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Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of the potential

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Abstract

In this paper we consider the system in $\mathbb{R}^3$

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi(x)u = u^p, \\ -\Delta\phi = u^2, \end{cases} \tag{0.1}$$

for $p \in (1,5)$. We prove the existence of multi-bump solutions whose bumps concentrate around a local minimum of the potential $V(x)$. We point out that such solutions do not exist in the framework of the usual Nonlinear Schrödinger Equation.

1. Introduction and main results

Recently, many papers have studied different versions of the Schrödinger-Poisson-X^α problem:

$$-\frac{\hbar^2}{2m}\Delta u + V(x)u + \left(u^2 \star \frac{1}{4\pi|x|}\right)u = |u|^{p-1}u, \;\; x \in \mathbb{R}^3, \tag{1.1}$$

where $V(x)$ is an external potential and $p \in (1,5)$. The interest on this problem stems from the Slater approximation of the exchange term in the Hartree-Fock model, see [24]. In this framework, $p = 5/3$; however, other exponents have been used in different approximations, which have been referred to as X^α type approximations, see [21]. From another point of view, this equation has been proposed in [5] under the name of Schrödinger-Maxwell equation. For more information on the relevance of this model and its deduction, we refer to [5, 6, 7, 8, 21, 25].

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From the mathematical point of view, problem (1.1) presents an interaction between two different kind of nonlinear terms: a repulsive nonlocal term and an attractive local term. This, and related problems, have been much studied recently by using variational methods, see [3, 4, 9, 10, 16, 17, 22, 23, 26, 27].

If we define $\phi_u = u^2 \star \frac{1}{4\pi|x|}$ and $\varepsilon^2 = \frac{\hbar^2}{2m}$, the equation (1.1) can be rewritten as a system in the form:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, \\ -\Delta\phi = u^2. \end{cases} \tag{1.2}$$

In this paper we are concerned with the semiclassical limit for the system (1.2), namely the problem of finding non trivial solutions $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and studying their asymptotic behavior as $\varepsilon \to 0$. Such solutions are usually referred to as *semiclassical states.*

A large number of papers study the semiclassical states for the following nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-1}u, \qquad x \in \mathbb{R}^3. \tag{1.3}$$

For the problem (1.3) spike solutions are found around the critical points of the potential V, see for instance [1, 20]. These are solutions that concentrate (as $\varepsilon \to 0$) around a unique point, and tend to zero uniformly outside a ball centered at this point. For instance in [20] Yanyan Li proved the existence of positive solutions concentrating near C^1 stable critical points of V. Moreover, Li proves also the existence of multi-bump solutions, namely, solutions concentrating around different critical points of V. Other results in this direction were given in [13, 14]. However, in the previous papers the bumps are well separated and so the interactions among the different bumps are neglected.

In [15] the authors prove the existence of multi-bump solutions for (1.3) whose bumps tend to a point of local maximum of V. Here the interactions among the bumps do play a role. In a certain sense, each bump has an attractive effect on the other bumps, whereas the potential has a repulsive effect (around its local maximum). The multi-bump solution exists due to a balance between the two effects. The authors also show that multi-bump solutions do not exist around nondegenerate local minima. In this case, both effects would be attractive and no balance could be possible.

With respect to (1.2), the existence of single-bump solutions near critical points of V has been recently proved, see [19]. Other concentration phenomena have been proved for this system even with the absence of the potential, see [11, 12].

In this paper we prove the existence of positive solutions with K interacting bumps around local minima of the potential V. These solutions appear because of the effect of the Poisson term in our equation. Indeed, the Poisson term implies a repulsive effect among the bumps which balance the attractive effect of the potential V.

We assume that:

(V1) V has a local strict minimum point in P_0, namely there exists a bounded open set $\mathcal{U}$ such that $P_0 \in \mathcal{U}$ and

$$V(P_0) = \min_{x \in \bar{\mathcal{U}}} V(x) < V(P), \qquad \forall\, P \in \mathcal{U} \setminus \{P_0\}$$

Up to a translation and dilatation, we can assume $P_0 = 0$, $V(0) = 1$.

(V2) $V(x) = 1 + |g(x)|^\alpha$ for any $x \in \mathcal{U}$, where $g : \mathcal{U} \to \mathbb{R}$ is a $C^{2,1}$ function and $\alpha > 2$.

In particular, there holds:

(V2') $V(x) \le 1 + C|x|^\alpha$ for $x \in \mathcal{U}$ and some $C > 0$.

(V3) $\inf V > 0$.

Observe that under the above conditions the local minimum must be degenerate. We point out that conditions (V1)-(V2')-(V3) are sufficient for most of our arguments. We need condition (V2) for technical reasons, to be able to rule out possible undesired oscillations of the derivatives of V near 0.

Let us denote by U the unique positive radial solution in $H^1(\mathbb{R}^3)$ of the problem (see [18]):

$$-\Delta U + U = U^p. \tag{1.4}$$

Our main result is the following.

Theorem 1.1. *Assume that V satisfies* (V1), (V2) *and* (V3) *and suppose $p \in (1, 5)$. Then for any positive integer $K \in \mathbb{Z}$, there exists $\varepsilon_K > 0$ such that for any $\varepsilon < \varepsilon_K$ there exists a positive solution u_ε of* (1.2) *with K bumps converging to* 0. *More specifically, there exists $Q_1^\varepsilon, \dots Q_k^\varepsilon \in \mathbb{R}^3$ such that:*

1. *$Q_i^\varepsilon \to 0$, $\varepsilon^{-1}|Q_i^\varepsilon| \to +\infty$ as $\varepsilon \to 0$.*
2. *Defining $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, we have that $\tilde{u}_\varepsilon(x) = \sum_{i=1}^K U(x - \varepsilon^{-1}Q_i^\varepsilon) + o(1)$, as $\varepsilon \to 0$.*

The proof uses a singular perturbation method, based on a Lyapunov-Schmidt reduction. We point out that the distance between the bumps is different from that of the multi-bump solutions of [15], and this is caused because the different balance involving the Poisson term.

The paper is organized as follows. Section 2 is devoted to some notations and to the variational setting of the problem. In Section 3 we introduce the Lyapunov-Schmidt reduction and solve the auxiliary equation. Finally, in Section 4 the reduced functional is studied, solving the bifurcation equation. This completes the proof of Theorem 1.1.

