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L^2 boundedness for maximal commutators with rough variable kernels

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Abstract

For $b \in BMO(\mathbb{R}^n)$ and $k \in \mathbb{N}$, the k-th order maximal commutator of the singular integral operator T with rough variable kernels is defined by

$$T^*_{b,k} f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x, x-y)}{|x-y|^n} (b(x)-b(y))^k f(y)\,dy \right|.$$

In this paper the authors prove that the k-th order maximal commutator $T^*_{b,k}$ is a bounded operator on $L^2(\mathbb{R}^n)$ if Ω satisfies the same conditions given by Calderón and Zygmund. Moreover, the L^2-boundedness of the k-th order commutator of the rough maximal operator M_Ω with variable kernel, which is defined by

$$M_{\Omega;b,k} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x, x-y)||b(x)-b(y)|^k |f(y)|\,dy,$$

is also given here. These results obtained in this paper are substantial improvement and extension of some known results.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ and $d\sigma$ be the area element on S^{n-1}. A function Ω defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q \geq 1$, if Ω satisfies the following conditions:

(i) for any $x,\ z \in \mathbb{R}^n$ and $\lambda > 0$, $\Omega(x, \lambda z) = \Omega(x, z)$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} := \sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^q \, d\sigma(z') \right)^{1/q} < \infty$, where $z' = z/|z|$, for any $z \in \mathbb{R}^n \setminus \{0\}$.

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If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(x, z')\, d\sigma(z') = 0 \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{1.1}$$

then the singular integral operator with variable kernel is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\, dy.$$

In 1955, Calderón and Zygmund [5] investigated the L^2 boundedness of the operator T. They found that these operators are relevant in the second order linear elliptic equations with variable coefficients. In [5], Calderón and Zygmund obtained the following result (see also [6]):

Theorem A (see [5]) *If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$ and satisfies* (1.1), *then there is a constant $C > 0$ such that $\|Tf\|_{L^2} \le C\|f\|_{L^2}$.*

Remark 1.1 In [5], Calderón and Zygmund showed that the condition $q > 2(n-1)/n$ is optimal in the sense that the L^2-boundedness of T fails if $q \le 2(n-1)/n$.

It is well known that maximal singular integral operators play a key role in studying the convergence of the singular integral operators almost everywhere. The mapping properties of the maximal singular integrals with convolution kernels have been extensively studied (see [25], [15] and [18], for example). In 1980, Aguilera and Harboure [1] considered the L^2 boundedness of the maximal singular integral operator T^* with variable kernel, where T^* is defined by

$$T^*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)dy \right| = \sup_{\varepsilon>0} |T_\varepsilon f(x)|,$$

where

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)dy.$$

Theorem B (see [1]) *If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 4(n-1)/(2n-1)$ and satisfies* (1.1), *then there is a constant $C > 0$ such that $\|T^*f\|_{L^2} \le C\|f\|_{L^2}$.*

In 1985, using spherical harmonic expansions of the kernel, Cowling and Mauceri [13] proved that the conclusion of Theorem B still holds for $q > 2(n-1)/n$. The same conclusion was also obtained by Christ, Duoandikoetxea and Rubio de Francia [10] by the method of rotations and mixed norm estimates in 1986.

Theorem C (see [13] or [10]) *If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$ and satisfies* (1.1), *then T^* is a bounded operator on $L^2(\mathbb{R}^n)$.*

Obviously, the range of q in Theorem C is also optimal by Remark 1.1.

In the present paper, we will discuss the L^2-boundedness of the maximal commutator of the singular integral with variable kernel. Let us recall some background. The commutators of the Hilbert transform were first introduced by Calderón in [3] and play an important role in the study of the Cauchy integral along Lipschitz curves (see also [4]). Motivated by the work of Calderón on commutators, in their famous paper [11] Coifman, Rochberg and Weiss discussed the commutator $[b, T]$ generated by a classical Calderón-Zygmund singular integral operator T and a function b, which is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x),$$

where $b \in BMO(\mathbb{R}^n)$, that is,

$$\|b\|_* := \sup_{\text{cube } Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(y) - b_Q|dy < \infty$$

with $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$. The authors of [11] gave a characterization of L^p-boundedness of the commutators generated by the Riesz transforms R_j $(j = 1, \ldots, n,)$. Using this characterization, Coifman, Rochberg and Weiss got a decomposition theorem of the real Hardy spaces.

These commutators are of interest in harmonic analysis and PDE's. For example, the commutators have some important applications in the theory of non-divergent elliptic equations with discontinuous coefficients (see [2], [8], [9] and [14]). Moreover, there is also an interesting connection between the nonlinear commutator, considered by Rochberg and Weiss in [24], and Jacobian mappings of vector functions. They have been applied in the study of nonlinear partial differential equations (see [19], [21], [12], [23] and Iwaniec's nice survey paper [22]).

The commutators of the singular integral operators with variable kernel arise naturally in the study of PDE's with variable coefficients. In 1991, to study interior $W^{2,2}$ estimates for nondivergence elliptic second order equation with discontinuous coefficients, Chiarenza, Frasca and Longo [8] (see also [9]) proved the $L^2(\mathbb{R}^n)$ boundedness of the commutator for the singular integral with variable kernel. For $k \in \mathbb{N}$, the k-th order commutator of T with variable kernel is defined by

$$T_{b,k}f(x) := \lim_{\varepsilon\to 0}\underbrace{[b, \cdots [b, T_\varepsilon]]}_{k} f(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n}(b(x)-b(y))^k f(y)\, dy.$$

For simplicity, we denote $T_{b,1}$ by T_b below. Clearly, $T_{b,k}$ is also a natural generalization of the commutator of the classical Calderón-Zygmund singular integral operator with convolution kernel.

Theorem D (see [8]) *If* $\Omega \in L^\infty(\mathbb{R}^n) \times C^\infty(S^{n-1})$ *and satisfies* (1.1), *then* T_b *is a bounded operator on* $L^2(\mathbb{R}^n)$ *for* $b \in BMO(\mathbb{R}^n)$.

Recently, Chen and Ding [7] proved that the conclusion of Theorem D holds still after removing this stronger smoothness condition assumed on Ω in its second variate.

Theorem E (see [7]) *If* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ *for* $q > 2(n-1)/n$ *and satisfies* (1.1), *then for* $b \in BMO(\mathbb{R}^n)$ *and* $k \in \mathbb{N}$, *there is a constant* $C > 0$ *such that* $\|T_{b,k}f\|_{L^2} \le C\|b\|_*^k\|f\|_{L^2}$.

Theorem E shows that the size condition of Ω in Theorem A is enough for the L^2 boundedness of higher order commutator of the singular integral with rough variable kernel. Inspired by Theorem E, a natural problem is whether or not the higher order maximal commutator $T^*_{b,k}$ of the singular integral T with rough variable kernel is bounded on $L^2(\mathbb{R}^n)$ under the same conditions of Theorem E, where $T^*_{b,k}$ is defined by

$$\begin{aligned} T^*_{b,k}f(x) \ &:= \sup_{\varepsilon>0} |\underbrace{[b, \cdots [b, T_\varepsilon]]}_{k} f(x)| \\ &= \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x, x-y)}{|x-y|^n}(b(x)-b(y))^k f(y)dy \right|. \end{aligned}$$

Note that the case $k = 0$ recaptures the maximal singular integral operator T^* with variable kernel.

In this paper we will give a positive answer to the above problem. Our main result is following:

Theorem 1 *Suppose that* $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ *for* $q > 2(n-1)/n$ *satisfies* (1.1). *Then for* $b \in BMO(\mathbb{R}^n)$ *and* $k \in \mathbb{N}$, *there is a constant* $C > 0$ *such that* $\|T^*_{b,k}f\|_{L^2} \le C\|b\|_*^k\|f\|_{L^2}$.

