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Strong A_∞–weights are A_∞–weights on metric spaces

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Abstract

We prove that every strong A_∞–weight is a Muckenhoupt weight in Ahlfors–regular metric measure spaces that support a Poincaré inequality. We also explore the relations between various definitions for A_∞-weights in this setting, since some of these characterizations are needed in the proof of the main result.

1. Introduction

The purpose of this paper is to study strong A_∞–weights and A_∞–weights in Ahlfors-regular metric measure spaces. In particular, we answer to a question proposed by Costea in [7], and show that every strong A_∞–weight is an A_p–weight for some $p < \infty$ also in general metric setting. The space is assumed to be Ahlfors–regular and satisfy a weak $(1, 1)$–Poincaré inequality. We thus extend the result by Semmes [18] from $\mathbb{R}^n$ to general metric spaces. The Euclidean proof used extensively the linear structure of $\mathbb{R}^n$, for example convolutions and lines parallel to the coordinate axes. These tools are naturally not available in the metric setting. However, they can be replaced by more general methods. This shows, in particular, that the geometry of $\mathbb{R}^n$ is not crucial to the result.

Strong A_∞-weights were first introduced in $\mathbb{R}^n$ by David and Semmes in [8] and [18] when trying to characterize the subclass of A_∞–weights that are comparable to the Jacobian determinants of quasiconformal mappings. Later they have studied strong A_∞–weights, for example, in [19]. See also Bonk, Heinonen and Saksman [4] and [5] and Heinonen and Koskela [14] for further results concerning the quasiconformal Jacobian problem. Recently,

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strong A_∞-weights have been studied, for example, by Costea in [7] and [6]. In [7] he studies connections between strong A_∞–weights and Besov and Morrey spaces, and in [6] he extends the results to the metric setting. Strong A_∞–weights turn out to be useful in various applications, such as in studying elliptic partial differential equations, weighted Sobolev inequalities and Mumford–Shah type functionals. See, for example, [1], [3], [9], [10], [11], [15] and [16].

In Euclidean spaces, there are several equivalent characterizations for A_∞–weights. For example, a weight is an A_∞–weight if and only if it satisfies the reverse Hölder inequality or belongs to the class A_p for some finite p. Some of these relations are needed in proving that strong A_∞–weights are A_∞–weights. However, in more general spaces, all of these conditions are not necessarily equivalent, and, in particular, the class of A_∞–weights can be strictly larger than the union of A_p–classes, see Strömberg and Torchinsky [21]. In the last section of the paper, following [21], we study the relations between five different conditions in general metric spaces, and, in particular, we show that strong A_∞-weights satisfy all of them. Furthermore, we give some examples of weights that only satisfy some of the characterizations.

2. Preliminaries

2.1. Assumptions on the measure

Let (X, d, μ) be a metric measure space, where μ is Borel regular. We assume that the space is Ahlfors Q–regular with $Q \geq 1$, i.e. there exists $c_A \geq 1$ such that

$$\frac{1}{c_A} r^Q \leq \mu(B(x,r)) \leq c_A r^Q$$

for all $x \in X$ and $r > 0$. Notice that such a measure is always *doubling*, that is, there exists a constant $c_D \geq 1$ such that

$$\mu(B(x,2r)) \leq c_D \mu(B(x,r))$$

for all $x \in X$ and $r > 0$. Later, λB denotes the ball with the same center as B but λ times its radius.

2.2. Modulus of a curve family and Newtonian spaces

Let $1 \leq p < \infty$. For a given curve family Γ in X, we define the p–modulus of Γ by

$$\operatorname{mod}_p(\Gamma) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0,\infty]$ satisfying

$$\int_\gamma \rho \, ds \geq 1 \tag{2.1}$$

for all rectifiable curves $\gamma \in \Gamma$. We recall that a curve is rectifiable if its length is finite.

Let u be a real–valued function on X. We recall that a nonnegative Borel measurable function g on X is said to be an *upper gradient* of u if for all rectifiable curves γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds. \tag{2.2}$$

If the above property fails only for a set of curves that is of zero p–modulus then g is said to be a *p–weak upper gradient* of u. Every function u that has a p–integrable p–weak upper gradient has a *minimal p–integrable p–weak upper gradient* denoted g_u.

Finally, we recall that the Newtonian space $N^{1,p}(X)$ is the collection of all p–integrable functions u on X that have a p–integrable p–weak upper gradient g on X. For the precise definition; see, for example, [20].

2.3. Poincaré inequality

We assume that X satisfies a weak $(1,1)$–Poincaré inequality, i.e., there exists constants $c_P, \lambda > 0$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c_P r \fint_{B(x,\lambda r)} g_u \, d\mu$$

for all $u \in N^{1,1}(X)$, $x \in X$ and $r > 0$.

The following estimate is a consequence of the Poincaré inequality; see for example Lemma 3.3 in [3] for a proof.

Lemma 2.3. *Let (X, d, μ) be a metric measure space, where μ is doubling and X supports a weak $(1,1)$–Poincaré inequality. Let Γ be a curve family consisting of all rectifiable curves joining $B(x_0, r)$ and $X \setminus B(x_0, 2r)$. Then*

$$\operatorname{mod}_1(\Gamma) \geq C\mu(B(x_0, r))/r.$$

The constant $C > 0$ depends only on c_D and c_P.

Next we define A_p– and strong A_∞–weights.