2. Preliminaries

As mentioned in the introduction, we denote by U the unique positive radial solution in $H^1(\mathbb{R}^3)$ of the problem

$$-\Delta U + U = U^p.$$

This solution satisfies the following decay property (see [18]):

$$\lim_{r\to+\infty} U(r)re^r = C > 0, \qquad \lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1, \qquad r = |x|.$$

for some constant C.

The function U is a critical point of the C^2 functional $I_0 : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx, \tag{2.1}$$

where $\|\cdot\|$ denotes the usual norm in $H^1(\mathbb{R}^3)$. Furthermore the solution U is nondegenerate (up to translations). More specifically, there holds:

Lemma 2.1. *Define the operator $Q : H^1(\mathbb{R}^3) \to \mathbb{R}$ as*

$$Q[\nu] := I_0''(U)[\nu,\nu] = \int_{\mathbb{R}^3} \Big[|\nabla\nu|^2 + \nu^2 - pU^{p-1}\nu^2\Big]\, dx.$$

We denote $U_k = \frac{\partial U}{\partial x_k}$. Then there hold:

- $Q[U] = (1-p)\|U\|^2 < 0.$
- $Q[\frac{\partial U}{\partial x_j}] = 0,\ j = 1,2,3.$
- $Q[\nu] \geq C\|\nu\|^2$ *for all* $\nu \perp U, \nu \perp \frac{\partial U}{\partial x_j}$, $j = 1,2,3.$

For a proof see for instance [2, Lemma 8.6].

It is convenient to make the change of variable $x \mapsto \varepsilon x$ and so we arrive to the problem:

$$-\Delta u + V(\varepsilon x)u + \varepsilon^2\phi_u u = u^p, \qquad u \in H^1(\mathbb{R}^3), \qquad u > 0. \tag{2.2}$$

Here $\phi_u \in D^{1,2}(\mathbb{R}^3)$, and

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2\, dx = \int_{\mathbb{R}^3} \phi_u u^2\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\, dx\, dy.$$

In general, given $f \in L^{6/5}$, the solution of the problem $-\Delta\phi = f$ belongs to $D^{1,2}(\mathbb{R}^3)$ and:

$$\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla\psi = \int_{\mathbb{R}^3} f\psi \leq \|\psi\|_{L^6}\|f\|_{L^{6/5}} \leq C\|\psi\|_{D^{1,2}}\|f\|_{L^{6/5}}.$$

Therefore, $\|\phi\|_{D^{1,2}} \leq C\|f\|_{6/5}$.

Moreover, it is well-known (see [5], for example) that the solutions of (2.2) correspond to positive critical points of the C^2 functional $I_\varepsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$,

$$I_\varepsilon(u) = = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right] dx + \frac{\varepsilon^2}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx. \tag{2.3}$$

Finally, let us compute the derivatives of V. By using (V2)

$$\begin{aligned} V_{x_i}(x) &= \alpha|g(x)|^{\alpha-2}g(x)g_{x_i}(x),\\ V_{x_i x_j}(x) &= \alpha(\alpha-1)|g(x)|^{\alpha-2}g_{x_i}(x)g_{x_j}(x) + \alpha|g(x)|^{\alpha-2}g(x)g_{x_i x_j}(x). \end{aligned} \tag{2.4}$$

In particular $V \in C^{2,\gamma}(\mathcal{U})$, where $\gamma = \min\{1, \alpha - 2\}$.

3. The Lyapunov-Schmidt reduction. The auxiliary equation

In this section we begin the Lyapunov-Schmidt for the proof of Theorem 1.1. This will be made around an appropriate set of "approximating solutions". For any $K \in \mathbb{N}$, we define

$$\Lambda_\varepsilon = \left\{\mathbf{P} \in \mathbb{R}^{3K} : |P_i - P_j| \geq \varepsilon^{\frac{2-\alpha}{\alpha+1}+\delta},\ i \neq j,\ V(\varepsilon P_i) \leq 1 + \varepsilon^{\frac{3\alpha}{\alpha+1}-\delta},\ \varepsilon P_i \in \mathcal{U}\right\}$$

where $\delta > 0$ is chosen small enough so that $\frac{3\alpha}{\alpha+1} - \delta > 2$ (this is possible since $\alpha > 2$). Observe that $\frac{2-\alpha}{\alpha+1} + \delta < 0$ and Λ_ε is not empty for ε small enough.

Fix $\mathbf{P} = (P_1, \ldots, P_K) \in \Lambda_\varepsilon$. Setting $z_{P_i}(x) = U(x - P_i)$, we define the manifold of "approximate solutions":

$$\mathcal{Z} = \left\{z_{\mathbf{P}}(x) = \sum_{i=1}^{K} z_{P_i}(x) \ :\ \mathbf{P} \in \Lambda_\varepsilon\right\}.$$

This section is devoted to the proof of the next result:

Proposition 3.1. *Assume that V satisfies* (V1), (V2) *and* (V3) *and suppose $p \in (1,5)$. Then for any positive integer $K \in \mathbb{Z}$, there exists $\varepsilon_K > 0$ such that for any $\varepsilon < \varepsilon_K$ there exists a positive solution u_ε of* (2.2), *and $z_\varepsilon \in \mathcal{Z}$ such that $\|u_\varepsilon - z_\varepsilon\| = O(\varepsilon^2)$.*

It is easy to check that Proposition 3.1 implies Theorem 1.1.

The proof uses a Lyapunov-Schmidt reduction. For every $z \in \mathcal{Z}$, we define $W = W_{z,\varepsilon} = (T_z\mathcal{Z})^{\perp}$ and $P : H^1(\mathbb{R}^3) \to W$ the orthogonal projection onto W. Our approach is to find a pair $z \in \mathcal{Z}$, $w \in W$, $\|w\| = O(\varepsilon^2)$, such that $I'_\varepsilon(z + w) = 0$. Equivalently:

$$\left\{ \begin{array}{l} \text{a) } PI'_\varepsilon(z+w) = 0, \\ \\ \text{b) } (\mathcal{I} - P)I'_\varepsilon(z+w) = 0. \end{array} \right. \tag{3.1}$$

The first equation above is called auxiliary equation, and the second one receives the name of bifurcation equation.