It is not difficult to check that the following inequality holds for the commutator $T^*_{b,k}$:

$$T^*_{b,k}f(x) \le \sup_{l\in\mathbb{Z}} |T^{2^l}_{b,k}f(x)| + M_{\Omega;b,k}f(x), \tag{1.2}$$

where

$$T^{2^l}_{b,k}f(x) := \underbrace{[b, \cdots [b, T_{2^l}]}_{k} f(x) = \int_{|x-y|>2^l} \frac{\Omega(x, x-y)}{|x-y|^n}(b(x)-b(y))^k f(y)dy$$

and

$$M_{\Omega;b,k}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x, x-y)||b(x)-b(y)|^k|f(y)|dy.$$

The latter is called the k-th order commutator of the maximal operator with rough variable kernel. Thus, to obtain Theorem 1, it is necessary to discuss the L^2-boundedness of $M_{\Omega;b,k}$ in (1.2). Moreover, the L^2-boundedness of $M_{\Omega;b,k}$ has its significance and interest independently.

Theorem 2 *If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$. Then for $b \in BMO(\mathbb{R}^n)$ and $k \in \mathbb{N}$, $M_{\Omega;b,k}$ is a bounded operator on $L^2(\mathbb{R}^n)$.*

Remark 1.2 Note that no smoothness is required on Ω in Theorems 1 and 2. In this sense, the results both of Theorems 1 and 2 are new even for the maximal commutators of singular integrals with convolution kernel.

Remark 1.3 By Remark 1.1, the condition $q > 2(n-1)/n$ in Theorems 1 and 2 are optimal for the L^2-boundedness of the higher order commutators $T^*_{b,k}$ and $M_{\Omega;b,k}$.

Throughout this paper, for convenience, we use the notations $L_{b,k}$ or $(K)_{b,k}$ alternately to denote the k-th commutators generated by a function b and a convolution operator L with its integral kernel K. That is,

$$L_{b,k}f(x) := \underbrace{[b, \cdots [b, L]]}_{k} f(x) =: (K)_{b,k}f(x).$$

The notations " $\widehat{\ }$ " and " $\vee$ " denote the Fourier transform and the inverse Fourier transform, respectively. The letter C will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. $|E|$ denotes the Lebesgue measure of the measurable set E in $\mathbb{R}^n$. As usual, for $p \geq 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

2. Proof of Theorem 2

In this section, we will give the proof of Theorem 2. In this proof, we need to use the boundedness of the maximal operator with rough variable kernel M_Ω, which is defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x, x-y)|\, |f(y)| dy.$$

We hence show first a mapping property of M_Ω. Note that M_Ω is a version of the Hardy-Littlewood maximal operator with variable kernel. We therefore write the L^p-boundedness of M_Ω as a theorem, although its proof is simple.

Theorem 3. *For $1 < p \leq \max\{2, (n+1)/2\}$ and $q > p(n-1)/(p-1)n$, if $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, then there is a constant $C > 0$ such that $\|M_\Omega f\|_{L^p} \leq C\|f\|_{L^p}$.*

Remark 2.1 If we take $p = 2$ then $q > 2(n-1)/n$, which is just the same kernel condition as in Theorem A.

Before showing Theorem 3, we give some notations and a lemma. For $y' \in S^{n-1}$, the Hardy-Littlewood maximal operator along direction y' is defined by

$$\mathfrak{M}f(x, y') = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - ty')|dt, \qquad (x, y') \in \mathbb{R}^n \times S^{n-1}.$$

For $1 \le p, q \le \infty$, the mixed norm space $L^p(L^q)(\mathbb{R}^n \times S^{n-1})$ is defined by

$$L^p(L^q)(\mathbb{R}^n \times S^{n-1}) :=$$
$$:= \Big\{ F : \|F\|_{L^p(L^q)} = \Big(\int_{\mathbb{R}^n} \Big(\int_{S^{n-1}} |F(x, y')|^q d\sigma(y') \Big)^{p/q} dx \Big)^{1/p} < \infty \Big\}.$$

Lemma 2.1 (see [10]) *The direct maximal operator* $\mathfrak{M}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^p(L^q)(\mathbb{R}^n \times S^{n-1})$ *for all* $1 < p \le \max\{2, (n + 1)/2\}$ *and* $q < p(n - 1)/(n - p)$.

Proof of Theorem 3. By the method of rotations, we can write

$$\begin{aligned} M_\Omega f(x) &\le \sup_{r>0} r^{-1} \int_0^r \int_{S^{n-1}} |\Omega(x, y')||f(x - ty')|d\sigma(y')dt \\ &\le 2 \int_{S^{n-1}} \mathfrak{M}f(x, y')|\Omega(x, y')|d\sigma(y'). \end{aligned}$$

Applying Hölder's inequality and Lemma 2.1 for $\Omega(x, y') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ and $q > p(n - 1)/(p - 1)n$ (equivalently, $q' < p(n - 1)/(n - p)$), we get

$$\begin{aligned} \|M_\Omega f\|_{L^p} &\le C \Big(\int_{\mathbb{R}^n} \Big(\int_{S^{n-1}} \mathfrak{M}f(x, y')|\Omega(x, y')|d\sigma(y') \Big)^p dx \Big)^{1/p} \\ &\le C \Big(\int_{\mathbb{R}^n} \Big(\int_{S^{n-1}} |\mathfrak{M}f(x, y')|^{q'} d\sigma(y') \Big)^{p/q'} \\ &\qquad \times \Big(\int_{S^{n-1}} |\Omega(x, y')|^q d\sigma(y') \Big)^{p/q} dx \Big)^{1/p} \\ &\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} \|\mathfrak{M}f\|_{L^p(L^{q'})} \le C\|f\|_{L^p}. \end{aligned}$$

Thus we complete the proof of Theorem 3. ■

Let us now turn to the proof of Theorem 2. Let us begin with recalling some known results.

Lemma 2.2 (see [7]) *Suppose that* $0 < \beta < 1$, $\ell \in \mathbb{Z}$, $m \in \mathbb{N}$. *Denote by* $\mathcal{H}_m$ *the space of surface spherical harmonics of degree* m *on* S^{n-1} *with its dimension* D_m. $\{Y_{m,j}\}_{j=1}^{D_m}$ *denotes a normalized complete system in* $\mathcal{H}_m$. *Let*

$$K_{\ell,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^n} \chi_{\{2^\ell < |x| \le 2^{\ell+1}\}}(x).$$

Then

$$|\widehat{K_{\ell,m,j}}(\xi)| = \begin{cases} Cm^{-\lambda-1}|2^\ell\xi||Y_{m,j}(\xi')|, & |2^\ell\xi| \le 1, \\ Cm^{-\lambda-1+\beta/2}|2^\ell\xi|^{-\beta/2}|Y_{m,j}(\xi')|, & |2^\ell\xi| > 1, \end{cases} \tag{2.1}$$

$$|\widehat{K_{\ell,m,j}}(\xi)| \le Cm^{-\lambda-1}|Y_{m,j}(\xi')|, \tag{2.2}$$

$$|\nabla\widehat{K_{\ell,m,j}}(\xi)| \le C2^\ell, \tag{2.3}$$

where $\lambda = (n-2)/2$ *and* $\xi' = \xi/|\xi|$.

Lemma 2.3 (see [20]) *Let* $\psi \in C_0^\infty(\mathbb{R}^n)$ *be a radial function such that* $supp(\psi) \subset \{\xi : 1/2 \le |\xi| \le 2\}$ *and* $\sum_{l\in\mathbb{Z}}\psi^2(2^{-l}\xi) = 1$ *for* $|\xi| \ne 0$. *Define the multiplier* S_l *by* $\widehat{S_l f}(\xi) = \psi(2^{-l}\xi)\widehat{f}(\xi)$. *For* $b \in BMO$ *and a nonnegative integer* k, *denote by* $S_{l;b,k}$ *the* k*-th order commutator of* S_l. *Then, for* $1 < p < \infty$

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}|S_{l;b,k}f|^2\Big)^{1/2}\Big\|_{L^p} \le C(n,k,p)\|b\|_*^k\|f\|_{L^p}.$$

Proof of Theorem 2. By Hölder's inequality, we split $\|M_{\Omega;b,k}f\|_{L^2}$ into two parts,

$$\begin{aligned}
\|M_{\Omega;b,k}f\|_{L^2}^2 &\le \int_{\mathbb{R}^n}\Big(\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x,x-y)||b(x)-b(y)|^{2k}|f(y)|dy\Big) \\
&\qquad \times\Big(\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x,x-y)||f(y)|dy\Big)dx \\
&= \int_{\mathbb{R}^n}(M_{\Omega;b,2k}f(x))\cdot(M_\Omega f(x))dx \\
&\le \|M_{\Omega;b,2k}f\|_{L^2}\|M_\Omega f\|_{L^2}.
\end{aligned} \tag{2.4}$$

