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The C^m Norm of a Function with Prescribed Jets I

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Abstract

We prove a variant of the classical Whitney extension theorem, in which the C^m-norm of the extending function is controlled up to a given, small percentage error.

0. Introduction

Here and in [16], we compute the least possible (infimum) C^m norm of a function F having prescribed Taylor polynomials at N given points of $\mathbb{R}^n$. Moreover, given $\epsilon > 0$, we exhibit such an F, whose C^m norm is within ϵ of the least possible. Our computation consists of an algorithm, to be implemented on an (idealized) digital computer. The algorithm works, thanks to a variant of the classical Whitney extension theorem, in which we control the C^m norm of the extending function up to an ϵ percentage error. This paper gives the variant of Whitney's theorem, while [16] presents the algorithm and the rest of the mathematics behind it. The number of operations used by our algorithm is $C(\epsilon)N \log N$, where N is the number of given points, and $C(\epsilon)$ grows rapidly as ϵ tends to zero.

To state our results precisely, we set up notation. Fix $m, n \geq 1$. We pick a norm on $C^m(\mathbb{R}^n)$, subject to restrictions to be spelled out in the next section. We write $\| F \|_{C^m(\mathbb{R}^n)}$ to denote the norm of F. Given $x \in \mathbb{R}^n$ and $F \in C^m(\mathbb{R}^n)$, we write $J_x(F)$ to denote the $m^{\underline{th}}$ order Taylor polynomial of F at x. Thus, $J_x(F)$ belongs to $\mathcal{P}$, the vector space of all (real) $m^{\underline{th}}$ degree polynomials on $\mathbb{R}^n$.

Let $E \subset \mathbb{R}^n$. We write $\#(E)$ for the number of points in E. (If E is infinite, then $\#(E) = +\infty$.) A Whitney field on E is a family $\vec{P} = (P^x)_{x \in E}$ of polynomials $P^x \in \mathcal{P}$, indexed by $x \in E$. If $\vec{P} = (P^x)_{x \in E}$ is a Whitney field and $S \subset E$ is a subset, then in an obvious way we can define the restriction $\vec{P}|_S$ of $\vec{P}$ to S.

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We say that a function $F \in C^m(\mathbb{R}^n)$ agrees with a Whitney field $\vec{P} = (P^x)_{x \in E}$, provided $J_x(F) = P^x$ for each $x \in E$. We define a C^m-norm on Whitney fields, by setting

$$\| \vec{P} \| = \inf\{\| F \|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n),\ F \text{ agrees with } \vec{P}\}.$$

(If there exists no such F, then we define $\| \vec{P} \| = +\infty$; this can happen only when E is infinite.) Similarly, for a function $f : E \to \mathbb{R}$, we define the C^m-norm

$$\| f \| = \inf\{\| F \|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n),\ F = f \text{ on } E\},$$

with $\| f \| = +\infty$ if there is no such F.

We are concerned with the following questions.

Problem 1: Compute the norm $\| \vec{P} \|$ of a given Whitney field $\vec{P}$ on a finite set. Given $\epsilon > 0$, exhibit a function $F_\epsilon \in C^m(\mathbb{R}^n)$ that agrees with $\vec{P}$, and satisfies $\| F_\epsilon \|_{C^m(\mathbb{R}^n)} \leq (1+\epsilon) \| \vec{P} \|$.

Problem 2: Compute the norm $\| f \|$ of a given function $f : E \to \mathbb{R}$ (E finite). Given $\epsilon > 0$, exhibit a function $F_\epsilon \in C^m(\mathbb{R}^n)$, such that $F_\epsilon = f$ on E, and $\| F_\epsilon \|_{C^m(\mathbb{R}^n)} \leq (1+\epsilon) \| f \|$.

In this paper and [16], we give an efficient solution of Problem 1, and an inefficient solution of Problem 2.

From previous work, it is known how to compute the "order of magnitude" of the norm in Problems 1 and 2. That is, one can give upper and lower bounds for $\| \vec{P} \|$ or $\| f \|$, that differ by a constant factor depending only on m, n and the choice of the norm on $C^m(\mathbb{R}^n)$. We review the previous work, then state our results on Problem 1, and finally return to Problem 2.

The order of magnitude of the norm of a Whitney field is provided by the classical Whitney extension theorem [22, 26, 27], which we now recall, in the case of finite sets E.

Theorem 1. *Let $\vec{P} = (P^x)_{x \in E}$ be a Whitney field on a finite set. Let $M \geq 0$ be the smallest number for which we have*

$$|\partial^\alpha P^x(x)| \leq M \quad \textit{for} \quad |\alpha| \leq m, x \in E; \qquad \textit{and}$$
$$|\partial^\alpha (P^x - P^y)(y)| \leq M|x-y|^{m-|\alpha|} \quad \textit{for} \quad |\alpha| \leq m-1, x, y \in E.$$

Then

$$cM \leq \| \vec{P} \| \leq CM,$$

where c and C depend only on m, n and the choice of norm on $C^m(\mathbb{R}^n)$.

The proof of Theorem 1 is constructive; it exhibits a function $F \in C^m(\mathbb{R}^n)$ that agrees with $\vec{P}$ and satisfies $\| F \|_{C^m(\mathbb{R}^n)} \leq C \| \vec{P} \|$, with C as in Theorem 1.

The order of magnitude of the C^m-norm of a function on a finite set E is computable, thanks to the following result.

Theorem 2. *Let $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^n$ finite. Then*

$$\| f \| \leq C \cdot \max\{\| (f|_S) \| : S \subset E, \#(S) \leq k^{\#}\}.$$

Here, $k^{\#}$ depends only on m, n; and C depends only on m, n, and the choice of the norm on $C^m(\mathbb{R}^n)$.

Elementary linear algebra provides upper and lower bounds for $\| (f|_S) \|$ that differ by a factor depending only on $\#(S), m, n$ and the choice of the norm in $C^m(\mathbb{R}^n)$. In particular, we can compute the "order of magnitude" of $\| (f|_S) \|$ when $\#(S) \leq k^{\#}$. Therefore, Theorem 2 provides the order of magnitude of $\| f \|$, for any function f defined on a finite set. See [10], and also Fefferman-Klartag [17,18]. Theorem 2 was conjectured by Y. Brudnyi and P. Shvartsman, and proven by them [6] for $m = 2$, with an optimal $k^{\#}$. (It is trivial for $m = 1$.) The general case was proven in Fefferman [10]. See Brudnyi-Shvartsman [3,...8], Fefferman [9...15], Fefferman-Klartag [17, 18], Bierstone-Milman-Pawłucki [1, 2], Whitney [27, 28, 29], Glaeser [21], Shvartsman [23, 24, 25], and Zobin [30, 31] for several related results and conjectures. The proof of Theorem 2 is again constructive.

Theorem 1 can be reformulated to look like Theorem 2. In fact, the following result is easily seen to be equivalent to Theorem 1.

Theorem 1′. *Let $\vec{P} = (P^x)_{x \in E}$ be a Whitney field on a finite set. Then*

$$\| \vec{P} \| \leq C \cdot \max\{\| (\vec{P}|_S) \| : S \subset E, \#(S) \leq 2\},$$

with C depending only on m, n and the choice of the norm on $C^m(\mathbb{R}^n)$.

One computes the order of magnitude of $\| (\vec{P}|_S) \|$ for $\#(S) \leq 2$ by using the case $\#(E) \leq 2$ of Theorem 1, which is a triviality.

We are ready to state our first result on Problem 1.

Theorem 3. *Let $\epsilon > 0$, and let $\vec{P}$ be a Whitney field on a finite set E. Then*

$$\| \vec{P} \| \leq (1 + \epsilon) \cdot \max\{\| (\vec{P}|_S) \| : S \subset E, \#(S) \leq k^{\#}(\epsilon)\},$$

where $k^{\#}(\epsilon)$ depends only on ϵ, m, n, and on the choice of the norm on $C^m(\mathbb{R}^n)$.