Our intention now is to find a solution $w \in W$ of the auxiliary equation for any $z \in \mathcal{Z}$. We begin with some estimates:

Proposition 3.2. *There exists $C = C(K) > 0$ such that for all $\varepsilon > 0$ small and any $\mathbf{P} \in \Lambda_\varepsilon$, we have*

$$\|I'_\varepsilon(z_{\mathbf{P}})\| \leq C\varepsilon^2. \tag{3.2}$$

Proof. Taking into account that z_{P_i} are solutions of (1.4), we have:

$$I'_\varepsilon(z_{\mathbf{P}})[v] = \underbrace{\int_{\mathbb{R}^3} [V(\varepsilon x) - 1] z_{\mathbf{P}} v \, dx}_{(I)} + \underbrace{\varepsilon^2 \int_{\mathbb{R}^3} \phi_{z_{\mathbf{P}}} z_{\mathbf{P}} v \, dx}_{(II)} - \underbrace{\int_{\mathbb{R}^3} \Big[|z_{\mathbf{P}}|^p - \sum_{i=1}^{K} z_{P_i}^p \Big] v \, dx}_{(III)}$$

Let us evaluate separately the various terms. The second term can be easily estimated (see Section 2):

$$(II) \quad \leq \quad \|z_{\mathbf{P}}\|^3 \cdot \|v\| \leq C\, K\, \|v\|. \tag{3.3}$$

For (I), it suffices to estimate

$$\int_{\mathbb{R}^3} [V(\varepsilon x) - 1] z_{P_i} v \, dx \leq \underbrace{\int_{\mathbb{R}^3} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i} v \, dx}_{(A)} + \underbrace{\int_{\mathbb{R}^3} [V(\varepsilon P_i) - 1] z_{P_i} v \, dx}_{(B)}.$$

By the definition of Λ_ε, we get that $(B) = o(\varepsilon^2)$. Let us estimate (A) by splitting the integral in two parts:

$$\int_{\mathbb{R}^3} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i} v \, dx = \int_{|x-P_i| > \varepsilon^{-1}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i} v \, dx + \int_{|x-P_i| < \varepsilon^{-1}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i} v \, dx.$$

Since V is bounded in L^∞, we use Hölder estimate and the change $y = x - P_i$, to conclude

$$\int_{|x-P_i|>\varepsilon^{-1}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i} v \, dx \leq C \Big(\int_{|y|>\varepsilon^{-1}} U^2(y)\, dy \Big)^{1/2} \|v\|_{L^2} = o(\varepsilon^M) \|v\|_{L^2}$$

for any $M > 0$, thanks to the exponential decay of U.

Observe that if $|x - P_i| < \varepsilon^{-1}$, εP_i belongs to $\mathcal{U}$ and $d(\varepsilon x, \mathcal{U}) \leq 1$. We use a Taylor expansion:

$$\begin{aligned} (3.4) \qquad \int_{|x-P_i|<\varepsilon^{-1}} & | V(\varepsilon x) - V(\varepsilon P_i) | z_{P_i} |v| \, dx \leq \\ & \leq \int_{\mathbb{R}^3} \big(\varepsilon |\nabla V(\varepsilon P_i)| \, |x - P_i| + C\varepsilon^2 |x - P_i|^2 \big) z_{P_i} |v| \, dx. \end{aligned}$$

Again by the exponential decay of U, $\| \, |x - P_i|^m z_{P_i} \|_{L^2}$ is uniformly bounded for any $m > 0$. So it suffices to estimate $|\nabla V(\varepsilon P_i)|$.

Recall that $\varepsilon P_i \in \Lambda_\varepsilon$, and so $V(\varepsilon P_i) = 1 + |g(\varepsilon P_i)|^\alpha \leq 1 + \varepsilon^{\frac{3\alpha}{\alpha+1} - \delta}$. By (2.4),

$$|V_{x_i}(x)| \leq C |g(x)|^{\alpha - 1} \leq C \varepsilon^{\left(\frac{3\alpha}{\alpha+1} - \delta\right) \frac{\alpha-1}{\alpha}}.$$

Observe that $\frac{3\alpha}{\alpha+1} - \delta > 2 > \frac{\alpha}{\alpha-1}$. Therefore, $\nabla V(\varepsilon P_i) = o(\varepsilon)$.

Finally we consider (III). These estimates have been done in [15]; we sketch here the proof for the sake of completeness. Let us define $\rho_\varepsilon = \varepsilon^{\frac{2-\alpha}{\alpha+1} + \delta}$ and divide $\mathbb{R}^3$ in $K + 1$ regions:

$$\Omega_i = \{ x \in \mathbb{R}^3 : 2|x - P_i| \leq \rho_\varepsilon \} \quad \text{for } i = 1 \dots K, \quad \Omega_0 = \mathbb{R}^3 \setminus \left(\cup_{i=1}^K \Omega_i \right).$$

We now use the $C^{1,\sigma}$ regularity of the function $f(u) = u^p$, where $\sigma = \min\{1, p-1\}$:

$$\begin{aligned} \int_{\Omega_j} & \Big| \Big(\sum_{i=1}^K z_{P_i} \Big)^p - z_{P_j}^p - \sum_{i \neq j} z_{P_i}^p \Big| |v| \, dx \\ & \leq \int_{\Omega_j} \Big[p z_{P_j}^{p-1} \Big(\sum_{i \neq j} z_{P_i} \Big) + C \Big(\sum_{i \neq j} z_{P_i} \Big)^{1+\sigma} + \sum_{i \neq j} z_{P_i}^p \Big] |v| \, dx \\ & \leq C \int_{\Omega_j} \Big(\sum_{i \neq j} z_{P_i} \Big) |v| \, dx. \end{aligned}$$

The last inequality is due to the fact that in Ω_j, $z_{P_i} \leq 1$. Indeed, defining $\rho_\varepsilon = \varepsilon^{\frac{2-\alpha}{\alpha+1} + \delta}$ and using the exponential decay of U, we have

$$\int_{\Omega_j} z_{P_i}^2(x) \, dx \leq \int_{2|x|>\rho_\varepsilon} U^2(y) \, dy \leq C \int_{2r>\rho_\varepsilon} e^{-2y} \, dr = C e^{-\rho_\varepsilon}.$$

On the other hand,

$$\int_{\Omega_0} \Big| \Big(\sum_{i=1}^{K} z_{P_i} \Big)^p - \sum_{i=1}^{K} z_{P_i}^p \Big| |v| \le C \int_{\Omega_0} \sum_{i=1}^{K} z_{P_i}^p |v|,$$

$$\int_{\Omega_0} z_{P_i}^{2p}(x)\, dx \le \int_{2|x|>\rho_\varepsilon} U^{2p}(y)\, dy \le Ce^{-p\rho_\varepsilon}.$$