2.4. A_p–weights

Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let ω be a nonnegative function on X. We say that ω is an A_p*–weight*, and write $\omega \in A_p$ if there exists a constant $c_\omega > 0$ such that

$$\left(\fint_B \omega \, d\mu \right) \left(\fint_B \omega^{1-q} \, d\mu \right)^{p-1} \leq c_\omega$$

for all balls B in X.

We say that ω is an A_1*–weight*, and write $\omega \in A_1$ if there exists a constant $c_\omega > 0$ such that

$$\fint_B \omega \, d\mu \leq c_\omega \operatorname*{ess\,inf}_B \omega$$

for all balls B in X.

Finally, ω is an A_∞*–weight* and we write $\omega \in A_\infty$ if there exists constants $c_\omega > 0$ and $\delta > 0$ such that

$$\frac{\int_E \omega \, d\mu}{\int_B \omega \, d\mu} \leq c_\omega \left(\frac{\mu(E)}{\mu(B)} \right)^\delta$$

for all balls B in X and all measurable subsets E of B.

We will discuss the relations between these definitions in Section 4.

2.5. Strong A_∞–weight

Let ν be a doubling measure on X. We associate to ν the quasi–distance $\delta_\nu(x, y)$ on X defined by

$$\delta_\nu(x, y) = [\nu(B(x, d(x, y))) + \nu(B(y, d(x, y)))]^{1/Q}.$$

We say that ν is a *metric doubling measure*, if there exists a distance function $\delta : X \times X \to [0, \infty)$ and a finite constant $C > 0$ such that

$$\frac{1}{C} \delta(x, y) \leq \delta_\nu(x, y) \leq C \delta(x, y) \tag{2.4}$$

for all $x, y \in X$. Moreover, $\omega \in L^1_{loc}(X)$ is called a *strong* A_∞*–weight*, $\omega \in SA_\infty$, if it is a density of a metric doubling measure, i.e.

$$d\nu = \omega \, d\mu. \tag{2.5}$$

Notice that the property (2.4) can be characterized in the following way, which will be useful later.

Lemma 2.6. *The condition* (2.4) *holds if and only if there is a constant* $C>0$ *such that for any finite sequence* $x_1, x_2, \dots, x_k$ *of points in* X, *we have*

$$\delta_\nu(x_1, x_k) \le C \sum_{j=1}^{k-1} \delta_\nu(x_j, x_{j+1}). \tag{2.7}$$

Proof. It is immediate that (2.4) implies (2.7). To prove the converse, define

$$\delta(x,y) = \inf \sum_{i=0}^{N-1} \delta_\nu(z_i, z_{i+1}),$$

where the infimum is taken over all finite sequences $z_0 = x, z_1, \dots, z_N = y$. Clearly $\delta(x,y) \le \delta_\nu(x,y)$ for all $x, y \in X$, and (2.7) implies that $\delta_\nu(x,y) \le C\delta(x,y)$. It is also easy to check that $\delta(\cdot,\cdot)$ is a distance function. ∎

The following theorems give some examples of strong A_∞-weights.

Theorem 2.8. *Every* A_1*–weight is a strong* A_∞*–weight.*

Proof. Notice first that the statement of Lemma 2.6 holds if and only if it holds with the additional restriction that $x_j \in B(x_1, 2d(x_1, x_k))$ for all j. The proof is similar to the Euclidean case and can be found in [18].

Assume then, that $\omega \in A_1$ and $d\nu = \omega\, d\mu$. Let $x_1, \dots, x_k$ be given and assume, that $x_j \in B(x_1, 2d(x_1, x_k))$ for all j. Write $B = B(x_1, d(x_1, x_k))$ and $B_j = B(x_j, d(x_j, x_{j+1}))$ for $j = 1, \dots, k-1$.

Notice also, that since ν is doubling, it readily follows from the definition of δ_ν that for all $x, y \in X$, we have

$$\nu(B(x, d(x,y)))^{1/Q} \le \delta_\nu(x,y) \le C\nu(B(x, d(x,y)))^{1/Q}, \tag{2.9}$$

where C depends only on the doubling constant. Then (2.9) implies that

$$\begin{aligned}\delta_\nu(x_j, x_{j+1}) &\ge \Big(\int_{B_j} \omega\, d\mu\Big)^{1/Q} \ge \big(\operatorname*{ess\,inf}_{B_j} \omega\big)^{1/Q} \mu(B_j)^{1/Q} \\ &\ge \big(\operatorname*{ess\,inf}_{6B} \omega\big)^{1/Q} \mu(B_j)^{1/Q}.\end{aligned}$$

Summing the above inequality over $j = 1, \dots, k-1$ we get

$$\begin{aligned}\sum_{j=1}^{k-1} \delta_\nu(x_j, x_{j+1}) &\ge \big(\operatorname*{ess\,inf}_{6B} \omega\big)^{1/Q} \sum_{j=1}^{k-1} \mu(B_j)^{1/Q} \\ &\ge C\big(\operatorname*{ess\,inf}_{6B} \omega\big)^{1/Q} \sum_{j=1}^{k-1} d(x_j, x_{j+1}) \ge C\big(\operatorname*{ess\,inf}_{6B} \omega\big)^{1/Q} d(x_1, x_k) \\ &\ge C\big(\operatorname*{ess\,inf}_{6B} \omega\big)^{1/Q} \mu(B)^{1/Q},\end{aligned}$$

where we used the triangle inequality and the Ahlfors regularity of μ. Moreover, since ω is an A_1–weight, and μ is doubling, we get

$$\begin{aligned}\big(\operatorname*{ess\,inf}_{6B}\omega\big)^{1/Q}\mu(B)^{1/Q} &\geq C\bigg(\fint_{6B}\omega\,d\mu\bigg)^{1/Q}\mu(B)^{1/Q}\\ &\geq C\bigg(\int_B\omega\,d\mu\bigg)^{1/Q}\geq C\delta_\nu(x_1,x_k).\end{aligned}$$