Thus, the computation of the norm in Problem 1 is reduced to the case $\#(E) \leq k^{\#}(\epsilon)$. In fact, we can do a bit better, by reducing the problem to subsets $S \subset E$, with $\#(S) \leq k^{\#}(\epsilon)$, that also satisfy a favorable geometrical condition. We call a set $S \subset \mathbb{R}^n$ an ϵ-testing set, provided it satisfies

- $\#(S) \leq k^{\#}(\epsilon)$, and
- $|x - y| \geq c(\epsilon) \cdot \mathrm{diam}\,(S)$ for any two distinct points $x, y \in S$.

Here, $\mathsf{diam}(S)$ denotes the diameter of S, $k^{\#}(\epsilon)$ is as in Theorem 3, and $c(\epsilon)$ is a small enough constant determined by ϵ, m, n and the choice of the norm on $C^m(\mathbb{R}^n)$.

We will prove the following sharper version of Theorem 3.

Theorem 4. *Let* $\epsilon > 0$*, and let* $\vec{P}$ *be a Whitney field on a finite set* E*. Then*

$$\| \vec{P} \| \leq (1+\epsilon) \cdot \max\{\| (\vec{P}|_S) \|: S \subset E,\ S \text{ an } \epsilon\text{-testing set}\}.$$

Thus, the computation of the norm in Problem 1 is reduced to the case of a Whitney field on an ϵ-testing set. This case is unfortunately non-trivial. Already, one needs an idea in order to compute $\| \vec{P} \|$ exactly (not just the order of magnitude) for a Whitney field $\vec{P}$ on a single point. We show in [16] how to compute $\| \vec{P} \|$ up to a percentage error at most ϵ, in the case of a Whitney field $\vec{P}$ on an ϵ-testing set. That computation is done by an algorithm that requires $C(\epsilon)$ steps, with $C(\epsilon)$ depending only on ϵ, m, n and the choice of the norm on $C^m(\mathbb{R}^n)$.

Our methods are constructive. In computing $\| \vec{P} \|$ to within a percentage error at most ϵ, we produce along the way a function $F_\epsilon \in C^m(\mathbb{R}^n)$ that agrees with $\vec{P}$ and satisfies $\| F \|_{C^m(\mathbb{R}^n)} \leq (1+\epsilon) \| \vec{P} \|$. Thus, our result solves Problem 1.

In [16], we discuss also some computer-science issues arising in the implementation of our algorithm. For a Whitney field $\vec{P}$ on a set with N elements, we can compute $\| \vec{P} \|$ to within a percentage error ϵ, using $C(\epsilon)N \log N$ operations and $C(\epsilon)N$ storage. Also, in a sense to be made precise as in Fefferman-Klartag [17, 18], a function F_ϵ as in Problem 1 may be computed in $C(\epsilon)N \log N$ operations and $C(\epsilon)N$ storage. Here again, $C(\epsilon)$ depends only on ϵ, m, n and the choice of the C^m-norm. Compare with Fefferman-Klartag [17, 18].

We provide an oversimplified sketch of the proof of Theorem 4. Recalling the definition of $\| \vec{P} \|$ for a Whitney field $\vec{P}$, we see that Theorem 4 amounts to the following statement.

Let $\vec{P} = (P^x)_{x \in E}$ be a Whitney field on a finite set. Assume that, for any ϵ-testing set $S \subset E$, there exists $F^S \in C^m(\mathbb{R}^n)$, such that

(1) $\| F^S \|_{C^m(\mathbb{R}^n)} \leq 1$ and F^S agrees with $\vec{P}|_S$.

Then there exists $\tilde{F} \in C^m(\mathbb{R}^n)$, such that

(2) $\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq 1 + C\epsilon$, and

(3) $\tilde{F}$ agrees with $\vec{P}$.

To prove this, we modify Whitney's classical proof of Theorem 1. We recall the main steps in Whitney's argument:

- Partition $\mathbb{R}^n \setminus E$ into Whitney cubes $\{Q_\mu\}$, with $\delta_\mu =$ diameter (Q_μ) comparable to the distance from Q_μ to E.
- Introduce a partition of unity $1 = \sum_\mu \theta_\mu$ on $\mathbb{R}^n \setminus E$, with each θ_μ supported near Q_μ, and satisfying estimates

(4) $|\partial^\alpha \theta_\mu| \leq C \cdot \delta_\mu^{-|\alpha|}$ on $\mathbb{R}^n$, for $|\alpha| \leq m$.

- Define $F = \sum_\mu \theta_\mu \cdot P^{x_\mu}$ on $\mathbb{R}^n \setminus E$, $F(x) = (P^x)(x)$ on E, where P^{x_μ} arises from our given Whitney field $\vec{P} = (P^x)_{x \in E}$, by taking $x_\mu \in E$ as close as possible to Q_μ.
- The above F belongs to C^m and agrees with $\vec{P}$. Moreover, the C^m-norm of F is bounded a-priori.

We now sketch the modifications of the above steps needed to prove Theorem 4. In place of Whitney's θ_μ, we introduce a "gentle partition of unity",

$$1 = \sum_{\ell,\nu} \chi_\nu^\ell \quad \text{on } \mathbb{R}^n \setminus E\,.$$

In place of (4), the χ_ν^ℓ satisfy

(5) $|\partial^\alpha \chi_\nu^\ell(x)| \leq C\epsilon \cdot \delta_\mu^{-|\alpha|}$ for $0 < |\alpha| \leq m$, $x \in Q_\mu$.

Note the extra "ϵ" on the right-hand side. Each χ_ν^ℓ is supported in a cube Q_ν^ℓ. The cubes Q_ν^ℓ are bigger than Whitney's Q_μ.

In place of Whitney's P^{x_μ}, we use a function $F_\nu^\ell \in C^m(\mathbb{R}^n)$, obtained as follows. For each χ_ν^ℓ, we pick out an ϵ-testing set $S_\nu^\ell \subset E \cap Q_\nu^\ell$. (Essentially, S_ν^ℓ is the largest possible such ϵ-testing set.) Our function F_ν^ℓ then arises by applying the hypothesis (1) to our ϵ-testing set S_ν^ℓ. Thus,

(6) $\| F_\nu^\ell \|_{C^m(\mathbb{R}^n)} \leq 1$, and

(7) F_ν^ℓ agrees with $\vec{P}|_{S_\nu^\ell}$.

Following Whitney, we define

$$\tilde{F} = \sum_{\ell,\nu} \chi_\nu^\ell \cdot F_\nu^\ell \text{ on } \mathbb{R}^n \setminus E\,;\ \tilde{F}(x) = (P^x)(x) \text{ for } x \in E\,.$$

Then $\tilde{F}$ belongs to $C^m(\mathbb{R}^n)$, and it satisfies the desired properties (2) and (3). A bit more precisely, (2) follows from (5) and (6), with the extra "ϵ" in (5) providing crucial help. In proving (3), we obtain crucial help from a geometric property of the S_ν^ℓ, namely,

(8) Any given point in $\operatorname{supp} \chi_\nu^\ell$ lies quite close to some point of S_ν^ℓ.

Without (8), we would be in trouble, because the Q_ν^ℓ are too big. We are able to arrange (8), because S_ν^ℓ is allowed to contain $k^\#(\epsilon)$ points, not just two.

Since $\tilde{F}$ has the desired properties (2) and (3), the proof of Theorem 4 is complete. However, we stress that the above discussion is oversimplified. For instance, what we really prove below is not Theorem 4, but rather a generalization to Whitney fields on (possibly infinite) compact sets. See Sections 1…6 below for full details. This concludes our introductory remarks on the proof of Theorem 4.

Returning to Problem 2, an optimist might speculate as follows, motivated by Theorems 2 and 3.

Conjecture. *Let* $\epsilon > 0$*, and let* $f : E \to \mathbb{R}$*, with* $E \subset \mathbb{R}^n$ *finite. Then*

$$\| f \| \leq (1+\epsilon) \cdot \max\{\| (f|_S) \| : S \subset E\, , \#(S) \leq k^\#(\epsilon)\}\, ,$$

with $k^\#(\epsilon)$ *depending only on* ϵ, m, n *and the choice of the* C^m*-norm.*

Even an optimist might prefer to restrict attention to a single, favorable norm on $C^m(\mathbb{R}^n)$. In fact, the above conjecture is false; see Fefferman-Klartag [19]. Thus, an efficient solution of Problem 2 will require new ideas.