This concludes the estimate (III). ■

Now we are concerned with the invertibility of $I''_\varepsilon(z_{\mathbf{P}})$ on $W = (T_{z_{\mathbf{P}}}(\mathcal{Z}))^\perp$. First we observe that $T_{z_{\mathbf{P}}}\mathcal{Z}$ is spanned by the functions $\dot{z}_{i,j} := \frac{\partial U}{\partial x_j}(x - P_i)$, with $i = 1, \dots, K$ and $j = 1, 2, 3$. Recall that P denotes the orthogonal projection onto W; me decompose: $W = A \oplus B$ where

$$A = \langle \{Pz_{P_i}\}_{i=1\dots K} \rangle \text{ and } B = (A \oplus T_{z_{\mathbf{P}}}\mathcal{Z})^\perp$$

Proposition 3.3. *For ε small and any $\mathbf{P} \in \Lambda_\varepsilon$, $PI''_\varepsilon(z_{\mathbf{P}}) : W \to W$ is invertible and $\|[PI''_\varepsilon(z_{\mathbf{P}})]^{-1}\| \le \bar{C}$.*

The above result follows directly from the following lemma (see [2]):

Lemma 3.4. *For all $\varepsilon > 0$ sufficiently small there exist two positive constants C_1, C_2 such that*

(a) $I''_\varepsilon(z_{\mathbf{P}})[u,u] \le -C_1\|u\|^2$, *for all* $u \in A$;

(b) $I''_\varepsilon(z_{\mathbf{P}})[u,u] \ge C_2\|u\|^2$, *for all* $u \in B$.

Proof. Let be $u \in A$. Then

$$u = \sum_{i=1}^{K} \lambda_i P z_{P_i}, \qquad \lambda_i \in \mathbb{R}, \quad i = 1, \dots, K.$$

For $i = 1, \dots, K$, Pz_{P_i} are orthogonal to $T_{z_{\mathbf{P}}}(\mathcal{Z})$. Hence we can write

$$Pz_{P_i} = z_{P_i} - \psi_i, \qquad i = 1, \dots, K$$

where ψ_i are given by

$$\psi_i = \sum_{\substack{l,j \\ l \ne i}} (z_{P_i}, \dot{z}_{l,j}) \frac{\dot{z}_{l,j}}{||\dot{z}_{l,j}||^2}.$$

The functions $\dot{z}_{l,j}$ satisfy $-\Delta \dot{z}_{l,j} + \dot{z}_{l,j} = pz_{P_l}^{p-1}\dot{z}_{l,j}$. Since for $i \ne l$, $|P_i - P_l| \to +\infty$ as $\varepsilon \to 0$, after an integration by parts, we get $(z_{P_i}, \dot{z}_{l,j}) = o(1)$ as $\varepsilon \to 0$. This implies $\|\psi_i\| = o(1)$ as $\varepsilon \to 0$ for $i = 1, \dots, K$.

We now apply the bilinear form given by $I''_\varepsilon(z_{\mathbf{P}})$ to obtain

$$I''_\varepsilon(z_{\mathbf{P}})[u,u] = \underbrace{I''_\varepsilon(z_{\mathbf{P}})\Big[\sum_{i=1}^K \lambda_i z_{P_i}, \sum_{i=1}^K \lambda_i z_{P_i}\Big]}_{(I)} + \underbrace{I''_\varepsilon(z_{\mathbf{P}})\Big[\sum_{i=1}^K \lambda_i \psi_i, \sum_{i=1}^K \lambda_i \psi_i\Big]}_{(II)}$$
$$+2\underbrace{I''_\varepsilon(z_{\mathbf{P}})\Big[\sum_{i=1}^K \lambda_i z_{P_i} \sum_{i=1}^K \lambda_i \psi_i\Big]}_{(III)}.$$

We observe that $I''_\varepsilon(z_{\mathbf{P}})$ maps bounded sets onto bounded sets, then since $z_{\mathbf{P}}$ is bounded

$$(II) \le \|I''_\varepsilon(z_{\mathbf{P}})\| \sum_{i=1}^K \lambda_i^2 \|\psi_i\|^2 \le C \sum_{i=1}^K \lambda_i^2 \|\psi_i\|^2 = o(1).$$

In the same way we obtain

$$(II) \le \|I''_\varepsilon(z_{\mathbf{P}})\| \sum_{i=1}^K \lambda_i^2 \|\psi_i\|^2 \le C \sum_{i=1}^K \lambda_i^2 \|\psi_i\| = o(1).$$

Furthermore, by making simple computations one finds

$$(I) = \sum_{i=1}^K \lambda_i^2 \Big(\int_{\mathbb{R}^3} [|\nabla z_{P_i}|^2 + z_{P_i}^2 - p z_{P_i}^{p+1}]\,dx \Big)$$
$$+ \sum_{i=1}^K \lambda_i^2 \underbrace{\Big(\int_{\mathbb{R}^3} [V(\varepsilon x) - 1] z_{P_i}^2\,dx \Big)}_{(A)}$$
$$+ 2 \sum_{i\neq j} \lambda_i \lambda_j \underbrace{\Big(\int_{\mathbb{R}^3} [\nabla z_{P_i} \nabla z_{P_j} + V(\varepsilon x) z_{P_i} z_{P_j}]\,dx \Big)}_{(B)}$$
$$\underbrace{+ \varepsilon^2 \int_{\mathbb{R}^3} \phi_{z_{\mathbf{P}}} \Big(\sum_{i=1}^K \lambda_i z_{P_i} \Big)^2 dx}_{(C)} + \underbrace{2\varepsilon^2 \int_{\mathbb{R}^3} \widetilde{\phi} \cdot z_{\mathbf{P}} \Big(\sum_{i=1}^K \lambda_i z_{P_i} \Big) dx}_{(D)}$$
$$\underbrace{- p \int_{\mathbb{R}^3} \Big[\Big| \sum_{i=1}^K z_{P_i} \Big|^{p-1} \Big(\sum_{i=1}^K \lambda_i z_{P_i} \Big)^2 - \sum_{i=1}^K \lambda_i^2 z_{P_i}^{p+1} \Big] dx}_{(E)}$$

where $\widetilde{\phi}$ solves $-\Delta\widetilde{\phi} = \big(\sum_{i=1}^K \lambda_i z_{P_i}\big) z_{\mathbf{P}}$.