Applying Theorem 3 with $p = 2$ and $q > 2(n-1)/n$, we obtain that

$$\|M_\Omega f\|_{L^2} \le C\|f\|_{L^2}. \tag{2.5}$$

By (2.4) and (2.5), to prove Theorem 2, it suffices to show

$$\|M_{\Omega;b,2k}f\|_{L^2} \le C\|b\|_*^{2k}\|f\|_{L^2}. \tag{2.6}$$

Let

$$\Omega_0(x,y') = |\Omega(x,y')| - \frac{\|\Omega(x,\cdot)\|_{L^1(S^{n-1})}}{\sigma(S^{n-1})}.$$

It is easy to check that $\Omega_0(x, y') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$ and satisfies (1.1). Thus

$$\begin{aligned} M_{\Omega;b,2k}f(x) &\le C \sup_{l\in\mathbb{Z}} \int_{2^{l-1}<|x-y|\le 2^l} \frac{|\Omega(x, x-y)|}{|x-y|^n}(b(x)-b(y))^{2k}|f(y)|dy \\ &= C \sup_{l\in\mathbb{Z}} \int_{2^{l-1}<|x-y|\le 2^l} \frac{\Omega_0(x, x-y)}{|x-y|^n}(b(x)-b(y))^{2k}|f(y)|dy \\ &\quad + C \sup_{l\in\mathbb{Z}} \int_{2^{l-1}<|x-y|\le 2^l} \frac{\|\Omega(x,\cdot)\|_{L^1(S^{n-1})}}{\sigma(S^{n-1})|x-y|^n}(b(x)-b(y))^{2k}|f(y)|dy \\ &:= N_1 + N_2. \end{aligned}$$

Define the k-th order commutator $M_{b,k}$ formed by the Hardy-Littlewood maximal operator M and a BMO function b by

$$M_{b,k}f(x) = \sup_{r>0} r^{-n} \int_{|x-y|<r} |b(x)-b(y)|^k |f(y)|dy.$$

Applying Theorem 2.4 in [17] with $\alpha = \beta \equiv 1$, we know that

$$\|M_{b,k}f\|_{L^2} \le C\|b\|_*^k \|f\|_{L^2}. \tag{2.7}$$

Without loss of generality, we can assume that $\|b\|_* = 1$. Observe that for any $x \in \mathbb{R}^n$, we have

$$\|N_2\|_{L^2} \le C\|M_{b,2k}f\|_{L^2} \le C\|f\|_{L^2}.$$

Therefore, to show (2.6), it remains to give the following estimate of N_1:

$$\|N_1\|_{L^2} \le C\|f\|_{L^2}. \tag{2.8}$$

As in [6], by a limit argument we may reduce the proof of Theorem 1 to the case of $f \in C_0^\infty(\mathbb{R}^n)$ and

$$\Omega_0(x, z') = \sum_{m\ge 0} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(z')$$

is a finite sum. Notice that $\Omega_0(x, z')$ satisfies (1.1), so $a_{0,j} \equiv 0$. Denote

$$a_m(x) = \left(\sum_{j=1}^{D_m} |a_{m,j}(x)|^2 \right)^{1/2} \quad \text{and} \quad d_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}. \tag{2.9}$$

Then

$$\sum_{j=1}^{D_m} d_{m,j}^2(x) = 1, \tag{2.10}$$

and

$$\Omega_0(x,z')=\sum_{m\geq 1}a_m(x)\sum_{j=1}^{D_m}d_{m,j}(x)Y_{m,j}(z'). \tag{2.11}$$

Let

$$F_{l,m,j;b,2k}f(x)=\int_{2^{l-1}<|x-y|\leq 2^l}\frac{Y_{m,j}(x-y)}{|x-y|^n}(b(x)-b(y))^{2k}|f(y)|dy.$$

Using Hölder's inequality twice and (2.10), we get for $0<\theta<1$,

$$\begin{aligned} N_1^2 &= C\bigg(\sup_{l\in\mathbb{Z}}\bigg|\int_{2^l<|x-y|\leq 2^{l+1}}\frac{\Omega_0(x,x-y)}{|x-y|^n}(b(x)-b(y))^{2k}|f(y)|dy\bigg|\bigg)^2\\ &\leq C\bigg\{\sum_{m=1}^{\infty}a_m^2(x)m^{-\theta}\bigg\}\bigg\{\sum_{m=1}^{\infty}m^{\theta}\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}}|F_{l,m,j;b,2k}f(x)|)^2\bigg\}. \end{aligned} \tag{2.12}$$

By [6, p. 230], for $q>2(n-1)/n$, if we take $0<\theta<1$ and close to 1 sufficiently, then

$$\begin{aligned} \bigg(\sum_{m\geq 1}a_m^2(x)m^{-\theta}\bigg)^{1/2} &\leq C\bigg(\int_{S^{n-1}}|\Omega_0(x,z')|^q d\sigma(z')\bigg)^{1/q}\\ &\leq C\|\Omega_0\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})}. \end{aligned} \tag{2.13}$$

By (2.12) and (2.13)

$$\begin{aligned} \|N_1\|_{L^2}^2 &\leq C\sum_{m=1}^{\infty}m^{\theta}\bigg\|\bigg(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}}|F_{l,m,j;b,2k}f(x)|)^2\bigg)^{1/2}\bigg\|_{L^2}^2\\ &\leq C\sum_{m=1}^{\infty}m^{\theta}\bigg\|\bigg(\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}|F_{l,m,j;b,2k}f(x)|^2\bigg)^{1/2}\bigg\|_{L^2}^2. \end{aligned}$$

Clearly, (2.8) will follow if we can show that there exists $0<\beta<(1-\theta)/2$ such that

$$\bigg\|\bigg(\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}|F_{l,m,j;b,2k}f(x)|^2\bigg)^{1/2}\bigg\|_{L^2}\leq Cm^{-1+\beta}\|f\|_{L^2}. \tag{2.14}$$

Let us take a radial function $\psi\in C_0^\infty$ such that $0\leq\psi\leq 1$, $\mathrm{supp}(\psi)\subset\{\xi:1/2\leq|\xi|\leq 2\}$ and $\sum_{i\in\mathbb{Z}}\psi^2(2^{-i}\xi)=1$ for $|\xi|\neq 0$. Define the multiplier S_i by $\widehat{S_if}(\xi)=\psi(2^{-i}\xi)\hat{f}(\xi)$. For $l\in\mathbb{Z}$, $m\in\mathbb{N}$ and $j=1,\dots,D_m$, set

$$\begin{aligned} E_{l,m,j}^i(\xi) &= \widehat{K_{l,m,j}}(\xi)\psi(2^{l-i}\xi),\\ F_{l,m,j}f(x) &= K_{l,m,j}*f(x)\\ \text{and}\quad \widehat{F_{l,m,j}^i f}(\xi) &= E_{l,m,j}^i(\xi)\hat{f}(\xi), \end{aligned}$$

where and in the sequel, $K_{l,m,j}$ is defined in Lemma 2.2. Define by the operator $F_{l,m,j;b,k}$ and $F^i_{l,m,j;b,k}$ the k-th order commutators of $F_{l,m,j}$ and $F^i_{l,m,j}$, respectively. Then

$$F_{l,m,j;b,2k}f(x) = \sum_{i\in\mathbb{Z}} (F^i_{l,m,j}S_{i-l})_{b,2k}f(x). \tag{2.15}$$

By (2.15) and Minkowski inequality,

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}} |(F_{l,m,j;b,2k}f(x)|^2\Big)^{1/2}\Big\|_{L^2} = \\
&= \Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}\Big|\sum_{i\in\mathbb{Z}}(F^i_{l,m,j}S_{i-l})_{b,2k}f(x)\Big|^2 dx\Big)^{1/2} \\
&\le \sum_{i\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|(F^i_{l,m,j}S_{i-l})_{b,2k}f(x)|^2dx\Big)^{1/2} \\
&:= R.
\end{aligned} \tag{2.16}$$