It would be interesting to find the best $k^\#(\epsilon)$ in Theorems 3 and 4. I don't even know whether it really depends on ϵ.

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1. Picking a Norm on $C^m(\mathbb{R}^n)$

In this section, we define the class of C^m-norms for which our results are valid. For each $x \in \mathbb{R}^n$, we suppose we are given a norm $P \mapsto |P|_x$ on the vector space $\mathcal{P}$.

We make the following assumptions on our norms $|\cdot|_x$.

The Bounded Distortion Property. *There exist constants* $\bar{c}_0, \bar{C}_0 > 0$*, for which we have*

$$\bar{c}_0 |P|_x \leq \max_{|\alpha| \leq m} |\partial^\alpha P(x)| \leq \bar{C}_0 |P|_x \text{ for all } P \in \mathcal{P},\ x \in \mathbb{R}^n\, .$$

Approximate Translation-Invariance. *If* $P \in \mathcal{P}$ *and* $\tau \in \mathbb{R}^n$*, we define the translate* $P_\tau \in \mathcal{P}$ *by setting* $P_\tau(z) = P(z - \tau)$ *for* $z \in \mathbb{R}^n$*. We assume that*

$$|P_\tau|_{x+\tau} \leq (1 + \bar{C}_1 |\tau|) \cdot |P|_x \textit{ for any } P \in \mathcal{P},\ x \in \mathbb{R}^n,\ |\tau| \leq 1\, .$$

Here, $\bar{C}_1$ *is a constant, independent of* x, τ, P.

These properties obviously hold, e.g., for

$$|P|_x = \max_{|\alpha| \le m} |\partial^\alpha P(x)|,$$

$$\text{or for } |P|_x = \max_{0 \le k \le m} \left(\sum_{i_1 \cdots i_k = 1}^{n} \left| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} P(x) \right|^2 \right)^{1/2}.$$

For a locally C^m function F defined on an open set $\Omega \subset \mathbb{R}^n$, we then define

(1) $\| F \|_{C^m(\Omega)} = \sup_{x \in \Omega} |J_x(F)|_x$.

We write $C^m(\Omega)$ for the space of real-valued locally C^m functions on Ω for which the norm (1) is finite.

Our results hold for C^m-norms of the form (1). This excludes, for example, the C^m-norm $\| F \| = \sum_{|\alpha| \le m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$, since it is not given as the sup on x of a single expression.

We say that a constant C is controlled if it depends only on $m, n, \bar{c}_0, \bar{C}_0$ and $\bar{C}_1$ in the Bounded Distortion and Approximate Translation-Invariance Properties. We write c, C, C', etc., to denote controlled constants.

The above conventions remain in force throughout our paper. In particular, we always assume that the norms $| \cdot |_x$ are given, and that our C^m-norm has the form (1).

We close this section with two elementary consequences of the Bounded Distortion and Approximate Translation-Invariance Properties.

(2) Let $F \in C^m(\Omega)$, let $U \subset \mathbb{R}^n$ be open, let $\tau \in \mathbb{R}^n$ with $|\tau| \le 1$, and suppose that the translate $U - \tau$ is contained in Ω. Define a function F_τ on U by setting $F_\tau(z) = F(z - \tau)$ for $z \in U$. Then

$$\| F_\tau \|_{C^m(U)} \le (1 + \bar{C}_1 |\tau|) \cdot \| F \|_{C^m(\Omega)} .$$

(3) Suppose $P \in \mathcal{P}, x \in \mathbb{R}^n$, $|\tau| \le 1$. If $|P|_x \le 1$, then also $|P|_{x+\tau} \le 1 + C|\tau|$.

To see (3), we note that $|\partial^\alpha P(x)| \le C$ for $|\alpha| \le m$, by the Bounded Distortion Property. Since $|\tau| \le 1$, it follows that $|\partial^\alpha (P_\tau - P)(x + \tau)| \le C'|\tau|$ for $|\alpha| \le m$, with P_τ as in the Approximate Translation-Invariance Property. Another application of the Bounded Distortion Property therefore gives $|P_\tau - P|_{x+\tau} \le C''|\tau|$. Since also $|P_\tau|_{x+\tau} \le 1 + \bar{C}_1 |\tau|$ by Approximate Translation-Invariance, our desired result (3) follows.

2. Gentle Partitions of Unity

In this section, we will be discussing functions F defined on an open set $\Omega \subset \mathbb{R}^n$. By the support of F, we mean the set of points x in Ω, such that F is not identically zero on any ball centered at x.

We suppose we are given an open cover $\{U_\nu\}$ of Ω, and a collection of functions $F_\nu \in C^m(U_\nu)$, each with norm at most M. We want to patch together the F_ν by using a partition of unity $\sum_\nu \chi_\nu = 1$ on Ω, with each χ_ν supported in U_ν.

We hope that $F = \sum_\nu \chi_\nu F_\nu$ will have norm at most $(1+\epsilon)M$ in $C^m(\Omega)$. Our next result, proven by a variant of Whitney's original argument in [27], shows that this holds, provided $F_\nu - F_\mu$ satisfies favorable estimates on supp $\chi_\nu \cap$ supp χ_μ, and provided the χ_ν form a "gentle partition of unity".

The estimates needed for $F_\nu - F_\mu$, and for χ_ν, involve a "lengthscale" $\delta(x) > 0$, defined for $x \in \Omega$. In our applications below, we will take $\Omega = \mathbb{R}^n \setminus E$ and $\delta(x)$ comparable to $\mathrm{dist}(x, E)$ for $x \in \Omega$. (Recall that our Whitney field $\vec{P} = (P^x)_{x \in E}$ is defined on a finite set E.)

The precise statement of our lemma on "gentle partitions of unity" is as follows.

Lemma GPU. *Let $\{U_\nu\}$ be an open cover of an open set $\Omega \subset \mathbb{R}^n$; and let $\delta(x) > 0$ be defined for $x \in \Omega$. Suppose that, for each ν, we are given a function $F_\nu \in C^m(U_\nu)$, and a function $\chi_\nu \in C^m(\Omega)$.*

Let ϵ, M, A_0, A_1 be positive real numbers. Assume that the following conditions are satisfied.

(GPU1) *Any given $x \in \Omega$ belongs to* supp χ_ν *for at most A_0 distinct ν.*

(GPU2) $\sum_\nu \chi_\nu = 1$ *on Ω.*

(GPU3) $\chi_\nu \geq 0$ *on Ω.*

(GPU4) $\mathrm{supp}\, \chi_\nu \subset U_\nu$.

(GPU5) $|\partial^\alpha \chi_\nu(x)| \leq \epsilon \cdot (\delta(x))^{-|\alpha|}$ *for* $0 < |\alpha| \leq m, x \in \Omega$.

(GPU6) $|J_x(F_\nu)|_x \leq M$ *for all* $x \in \mathrm{supp}\, \chi_\nu$.

(GPU7) $|\partial^\alpha (F_\nu - F_\mu)(x)| \leq A_1 M \cdot (\delta(x))^{m-|\alpha|}$ *for* $|\alpha| \leq m-1$, $x \in \mathrm{supp}\, \chi_\nu \cap \mathrm{supp}\, \chi_\mu$.

Then the function $F = \sum_\nu \chi_\nu F_\nu$ belongs to $C^m(\Omega)$, and satisfies

$$\| F \|_{C^m(\Omega)} \leq (1 + A\epsilon) \cdot M,$$

where A depends only on A_0 in (GPU1)*, A_1 in* (GPU7)*, $\bar{c}_0$ and $\bar{C}_0$ in the Bounded Distortion Property*, and m, n.

Proof. Clearly, $F \in C^m_{\ell oc}(\Omega)$. We must show that

(1) $|J_x(F)|_x \le (1 + A\epsilon)M$ for all $x \in \Omega$, with A as above.