Reasoning as in the proof of Proposition 3.2, we obtain that $(A) = o(1)$, $(B) = o(1)$, $(C) = o(1)$, $(D) = o(1)$. Moreover

$$(E) \leq C(\lambda_i) \int_{\mathbb{R}^3} \left[|z_{\mathbf{P}}|^{p+1} - \sum_{i=1}^{K} z_{P_i}^{p+1} \right] dx.$$

Then $(E) = o(1)$ as $\varepsilon \to 0$ (see Proposition 3.2). At the end

$$I''_\varepsilon(z_{\mathbf{P}})[u,u] = \sum_{i=1}^{K} \lambda_i^2 I''_0(z_{P_i})[z_{P_i}, z_{P_i}] + o(1).$$

Therefore, using Lemma 2.1 we have, for ε small, that

$$I''_\varepsilon(z_{\mathbf{P}})[u,u] \leq (1-p) \sum_{i=1}^{K} \lambda_i^2 \|z_{P_i}\|^2 < -C_1 < 0.$$

So $I''_\varepsilon(z_{\mathbf{P}})$ is negative definite on A. We now prove that $I''_\varepsilon(z_{\mathbf{P}})$ is positive definite on B.

Choose an arbitrary $u \in B$. For simplicity, assume that $\|u\| = 1$. We denote by $\hat{\phi}$ the solution of $-\Delta\hat{\phi} = z_{\mathbf{P}} u$. Since $z_{\mathbf{P}}$ and u are bounded, it is easy to see that, for ε small enough,

$$\varepsilon^2 \int_{\mathbb{R}^3} \left[\phi_{z_{\mathbf{P}}} u^2 + 2\hat{\phi} z_{\mathbf{P}} u \right] dx = \int_{\mathbb{R}^3} \left[V(\varepsilon x) - 1 \right] u^2 dx = o(1).$$

Then

$$\begin{aligned} I''_\varepsilon(z_{\mathbf{P}})[u,u] &= \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(\varepsilon x) u^2 + \varepsilon^2 \phi_{z_{\mathbf{P}}} u^2 + 2\varepsilon^2 \hat{\phi} z_{\mathbf{P}} u - p z_{\mathbf{P}}^{p-1} u^2 \right] dx \\ &= \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2 - p z_{\mathbf{P}}^{p-1} u^2 \right] dx + o(1). \end{aligned}$$

As done in Proposition 3.2 it can be proved that

$$\int_{\mathbb{R}^3} z_{\mathbf{P}}^{p-1} u^2 dx = \int_{\mathbb{R}^3} \sum_{i=1}^{K} z_{P_i}^{p-1} u^2 dx + o(1).$$

Hence

$$I''_\varepsilon(z_{\mathbf{P}})[u,u] = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2 - p \sum_{i=1}^{K} z_{P_i}^{p-1} u^2 \right] dx + o(1). \tag{3.5}$$

We need to estimate the integral in (3.5). In order to do this, we use the following technical result:

Claim: for ε small there exists $R \in \left(\varepsilon^{\frac{\theta}{2}}, \frac{1}{2}\varepsilon^{\theta}\right)$, with $\theta = \frac{2-\alpha}{\alpha+1} + \delta < 0$, such that

$$\sum_{i=1}^{K} \int_{R<|x-P_i|<R+1} [|\nabla u|^2 + u^2]\, dx < 4\varepsilon^{-\theta}. \tag{3.6}$$

To prove this we remark that from $\|u\| = 1$ it follows

$$\sum_{i=1}^{K} \sum_{R \in (\varepsilon^{\frac{\theta}{2}}, \frac{1}{2}\varepsilon^{\theta})} \int_{R<|x-P_i|<R+1} [|\nabla u|^2 + u^2]\, dx \le 1 \qquad R \in \mathbb{N}.$$

Since, for ε small, the above sum has more than $\frac{\varepsilon^{\theta}}{4}$ summands, then, it is always possible to choose $R \in \mathbb{N}$, $R \in \left(\varepsilon^{\frac{\theta}{2}}, \frac{1}{2}\varepsilon^{\theta}\right)$ such that the claim holds.

Let us fix R such that (3.6) is satisfied and define the smooth cut-off functions $\chi_i : \mathbb{R} \to [0,1]$, $i = 1, \ldots, K$ by setting

$$\chi_i(x) := \begin{cases} 1 & |x - P_i| < R \\ 0 & |x - P_i| > R + 1 \\ |\nabla \chi_i(x)| \le 2 & \forall\, x \in \mathbb{R}^3. \end{cases}$$

Define also

$$\chi_0(x) = 1 - \sum_{i=1}^{K} \chi_i(x).$$

Then we can decompose $u = \sum_{i=0}^{K} u_i$ where $u_i = u\chi_i$. From (3.6) it follows that for $i \neq j$ $(u_i, u_j) = o(1)$. Thus

$$1 = \|u\|^2 = \sum_{i=0}^{K} \|u_i\|^2 + o(1).$$

Using again (3.6), we obtain that $(z_{P_i}, u_j) = o(1)$ for $i > 0$, $i \neq j$. Since $u \in B$ we have $(z_{P_i}, u) = o(1)$. Then for $i = 1, \ldots, K$

$$(z_{P_i}, u) = \sum_{j=0}^{K} (z_{P_i}, u_j) = (z_{P_i}, u_i) + o(1).$$

Hence $(z_{P_i}, u_i) = o(1)$. Finally, for $i = 1, \ldots, K$, since $u \perp \dot{z}_{i,j}$, reasoning as above, we find also $(\dot{z}_{i,j}, u_l) = o(1)$ for all i, j, l.

By using the above properties and Lemma 2.1 we obtain

$$\begin{aligned} I''_\varepsilon(z_{\mathbf{P}})[u,u] &= \int_{\mathbb{R}^3} \Big[|\nabla u|^2 + u^2 - p\sum_{i=1}^K z_{P_i}^{p-1} u^2\Big]\,dx + o(1) \\ &= \sum_{i=1}^K \int_{\mathbb{R}^3} \big[|\nabla u_i|^2 + u_i^2 - p z_{P_i}^{p-1} u_i^2\big]\,dx + \|u_0\|^2 + o(1) \\ &\geq C\sum_{i=1}^K \|u_i\|^2 + \|u_0\|^2 + o(1) \\ &\geq C_2\Big(\sum_{i=0}^K \|u_i\|^2\Big) + o(1) \\ &\geq C_2 > 0. \end{aligned}$$