With the aid of the formula

$$(b(x)-b(y))^k = \sum_{u=0}^{k} C^u_k(b(x)-b(z))^u(b(z)-b(y))^{k-u}, \quad x,y,z\in\mathbb{R}^n, \tag{2.17}$$

it is easy to check that

$$(F^i_{l,m,j}S_{i-l})_{b,2k}f(x) = \sum_{\alpha=0}^{2k} C^\alpha_{2k}F^i_{l,m,j;b,\alpha}(S_{i-l;b,2k-\alpha}f)(x). \tag{2.18}$$

Let

$$F^i_{l,m;b,\alpha}f(x) := \Big(\sum_{j=1}^{D_m}|F^i_{l,m,j;b,\alpha}f(x)|^2\Big)^{1/2}.$$

Then, applying Minkowski inequality and by (2.18),

$$\begin{aligned}
R &= \sum_{i\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|\sum_{\alpha=0}^{2k}C^\alpha_{2k}F^i_{l,m,j;b,\alpha}(S_{i-l;b,2k-\alpha}f)(x)|^2dx\Big)^{1/2} \\
&\le C\sum_{i\in\mathbb{Z}}\sum_{\alpha=0}^{2k}C^\alpha_{2k}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|F^i_{l,m,j;b,\alpha}(S_{i-l;b,2k-\alpha}f)(x)|^2dx\Big)^{1/2} \\
&= C\sum_{i\in\mathbb{Z}}\sum_{\alpha=0}^{2k}C^\alpha_{2k}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}|F^i_{l,m;b,\alpha}(S_{i-l;b,2k-\alpha}f)(x)|^2dx\Big)^{1/2}.
\end{aligned} \tag{2.19}$$

Hence, if there exists $0 < v_0 < 1$ such that

$$\|F^i_{l,m;b,\alpha} f\|_{L^2} \le C m^{-1+\beta} 2^{-\beta v_0 |i|/2} \|f\|_{L^2}, \tag{2.20}$$

then we may get (2.14). In fact, by (2.16), (2.19) and Lemma 2.3, we have

$$\begin{aligned}
&\Big\| \Big(\sum_{j=1}^{D_m} \sum_{l\in\mathbb{Z}} |(F_{l,m,j;b,2k} f(x)|^2 \Big)^{1/2} \Big\|_{L^2} \\
&\le C \sum_{i\in\mathbb{Z}} \sum_{\alpha=0}^{2k} C^\alpha_{2k} m^{-1+\beta} 2^{-v_0\beta|i|/2} \Big(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} |(S_{i-l;b,2k-\alpha} f)(x)|^2 dx \Big)^{1/2} \\
&= C \sum_{i\in\mathbb{Z}} \sum_{\alpha=0}^{2k} C^\alpha_{2k} m^{-1+\beta} 2^{-v_0\beta|i|/2} \Big\| \Big(\sum_{l\in\mathbb{Z}} |S_{i-l;b,2k-\alpha} f|^2 \Big)^{1/2} \Big\|_{L^2} \\
&\le C m^{-1+\beta} \|f\|_{L^2}.
\end{aligned}$$

Thus, to finish the proof of Theorem 2, it remains to verify (2.20). Define the operator $\tilde{F}^i_{l,m,j}$ by

$$\widehat{\tilde{F}^i_{l,m,j} f}(\xi) = E^i_{l,m,j}(2^{-l}\xi)\hat{f}(\xi).$$

Denote by $\tilde{F}^i_{l,m,j;b,\alpha}$ the α-th order commutator of $\tilde{F}^i_{l,m,j}$. Let

$$\tilde{F}^i_{l,m;b,\alpha} f(\xi) := \Big(\sum_{j=1}^{D_m} |\tilde{F}^i_{l,m,j;b,\alpha} f(\xi)|^2 \Big)^{1/2}.$$

Applying Lemma 2.2, we have

$$\begin{aligned}
|\widehat{K_{l,m,j}}(\xi)| &\le C m^{-\lambda-1+\beta/2} \min\{|2^l\xi|, |2^l\xi|^{-\beta/2}\} |Y_{m,j}(\xi')|, \\
|\widehat{K_{l,m,j}}(\xi)| &\le C m^{-\lambda-1} |Y_{m,j}(\xi')|, \\
|\nabla \widehat{K_{l,m,j}}(\xi)| &\le C 2^l.
\end{aligned}$$

Note that $\mathrm{supp}(E^i_{l,m,j}(2^{-l}\cdot)) \subset \{\xi : 2^{i-1} \le |\xi| \le 2^{i+1}\}$, then we get

$$\begin{aligned}
|E^i_{l,m,j}(2^{-l}\xi)| &\le C m^{-\lambda-1+\beta/2} \min\{2^i, 2^{-\beta i/2}\} |Y_{m,j}(\xi')|, \\
|E^i_{l,m,j}(2^{-l}\xi)| &\le C m^{-\lambda-1} |Y_{m,j}(\xi')|, \\
|\nabla E^i_{l,m,j}(2^{-l}\xi)| &\le C.
\end{aligned}$$

Using Lemma 2.3 in [7] with $\delta = 2^i$ and $s = 0$, we know that for any fixed $0 < v < 1$ and nonnegative integer α

$$\|\tilde{F}^i_{l,m;b,\alpha}\|_{L^2} \le C m^{(-1+\beta/2)v} 2^{-\beta|i|v/2} \|f\|_{L^2}.$$

For fixed $0 < \beta < (1-\theta)/2$, we can find $0 < v_0 < 1$ such that $v_0(-1+\beta/2) \leq -1+\beta$. Hence

$$\|\tilde{F}^i_{l,m;b,\alpha}\|_{L^2} \leq Cm^{-1+\beta}2^{-\beta|i|v_0/2}\|f\|_{L^2},$$

which implies (2.20) by dilation-invariance. Therefore we have completed the proof of Theorem 2. ■

3. Proof of Theorem 1

This section is divided into two parts. In Subsection 3.1, we give a lemma which plays a key role in the proof of Theorem 1. In Subsection 3.2, we will finish the proof of Theorem 1.

3.1. Key lemma

Lemma 3.1.1 *For $0 < \delta < \infty, m \in \mathbb{N}$, $s \in \mathbb{N}$ and $j = 1, \ldots, D_m$, take $B_{s,\delta,m,j} \in C_0^\infty(\mathbb{R}^n)$ such that* $\operatorname{supp}(B_{s,\delta,m,j}) \subset \{\xi : \delta/2 \leq |\xi| \leq 2\delta\}$. *Let $T_{s,\delta,m,j}$ be the multiplier operators defined by*

$$\widehat{T_{s,\delta,m,j}f}(\xi) = B_{s,\delta,m,j}(\xi)\hat{f}(\xi), \quad j = 1, \ldots, D_m.$$

Moreover, for $b \in BMO$ and $k \in \mathbb{N}$, denote by $T_{s,\delta,m,j;b,k}$ the k-th order commutator of $T_{s,\delta,m,j}$ and

$$T_{s,\delta,m;b,k}f(x) = \left(\sum_{j=1}^{D_m}(T_{s,\delta,m,j;b,k}f(x))^2\right)^{1/2}.$$

If for some constant $0 < \beta < 1$, $B_{s,\delta,m,j}$ satisfies the following conditions:

$$|B_{s,\delta,m,j}(\xi)| \leq C2^{-\beta s/2}m^{-\lambda-1+\beta/2}\min\{\delta, \delta^{-\beta/2}\}|Y_{m,j}(\xi')|, \tag{3.1.1}$$

$$|B_{s,\delta,m,j}(\xi)| \leq Cm^{-\lambda-1}|Y_{m,j}(\xi')|, \tag{3.1.2}$$

$$|\nabla B_{s,\delta,m,j}(\xi)| \leq C2^s, \tag{3.1.3}$$

then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, k, v)$ such that

$$\|T_{s,\delta,m;b,k}f\|_{L^2} \leq C2^{-\beta sv/2}m^{(-1+\beta/2)v}\min\{\delta^v, \delta^{-\beta v/2}\}\|b\|_*^k\|f\|_{L^2}. \tag{3.1.4}$$

Proof. We may assume that $\|b\|_* = 1$. Let us consider a $C_0^\infty(\mathbb{R}^n)$ radial function ϕ, such that $\operatorname{supp}\phi \subset \{x : 1/2 \leq |x| \leq 2\}$ and $\sum_{l\in\mathbb{Z}}\phi(2^{-l}|x|) = 1$ for any $|x| > 0$. Denote $\phi_0(x) = \sum_{l=-\infty}^{0}\phi(2^{-l}|x|)$ and $\phi_l(x) = \phi(2^{-l}|x|)$ for positive integer l. Then $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp}\phi_0 \subset \{x : 0 < |x| \leq 2\}$.