For the proof of (1), we write A, A', A'', etc., to denote constants depending only on A_0, $A_1, \bar{c}_0$, $\bar{C}_0, m, n$, as in the statement of Lemma GPU. These constants need not be the same from one occurrence to the next.

To prove (1), we fix $x \in \Omega$, and write

(2) $J_x(F) = \sum_\nu J_x(\chi_\nu F_\nu) = \sum_\nu \chi_\nu(x) J_x(F_\nu) + \sum_\nu \mathcal{E}_\nu$, where

(3) $\mathcal{E}_\nu = [J_x(\chi_\nu F_\nu) - \chi_\nu(x) J_x(F_\nu)] \in \mathcal{P}$.

Since $|\cdot|_x$ is a norm, our assumptions (GPU2,3,6) imply

(4) $|\sum_\nu \chi_\nu(x) J_x(F_\nu)|_x \le M$.

We next study the $\mathcal{E}_\nu$, for those ν such that $x \in \operatorname{supp} \chi_\nu$. For $|\alpha| \le m$, we have

(5) $\partial^\alpha \mathcal{E}_\nu(x) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!\,\gamma!} \partial^\beta \chi_\nu(x) \cdot \partial^\gamma F_\nu(x)$.

Fix μ such that $x \in \operatorname{supp} \chi_\mu$. (Such a μ exists, by (GPU2).) Another application of (GPU2) gives

$$\sum_\nu \partial^\beta \chi_\nu(x) = 0 \quad \text{for } \beta \neq 0.$$

Consequently, (5) may be rewritten in the form

(6) $\partial^\alpha \mathcal{E}_\nu(x) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!\,\gamma!} \partial^\beta \chi_\nu(x) \cdot \partial^\gamma (F_\nu - F_\mu)(x)$.

If $\delta(x) \le 1$, then (GPU5,7) and (6) together imply that

$$\begin{aligned} |\partial^\alpha \mathcal{E}_\nu(x)| &\le \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!} \left[\epsilon \cdot (\delta(x))^{-|\beta|}\right] \cdot \left[A_1 M \cdot (\delta(x))^{m-|\gamma|}\right] \\ &\le A\epsilon M \cdot (\delta(x))^{m-|\alpha|} \le A\epsilon M \text{ for } |\alpha| \le m. \end{aligned}$$

On the other hand, if $\delta(x) > 1$, then by (GPU5) and (5), together with (GPU6) and the Bounded Distortion Property, we have

$$|\partial^\alpha \mathcal{E}_\nu(x)| \le \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!} [\epsilon \cdot (\delta(x))^{-|\beta|}] \cdot [AM] \le A'\epsilon M \quad \text{for } |\alpha| \le m.$$

Thus, in either case, we have

$$|\partial^\alpha \mathcal{E}_\nu(x)| \leq A\epsilon M \quad \text{for } |\alpha| \leq m\,.$$

Consequently, the Bounded Distortion Property gives

$$|\mathcal{E}_\nu|_x \leq A'\epsilon M \quad \text{for each } \nu \text{ such that } x \in \operatorname{supp}\chi_\nu.$$

Together with (GPU1), this implies

(7) $|\sum_\nu \mathcal{E}_\nu|_x \leq A''\epsilon M.$

Our desired conclusion (1) follows at once from (2), (4) and (7). The proof of the Lemma is complete. ■

3. Testing Sets

Let $0 < \epsilon < 1/2$. An ϵ-testing set is a subset $S \subset \mathbb{R}^n$, such that

(1) $\#(S) \leq \left(\frac{2}{\epsilon} \cdot e^{2/\epsilon}\right)^n$, and

(2) $|x - y| \geq \hat{c}\,\epsilon\, e^{-2/\epsilon} \operatorname{diam}(S)$, for any two distinct points $x, y \in S$.

Here, $\hat{c}$ is a small enough controlled constant.

The above definition differs slightly from the notion of an ϵ-testing set, given in the Introduction. (Here, we specify $k^\#(\epsilon)$ and $c_0(\epsilon)$.) We use our present definition in Sections 4, 5 and 6 below.

The following elementary observation, essentially a special case of Vitali's covering lemma, will be useful in the proof of our main result.

Lemma. *Let Q be a cube of sidelength δ_Q, let $E \subset \mathbb{R}^n$, and let $0 < \epsilon < 1/2$. Then there exists an ϵ-testing set $S \subset E \cap Q$, such that any point of $E \cap Q$ lies within distance $C\epsilon e^{-2/\epsilon}\delta_Q$ from some point of S.*

Proof. Subdivide Q into a grid of cubes $\{Q_\nu\}$ of sidelength between $\frac{\epsilon}{2}e^{-2/\epsilon}\delta_Q$ and $\epsilon e^{-2/\epsilon}\delta_Q$. The number of such Q_ν is at most $\left(\frac{2}{\epsilon}\, e^{2/\epsilon}\right)^n$. In each non-empty $E \cap Q_\nu$, we pick a "representative" $\hat{y}_\nu$. Let $\hat{S}$ be the set of all the representatives picked above. Then

(3) $\hat{S} \subset E \cap Q$,

(4) $\#(\hat{S}) \leq \left(\frac{2}{\epsilon}\, e^{2/\epsilon}\right)^n$, and

(5) Any $x \in E \cap Q$ lies within distance $C\epsilon\, e^{-2/\epsilon}\delta_Q$ from some $\hat{y} \in \hat{S}$.

(In fact, we can take $\hat{y}$ to be the representative picked for the $E \cap Q_\nu$ that contains x.)

Unfortunately, $\hat{S}$ may not satisfy (2). Therefore, we proceed as follows. Let $y_1, y_2, \dots, y_L$ be an enumeration of $\hat{S}$. By induction on $\ell (1 \leq \ell \leq L)$, we decide whether to discard y_ℓ, according to the following rule:

We discard y_ℓ if and only if $|y_\ell - y_{\ell'}| < \epsilon\, e^{-2/\epsilon} \delta_Q$ for some $\ell' < \ell$ for which we did not discard $y_{\ell'}$.

Let S be the set of all the y_ℓ that were not discarded. Evidently,

(6) $S \subset \hat{S} \subset E \cap Q$, and

(7) $|y - y'| \geq \epsilon\, e^{-2/\epsilon}\, \delta_Q$ for any two distinct points $y, y' \in S$.

We claim that

(8) Any $x \in E \cap Q$ satisfies $|x - y| \leq C\epsilon\, e^{-2/\epsilon} \delta_Q$ for some $y \in S$.

To see (8), let $\hat{y} \in \hat{S}$ be as in (5). If $\hat{y} \in S$, then (8) holds, with $y = \hat{y}$. On the other hand, if $\hat{y} \notin S$, then we have $|\hat{y} - y| < \epsilon\, e^{-2/\epsilon} \delta_Q$ for some $y \in S$; consequently,

$$|x - y| \leq |x - \hat{y}| + |\hat{y} - y| \leq C\epsilon\, e^{-2/\epsilon} \delta_Q + \epsilon\, e^{-2/\epsilon} \delta_Q \leq C'\epsilon\, e^{-2/\epsilon} \delta_Q\,,$$

and again (8) holds. Thus, (8) holds in all cases.

Since $S \subset Q$, we have $\mathsf{diam}\,(S) \leq C\delta_Q$. Therefore, (4) and (7) imply (1) and (2) for $\hat{c}$ small enough. Thus, S is an ϵ-testing set. We know also that $S \subset E \cap Q$, and that (8) holds. Thus, S satisfies all the conclusions of the Lemma. ■

Note that any $S \subset \mathbb{R}^n$ with $\#(S) \leq 2$ is an ϵ-testing set.

4. The Main Result

Our main result is the following analogue of Theorem 4 for general compact sets. Here, we use the notion of an "ϵ-testing set" from Section 3.