∎

With this estimates in hand we can now solve the auxiliary equation. Consider $z = z_{\mathbf{P}} \in \mathcal{Z}$ fixed, and define

$$B_\varepsilon = \big\{u \in W : \|u\| \leq 2\bar{C}\|I'_\varepsilon(z)\|\big\},$$

where $\bar{C}$ is the positive constant given by Proposition 3.3. So, the solutions of the auxiliary equations are fixed points of the map $S_\varepsilon : W \to W$

$$S_\varepsilon(w) = w - [PI''_\varepsilon(z)]^{-1}[PI'_\varepsilon(z+w)].$$

It is easy to check that $\|S_\varepsilon(0)\| \leq \bar{C}\|I'_\varepsilon(z)\|$. We now compute the derivative of S_ε:

$$S'_\varepsilon(w)[v] = v - [PI''_\varepsilon(z)]^{-1} PI''_\varepsilon(z+w)[v] = [PI''_\varepsilon(z)]^{-1}\left(PI''_\varepsilon(z) - PI''_\varepsilon(z+w)\right)[v].$$

Now observe that I''_ε is uniformly continuous in bounded sets, so

$$\|PI''_\varepsilon(z+w) - PI''_\varepsilon(z)\| \to 0 \qquad (\varepsilon \to 0)$$

uniformly in $z \in \mathcal{Z}$ and $w \in B_\varepsilon$ (recall Proposition 3.2).

This implies that $\|S'_\varepsilon(w)\| = o(1)$ for any $w \in B_\varepsilon$. Therefore, S_ε is a contraction and, by using the mean value theorem, $S_\varepsilon(B_\varepsilon) \subset B_\varepsilon$. We make use of the Banach contraction theorem to find a unique fixed point $w = w_{\varepsilon,z} \in B_\varepsilon$ of S_ε. Moreover one has

$$\|w_{\varepsilon,z}\| \leq 2\bar{C}\|I'_\varepsilon(z)\| \leq C\varepsilon^2 \tag{3.7}$$

4. The reduced functional

In this section we will find a solution for the bifurcation equation among the set of solutions of the auxiliary equation, which is:

$$\bar{\mathcal{Z}} = \{z + w_{\varepsilon,z} : z \in \mathcal{Z},\ w_{\varepsilon,z} \text{ solves } (3.1)(a), \text{ and satisfies } (3.7)\}.$$

By the Implicit Function Theorem it is easy to check that $\bar{\mathcal{Z}}$ is a C^1 manifold. Moreover, it is well-known (see [2], for example) that $\bar{\mathcal{Z}}$ is a natural constraint for I_ε for ε small. In other words, critical points of $I_\varepsilon|_{\bar{\mathcal{Z}}}$ are solutions of the bifurcation equation (3.1) (b), and hence solutions of (2.2).

So, let us define the reduced functional as the restriction of the functional I_ε to the natural constraint $\bar{\mathcal{Z}}$, namely $\Phi_\varepsilon : \Lambda_\epsilon \to \mathbb{R}$, $\Phi_\varepsilon(\mathbf{P}) = I_\varepsilon(z_{\mathbf{P}} + w_{\varepsilon,z_{\mathbf{P}}})$, and we look for critical points of Φ_ε. Using the information on $\|w_{\varepsilon,z_{\mathbf{P}}}\|$, we will be able to find an expansion of $\Phi_\varepsilon(\mathbf{P})$.

First of all, since I''_ε maps bounded sets onto bounded sets, we have

$$\Phi_\varepsilon(\mathbf{P}) = I_\varepsilon(z_{\mathbf{P}}) + I'_\varepsilon(z_{\mathbf{P}})[w_{\varepsilon,z_{\mathbf{P}}}] + O(\|w_{\varepsilon,\mathbf{P}}\|^2).$$

Using Proposition 3.2 and (3.7) we deduce

$$\Phi_\varepsilon(\mathbf{P}) = I_\varepsilon(z_{\mathbf{P}}) + O(\varepsilon^4). \tag{4.1}$$

So we have to compute $I_\varepsilon(z_{\mathbf{P}})$. Preliminary lemmas are in order.

Lemma 4.1. *For $\beta = 1, 2$ and $F : \mathbb{R}^3 \to \mathbb{R}$ such that $(1+|y|^{\beta+1})F \in L^1 \cap L^\infty$ set*

$$\Psi_\beta[F](x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|^\beta} F(y)\, dy.$$

Then there exist two positive constants $C = C(\beta, F)$ and $C' = C'(\beta, F)$ such that

$$\left| \Psi_\beta[F] - \frac{C}{|x|^\beta} \right| \le \frac{C'}{|x|^{\beta+1}}, \qquad \forall\, x \neq 0. \tag{4.2}$$

For a proof see [11]. Now, thanks to the the exponential decay of U the following estimate holds (see Lemma 2.1 of [15]):

Lemma 4.2. *For ε sufficiently small and $\mathbf{P} \in \Lambda_\varepsilon$, we have*

$$\int_{\mathbb{R}^3} z_{P_i}^p z_{P_j}\, dx = (\eta + o(1))e^{-|P_1 - P_2|}$$

where

$$\eta = \int_{\mathbb{R}^3} U^p(x) e^{-x_1}\, dx > 0.$$

We are now in position to find an expansion of $I_\varepsilon(z_{\mathbf{P}})$.

Proposition 4.3. *For any* $\mathbf{P} = (P_1, \dots, P_K) \in \Lambda_\varepsilon$ *and* $\varepsilon > 0$ *sufficiently small we have*

$$I_\varepsilon(z_{\mathbf{P}}) = C_0 + \varepsilon^2 C_1 + C_2 \sum_{i=1}^{K} V(\varepsilon P_i) + C_3 \varepsilon^2 \sum_{i \neq j} \frac{1}{|P_i - P_j|} + o(\varepsilon^{\frac{3\alpha}{\alpha+1} - \delta}) \tag{4.3}$$

where

$$C_0 = K \cdot \left(\frac{1}{2} \int_{\mathbb{R}^3} |\nabla U|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} U^{p+1} \, dx \right),$$
$$C_1 = \frac{K}{4} \int_{\mathbb{R}^3} \frac{U^2(x) U^2(y)}{|x-y|} \, dx \, dy,$$
$$C_2 = \frac{1}{2} \int U^2 \, dx,$$

and C_3 *is a positive constant given by* Lemma 4.1, *which depends only on* U.