The proof follows now from Lemma 2.6. ■

Theorem 2.10. *Let (X,d_X,μ_X) and (Y,d_Y,μ_Y) be locally compact Ahlfors regular metric measure spaces such that X supports a weak $(1,p)$–Poincaré inequality for some $p<Q$ and let $f:X\to Y$ be a quasisymmetric mapping. Then the Jacobian of f is a strong A_∞-weight.*

Proof. We write

$$L(x,r)=\sup_{d_X(x,y)\leq r}d_Y(f(x),f(y)),\quad l(x,r)=\inf_{d_X(x,y)\geq r}d_Y(f(x),f(y)),$$

and recall that f is quasisymmetric, if it is a homeomorphism with a positive and finite constant K such that $L(x,r)\leq K\,l(x,r)$ for all $x\in X$ and $r>0$. Moreover, remember that the generalized Jacobian is defined as

$$J_f(x)=\lim_{r\to0}\frac{\mu_Y(f(B(x,r)))}{\mu_X(B(x,r))}.$$

By the Lebesgue-Radon-Nikodym theorem, the limit exists almost everywhere, and

$$\int_E J_f\,d\mu_X=\mu_Y(f(E)) \tag{2.11}$$

for all measurable $E\subset X$, since the measures are absolutely continuous.

Consider $d\nu=J_f\,d\mu_X$. Let $x,y\in X$ and write $d(x,y)=r$. Then

$$\begin{aligned}d_Y(f(x),f(y))^Q &\leq L(x,r)^Q\\ &\leq K^Q l(x,r)^Q\\ &\leq CK^Q\mu_Y(B(f(x),l(x,r)))\\ &\leq CK^Q\mu_Y(f(B(x,r)))\\ &= CK^Q\int_{B(x,r)}J_f\,d\mu_X\\ &\leq CK^Q\delta_\nu(x,y)^Q.\end{aligned}$$

Here we used the Ahlfors regularity and quasisymmetricity. On the other hand,

$$\begin{aligned}\delta_\nu(x,y)^Q &= \int_{B(x,r)} J_f\,d\mu_X + \int_{B(y,r)} J_f\,d\mu_X\\ &= \mu_Y(f(B(x,r))) + \mu_Y(f(B(y,r)))\\ &\le \mu_Y(B(f(x),L(x,r))) + \mu_Y(B(f(y),L(y,r)))\\ &\le C\big(L(x,r)^Q + L(y,r)^Q\big)\\ &\le CK^Q\big(l(x,r)^Q + l(y,r)^Q\big)\\ &\le 2CK^Q d_Y(f(x),f(y))^Q.\end{aligned}$$

Since $d_Y(f(\cdot),f(\cdot))$ is a distance on X, the claim follows. ■

3. Main result

In this section, we show that metric doubling measures have A_∞-densities in Ahlfors regular metric spaces. In particular, this implies that strong A_∞–weights are A_∞-weights in this setting. First, we recall the Gehring lemma. A proof can be found, for example, in [17], [22] and [2].

Theorem 3.1. *Let $1 < p < \infty$ and assume that $f \in L^1_{loc}(X)$ is nonnegative and defines a doubling measure. If there exists a constant c such that f satisfies the reverse Hölder inequality*

$$\left(\fint_B f^p\,d\mu\right)^{1/p} \le c \fint_B f\,d\mu \tag{3.2}$$

for all balls B of X, then there exists positive constants ε and c_ε such that

$$\left(\fint_B f^{p+\varepsilon}\,d\mu\right)^{1/(p+\varepsilon)} \le c_\varepsilon \fint_B f\,d\mu \tag{3.3}$$

for all balls B of X. The constant c_ε as well as ε depend only on the doubling constant, p, and on the constant in (3.2).

Now we are ready to state our main result.

Theorem 3.4. *Suppose that ν is a metric doubling measure. Then ν has an A_∞–density.*

Proof. Let ν be a metric doubling measure. First, we construct a set of measures $\{\nu_t\}_{t>0}$ that approximate ν. Then we show that the weights $\{\omega_t\}_{t>0}$ related to measures $\{\nu_t\}_{t>0}$ as in (2.5) satisfy a reverse Hölder inequality with uniform constants.

Fix $t > 0$. Let $\{B_i^t = B(x_i, t)\}_{i=1}^{\infty}$ be a collection of balls such that

$$X = \bigcup_{i=1}^{\infty} B_i^t$$

and

$$B(x_i, t/5) \cap B(x_j, t/5) = \emptyset \text{ for all } i \neq j.$$

Note that the doubling property of μ implies that

$$\sum_{i=1}^{\infty} \chi_{2B_i^t} < C. \tag{3.5}$$

To construct a partition of unity, we define cut–off functions

$$\widetilde{\phi}_i^t(x) = \begin{cases} 1, & x \in B_i^t, \\ 2 - \operatorname{dist}(x, x_i)/t, & x \in 2B_i^t \setminus B_i^t, \\ 0, & x \in X \setminus 2B_i^t, \end{cases}$$

and we set

$$\phi_i^t = \frac{\widetilde{\phi}_i^t}{\sum_{i=1}^{\infty} \widetilde{\phi}_i^t}.$$