The $(1 + \epsilon)$-Whitney Theorem. *Let $0 < \epsilon < \tilde{c}$, for a small enough controlled constant $\tilde{c}$. Let $\vec{P} = (P^x)_{x \in E}$ be a Whitney field on a compact set $E \subset \mathbb{R}^n$.*

(A) *Suppose $\| (\vec{P}|_S) \| < 1$ for every ϵ-testing set $S \subset E$. Then there exists $\tilde{F} \in C^m(\mathbb{R}^n \setminus E)$, such that*

- $\| \tilde{F} \|_{C^m(\mathbb{R}^n \setminus E)} \leq 1 + C\epsilon$, *and*

- $|\partial^\alpha (\tilde{F} - P^y)(x)| \leq C|x - y|^{m - |\alpha|}$ *for* $|\alpha| \leq m$, $x \in \mathbb{R}^n \setminus E$, $y \in E$.

(B) *In addition to the assumption of part (A), suppose there exists a function* $F_0 \in C^m(\mathbb{R}^n)$ *that agrees with* $\vec{P}$. *Then there exists* $F \in C^m(\mathbb{R}^n)$, *such that*

- $\| F \|_{C^m(\mathbb{R}^n)} \leq 1 + C\epsilon$, *and*
- F *agrees with* $\vec{P}$.

Trivially, if E is finite, then there exists a function F_0 as in part (B). Hence, the $(1+\epsilon)$-Whitney theorem immediately implies Theorems 3 and 4 in the Introduction (with the definition of an ϵ-testing set given there). For general compact E, the classical Whitney extension theorem tells us whether there exists an F_0 as in part (B).

5. Analysis on $\mathbb{R}^n \setminus E$

In this section, we prove part (A) of the $(1+\epsilon)$-Whitney theorem. We suppose ϵ, $\vec{P} = (P^x)_{x \in E}$ satisfy the hypotheses of part (A). In particular, for every ϵ-testing set $S \subset E$, there exists $F^S \in C^m(\mathbb{R}^n)$, with

(1) $\| F^S \|_{C^m(\mathbb{R}^n)} \leq 1$, and

(2) $J_x(F^S) = P^x$ for all $x \in S$.

From (1) and the Bounded Distortion Property, we have

(3) $|\partial^\alpha F^S(x)| \leq C$ for $|\alpha| \leq m$, $x \in \mathbb{R}^n$.

Recall that any set $S \subset \mathbb{R}^n$ with at most two points is an ϵ-testing set. Hence, (1), (2), (3) and Taylor's theorem yield

(4) $|P^x|_x \leq 1$ for all $x \in E$,

(5) $|\partial^\alpha P^x(x)| \leq C$ for $|\alpha| \leq m$, $x \in E$, and

(6) $|\partial^\alpha (P^x - P^y)(y)| \leq C|x-y|^{m-|\alpha|}$ for $|\alpha| \leq m$, $x, y \in E$.

Now let

(7) $\Omega = \mathbb{R}^n \setminus E$.

We prepare to set up a gentle partition of unity on Ω. The proof of the classical Whitney extension theorem gives a function $\delta(x)$, defined on Ω, with the following properties.

(8) $c\delta(x) < \operatorname{dist}(x, E) < C\delta(x)$ for all $x \in \Omega$.

(9) $|\partial^\alpha \delta(x)| \leq C(\delta(x))^{1-|\alpha|}$ for $|\alpha| \leq m$, $x \in \Omega$.

Here, of course, $\operatorname{dist}(x, E)$ denotes the distance from x to E, and (9) includes the assertion that the function $\delta(x)$ belongs to $C^m_{\ell oc}(\Omega)$. See [26, p. 171].

We fix a C^m partition of unity

(10) $1 = \sum_{-\infty<\ell<\infty} \chi_\ell(t)$ on $\mathbb{R}$, where:

(11) $\chi_\ell(t) \geq 0$ for all $t \in \mathbb{R}$, $\ell \in \mathbb{Z}$;

(12) $\mathrm{supp}\, \chi_\ell \subset (\ell-1, \ell+1)$ for each $\ell \in \mathbb{Z}$; and

(13) $\left| \left(\frac{d}{dt}\right)^k \chi_\ell(t) \right| \leq C$ for $k \leq m$, $t \in \mathbb{R}$, $\ell \in \mathbb{Z}$.

Note that the function $\chi_\ell(\epsilon \log \delta(x))$, defined for $x \in \Omega$, has the following properties.

(14) $\chi_\ell(\epsilon \log \delta(x)) \geq 0$ for all $x \in \Omega$.

(15) $\mathrm{supp}\, \chi_\ell(\epsilon \log \delta(x)) \subset \{x \in \Omega : e^{\frac{\ell-1}{\epsilon}} < \delta(x) < e^{\frac{\ell+1}{\epsilon}}\}$

$$\subset \{x \in \Omega : c\, e^{\frac{\ell-1}{\epsilon}} < \mathrm{dist}\,(x, E) < C\, e^{\frac{\ell+1}{\epsilon}}\}.$$

(See (8).) Also, we will check that

(16) $|\partial^\alpha[\chi_\ell(\epsilon \log \delta(x))]| \leq C\epsilon \cdot (\delta(x))^{-|\alpha|}$ for $0 < |\alpha| \leq m$, $x \in \Omega$.

To see (16), we note first that $\partial^\alpha[\chi_\ell(\epsilon \log \delta(x))]$ is a sum of terms of the form

$$\prod_{\nu=1}^{r} [\partial^{\alpha_\nu}(\epsilon \log \delta(x))] \cdot \chi_\ell^{(r)}(\epsilon \log \delta(x)),$$

where $\alpha_1 + \cdots + \alpha_r = \alpha$, each $\alpha_\nu \neq 0$, and $\chi_\ell^{(r)}$ denotes the $r^{\underline{th}}$ derivative of χ_ℓ. Next, observe that each factor $[\partial^{\alpha_\nu}(\epsilon \log \delta(x))]$ is a sum of terms of the form

$$\epsilon \frac{[\partial^{\beta_1}\delta(x)] \cdots [\partial^{\beta_s}\delta(x)]}{(\delta(x))^s}, \text{ with } \beta_1 + \cdots + \beta_s = \alpha_\nu.$$

Consequently, $\partial^\alpha[\chi_\ell(\epsilon \log \delta(x))]$ $(\alpha \neq 0)$ is a sum of terms of the form

(17) $\epsilon^r \frac{[\partial^{\beta_1}\delta(x)] \cdots [\partial^{\beta_s}\delta(x)]}{(\delta(x))^s} \cdot \chi_\ell^{(r)}(\epsilon \log \delta(x))$, with $r \geq 1$ and $\beta_1 + \cdots + \beta_s = \alpha$.

Each term (17) is bounded by $C\epsilon \cdot (\delta(x))^{-|\alpha|}$, thanks to (9) and (13). This completes the proof of (16). ■

Next, for each $\ell \in \mathbb{Z}$, we fix a partition of unity

(18) $1 = \sum_\nu \theta_\nu^\ell$ on $\mathbb{R}^n$,

where the θ^ℓ_ν are C^m functions with the following properties.

(19) $\theta^\ell_\nu \geq 0$ on $\mathbb{R}^n$, for each ℓ, ν.

(20) Each θ^ℓ_ν is supported in a cube Q^ℓ_ν of side $\frac{1}{3\epsilon}\, e^{\frac{\ell+1}{\epsilon}}$.

(21) $|\partial^\alpha \theta^\ell_\nu(x)| \leq C \cdot \left(\frac{1}{\epsilon}\, e^{\frac{\ell+1}{\epsilon}}\right)^{-|\alpha|}$ for $|\alpha| \leq m$, $x \in \mathbb{R}^n$, any ℓ, ν.

(22) For fixed $\ell \in \mathbb{Z}$, any given $x \in \mathbb{R}^n$ belongs to at most C of the cubes Q^ℓ_ν.

We are now ready to define our gentle partition of unity. For each ℓ, ν, we define

(23) $\chi^\ell_\nu(x) = \theta^\ell_\nu(x) \cdot \chi_\ell(\epsilon \log \delta(x))$ for $x \in \Omega$. Thus,

(24) $\chi^\ell_\nu \in C^m(\Omega)$.

We check that

(25) Any given $x \in \Omega$ belongs to $\mathsf{supp}\ \chi^\ell_\nu$ for at most C distinct (ℓ, ν).