Let $K_{s,\delta,m,j}(x) = (B_{s,\delta,m,j})^\vee(x)$. Denote $K^l_{s,\delta,m,j}(x) = K_{s,\delta,m,j}(x)\phi_l(x)$ for $l = 0, 1, \ldots$ We have

$$K_{s,\delta,m,j}(x) = \sum_{l=0}^{\infty} K^l_{s,\delta,m,j}(x).$$

Denote by $T^l_{s,\delta,m,j}$ and $T^l_{s,\delta,m,j;b,k}$ the convolution operator with kernel $K^l_{\delta,m,j}$ and the k-th order commutator of $T^l_{s,\delta,m,j}$ and b, respectively. Then by Minkowski's inequality

$$\begin{aligned}\|T_{s,\delta,m;b,k}f\|_{L^2} &= \Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big|\sum_{l=0}^{\infty}T^l_{s,\delta,m,j;b,k}f(x)\Big|^2dx\Big)^{1/2}\\ &\le \sum_{l=0}^{\infty}\Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big|T^l_{s,\delta,m,j;b,k}f(x)\Big|^2dx\Big)^{1/2}\\ &:= \sum_{l=0}^{\infty}\|T^l_{s,\delta,m;b,k}f\|_{L^2},\end{aligned} \tag{3.1.5}$$

where $T^l_{s,\delta,m;b,k}f(x) = \big(\sum_{j=1}^{D_m}\big|T^l_{s,\delta,m,j;b,k}f(x)\big|^2\big)^{1/2}$. It is easy to see that (3.1.4) is the consequence of (3.1.5) and the following Claim 1.

Claim 1: For any fixed $0 < v < 1$, there exists $\gamma > 0$ such that

$$\|T^l_{s,\delta,m;b,k}f\|_{L^2} \le C2^{-\beta sv/2}m^{(-1+\beta/2)v}2^{-l\gamma}\min\{\delta^v, \delta^{-\beta v/2}\}\|f\|_{L^2}. \tag{3.1.6}$$

Below we show Claim 1 by an almost orthogonality decomposition. For $l \ge 0$, we decompose $\mathbb{R}^n = \bigcup_{d=-\infty}^{\infty} Q_d$, where Q_d's are non-overlapping cubes with side length 2^l. Set $f_d = f\chi_{Q_d}$. Then $f(x) = \sum_{d=-\infty}^{\infty} f_d(x)$ for a.e. $x \in \mathbb{R}^n$. It is obvious that $\mathrm{supp}(T^l_{s,\delta,m,j;b,k}f_d) \subset 10nQ_d$ since $\mathrm{supp}(K^l_{s,\delta,m,j}) \subset \{x : |x| \le 2^{l+2}\}$. Moreover, the sets in the family $\{\mathrm{supp}(T^l_{s,\delta,m,j;b,k}f_d)\}_{d=-\infty}^{\infty}$ have bounded overlaps. So we have the following almost orthogonality property:

$$\|T^l_{s,\delta,m,j;b,k}f\|^2_{L^2} \le C\sum_{d=-\infty}^{\infty}\|T^l_{s,\delta,m,j;b,k}f_d\|^2_{L^2}.$$

Thus

$$\begin{aligned}\|T^l_{s,\delta,m;b,k}f\|^2_{L^2} &= \sum_{j=1}^{D_m}\|T^l_{s,\delta,m,j;b,k}f\|^2_{L^2}\\ &\le C\sum_{d=-\infty}^{\infty}\sum_{j=1}^{D_m}\|T^l_{s,\delta,m,j;b,k}f_d\|^2_{L^2} = C\sum_{d=-\infty}^{\infty}\|T^l_{s,\delta,m;b,k}f_d\|^2_{L^2}.\end{aligned}$$

Hence, it is suffices to verify (3.1.6) for a function f with $\mathrm{supp}f \subset Q$, where Q has its side length 2^l.

Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, φ identically one on $50nQ$, and supp$\varphi \subset 100nQ$. Set $\widetilde{Q} = 200nQ$, and $b_{\widetilde{Q}} = |\widetilde{Q}|^{-1} \int_{\widetilde{Q}} b(y)\,dy$. Let

$$\widetilde{b}(x) = (b(x) - b_{\widetilde{Q}})\varphi(x).$$

It is easy to see that

$$T^l_{s,\delta,m,j;b,k} f(x) = \sum_{\mu=0}^{k} C_k^\mu \widetilde{b}^\mu(x) T^l_{s,\delta,m,j}(\widetilde{b}^{k-\mu} f)(x).$$

Denote

$$T^l_{s,\delta,m} f(x) = \Big(\sum_{j=1}^{D_m} \big|T^l_{s,\delta,m,j} f(x)\big|^2\Big)^{1/2},$$

then we have

$$\begin{aligned}\|T^l_{s,\delta,m;b,k} f\|^2_{L^2} &= \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \Big| \sum_{\mu=0}^{k} C_k^\mu \widetilde{b}^\mu(x) T^l_{s,\delta,m,j}(\widetilde{b}^{k-\mu} f)(x)\Big|^2 dx \\ &\le C \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \sum_{\mu=0}^{k} \Big|\widetilde{b}^\mu(x) T^l_{s,\delta,m,j}(\widetilde{b}^{k-\mu} f)(x)\Big|^2 , dx \\ &\le C \sum_{\mu=0}^{k} \int_{\mathbb{R}^n} |\widetilde{b}^\mu(x)|^2 \sum_{j=1}^{D_m} |T^l_{s,\delta,m,j}(\widetilde{b}^{k-\mu} f)(x)|^2\,dx \\ &= C \sum_{\mu=0}^{k} \|\widetilde{b}^\mu T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|^2_{L^2}.\end{aligned} \tag{3.1.7}$$

Thus, in order to prove Claim 1, by (3.1.7) we only need to show the following

Claim 2: For any fixed $0 < v < 1$, there exists $\gamma > 0$ such that for a function f supported in Q with side length 2^l

$$\|\widetilde{b}^\mu T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} \le C 2^{-\beta s v/2} m^{(-1+\beta/2)v} 2^{-l\gamma} \min\{\delta^v, \delta^{-\beta v/2}\} \|f\|_{L^2}. \tag{3.1.8}$$

However, Claim 2 can be reduced from the following

Claim 3: For $g \in L^{q'}(\mathbb{R}^n)$, $1 \le q' \le 2$ (hence $2 \le q \le \infty$) and $0 < t < 1$

$$\begin{aligned}\|T^l_{s,\delta,m} g\|_{L^q} &\le C 2^{\frac{2ts}{q}} 2^{-\frac{\beta(1-t)s}{q}} 2^{-\frac{2tl}{q}} m^{\frac{(-2+\beta)(1-t)}{q} - (1-\frac{2}{q}) + \frac{2t\lambda}{q}} \delta^{n(1-\frac{2}{q})} \\ &\quad \times \big(\min\{\delta, \delta^{-\beta}\}\big)^{\frac{2(1-t)}{q}} \|g\|_{L^{q'}}.\end{aligned} \tag{3.1.9}$$

In fact, notice that for any $1 < \sigma < \infty$ and $\mu = 0, 1, \dots, k$,

$$\|\widetilde{b}^\mu\|_{L^\sigma} \le C \|b\|_*^\mu |Q|^{1/\sigma} \le C 2^{nl/\sigma}.$$