Let

$$a_i^t = \frac{\int_X \phi_i^t \, d\nu}{\int_X \phi_i^t \, d\mu}.$$

Since both μ and ν are doubling, and

$$\frac{1}{C} \chi_{B_i^t} \leq \phi_i^t \leq \chi_{2B_i^t},$$

we have

$$\frac{1}{C} \frac{\nu(B_i^t)}{\mu(B_i^t)} \leq \frac{1}{C} \frac{\nu(B_i^t)}{\mu(2B_i^t)} \leq a_i^t \leq C \frac{\nu(2B_i^t)}{\mu(B_i^t)} \leq C \frac{\nu(B_i^t)}{\mu(B_i^t)}. \tag{3.6}$$

Finally, we define the measures ν_t, $t > 0$ as

$$\nu_t(A) = \sum_{i=1}^{\infty} a_i^t \int_A \phi_i^t \, d\mu.$$

Thus

$$d\nu_t = \omega_t \, d\mu = \Big(\sum_{i=1}^{\infty} a_i^t \phi_i^t \Big) d\mu.$$

The doubling property of μ and ν together with (3.6) imply that for every $x, y \in X$ such that $d(x,y) \le 2t$, we have

$$\frac{1}{C}\omega_t(x) \le \omega_t(y) \le C\omega_t(x). \tag{3.7}$$

More precisely, we have

$$\frac{1}{C}\frac{\nu(B(x,t))}{\mu(B(x,t))} \le \omega_t(y) \le C\frac{\nu(B(x,t))}{\mu(B(x,t))}, \tag{3.8}$$

where C depends only on the doubling constants of μ and ν.

Now fix a ball $B(x_0, r_0)$ in X. If $t \ge r_0$, we have by (3.7)

$$\Big(\fint_{B(x_0,r_0)} \omega_t \, d\mu\Big)^{1/Q} \le C\omega_t(x_0)^{1/Q} \le C \fint_{B(x_0,r_0)} \omega_t^{1/Q}\, d\mu. \tag{3.9}$$

Now consider the case $t < r_0$. By the definition of ω_t and the fact that $\operatorname{diam}(\operatorname{supp}(\phi_i^t)) \le 4t$, we have

$$\begin{aligned}\int_{B(x_0,r_0)} \omega_t\, d\mu &= \int_{B(x_0,r_0)} \Big(\sum_{i=1}^{\infty} a_i^t \phi_i^t\Big) d\mu \le \sum_{i\in I} \frac{\int_X \phi_i^t d\nu}{\int_X \phi_i^t d\mu} \int_X \phi_i^t\, d\mu \\ &\le \int_{B(x_0,r_0+4t)} 1 d\nu = \nu(B(x_0, r_0+4t))\end{aligned} \tag{3.10}$$

where $i \in I$ if $\operatorname{supp}(\phi_i^t) \cap B(x_0,r_0) \ne \emptyset$.

Let Γ be the set of all rectifiable curves $\gamma : [0, L] \to X$ parametrized by arc length such that $\gamma(0) \in B(x_0, r_0/2)$ and $\gamma(L) \in \partial B(x_0, r_0)$. Fix $\gamma \in \Gamma$. Let k be the integer part of L/t. Using the previous estimate, Lemma 2.6 and the doubling property of ν, we obtain

$$\begin{aligned}\int_0^L \omega_t(\gamma(s))^{1/Q}\, ds &= \sum_{j=1}^{k} \int_{(j-1)t}^{jt} \omega_t(\gamma(s))^{1/Q}\, ds + \int_{kt}^{L} \omega_t(\gamma(s))^{1/Q}\, ds \\ &\ge 1/C \sum_{j=1}^{k} \nu(B(\gamma(jt), t))^{1/Q} \ge 1/C \sum_{j=1}^{k} \delta_\nu(\gamma((j-1)t), \gamma(jt)) \\ &\ge 1/C\delta_\nu(\gamma(0), \gamma(L)).\end{aligned}$$

Thus for every $\gamma \in \Gamma$ we have

$$\nu(B(x_0, r_0))^{1/Q} \le C \int_\gamma \omega_t^{1/Q}\, ds.$$

If we define

$$\rho = \frac{C}{\nu(B(x_0,r_0))^{1/Q}} \omega_t^{1/Q} \chi_{B(x_0,r_0)},$$

then ρ satisfies (2.1) for every $\gamma \in \Gamma$ and consequently

$$\operatorname{mod}_1(\Gamma) \leq \int_X \rho \, d\mu = \frac{C}{\nu(B(x_0,r_0))^{1/Q}} \int_{B(x_0,r_0)} \omega_t^{1/Q} \, d\mu.$$

This combined with (3.10) and Lemma 2.3 gives

$$\begin{aligned} \left(\fint_{B(x_0,r_0)} \omega_t \, d\mu \right)^{1/Q} &\leq C \, \frac{\nu(B(x_0,r_0+4t))^{1/Q}}{\mu(B(x_0,r_0))^{1/Q}} \\ &\leq C \, \frac{\nu(B(x_0,r_0))^{1/Q}}{\mu(B(x_0,r_0))^{1/Q}} \leq C \fint_{B(x_0,r_0)} \omega_t^{1/Q} \, d\mu, \end{aligned} \tag{3.11}$$

where C is independent of t, x_0 and r_0. The first two inequalities above follow from the definition of ν_t and the doubling property of ν. Here we also used the Ahlfors regularity of μ. By combining (3.9) and (3.11), we obtain

$$\left(\fint_{B(x_0,r_0)} \omega_t \, d\mu \right)^{1/Q} \leq C \fint_{B(x_0,r_0)} \omega_t^{1/Q} \, d\mu$$

for all $t > 0$.