To see (25), we note that $x \in \mathsf{supp}\, \chi^\ell_\nu$ implies $e^{\frac{\ell-1}{\epsilon}} < \delta(x) < e^{\frac{\ell+1}{\epsilon}}$, which holds for at most two distinct ℓ when x is fixed. (See (15).)

On the other hand, since $\mathsf{supp}\, \chi^\ell_\nu \subseteq \mathsf{supp}\, \theta^\ell_\nu \subseteq Q^\ell_\nu$ (see (20) and (23)), it follows from (22) that $x \in \mathsf{supp}\, \chi^\ell_\nu$ for at most C distinct ν, once x and ℓ are fixed. This completes the proof of (25). ■

In view of (25), the following formal calculation is justified, for any $x \in \Omega$.

(26)
$$\sum_{\ell,\nu} \chi^\ell_\nu(x) = \sum_\ell \chi_\ell(\epsilon \log \delta(x)) \cdot \sum_\nu \theta^\ell_\nu(x) \quad \text{(see (23))}$$
$$= \sum_\ell \chi_\ell(\epsilon \log \delta(x)) \text{ (see (18))} = 1 \quad \text{(see (10))}.$$

Next, note that

(27) $\chi^\ell_\nu \geq 0$ on Ω, for each ℓ, ν (see (11), (19), (23)), and

(28) $\mathsf{supp}\, \chi^\ell_\nu \subset \{x \in Q^\ell_\nu \cap \Omega : e^{\frac{\ell-1}{\epsilon}} < \delta(x) < e^{\frac{\ell+1}{\epsilon}}\}$ (see (15), (20), (23)).

Next, we check that

(29) $|\partial^\alpha \chi^\ell_\nu(x)| \leq C\epsilon \cdot (\delta(x))^{-|\alpha|}$ for $0 < |\alpha| \leq m$, $x \in \Omega$, any ℓ, ν.

To prove (29), we may restrict attention to $x \in \mathsf{supp}\, \chi^\ell_\nu$. Hence, $\delta(x) < e^{\frac{\ell+1}{\epsilon}}$ (see (28)), and therefore (21) implies that

(30) $|\partial^\alpha \theta^\ell_\nu(x)| \leq C\epsilon \cdot (\delta(x))^{-|\alpha|}$ for $0 < |\alpha| \leq m$.

We have also

(31) $0 \le \theta_\nu^\ell(x) \le 1$ (see (18), (19)), and $0 \le \chi_\ell(\epsilon \log \delta(x)) \le 1$ (see (10) and (11)).

The desired estimate (29) now follows from (16), (30) and (31), thanks to (23) and the product rule for derivatives.

This completes the proof of (29). ■

Properties (24)...(27) and (29) are hypotheses of Lemma GPU from Section 2. Thus, we have constructed our gentle partition of unity.

Next, we prepare to define functions $F_\nu^\ell \in C^m(\mathbb{R}^n)$, to be patched together using the gentle partition of unity. To do so, we apply the lemma from Section 3, to the cube $(Q_\nu^\ell)^*$, which has the same center as Q_ν^ℓ but three times the sidelength. Thus,

(32) sidelength $((Q_\nu^\ell)^*) = \frac{1}{\epsilon} e^{\frac{\ell+1}{\epsilon}}$ (see (20)),

and the lemma from Section 3 yields a set S_ν^ℓ with the following properties:

(33) S_ν^ℓ is an ϵ-testing set.

(34) $S_\nu^\ell \subset E \cap (Q_\nu^\ell)^*$.

(35) For any $y \in E \cap (Q_\nu^\ell)^*$, there exists $y' \in S_\nu^\ell$, such that $|y - y'| \le C\, e^{(\ell-1)/\epsilon}$. (See (32).)

To define F_ν^ℓ, we now apply (1), (2), (3) to the ϵ-testing set S_ν^ℓ. Thus, for each ℓ, ν, we have:

(36) $F_\nu^\ell \in C^m(\mathbb{R}^n)$, $\| F_\nu^\ell \|_{C^m(\mathbb{R}^n)} \le 1$,

(37) $J_y(F_\nu^\ell) = P^y$ for all $y \in S_\nu^\ell$, and

(38) $|\partial^\alpha F_\nu^\ell(x)| \le C$ for $|\alpha| \le m$, $x \in \mathbb{R}^n$.

To apply Lemma GPU from Section 2, we must estimate the difference $F_\nu^\ell - F_{\nu'}^{\ell'}$ on $\operatorname{supp} \chi_\nu^\ell \cap \operatorname{supp} \chi_{\nu'}^{\ell'}$.

Thus, let $x \in \operatorname{supp} \chi_\nu^\ell \cap \operatorname{supp} \chi_{\nu'}^{\ell'}$ be given. According to (28), we have

(39) $x \in Q_\nu^\ell \cap \Omega$ and $e^{(\ell-1)/\epsilon} < \delta(x) < e^{(\ell+1)/\epsilon}$, and similarly,

(40) $x \in Q_{\nu'}^{\ell'} \cap \Omega$, and $e^{(\ell'-1)/\epsilon} < \delta(x) < e^{(\ell'+1)/\epsilon}$.

We will check, using (35) and (39), that there exists

(41) $y \in S_\nu^\ell$, such that

(42) $|x - y| \le C\delta(x)$.

In fact, (39) and (8) produce a point $z \in E$, with $|z - x| \le C\delta(x) \le C\, e^{(\ell+1)/\epsilon}$. Since $x \in Q^\ell_\nu$, with the sidelength of Q^ℓ_ν equal to $\frac{1}{3}\, \epsilon^{-1}\, e^{(\ell+1)/\epsilon}$, we have $z \in E \cap (Q^\ell_\nu)^*$. (Here, we use our assumption that $\epsilon < \tilde{c}$ for a small enough $\tilde{c}$.) Hence (35) produces a point $y \in S^\ell_\nu$, with $|z - y| \le C\, e^{(\ell-1)/\epsilon} \le C\delta(x)$. Thus, $|x - y| \le |x - z| + |z - y| \le C\,\delta(x)$, and we obtain (41), (42). Similarly, there exists

(43) $y' \in S^{\ell'}_{\nu'}$ such that

(44) $|x - y'| \le C\delta(x)$.

Let y, y' be as in (41)...(44). By (37), (38), and Taylor's theorem, we have

$$|\partial^\alpha (F^\ell_\nu - P^y)(x)| \le C|x - y|^{m-|\alpha|} \text{ for } |\alpha| \le m\,.$$

Applying (42), we learn that

(45) $|\partial^\alpha (F^\ell_\nu - P^y)(x)| \le C \cdot (\delta(x))^{m-|\alpha|}$ for $|\alpha| \le m$.

Similarly,

(46) $|\partial^\alpha (F^{\ell'}_{\nu'} - P^{y'})(x)| \le C \cdot (\delta(x))^{m-|\alpha|}$ for $|\alpha| \le m$.

From (6), we have

$$|\partial^\alpha (P^y - P^{y'})(y)| \le C|y - y'|^{m-|\alpha|} \text{ for } |\alpha| \le m\,.$$

Applying (42), (44), we see that $|y - y'| \le C\delta(x)$, and therefore

$$|\partial^\alpha (P^y - P^{y'})(y)| \le C \cdot (\delta(x))^{m-|\alpha|} \text{ for } |\alpha| \le m\,.$$

In view of (42), this in turn implies that

(47) $|\partial^\alpha (P^y - P^{y'})(x)| \le C \cdot (\delta(x))^{m-|\alpha|}$ for $|\alpha| \le m$.

Combining (45), (46), (47), we find that

(48) $|\partial^\alpha (F^\ell_\nu - F^{\ell'}_{\nu'})(x)| \le C \cdot (\delta(x))^{m-|\alpha|}$ for $|\alpha| \le m$, $x \in \operatorname{supp} \chi^\ell_\nu \cap \operatorname{supp} \chi^{\ell'}_{\nu'}$.

This is our desired estimate for $F^\ell_\nu - F^{\ell'}_{\nu'}$.