Proof. We compute

$$\begin{aligned}
I_\varepsilon(z_{\mathbf{P}}) =& \sum_{i=1}^{K} I_\varepsilon(z_{P_i}) + \sum_{i \neq j} \int_{\mathbb{R}^3} \left[\nabla z_{P_i} \nabla z_{P_j} + V(\varepsilon x) z_{P_i} z_{P_j} \right] dx \\
&+ \frac{\varepsilon^2}{4} \sum_{i \neq j} \int_{\mathbb{R}^3} \phi_{z_{P_i}} z_{P_j}^2 dx + \frac{\varepsilon^2}{2} \sum_{l, i \neq j} \int_{\mathbb{R}^3} \phi_{i,j} z_l^2 dx \\
&+ \frac{\varepsilon^2}{4} \sum_{i \neq j} \int_{\mathbb{R}^3} \phi_{z_{\mathbf{P}}} z_{P_i} z_{P_j} dx - \frac{1}{p+1} \int_{\mathbb{R}^3} \left[|z_{\mathbf{P}}|^{p+1} - \sum_{i=1}^{K} |z_{P_i}|^{p+1} \right] dx \\
=& \sum_{i=1}^{K} I_\varepsilon(z_{P_i}) + \sum_{i \neq j} \int_{\mathbb{R}^3} z_{P_i}^p z_{P_j} dx + \frac{\varepsilon^2}{4} \sum_{i \neq j} \int_{\mathbb{R}^3} \phi_{z_{P_i}} z_{P_j}^2 dx \\
&+ \frac{\varepsilon^2}{2} \sum_{l, i \neq j} \int_{\mathbb{R}^3} \phi_{i,j} z_l^2 dx + \frac{\varepsilon^2}{4} \sum_{i \neq j} \int_{\mathbb{R}^3} \phi_{z_{\mathbf{P}}} z_{P_i} z_{P_j} dx \\
&- \frac{1}{p+1} \int_{\mathbb{R}^3} \left[|z_{\mathbf{P}}|^{p+1} - \sum_{i=1}^{K} |z_{P_i}|^{p+1} \right] dx + o(\varepsilon^{\frac{3\alpha}{\alpha+1} - \delta}).
\end{aligned}$$

Here $\phi_{i,j}$ are the solutions of $-\Delta \phi_{i,j} = z_{P_i} z_{P_j}$, $i \neq j$. Let us evaluate separately the various terms.

Claim: There holds:

$$I_\varepsilon(z_{P_i}) = \widetilde{C}_0 + \varepsilon^2 \widetilde{C_1} + C_2 V(\varepsilon P_i) + o(\varepsilon^{\frac{3\alpha}{\alpha+1} - \delta}) \tag{4.4}$$

where

$$\widetilde{C}_0 = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} dx, \quad \widetilde{C_1} = \frac{1}{4}\int_{\mathbb{R}^3} \phi_U U^2 dx,$$
$$C_2 = \frac{1}{2}\int_{\mathbb{R}^3} U^2 dx.$$

It suffices to estimate:

$$\int_{\mathbb{R}^3} [V(\varepsilon x) - V(\varepsilon P_i)] U^2(x - P_i)\, dx.$$

First, we split this integral expression in two terms

$$\int_{\mathbb{R}^3} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i}^2\, dx = \int_{|x-P_i|>\varepsilon^{-\tau}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i}^2\, dx + \int_{|x-P_i|<\varepsilon^{-\tau}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i}^2\, dx,$$

for some positive constant τ to be determined. Since V is bounded in L^∞, we use the change $y = x - P_i$, and the exponential decay of U to conclude

$$\int_{|x-P_i|>\varepsilon^{-\tau}} [V(\varepsilon x) - V(\varepsilon P_i)] z_{P_i}^2\, dx \le C \int_{|y|>\varepsilon^{-\tau}} U^2(y)\, dy = o(\varepsilon^M)$$

for any positive M.

We use a Taylor expansion:

$$\left| \int_{|x-P_i|<\varepsilon^{-\tau}} [V(\varepsilon x) - V(\varepsilon P_i) - \varepsilon \nabla V(\varepsilon P_i)\cdot (x - P_i)]\, z_{P_i}^2 \right| \le \tag{4.5}$$
$$\le \frac{\varepsilon^2}{2} \max\{\|D^2 V(\xi)\| :\ |\xi - \varepsilon P_i| < \varepsilon^{1-\tau}\} \int_{\mathbb{R}^3} |x - P_i|^2 z_{P_i}^2\, dx.$$

By using the radial symmetry of U,

$$\int_{|x-P_i|<\varepsilon^{-\tau}} \nabla V(\varepsilon P_i)\cdot (x - P_i) U(x - P_i)^2 = 0.$$

So, it suffices to estimate $\|D^2 V(\xi)\|$ for $|\xi - \varepsilon P_i| < \varepsilon^{1-\tau}$. First, observe that if $\tau < 1$ and ε is small enough, $\xi \in \mathcal{U}$.

Moreover, by the definition of Λ_ε, $V(\varepsilon P_i) = 1 + |g(\varepsilon P_i)|^\alpha \le 1 + \varepsilon^{\frac{3\alpha}{\alpha+1} - \delta}$. From this and (2.4) we have that

$$|V_{x_i x_j}(\varepsilon P_i)| \le C |g(x)|^{\alpha - 2} \le C \varepsilon^{\frac{\alpha-2}{\alpha}\left(\frac{3\alpha}{\alpha+1} - \delta\right)}.$$

On the other hand, since $V \in C^{2,\gamma}$ (recall, $\gamma = \min\{1, \alpha - 2\}$):

$$\left|V_{x_i x_j}(\xi) - V_{x_i x_j}(\varepsilon P_i)\right| \le C\varepsilon^{\gamma(1-\tau)}.$$

Therefore,

$$|V_{x_i x_j}(\xi)| \le C\varepsilon^{\min\{\frac{\alpha-2}{\alpha}\left(\frac{3\alpha}{\alpha+1}-\delta\right),\ \gamma(1-\tau)\}}.$$

By direct computation, $2 + \frac{\alpha-2}{\alpha}\left(\frac{3\alpha}{\alpha+1} - \delta\right) > \frac{3\alpha}{\alpha+1} - \delta$. Moreover, $2 + 1 = 3 > \frac{3\alpha}{\alpha+1}$ and $2 + \alpha - 2 = \alpha > \frac{3\alpha}{\alpha+1}$. Then, we can choose $\tau > 0$ small enough such that $2 + \gamma(1-\tau) > \frac{3\alpha}{\alpha+1} - \delta$. This concludes the proof of the claim.