If we set $f = \omega_t^{1/Q}$, the Gehring lemma 3.1 now implies that there exists $\varepsilon > 0$ such that

$$\left(\fint_{B(x_0,r_0)} \omega_t^{1+\varepsilon/Q} \, d\mu \right)^{\frac{1}{1+\varepsilon/Q}} \leq C \fint_{B(x_0,r_0)} \omega_t \, d\mu,$$

with C independent of t, x_0 and r_0. By Lemma 4.2 in the next section this implies that ω_t is an A_∞–weight, and there exist $p > 1$ and $C > 0$, independent on t, such that

$$\frac{\nu_t(E)}{\nu_t(B)} \leq C \left(\frac{\mu(E)}{\mu(B)} \right)^{1/p} \tag{3.12}$$

for all balls B and measurable subsets $E \subset B$.

Next, we show that

$$\nu_t \to \nu$$

weakly in the sense of measures as $t \to 0$. In order to do that, fix an open set $U \subset X$. Denote

$$U_\varepsilon = \{x \in U \,:\, d(x, X \setminus U) > \varepsilon\}$$

and
$$I^t = \{i : 2B_i^t \subset U\} = \{i : \phi_i^t = 0 \text{ in } X \setminus U\}.$$
Note the fact that $\operatorname{diam}(2B_i^t) \le 4t$ implies that $i \in I^t$ whenever
$$\operatorname{supp}(\phi_i^t) \cap U_{4t} \neq \emptyset,$$
and, consequently,
$$\sum_{i \in I^t} \phi_i^t \ge \chi_{U_{4t}}.$$
Now by the definition of ν_t, we have
$$\begin{aligned}\nu_t(U) &= \sum_{i=1}^{\infty} \frac{\int_X \phi_i^t \, d\nu}{\int_X \phi_i^t \, d\mu} \int_U \phi_i^t \, d\mu \ge \sum_{i \in I^t} \frac{\int_X \phi_i^t \, d\nu}{\int_X \phi_i^t \, d\mu} \int_U \phi_i^t \, d\mu \\ &= \int_X \sum_{i \in I^t} \phi_i^t \, d\nu \ge \nu(U_{4t}).\end{aligned}$$
Thus
$$\liminf_{t \to 0} \nu_t(U) \ge \liminf_{t \to 0} \nu(U_{4t}) = \nu(U).$$
Since this holds for all open sets $U \subset X$, the claim follows.

Next we show that (3.12) holds true for ν. To this end, fix a ball B, a measurable set $E \subset B$ and an open set V such that $E \subset V \subset B$. Note that the weak convergence of ν_t implies that
$$\limsup_{t \to 0} \nu_t(S) \le \nu(S)$$
for all closed sets $S \subset X$, and by the doubling property of ν we have
$$\nu(\overline{B}) \le \nu(2B) \le c_D \nu(B)$$
for all balls $B \subset X$. Consequently,
$$\begin{aligned}\frac{\nu(E)}{\nu(B)} \le C \frac{\nu(V)}{\nu(2B)} \le C \frac{\nu(V)}{\nu(\overline{B})} &\le C \frac{\liminf_{t \to 0} \nu_t(V)}{\limsup_{t \to 0} \nu_t(\overline{B})} \\ &\le C \liminf_{t \to 0} \frac{\nu_t(V)}{\nu_t(B)} \le C \left(\frac{\mu(V)}{\mu(B)} \right)^{1/p}.\end{aligned}$$
Since μ is Borel regular, taking infimum over all such V finishes the proof of the claim. Finally, Lemma 4.2 implies that ν has an A_∞–density. ■

Corollary 3.13. *Every strong A_∞–weight is an A_∞–weight.*

4. Characterizations for A_∞-weights

There are several equivalent characterizations for A_∞-weights in the Euclidean setting. However, not all of them are necessarily equivalent in general metric spaces. In this section, we study the relationship between these conditions in metric spaces that are only assumed to satisfy the doubling condition. Most of these results can be found in [21], but for completeness, we have included the proofs here.

Recall the definitions of A_p–weights from Section 2.4. It follows immediately from the definitions that for every $1 < p < q < \infty$ we have

$$A_1 \subset A_p \subset A_q \subset A_\infty\,.$$

Moreover, in the Euclidean case,

$$A_\infty = \bigcup_{p<\infty} A_p\,.$$

Also in the metric setting, an A_p–weight is always an A_∞–weight, but there exist metric spaces, where the class of A_∞-weights is strictly larger than the union; see [21] and Example 4.1.

Next, we state five conditions that are equivalent in the Euclidean setting. For more definitions and the Euclidean case; see [12]. We consider a slightly more general situation first. Let ν be an arbitrary measure on X. We say that ν is a weighted measure with respect to μ is there exists $\omega \in L^1_{loc}(X)$ such that for every μ–measurable set $A \subset X$ we have

$$\nu(A) = \int_A \omega\, d\mu.$$

We define the conditions:

1. There are $0 < \varepsilon, \delta < 1$ such that for each ball B and each measurable set $E \subseteq B$, we have $\nu(E) \le (1-\delta)\nu(B)$ whenever $\mu(E) \le \varepsilon\mu(B)$.