We can now apply Lemma GPU from Section 2, to the partition of unity $\{\chi^\ell_\nu\}$, the open sets $U^\ell_\nu := \Omega$, and the functions $F^\ell_\nu \in C^m(\Omega)$, with $M = 1$, with $C\epsilon$ here in place of ϵ in Lemma GPU, and with the constants A_0, A_1 in Lemma GPU being controlled constants.

Let us check the hypotheses of Lemma GPU.

Evidently, the $U^\ell_\nu = \Omega$ form an open cover of Ω; and $\delta(x) > 0$ on Ω. For each ℓ, ν, we have F^ℓ_ν, $\chi^\ell_\nu \in C^m(\Omega)$, thanks to (36) and (24).

Also, hypotheses (GPU1,2,3,5,6,7) are immediate from our results (25), (26), (27), (29), (36), (48), respectively. Hypothesis (GPU4) is immediate from (28), since we take $U^\ell_\nu = \Omega$.

Thus, all the hypotheses of Lemma GPU hold here.

Applying Lemma GPU, we learn that the function

(49) $\tilde{F} = \sum_{\ell,\nu} \chi^\ell_\nu \cdot F^\ell_\nu$

belongs to $C^m(\Omega)$, and satisfies

(50) $\| \tilde{F} \|_{C^m(\Omega)} \leq 1 + C\epsilon$.

To complete the proof of part (A) of the $(1+\epsilon)$-Whitney theorem, it remains to show that

(51) $|\partial^\alpha(\tilde{F} - P^{\bar{y}})(x)| \leq C|x - \bar{y}|^{m-|\alpha|}$ for $|\alpha| \leq m$, $x \in \Omega$, $\bar{y} \in E$.

To prove this, we argue as follows.

Fix $x \in \Omega$, let $\bar{y}$ be a point of E closest to x, and let (ℓ, ν) be such that $x \in \operatorname{supp} \chi^\ell_\nu$. Then the proofs of (41), (42), (45) apply. Let y be as in (41), (42). Since $\delta(x) < C \operatorname{dist}(x, E) = C\,|x - \bar{y}|$ (see (8)), we conclude that

(52) $y \in S^\ell_\nu \subset E$, $|x - y| \leq C\,|x - \bar{y}|$, and

(53) $|\partial^\alpha(F^\ell_\nu - P^y)(x)| \leq C\,|x - \bar{y}|^{m-|\alpha|}$ for $|\alpha| \leq m$.

On the other hand, (6) and (52) give

$$|\partial^\alpha(P^y - P^{\bar{y}})(\bar{y})| \leq C\,|y - \bar{y}|^{m-|\alpha|} \leq C'|x - \bar{y}|^{m-|\alpha|} \text{ for } |\alpha| \leq m\,.$$

This in turn implies that

(54) $|\partial^\alpha(P^y - P^{\bar{y}})(x)| \leq C\,|x - \bar{y}|^{m-|\alpha|}$ for $|\alpha| \leq m$.

From (53) and (54), we conclude that

(55) $|\partial^\alpha(F^\ell_\nu - P^{\bar{y}})(x)| \leq C\,|x - \bar{y}|^{m-|\alpha|}$ for $|\alpha| \leq m$, $x \in \operatorname{supp} \chi^\ell_\nu$, $\bar{y}=$ point of E closest to x.

Moreover, with $x, \bar{y}, (\ell, \nu)$ as in (55), we have

(56) $|\partial^\alpha \chi^\ell_\nu(x)| \leq C \cdot (\delta(x))^{-|\alpha|} \leq C' \cdot (\operatorname{dist}(x, E))^{-|\alpha|} = C'|x - \bar{y}|^{-|\alpha|}$ for $|\alpha| \leq m$.

(Here, we use (26), (27), (29) and (8).)

Combining (55), (56), we learn that

$$|\partial^\alpha[\chi_\nu^\ell \cdot (F_\nu^\ell - P^{\bar{y}})](x)| \le C|x - \bar{y}|^{m-|\alpha|}$$

for $|\alpha| \le m$, $x \in \operatorname{supp} \chi_\nu^\ell$, $\bar{y}$ = a point of E closest to x.

Together with (25), this shows that

$$\left|\partial^\alpha\left[\sum_{\ell,\nu} \chi_\nu^\ell \cdot (F_\nu^\ell - P^{\bar{y}})\right](x)\right| \le C\,|x - \bar{y}|^{m-|\alpha|}$$

for $|\alpha| \le m$, $x \in \Omega$, $\bar{y}$ = a point of E closest to x.

In view of (26) and (49), this in turn yields

(57) $|\partial^\alpha(\tilde{F} - P^{\bar{y}})(x)| \le C\,|x - \bar{y}|^{m-|\alpha|}$

for $|\alpha| \le m$, $x \in \Omega$, $\bar{y}$ = point of E closest to x.

Finally, we pass from $\bar{y}$ in (57) to an arbitrary point $y \in E$. By definition of $\bar{y}$, we have

(58) $|x - \bar{y}| \le |x - y|$, hence also

(59) $|y - \bar{y}| \le 2|x - y|$.

Applying (6) and (59), we learn that

$$|\partial^\alpha(P^y - P^{\bar{y}})(y)| \le C\,|y - \bar{y}|^{m-|\alpha|} \le C'\,|x - y|^{m-|\alpha|} \text{ for } |\alpha| \le m\,,$$

and therefore

(60) $|\partial^\alpha(P^y - P^{\bar{y}})(x)| \le C''|x - y|^{m-|\alpha|}$ for $|\alpha| \le m$.

From (57) and (58), we have also

(61) $|\partial^\alpha(\tilde{F} - P^{\bar{y}})(x)| \le C\,|x - y|^{m-|\alpha|}$ for $|\alpha| \le m$.

Combining (60) and (61), we learn that

$$|\partial^\alpha(\tilde{F} - P^y)(x)| \le C\,|x - y|^{m-|\alpha|} \text{ for } |\alpha| \le m\,,\ x \in \Omega, y \in E\,.$$

This is precisely our desired estimate (51). The proof of part (A) of the $(1+\epsilon)$-Whitney theorem is complete. ■

Remark. Suppose E is finite; say $\#(E) = N$. Then, with a little extra care, we can arrange the arguments in this section to produce the following:

- The function $\tilde{F}$, constructed here on $\mathbb{R}^n \smallsetminus E$, extends trivially to a function $F \in C^m(\mathbb{R}^n)$, with norm at most $1 + C\epsilon$; and
- The construction of $\tilde{F}$ makes use of at most $C(\epsilon) \cdot N$ distinct ϵ-testing sets S.

Here, $C(\epsilon)$ depend only on ϵ, m, n and the constants $\bar{c}_0, \bar{C}_0$ in the Bounded Distortion Property; while C depends only on $m, n, \bar{c}_0, \bar{C}_0$. See [16].

6. Patching Near E

In this section we prove part (B) of the $(1+\epsilon)$-Whitney theorem. We recall the results and notation of the preceding section, and we assume in addition that $F_0 \in C^m(\mathbb{R}^n)$, with

(1) $J_y(F_0) = P^y$ for all $y \in E$.

Our plan is to patch together F_0 with $\tilde{F}$ from part (A), using a gentle partition of unity.

We start by analyzing F_0 near E. From (1), together with (5.4), we have

$$|J_y(F_0)|_y \le 1 \text{ for all } y \in E\,,$$

and therefore (1.3) yields

(2) $|J_y(F_0)|_x \le 1 + C\epsilon$ for $y \in E$, $|x-y| < \epsilon$.

Also, since $F_0 \in C^m(\mathbb{R}^n)$, we have

(3) $|\partial^\alpha(J_x(F_0) - J_y(F_0))(x)| \le C\omega(|x-y|) \cdot |x-y|^{m-|\alpha|}$
for $y \in E\,,\ |x-y| < \epsilon\,,\ |\alpha| \le m\,;$

where $\omega(\cdot)$ is the modulus of continuity of the $m^{\underline{\text{th}}}$ derivatives of F_0 on a large closed ball containing E.