We now continue the estimates of the remaining terms. From Lemma 4.2

$$\int_{\mathbb{R}^3} z_{P_i}^p z_{P_j}\,dx = (\eta + o(1))e^{-|P_i - P_j|} = o(\varepsilon^M) \tag{4.6}$$

for any $M > 0$. Now, by using the notations of Lemma 4.1, we have

$$\phi_{z_{P_i}} = \frac{1}{4\pi}\Psi_1[U^2](x - P_i).$$

If $i \neq j$, by (4.2)

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_{z_{P_i}} z_{P_j}^2\,dx &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \Psi_1[U^2](x - P_i)U^2(x - P_j)\,dx\\ &= C_3\int_{\mathbb{R}^3} \frac{1}{|x - P_i|}U^2(x - P_j)\,dx + O(1)\int_{\mathbb{R}^3} \frac{1}{|y + P_j - P_i|^2}U^2(y)\,dx\\ &= C_3\Psi_1(U^2)|P_i - P_j| + O(1)|P_i - P_j|^{-2}.\end{aligned}$$

From the definition of Λ_ε and since $\alpha > 2$, $|P_i - P_j|^{-2} = o(\varepsilon^{\frac{3\alpha}{\alpha+1}-\delta})$. Furthermore

$$\int_{\mathbb{R}^3} \phi_{z_{\mathbf{P}}} z_{P_i} z_{P_j}\,dx \le C\int_{\mathbb{R}^3} z_{P_i} z_{P_j}\,dx = o(\varepsilon^M)\ (i \neq j).$$

and, consequently,

$$\int_{\mathbb{R}^3} \phi_{i,j} z_{P_i}^2\,dx = -\int_{\mathbb{R}^3} \phi_{i,j}\Delta\phi_{z_{P_i}} = -\int_{\mathbb{R}^3} \Delta\phi_{i,j}\,\phi_{z_{P_i}} = \int_{\mathbb{R}^3} \phi_{z_{P_i}} z_{P_i} z_{P_j}\,dx = o(\varepsilon^M)$$

for any $M > 0$. Since $\mathbf{P} \in \Lambda_\varepsilon$, we have that for $i \neq j$

$$\frac{\varepsilon^2}{4}\int_{\mathbb{R}^3} \phi_{z_{P_i}} z_{P_j}^2\,dx = C_3\frac{\varepsilon^2}{|P_i - P_j|} + \varepsilon^2 O(|P_i - P_j|^{-2}) = C_3\frac{\varepsilon^2}{|P_i - P_j|} + o(\varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}).$$

Finally, arguing as in Proposition 3.2, we obtain

$$\int_{\mathbb{R}^3} \left[|z_{\mathbf{P}}|^{p+1} - \sum_{i=1}^{K} |z_{P_i}|^{p+1}\right] dx = o(\varepsilon^M) \tag{4.7}$$

for any $M > 0$. All previous estimates imply the expansion (4.3). ∎

From (4.1) and (4.3) we have the following expansion for the reduced functional

$$\Phi_\varepsilon(\mathbf{P}) = C_0 + \varepsilon^2 C_1 + C_2 \sum_{i=1}^{K} V(\varepsilon P_i) + C_3 \varepsilon^2 \sum_{i\neq j} \frac{1}{|P_i - P_j|} + o(\varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}). \tag{4.8}$$

Proposition 4.4. *For ε sufficiently small, the following minimization problem*

$$\min \{\Phi_\varepsilon(\mathbf{P}) \ : \ \mathbf{P} \in \Lambda_\varepsilon\} \tag{4.9}$$

has a solution $\mathbf{P}_\varepsilon \in \Lambda_\varepsilon$.

Proof. Since $\Phi_\varepsilon(\mathbf{P})$ is continuous in $\mathbf{P}$ in a compact set, the minimization problem has a solution. Let $\Phi_\varepsilon(\mathbf{P}^\varepsilon)$ be the minimum of Φ_ε where $\mathbf{P}^\varepsilon$ is in the closure of the set Λ_ε. We prove by energy comparison that $\mathbf{P}^\varepsilon$ is not on the boundary of Λ_ε. In order to do this, first we obtain an upper bound for $\Phi_\varepsilon(\mathbf{P}^\varepsilon)$. Let us choose

$$P_j^0 = \varepsilon^{\frac{2-\alpha}{\alpha+1}} X_j$$

where $X_j, \ \ j = 1, \dots, K$ are the K vortices of $K-$polygon centered at 0 with $|X_i - X_j| = 1$, $i \neq j$.
Then for ε small it is clear that $\varepsilon P_j^0 \in \mathcal{U}$. Moreover

$$|P_i^0 - P_j^0| = \varepsilon^{\frac{2-\alpha}{\alpha+1}} |X_i - X_j|$$

and

$$V(\varepsilon P_j^0) \leq 1 + C|\varepsilon P_j^0|^\alpha \leq 1 + C\varepsilon^{\frac{3\alpha}{\alpha+1}}.$$

Therefore, $\mathbf{P^0} = (P_1^0, \dots, P_K^0) \in \Lambda_\varepsilon$. Hence by (4.8) we obtain

$$\Phi_\varepsilon(\mathbf{P}^\varepsilon) = \min_{\mathbf{P}\in\Lambda_\varepsilon} \Phi_\varepsilon(\mathbf{P}) \leq \Phi_\varepsilon(\mathbf{P^0}) \leq C_0 + \varepsilon^2 C_1 + KC_2 + C_3 \varepsilon^{\frac{3\alpha}{\alpha+1}}. \tag{4.10}$$

If now $\mathbf{P}^\varepsilon$ is such that $|P_i^\varepsilon - P_j^\varepsilon| = \varepsilon^{\frac{2-\alpha}{\alpha+1}+\delta}$ for some $i \neq j$, then

$$\Phi_\varepsilon(\mathbf{P}^\varepsilon) \geq C_0 + \varepsilon^2 C_1 + KC_2 + C_3 \varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}. \tag{4.11}$$

If, instead, $\mathbf{P}^\varepsilon$ is such that $V(\varepsilon P_i^\varepsilon) = 1 + \varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}$ for some i, then

$$\Phi_\varepsilon(\mathbf{P}^\varepsilon) \geq C_0 + \varepsilon^2 C_1 + KC_2 + C_2 \varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}. \tag{4.12}$$

But both (4.11) and (4.12) are in contradiction with (4.10).

We remark that we have not considered the case $\varepsilon\mathbf{P}^\varepsilon \in \partial\mathcal{U}$, because this would be in contradiction with $V(\varepsilon P_j^\varepsilon) \leq 1 + \varepsilon^{\frac{3\alpha}{\alpha+1}-\delta}$ for ε small. ■

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