2. There are constants $c > 0$ and $p \ge 1$ such that
$$\frac{\nu(E)}{\nu(B)} \le c\left(\frac{\mu(E)}{\mu(B)}\right)^{1/p}$$
for each ball B and each measurable set $E \subseteq B$.

3. ν is a weighted measure with respect to μ, and there exist positive constants $\varepsilon\,, C$ such that the weight ω satisfies the reverse Hölder inequality
$$\left(\fint_B \omega^{1+\varepsilon}\, d\mu\right)^{1/(1+\varepsilon)} \le C \fint_B \omega\, d\mu$$
for all balls B.

4. ν is a weighted measure with respect to μ, and there exist constants $c > 0$, $p \geq 1$ such that
$$\frac{\nu(E)}{\nu(B)} \geq c\left(\frac{\mu(E)}{\mu(B)}\right)^p$$
for each ball B and each measurable set $E \subseteq B$.

5. ν is a weighted measure with respect to μ, and the weight ω is in A_p for some $p > 1$.

If only the measure μ is assumed to be doubling, we obtain the following relations between the conditions above in the metric setting:
$$(1) \Leftarrow (2) \Leftrightarrow (3) \Leftarrow (4) \Leftrightarrow (5).$$

First we make some immediate remarks. The condition (1) follows easily from (2), and, since (1) is symmetric with respect to ν and μ, it follows also from (4). Condition (4) follows from (5) by applying the Hölder inequality on
$$\mu(E) = \int_B \chi_E \omega^{1/p} \omega^{-1/p}\, d\mu.$$
In addition, (4) implies that ν is doubling.

The following example shows that in general metric spaces, the conditions (3) and (4) are not necessarily equivalent:

Example 4.1. Let $X = \{x \in \mathbb{R}^n : x_1 = 0 \text{ or } x_2 = 0\}$ with $n \geq 2$. We endow X with the l^∞-metric i.e. $d(x, y) = \max_{1\leq i\leq n} |x_i - y_i|$ and the $(n-1)$-dimensional Lebesgue measure μ_{n-1}. Clearly, the measure μ_{n-1} is doubling and the space (X, d, μ_{n-1}) satisfies a $(1, 1)$-Poincaré inequality.

Now let $\omega(x) = \chi_{\{x_1\neq 0\}}(x)$. This weight satisfies condition (2) with $p = 1$ and $c = 2$ (and therefore also (3)). This follows easily from the fact that any ball $B \subset X$ that intersects the set $\{x \in X : x_1 = x_2 = 0\}$ satisfies
$$\mu_{n-1}(\{x \in B : x_1 = 0\}) = \mu_{n-1}(\{x \in B : x_2 = 0\}) = \mu_{n-1}(B)/2.$$
However, it cannot satisfy condition (4) since it is not doubling.

Lemma 4.2. (2) $\Leftrightarrow$ (3)

Proof. We give a sketch of the proof. If we assume (2), the absolute continuity part in (3) is clear. We fix a ball B and write $E_\lambda = \{x \in B \colon \omega(x) > \lambda\}$. Then by (2), we have
$$\mu(E_\lambda) \leq \frac{1}{\lambda}\nu(E_\lambda) \leq \frac{c}{\lambda}\nu(B)\left(\frac{\mu(E_\lambda)}{\mu(B)}\right)^{1/p},$$

and, hence

$$\mu(E_\lambda) \le \min\{\mu(B), c\big(\nu(B)^q/(\lambda\mu(B)^{1/p})^q\big)\}.$$

Now (3) follows from

$$\int_B \omega^{1+\varepsilon}\, d\mu = (1+\varepsilon)\int_0^\infty \lambda^\varepsilon \mu(E_\lambda)\, d\lambda$$

with $0 < \varepsilon < q-1$.

On the other hand, (2) follows from (3) by applying first the Hölder and then the reverse Hölder inequality to $\nu(E) = \int_B \chi_E \omega\, d\mu$.

Note that the doubling property of μ is not needed here. ■

The proof of the following lemma is similar to the Euclidean case; see [12].

Lemma 4.3. (2) & (4) $\Rightarrow$ (5)

The proof of the following theorem is based on ideas in [21]. However, the proof we present here is organized in a different way and contains more details.

Theorem 4.4. *If ν is doubling, then* (1) $\Rightarrow$ (2) *and* (1) $\Rightarrow$ (4)*.*

In order to prove Theorem 4.4 we introduce the notion of telescoping sequences of sets.

Definition 4.5. Let $s > 0$. We say that $\{\mathcal{F}_k\}_{k=1}^{k_0}$ is a *s–telescoping sequence of collections of balls*, $\mathcal{F}_k = \{B_{i,k}\}_{i=1}^\infty$, provided that

- $B_{i,k} \cap B_{j,k} = \emptyset$, for each $i \neq j$ and k.
- For each $B \in \mathcal{F}_k$, $k = 1, 2, \dots, k_0 - 1$, there exists $\widetilde{B} \in \mathcal{F}_{k+1}$ such that
$$sB \subset s\widetilde{B}.$$

The following lemma is a standard covering argument, see for example Theorem 1.2 in [13]. We have formulated it here to emphasize the fact that the cover can be chosen in such a way that every ball in $\mathcal{F}$ is included in $5B$ for some $B \in \mathcal{G}$.