Then for any $2 < q_1, q_2 < \infty$ with $1/q_1 + 1/q_2 = 1/2$, applying Hölder's inequality twice and by (3.1.9), we get

$$\begin{aligned}
\|\widetilde{b}^{\mu} T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\leq \|\widetilde{b}^{\mu}\|_{L^{q_1}} \|T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^{q_2}} \\
&\leq C 2^{\frac{2ts}{q_2}} 2^{-\frac{\beta(1-t)s}{q_2}} 2^{-\frac{2tl}{q_2}} \delta^{n(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \\
&\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|\widetilde{b}^{\mu}\|_{L^{q_1}} \|\widetilde{b}^{k-\mu} f\|_{L^{q_2'}} \\
&\leq C 2^{\frac{2ts}{q_2}} 2^{-\frac{\beta(1-t)s}{q_2}} 2^{-\frac{2tl}{q_2}} \delta^{n(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \\
&\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|\widetilde{b}^{\mu}\|_{L^{q_1}} \|\widetilde{b}^{k-\mu}\|_{L^{2q_2/(q_2-2)}} \|f\|_{L^2} \\
&\leq C 2^{\frac{2ts}{q_2}} 2^{-\frac{\beta(1-t)s}{q_2}} 2^{-\frac{2tl}{q_2} + nl(1-\frac{2}{q_2})} m^{\frac{(-2+\beta)(1-t)}{q_2} - (1-\frac{2}{q_2}) + \frac{2t\lambda}{q_2}} \delta^{n(1-\frac{2}{q_2})} \\
&\quad \times \big(\min\{\delta, \delta^{-\beta/2}\}\big)^{\frac{2(1-t)}{q_2}} \|f\|_{L^2}.
\end{aligned} \tag{3.1.10}$$

Now, for any fixed $0 < v < 1$, we choose $q_2 > 2$ but close to 2 sufficiently and $t > 0$ but close to 0 sufficiently, such that q_2 and t satisfy:

$$\begin{gathered}
2t/q_2 > n(1 - 2/q_2), \\
2t/q_2 - \beta(1-t)/q_2 < -v\beta/2, \\
(-2+\beta)(1-t)/q_2 - (1 - 2/q_2) + 2t\lambda/q_2 < (-1+\beta/2)v.
\end{gathered}$$

Then we have

$$\begin{gathered}
\gamma := 2t/q_2 - n(1 - 2/q_2) > 0, \\
m^{(-2+\beta)(1-t)/q_2 - (1-2/q_2) + 2t\lambda/q_2} \leq m^{(-1+\beta/2)v}, \\
2^{\frac{2ts}{q_2}} 2^{-\frac{\beta(1-t)s}{q_2}} \leq 2^{-v\beta s/2}.
\end{gathered}$$

If $\delta \geq 1$, then by (3.1.10)

$$\begin{aligned}
\|\widetilde{b}^{\mu} T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\leq C 2^{-v\beta s/2} m^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{n(1-2/q_2)} \delta^{-\beta(1-t)/q_2} \|f\|_{L^2} \\
&\leq C 2^{-v\beta s/2} m^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{-\beta v/2} \|f\|_{L^2}.
\end{aligned} \tag{3.1.11}$$

If $0 < \delta < 1$, similar to the estimate of (3.1.11), we have

$$\begin{aligned}
\|\widetilde{b}^{\mu} T^l_{s,\delta,m}(\widetilde{b}^{k-\mu} f)\|_{L^2} &\leq C 2^{-v\beta s/2} m^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{n(1-2/q_2)} \delta^{2(1-t)/q_2} \|f\|_{L^2} \\
&\leq C 2^{-v\beta s/2} m^{(-1+\beta/2)v} 2^{-l\gamma} \delta^{v} \|f\|_{L^2}.
\end{aligned} \tag{3.1.12}$$

Thus Claim 2 follows from (3.1.11) and (3.1.12).

Hence, to finish the proof of Lemma 3.1.1, it remains to verify Claim 3.

First we consider the case where $q=\infty$. By the definition of $T^l_{s,\delta,m,j}$, we have

$$\begin{aligned}|T^l_{s,\delta,m}g(x)| &\le \Bigg(\sum_{j=1}^{D_m}\Big(\int_{\mathbb{R}^n}|K^l_{s,\delta,m,j}(x-y)||g(y)|\,dy\Big)^2\Bigg)^{1/2}\\ &\le \int_{\mathbb{R}^n}\Big(\sum_{j=1}^{D_m}|K^l_{s,\delta,m,j}(x-y)|^2\Big)^{1/2}|g(y)|\,dy\\ &\le \|g\|_{L^1}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{D_m}|\widehat{K^l_{s,\delta,m,j}}(\xi)|^2\Big)^{1/2}d\xi.\end{aligned}$$

Note that

$$\widehat{K^l_{s,\delta,m,j}}(\xi)=\widehat{K_{s,\delta,m,j}}*\widehat{\phi_l}(\xi) = \int_{\mathbb{R}^n}B_{s,\delta,m,j}(y)\widehat{\phi_l}(\xi-y)\,dy \tag{3.1.13}$$

and (see [6, p. 225, (2.6)])

$$(\sum_{j=1}^{D_m}|Y_{m,j}(x')|^2)^{1/2}\sim m^\lambda, \quad \text{for any} \quad x\neq 0, \tag{3.1.14}$$

by (3.1.2), we get

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{D_m}|\widehat{K^l_{s,\delta,m,j}}(\xi)|^2\Big)^{1/2}d\xi &\le \int_{\mathbb{R}^\kappa}\Bigg(\sum_{j=1}^{D_m}\Big|\int_{\mathbb{R}^n}B_{s,\delta,m,j}(y)\widehat{\phi_l}(\xi-y)\,dy\Big|^2\Bigg)^{1/2}d\xi\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(\sum_{j=1}^{D_m}|B_{s,\delta,m,j}(y)|^2\Big)^{1/2}\widehat{\phi_l}(\xi-y)\,dy\,d\xi\\ &\le \int_{\delta/2<|y|<2\delta}\Big(\sum_{j=1}^{D_m}|B_{s,\delta,m,j}(y)|^2\Big)^{1/2}dy\,\|\widehat{\phi_l}\|_{L^1}\\ &\le Cm^{-\lambda-1}\int_{\delta/2<|y|<2\delta}\Big(\sum_{j=1}^{D_m}|Y_{m,j}(y')|^2\Big)^{1/2}dy\\ &\le Cm^{-1}\delta^n,\end{aligned}$$

i.e.

$$\|T^l_{s,\delta,m}g\|_{L^\infty}\le Cm^{-1}\delta^n\|g\|_{L^1}. \tag{3.1.15}$$

For $q=2$, note that $\int_{\mathbb{R}^n}\widehat{\phi}(\eta)\,d\eta=\phi(0)=0$, then by (3.1.13) and (3.1.3) we have

$$\begin{aligned}|\widehat{K^l_{s,\delta,m,j}}(x)| &\le \int_{\mathbb{R}^n}|(B_{s,\delta,m,j}(x-2^{-l}y)-B_{s,\delta,m,j}(x))||\widehat{\phi}(y)|\,dy\\ &\le C2^{-l}\|\nabla B_{s,\delta,m,j}\|_{L^\infty}\int_{\mathbb{R}^n}|y||\widehat{\phi}(y)|\,dy\le C2^s2^{-l}.\end{aligned}$$

Therefore, applying Plancherel theorem and the fact (see [6, p. 226, (2.7)])

$$D_m \sim m^{2\lambda}, \tag{3.1.16}$$

we have

$$\begin{aligned} \|T^l_{s,\delta,m} g\|_{L^2} &\le \Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\widehat{K^l_{s,\delta,m,j}}(\xi)|^2 |\widehat{g}(\xi)|^2 \, d\xi \Big)^{1/2} \\ &\le C 2^s 2^{-l} m^\lambda \|g\|_{L^2}. \end{aligned} \tag{3.1.17}$$