In particular, this yields

$$|\partial^\alpha(J_x(F_0) - J_y(F_0))(x)| \le C\omega(|x-y|) \text{ for } |\alpha| \le m\,, |x-y| < \epsilon, y \in E\,,$$

and therefore

(4) $|J_x(F_0) - J_y(F_0)|_x \le C'\omega(|x-y|)$ for $|x-y| < \epsilon$, $y \in E$,

by the Bounded Distortion Property.

Since $\omega(t) \to 0$ as $t \to 0$, we can pick $r_1 > 0$ such that

(5) $\omega(|x-y|) < \epsilon$ for $|x-y| < r_1$.

We may take $r_1 < \epsilon$; we don't know how small r_1 might be. From (2), (4) and (5), we conclude that

(6) $|J_x(F_0)|_x \le 1 + C\epsilon$ for $\mathrm{dist}(x, E) < r_1$.

This is our desired estimate for F_0 near E.

Next, we estimate $\tilde{F} - F_0$ at points of Ω near E. Let $x \in \Omega$, and let $\bar{y}$ be a point of E closest to x. From (1), (3), (5), we obtain the estimate

(7) $|\partial^\alpha(F_0 - P^{\bar{y}})(x)| \le C\epsilon|x-\bar{y}|^{m-|\alpha|}$ for $|\alpha| \le m$,

provided $|x-\bar{y}| < r_1$.

On the other hand, the conclusion of part (A) of the $(1+\epsilon)$-Whitney theorem tells us that

(8) $|\partial^\alpha(\tilde{F} - P^{\bar{y}})(x)| \le C|x-\bar{y}|^{m-|\alpha|}$ for $|\alpha| \le m$.

Combining (7) and (8), we learn that

(9) $|\partial^\alpha(\tilde{F} - F_0)(x)| \le C|x-\bar{y}|^{m-|\alpha|}$ for $|\alpha| \le m$,

provided $|x-\bar{y}| < r_1$.

Since $|x-\bar{y}| = \mathrm{dist}\,(x,E)$, estimates (9) and (5.8) show that

(10) $|\partial^\alpha(\tilde{F} - F_0)(x)| \le C \cdot (\delta(x))^{m-|\alpha|}$ for $|\alpha| \le m$, $x \in \Omega$, $\delta(x) < r_2$,

where $r_2 > 0$ is small enough that $\delta(x) < r_2$ implies $\mathrm{dist}\,(x,E) < r_1$.

We then have also

(11) $|J_x(F_0)|_x \le 1 + C\epsilon$ for $x \in \Omega$, $\delta(x) < r_2$,

thanks to (6). Moreover, from part (A) of the $(1+\epsilon)$-Whitney theorem, we have

(12) $|J_x(\tilde{F})|_x \le 1 + C\epsilon$ for all $x \in \Omega$.

Estimates (10), (11), (12) will give us hypotheses (GPU6,7) of Lemma GPU.

Next, we define our gentle partition of unity.

Let $\chi(t)$ be a function on $[-\infty,\infty)$, with the following properties:

(13) $\chi(t) = 0$ for $t \ge -1$; $\chi(t) = 1$ for $t \le -2$; $0 \le \chi(t) \le 1$ for all t; and

(14) $\chi(t)$ is a C^m-function on $(-\infty,\infty)$, with $\left|\left(\frac{d}{dt}\right)^k \chi(t)\right| \le C$ for $k \le m$, $t \in (-\infty,\infty)$.

We define

(15) $\chi_{in}(x) = \chi\left(\epsilon \log \frac{\delta(x)}{r_2}\right)$ and $\chi_{out}(x) = 1 - \chi_{in}(x)$, for all $x \in \mathbb{R}^n$.

(If $x \in E$, we define $\delta(x) = 0$, $\chi_{in}(x) = 1$, $\chi_{out}(x) = 0$.) From (13) and (15), we see that

(16) $\chi_{in}(x) = 1$, $\chi_{out}(x) = 0$ for $\mathrm{dist}\,(x,E) < r_3$; and

(17) $\chi_{in}(x) = 0$, $\chi_{out}(x) = 1$ for $x \in \Omega$, $\delta(x) \ge e^{-1/\epsilon} r_2$.

Here, r_3 is a small enough positive number, such that

(18) $\mathrm{dist}\,(x,E) < r_3$ implies $\delta(x) < e^{-2/\epsilon} r_2$, for $x \in \Omega$. (See (5.8).)

Also from (13), (15), we obtain

(19) $0 \leq \chi_{\mathrm{in}}(x), \chi_{\mathrm{out}}(x) \leq 1$ for all $x \in \mathbb{R}^n$, and

(20) $\chi_{\mathrm{in}}(x) + \chi_{\mathrm{out}}(x) = 1$ for all $x \in \mathbb{R}^n$.

Finally, the proof of (5.16) applies here, to prove that $\chi_{\mathrm{in}}, \chi_{\mathrm{out}} \in C^m(\Omega)$, and

(21) $|\partial^\alpha \chi_{\mathrm{in}}(x)|, |\partial^\alpha \chi_{\mathrm{out}}(x)| \leq C\epsilon \cdot (\delta(x))^{-|\alpha|}$ for $0 < |\alpha| \leq m$, $x \in \Omega$.

We now define

(22) $F = \chi_{\mathrm{in}} \cdot F_0 + \chi_{\mathrm{out}} \cdot \tilde{F}$ on $\mathbb{R}^n$.

We first study F on Ω. Taking $U_{\mathrm{in}} = U_{\mathrm{out}} = \Omega$ and $M = 1 + C\epsilon$, we can check the hypotheses of Lemma GPU, with controlled constants for A_0, A_1 in that lemma. In fact:

$U_{\mathrm{in}}, U_{\mathrm{out}}$ form an open cover of the open set Ω;

$\delta(x) > 0$ for all $x \in \Omega$;

$F_0 \in C^m(U_{\mathrm{in}})$ and $\tilde{F} \in C^m(U_{\mathrm{out}})$;

χ_{in} and χ_{out} belong to $C^m(\Omega)$;

(GPU1) is obvious, since there are only two functions $\chi_{\mathrm{in}}, \chi_{\mathrm{out}}$ in our partition of unity;

(GPU2) is (20);

(GPU3) is immediate from (19);

(GPU4) holds, since, for Lemma GPU, $\mathrm{supp}\, \chi_{\mathrm{in}}$, $\mathrm{supp}\, \chi_{\mathrm{out}}$ are defined to be subsets of Ω, and we are taking $U_{\mathrm{in}} = U_{\mathrm{out}} = \Omega$.

(GPU5) is (21);

(GPU6) is immediate from (11), (12) and (17); finally,

(GPU7) is immediate from (10) and (17).

Thus, all the hypotheses of Lemma GPU hold. Applying that lemma, we learn that $F \in C^m(\Omega)$, and $\| F \|_{C^m(\Omega)} \leq 1 + C\epsilon$, i.e.,

(23) $|J_x(F)|_x \leq 1 + C\epsilon$ for all $x \in \Omega$.

On the other hand, (16) and (22) show that

(24) $F = F_0$ on $U = \{x \in \mathbb{R}^n : \mathrm{dist}\,(x, E) < r_3\}$.

In particular, $F \in C^m(U)$, and we have

(25) $J_x(F) = J_x(F_0) = P^x$ for all $x \in E$. (See (1).)

Consequently,

(26) $|J_x(F)|_x \leq 1$ for all $x \in E$,

thanks to (5.4).

Since $F \in C^m(\Omega)$ and $F \in C^m(U)$, with $\{\Omega, U\}$ an open cover of $\mathbb{R}^n$ (see (24)), we have

(27) $F \in C^m(\mathbb{R}^n)$.

From (23) and (26), we have $|J_x(F)|_x \leq 1 + C\epsilon$ for all $x \in \mathbb{R}^n$, i.e.,

(28) $\| F \|_{C^m(\mathbb{R}^n)} \leq 1 + C\epsilon$.

Our results (27), (28), (25) are the conclusions of part (B) of the $(1+\epsilon)$-Whitney theorem. The proof of part (B) is complete. ■

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