Lemma 4.6. *Every family $\mathcal{F}$ of balls of uniformly bounded diameter in a metric space X contains a disjointed subfamily $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

In fact, every ball B from $\mathcal{F}$ meets a ball from $\mathcal{G}$ with radius at least half that of B.

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. We show that (1) implies (2). Note that in this proof, c_D denotes a constant that only depends on the doubling constants of μ and ν, but it is not necessarily exactly the doubling constant.

Fix a ball $B_0 = B(x_0, R)$ and a measurable set $E \subset B_0$. To prove the assertion, we will construct a 5-telescoping sequence of collections of balls $\mathcal{F}_k$, $k = 1, 2, \dots, k_0$, where k_0 is an integer such that

$$(\varepsilon/c_D^2)^{k_0+2} < \mu(E)/\mu(B_0) \le (\varepsilon/c_D^2)^{k_0+1}, \tag{4.7}$$

and the following properties hold: If

$$E_k := \bigcup_{B\in\mathcal{F}_k} B \quad \text{and} \quad \widetilde{E}_k := \bigcup_{B\in\mathcal{F}_k} 5B, \tag{4.8}$$

then we have $E \subset \widetilde{E}_1$, $\widetilde{E}_{k_0} \subset 5B_0$, and

$$\nu(\widetilde{E}_{k-1} \cap B) \le (1-\delta)\nu(B) \tag{4.9}$$

for all $B \in \mathcal{F}_k$. Here δ is as in (1). Note that (4.7) implies that

$$k_0 \ge \frac{\log(\mu(E)/\mu(B_0))}{\log(\varepsilon/c_D^2)} - 2. \tag{4.10}$$

We may assume that $\mu(E)/\mu(B_0)$ is small enough so that k_0 is positive, because if $\mu(E)/\mu(B_0)$ is bigger than a fixed constant, choosing c big enough makes the right–hand side of (2) bigger than one.

Once such a telescoping sequence of collections of balls has been constructed, the conclusion follows since

$$\begin{aligned}\nu(\widetilde{E}_{k-1}) &= \nu(\widetilde{E}_{k-1} \cap E_k) + \nu(\widetilde{E}_{k-1} \setminus E_k)\\ &\le \sum_{B\in\mathcal{F}_k} \nu(\widetilde{E}_{k-1} \cap B) + \nu(\widetilde{E}_k \setminus E_k) \le (1-\delta/c_D)\nu(\widetilde{E}_k).\end{aligned} \tag{4.11}$$

Here we used the fact that E_k is a union of disjoint balls satisfying (4.9), $\widetilde{E}_{k-1} \subset \widetilde{E}_k$, and that

$$\nu(\widetilde{E}_k) \le c_D \nu(E_k).$$

The last estimate above follows from the doubling property of ν. Iterating (4.11), we obtain

$$\nu(E)/\nu(B_0) \le c_D\nu(E)/\nu(5B_0) \le c_D\nu(\widetilde{E}_1)/\nu(\widetilde{E}_{k_0}) \le c_D(1-\delta/c_D)^{k_0-1}.$$

Then (2) follows by (4.10) with constants c and p depending only on δ, ε and the doubling constants of μ and ν. Now it remains to construct the $\mathcal{F}_k$'s.

We start with $\mathcal{F}_1$. Let $x \in E$ be a Lebesgue point of X. Then we have

$$\lim_{r\to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = 1,$$

and hence there exists $r_\varepsilon > 0$ such that

$$\frac{\mu(E \cap B(x,r_\varepsilon))}{\mu(B(x,r_\varepsilon))} > \varepsilon. \tag{4.12}$$

On the other hand, for all $r > 0$,

$$\frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \le \frac{\mu(E)}{\mu(B(x,r))}, \tag{4.13}$$

where the right–hand side tends to zero as r tends to infinity since E is of finite measure. We set $r_x = 2^n r_\varepsilon$, where n is the smallest positive integer such that

$$\frac{\mu(E \cap B(x,r_x))}{\mu(B(x,r_x))} \le \varepsilon. \tag{4.14}$$

By (4.13) such an n exists, and by the choice of r_x and (4.12), it follows that

$$\frac{\mu(E \cap B(x,r_x/2))}{\mu(B(x,r_x/2))} > \varepsilon. \tag{4.15}$$

Now the doubling property of μ together with (4.14) and (4.15) implies that

$$\varepsilon\mu(B(x,r_x))/c_D < \mu(E \cap B(x,r_x)) \le \varepsilon\mu(B(x,r_x)). \tag{4.16}$$

Now let $\mathcal{F}_1$ be a pairwise disjoint subfamily of the balls $\{B(x,r_x)\}_{x\in E}$ given by the 5–covering Theorem 4.6. Note that we are actually only able to cover Lebesgue points of E but this is enough, since μ–almost every point is a Lebesgue point.

Now let E_1 and $\widetilde{E}_1$ be defined by (4.8). Next, we replace E by $\widetilde{E}_1$ and construct $\mathcal{F}_2$ the same way as we constructed $\mathcal{F}_1$. Moreover, we repeat the procedure k_0-1 times and construct $\mathcal{F}_k$ by replacing E above by $\widetilde{E}_{k-1}$, $k \ge 2$.

Next, we show that $\widetilde{E}_{k_0} \subset 5B_0$. Assume, by contradiction, that there exists $m \le k_0$ such that $\widetilde{E}_m \not\subseteq 5B_0$. Then there exist balls $B(x_k,r_k) \in \mathcal{F}_k$, $k = 1,2,\dots,m$ such that

$$x_{k+1} \in B(x_k, 5r_k), \qquad k = 1,2,\dots,m-1,$$

and $B(x_m, 5r_m) \not\subseteq 5B_0$.