Applying Plancherel theorem again, (3.1.1), (3.1.13) and (3.1.14) we see that

$$\begin{aligned} \|T^l_{s,\delta,m} g\|^2_{L^2} &\le \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\widehat{K^l_{s,\delta,m,j}}(\xi)|^2 |\widehat{g}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |B_{s,\delta,m,j} * \widehat{\phi_l}(\xi)|^2 |\widehat{g}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^n} \Big\{ \Big(\sum_{j=1}^{D_m} \Big| \int_{\mathbb{R}^n} B_{s,\delta,m,j}(\xi - y) \widehat{\phi_l}(y) \, dy \Big|^2 \Big)^{1/2} \Big\}^2 |\widehat{g}(\xi)|^2 d\xi \\ &\le C \int_{\mathbb{R}^n} \Big\{ \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{D_m} |B_{s,\delta,m,j}(\xi - y)|^2 \Big)^{1/2} |\widehat{\phi_l}(y)| \, dy \Big\}^2 |\widehat{g}(\xi)|^2 \, d\xi \\ &\le C 2^{-s\beta} m^{-2\lambda-2+\beta} \big(\min\{\delta, \delta^{-\beta/2}\} \big)^2 \\ &\quad \times \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \Big(\sum_{j=1}^{D_m} |Y_{m,j}((\xi - y)')|^2 \Big)^{1/2} |\widehat{\phi_l}(y)| \, dy \Big)^2 |\widehat{g}(\xi)|^2 d\xi \\ &\le C 2^{-s\beta} m^{-2+\beta} \big(\min\{\delta, \delta^{-\beta/2}\} \big)^2 \|\widehat{\phi_l}\|^2_{L^1} \|g\|^2_{L^2}. \end{aligned}$$

That is,

$$\|T^l_{s,\delta,m} g\|_{L^2} \le C 2^{-s\beta/2} m^{(-1+\beta/2)} \min\{\delta, \delta^{-\beta/2}\} \|g\|_{L^2}. \tag{3.1.18}$$

Hence, interpolating between estimates (3.1.17) and (3.1.18), for any $0 < t < 1$,

$$\begin{aligned} \|T^l_{s,\delta,m} g\|_{L^2} &\le C 2^{-tl} 2^{ts} 2^{-(1-t)s\beta/2} m^{t\lambda} m^{(-1+\frac{\beta}{2})(1-t)} \\ &\qquad \times \big(\min\{\delta, \delta^{-\beta/2}\} \big)^{1-t} \|g\|_{L^2}. \end{aligned} \tag{3.1.19}$$

Thus we obtain (3.1.9) for $2 \le q \le \infty$ by interpolating between (3.1.15) and (3.1.19). The proof of Lemma 3.1.1. is now completed. ■

Remark 3.1.1. When $k = 0$, Lemma 3.1.1 also holds; when $s = 0$, Lemma 3.1.1 is just the Lemma 2.3 in [7].

Remark 3.1.2. From the proof of Lemma 3.1.1, if we replace (3.1.1)-(3.1.3) by

$$|B_{s,\delta,m,j}(\xi)| \le Cm^{-\lambda-1}2^{-s}\min\{\delta,\delta^{-1}\}|Y_{m,j}(\xi')|, \tag{3.1.1$'$}$$

$$|B_{s,\delta,m,j}(\xi)| \le Cm^{-\lambda-1}2^{-s}|Y_{m,j}(\xi')|, \tag{3.1.2$'$}$$

$$|\nabla B^i_{s,\delta,m,j}(\xi)| \le C2^{-s}, \tag{3.1.3$'$}$$

then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, k, v)$ such that

$$\|T_{s,\delta,m;\,b,k}f\|_{L^2} \le C2^{-s}m^{-v}\min\{\delta^v,\delta^{-v}\}\|b\|_*^k\|f\|_{L^2}. \tag{3.1.4$'$}$$

Lemma 3.1.2 (see [7, (3.4)]) *For some* $0 < \beta < (1-\theta)/2$ *and* $0 < \theta < 1$

$$\left\|\left(\sum_{j=1}^{D_m}|T_{m,j;b,k}f(x)|^2\right)^{1/2}\right\|_{L^2} \le Cm^{-1+\beta}\|f\|_{L^2},$$

where

$$T_{m,j;b,k}f(x) = \text{p.v.}\int_{\mathbb{R}^n}\frac{Y_{m,j}(x-y)}{|x-y|^n}(b(x)-b(y))^kf(y)dy.$$

3.2. Proof of Theorem 1

We still assume that $\|b\|_* = 1$. By (1.2) and Theorem 2, it suffices to show that

$$\left\|\sup_{l\in\mathbb{Z}}\left|\int_{|x-y|>2^l}\frac{\Omega(x,x-y)}{|x-y|^n}(b(x)-b(y))^kf(y)dy\right|\right\|_{L^2} \le C\|f\|_{L^2}. \tag{3.2.1}$$

Similarly to the decomposition of $\Omega_0(x, z')$ in the proof of Theorem 2, we have

$$\Omega(x,z') = \sum_{m\ge1}a_m(x)\sum_{j=1}^{D_m}d_{m,j}(x)Y_{m,j}(z'),$$

where $a_m(x)$ and $d_{m,j}(x)$ satisfy (2.9) and (2.10). For $s \in \mathbb{Z}$, set

$$T_{s,m,j;b,k}f(x) = \int_{2^s<|x-y|\le2^{s+1}}\frac{Y_{m,j}(x-y)}{|x-y|^n}(b(x)-b(y))^kf(y)dy.$$

Then

$$\int_{|x-y|>2^l}\frac{\Omega(x,x-y)}{|x-y|^n}(b(x)-b(y))^kf(y)dy = \sum_{m=1}^{\infty}\sum_{j=1}^{D_m}\sum_{s=l}^{\infty}T_{s,m,j;b,k}f(x).$$

Using Hölder's inequality twice, by (2.13) we have

$$\begin{aligned}
&\left(\sup_{l\in\mathbb{Z}}\left|\int_{|x-y|>2^l}\frac{\Omega(x,x-y)}{|x-y|^n}(b(x)-b(y))^k f(y)dy\right|\right)^2\leq\\
&\quad\leq\left(\sum_{m=1}^{\infty}a_m^2(x)m^{-\theta}\right)\left(\sum_{m=1}^{\infty}m^{\theta}\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}\left|\sum_{s=l}^{\infty}T_{s,m,j;b,k}f(x)\right|^2\right) \qquad (3.2.2)\\
&\quad\leq C\|\Omega\|_{L^\infty\times L^q}\left(\sum_{m=1}^{\infty}m^{\theta}\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}\left|\sum_{s=l}^{\infty}T_{s,m,j;b,k}f(x)\right|^2\right),
\end{aligned}$$

where $0<\theta<1$ is defined by (2.13).

Hence, by (3.2.2) it is easy to see that, to get (3.2.1), it suffices to show that for some $0<\beta<(1-\theta)/2$,

$$\left\|\left(\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}\left|\sum_{s=l}^{\infty}T_{s,m,j;b,k}f(x)\right|^2\right)^{1/2}\right\|_{L^2}\leq Cm^{-1+\beta}\|f\|_{L^2}. \qquad (3.2.3)$$

We will give the proof of (3.2.3) by induction on the order k.

(i) Proof of (3.2.3) for $k=0$.

In this case, we need to show that for $0<\beta<(1-\theta)/2$

$$\left\|\left(\sum_{j=1}^{D_m}\left(\sup_{l\in\mathbb{Z}}\left|\sum_{s=l}^{\infty}T_{s,m,j}f(x)\right|\right)^2\right)^{1/2}\right\|_{L^2}\leq Cm^{-1+\beta}\|f\|_{L^2}. \qquad (3.2.4)$$

Take $\eta\in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\eta)\subset\{x:|x|\leq 1\}$ and $\eta(x)\equiv 1$ when $|x|\leq 1/2$. Let $\Phi_l\in\mathcal{S}(\mathbb{R}^n)$ be such that $\hat{\Phi}_l(\xi)=\eta(2^l\xi)$. Let

$$T_{m,j}f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{Y_{m,j}(x-y)}{|x-y|^n}f(y)dy \quad \text{for} \quad m\in\mathbb{N},\ j=1,2,\dots,D_m.$$

Then we have

$$\begin{aligned}
\left|\sum_{s=l}^{\infty}T_{s,m,j}f(x)\right| &\leq\left|\Phi_l*(T_{m,j}f(x))\right|+\left|\Phi_l*\left(\sum_{s=-\infty}^{l-1}T_{s,m,j}f(x)\right)\right|\\
&\quad+\left|(\delta-\Phi_l)*\left(\sum_{s=l}^{\infty}T_{s,m,j}f\right)(x)\right| \qquad (3.2.5)\\
&:=P_1+P_2+P_3,
\end{aligned}$$

where and in the sequel, δ denotes the Dirac function. Below we set up the estimate of (3.2.4) for $P_i\,(i=1,2,3)$, respectively. Firstly, we consider P_1.

