciences
33, rue Saint-Leu
80039 Amiens CEDEX 1, France
alberto.farina@u-picardie.fr

Enrico Valdinoci
Università di Roma Tor Vergata
Dipartimento di Matematica
via della ricerca scientifica, 1
I-00133 Rome, Italy
enrico.valdinoci@uniroma2.it

This work has been partially supported by MIUR *Variational Methods and Nonlinear Differential Equations* and GNAMPA *Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni.*