Since $E \subset B_0$, we also know that $B(x_1, r_1)$ intersects B_0. This implies that

$$\sum_{k=1}^{m} 5r_k > 4R_0. \tag{4.17}$$

Note also that since $\widetilde{E}_{m-1} \subset 5B_0$, also $x_m \in 5B_0$. Next, we need an estimate for the measure of $\widetilde{E}_k$. First, by (4.16), we obtain

$$\begin{aligned}\mu(\widetilde{E}_k) &= \mu\Big(\bigcup_{B\in\mathcal{F}_k} 5B\Big) \le c_D \sum_{B\in\mathcal{F}_k} \mu(B) \le \frac{c_D^2}{\varepsilon} \sum_{B\in\mathcal{F}_k} \mu(\widetilde{E}_{k-1}\cap B)\\ &= \frac{c_D^2}{\varepsilon}\mu(\widetilde{E}_{k-1}\cap E_k) \le \frac{c_D^2}{\varepsilon}\mu(\widetilde{E}_{k-1}).\end{aligned}$$

Write $\widetilde{E}_0 = E$. By iterating the above inequality and by using (4.7) we get

$$\mu(\widetilde{E}_k) \le \left(\frac{c_D^2}{\varepsilon}\right)^k \mu(E) \le \left(\frac{\varepsilon}{c_D^2}\right)^{k_0+1-k} \mu(B_0) \tag{4.18}$$

for $k = 0, 1, \ldots, k_0$.

The doubling property of μ and the fact that $x_k \in 5B_0$, $k = 1, 2, \ldots, m$, implies that there exists $s > 0$ depending only on the doubling constant of μ such that

$$\left(\frac{5r_k}{5R}\right)^s \le C\frac{\mu(5B_k)}{\mu(5B_0)} \le C\frac{\mu(\widetilde{E}_k)}{\mu(B_0)} \le C\left(\frac{\varepsilon}{c_D^2}\right)^{k_0+1-k} \le C\varepsilon^{k_0+1-k}$$

for $k = 1, 2, \ldots, m$. The last inequality above follows from (4.18). From this we deduce that

$$\sum_{k=1}^{m} 5r_k \le C\sum_{k=1}^{k_0} \varepsilon^{(k_0+1-k)/s} R \le C\varepsilon^{1/s} R$$

provided that ε is small enough. Here the constant C depends only on the doubling constant. If μ and ν satisfy (1) for some $0 < \varepsilon < 1$, they satisfy it for all smaller ε as well. Thus we can assume that ε is small enough to guarantee that the right–hand side of the above inequality is less than $4R$. However, this contradicts (4.17). Thus $\widetilde{E}_k \subset 5B_0$ for all $k = 1, 2, \ldots, k_0$.

Next, we verify that $\{\mathcal{F}_k\}_{k=1}^{k_0}$ is a telescoping sequence of collections of balls. First, by construction, the balls in $\mathcal{F}_k$ are pairwise disjoint for $k = 1, 2, \ldots, k_0$.

Finally, if $B(x,r) \in \mathcal{F}_{k-1}$, then $B(x,5r) \subset \widetilde{E}_{k-1}$. Hence, in the construction of $\mathcal{F}_k$, $r_x \geq 5r$, since (4.14) with E replaced by $\widetilde{E}_{k-1}$ cannot hold for any smaller radius. As $B(x,r_x)$ is one of the balls that is available when we use the covering argument to choose $\mathcal{F}_k$, we have $B(x,5r) \subset B(x,r_x) \subset 5B$ for some $B \in \mathcal{F}_k$. This shows that also the second condition for telescoping sequences holds and thus the collection is telescoping.

Since μ and ν satisfy condition (1), we conclude from (4.16) with E replaced by $\widetilde{E}_{k-1}$ that (4.9) holds for every $k = 1, 2, \ldots, k_0$. This completes the proof of (1) $\Rightarrow$ (2).

Since (1) is symmetric with respect to μ and ν, and (4) is (2) with the roles of μ and ν interchanged, we also get (1) $\Rightarrow$ (4). ■

If the function $r \mapsto \mu(B(x,r))$ is continuous for every $x \in X$, then it is rather easy to show that the condition (1) implies that ν is doubling and consequently the conditions (1)–(5) are all equivalent, see Theorem 17 on page 9 in [21]. In particular if X is a geodesic space and μ doubling, the conditions are equivalent. However, Example 4.1 shows that there are quite simple doubling metric measure spaces supporting a $(1,1)$-Poincaré inequality where $r \mapsto \mu(B(x,r))$ is not always continuous and all of the conditions are not equivalent.

However, a combination of the results in this section gives us the following:

Corollary 4.19. *If ν is a doubling measure, then the conditions* (1)–(5) *are equivalent. In particular, if $d\nu = \omega\, d\mu$ with $\omega \in SA_\infty$, then μ and ν satisfy all the conditions.*

Remark 4.20. If $p < \infty$ then for every $\omega \in A_p$, the measure $d\nu = \omega\, d\mu$ is doubling. Therefore, Lemma 4.3 and Theorem 4.4 imply that an A_∞–weight is an A_p–weight for some $p < \infty$ if and only if the weight defines a doubling measure.

Now, by Remark 4.20, we obtain the following.

Corollary 4.21. *Every strong A_∞–weight is an A_p–weight for some $p < \infty$.*

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