By [5], we have

$$
\begin{aligned}
\left\|\left(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}} P_1)^2\right)^{1/2}\right\|_{L^2}^2 &= \int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}|\Phi_l * T_{m,j}f(x)|^2dx \\
&\le C\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}|M(T_{m,j}f)(x)|^2dx \\
&\le C\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}|T_{m,j}f(x)|^2dx \\
&= C\sum_{j=1}^{D_m}\|T_{m,j}f(x)\|_{L^2}^2 \\
&\le Cm^{-2}\|f\|_{L^2}^2,
\end{aligned} \tag{3.2.6}
$$

where and in the sequel, M denotes the Hardy-Littlewood maximal operator. Secondly, we consider P_2. For $s \in \mathbb{Z}, m \in \mathbb{N}$ and $j = 1, 2, \ldots, D_m$, define

$$K_{s,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^n}\chi_{\{2^s<|x|\le 2^{s+1}\}}(x).$$

Then

$$
\begin{aligned}
\left\|\left(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}} P_2)^2\right)^{1/2}\right\|_{L^2}^2 &= \int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\left(\sup_{l\in\mathbb{Z}}\left|\sum_{s=-\infty}^{l-1}(\Phi_l * K_{s,m,j} * f)(x)\right|\right)^2dx \\
&\le \int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}\left|\sum_{s=-\infty}^{l-1}(\Phi_l * K_{s,m,j} * f)(x)\right|^2dx \\
&= \int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}\left|\sum_{s=1}^{\infty}(\Phi_l * K_{l-s,m,j} * f)(x)\right|^2dx \\
&\le \left(\sum_{s=1}^{\infty}\left(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}|(\Phi_l * K_{l-s,m,j} * f)(x)|^2dx\right)^{1/2}\right)^2 \\
&= \left(\sum_{s=1}^{\infty}\left(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}|\widehat{\Phi}_l(\xi)\widehat{K_{l-s,m,j}}(\xi)\hat{f}(\xi)|^2d\xi\right)^{1/2}\right)^2.
\end{aligned}
$$

Note that

$$\operatorname{supp}(\widehat{\Phi}_l\widehat{K_{l-s,m,j}}) \subset \{\xi : |2^l\xi| \le 1\}.$$

Applying Lemma 2.2, Plancherel theorem and (3.1.11), we have

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}} P_2)^2\Big)^{1/2}\Big\|_{L^2}\leq \\
&\quad\leq \sum_{s=1}^{\infty}\Big(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{\{l:2^l\leq|\xi|^{-1}\}}|\widehat{\Phi}_l(\xi)\widehat{K_{l-s,m,j}}(\xi)\hat{f}(\xi)|^2d\xi\Big)^{1/2}\\
&\quad\quad+\sum_{s=1}^{\infty}\Big(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{\{l:2^l>|\xi|^{-1}\}}|\widehat{\Phi}_l(\xi)\widehat{K_{l-s,m,j}}(\xi)\hat{f}(\xi)|^2d\xi\Big)^{1/2}\\
&\quad\leq Cm^{-1-\lambda}\sum_{s=1}^{\infty}2^{-s}\Big(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{\{l:2^l\leq|\xi|^{-1}\}}|2^l\xi|^2||Y_{m,j}(\xi)|^2|\hat{f}(\xi)|^2d\xi\Big)^{1/2}\\
&\quad\leq Cm^{-1-\lambda}\Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|Y_{m,j}(\xi)|^2|\hat{f}(\xi)|^2d\xi\Big)^{1/2}\\
&\quad\leq Cm^{-1}\|f\|_{L^2}.
\end{aligned}\tag{3.2.7}$$

Finally, we discuss P_3. By Minkowski inequality and Plancherel theorem, we get

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}} P_3)^2\Big)^{1/2}\Big\|_{L^2}^2=\\
&\quad=\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big(\sup_{l\in\mathbb{Z}}\Big|\sum_{s=l}^{\infty}((\delta-\Phi_l)*K_{s,m,j}*f)(x)\Big|\Big)^2dx\\
&\quad\leq\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}\Big|\sum_{s=0}^{\infty}((\delta-\Phi_l)*K_{s+l,m,j}*f)(x)\Big|^2dx\\
&\quad\leq\Big(\sum_{s=0}^{\infty}\Big(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}|((\delta-\Phi_l)*K_{s+l,m,j}*f)(x)|^2dx\Big)^{1/2}\Big)^2\\
&\quad=\Big(\sum_{s=0}^{\infty}\Big(\sum_{j=1}^{D_m}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}|1-\widehat{\Phi}_l(x)||\widehat{K_{s+l,m,j}}(x)||\widehat{f}(x)|^2dx\Big)^{1/2}\Big)^2.
\end{aligned}$$

Since $\mathrm{supp}((1-\widehat{\Phi}_l)\widehat{K_{s+l,m,j}})\subset\{\xi:|2^l\xi|>1/2\}$, by Lemma 2.2 we have

$$|(1-\widehat{\Phi}_l(\xi))\widehat{K_{s+l,m,j}}(\xi)|\leq Cm^{-1-\lambda+\beta/2}2^{-\beta s/2}|2^l\xi|^{-\beta/2}|Y_{m,j}(\xi')|.$$

Similarly to (3.2.7), it is easy to obtain that

$$\Big\|\Big(\sum_{j=1}^{D_m}(\sup_{l\in\mathbb{Z}} P_3)^2\Big)^{1/2}\Big\|_{L^2}\leq Cm^{-1+\beta/2}\|f\|_{L^2}.\tag{3.2.8}$$

By (3.2.6)–(3.2.8), we obtain (3.2.4) and hence (3.2.3) holds for $k=0$.

(ii) *Proof of* (3.2.3) *for* $k \in \mathbb{N}$.

In this case, we assume that (3.2.3) is true for all integers u with $0 \le u \le k-1$ and we will prove that (3.2.3) holds also for k.

Take $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) \equiv 1$ when $|x| \le 1/2$, $\mathrm{supp}(\eta) \subseteq \{x : |x| \le 1\}$. Let $\Phi_l \in \mathcal{S}(\mathbb{R}^n)$ be such that $\hat{\Phi}_l(\xi) = \eta(2^l \xi)$. Write

$$\begin{aligned}\sum_{s=l}^{\infty} T_{s,m,j;b,k} f(x) &\le |\Phi_l * T_{m,j;b,k} f(x)| + \Big|\Phi_l * \sum_{s=-\infty}^{l-1} T_{s,m,j;b,k} f(x)\Big| \\ &\quad + \Big|\sum_{s=l}^{\infty} T_{s,m,j;b,k} f(x) - \Phi_l * \big(\sum_{s=l}^{\infty} T_{s,m,j;b,k} f\big)(x)\Big| \\ &:= I + II + III. \end{aligned} \tag{3.2.9}$$

We need to prove that (3.2.3) is true for I, II and III, respectively. First we consider I. By Lemma 3.1.2, for $0 < \beta < (1-\theta)/2$ and $0 < \theta < 1$, we have

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{D_m} (\sup_{l\in\mathbb{Z}} I)^2\Big)^{1/2}\Big\|_{L^2}^2 &= \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \sup_{l\in\mathbb{Z}} |\Phi_l * T_{m,j;b,k} f(x)|^2 dx \\ &\le C \sum_{j=1}^{D_m} \int_{\mathbb{R}^n} |M(T_{m,j;b,k} f)(x)|^2 dx \\ &\le C \sum_{j=1}^{D_m} \int_{\mathbb{R}^n} |T_{m,j;b,k} f(x)|^2 dx \\ &\le C m^{-2+2\beta} \|f\|_{L^2}^2. \end{aligned} \tag{3.2.10}$$

Now we consider II. Denote by G_l and $G_{l;b,u}$ the convolution operator with kernel Φ_l and the u-th order commutator of G_l, respectively. Applying formula (2.17) we can write

$$II = \Big|\big(\Phi_l * \big(\sum_{s=-\infty}^{l-1} K_{s,m,j}\big)\big)_{b,k} f(x) - \sum_{u=0}^{k-1} C_k^u G_{l;b,k-u}\big(\sum_{s=-\infty}^{l-1} T_{s,m,j;b,u} f\big)(x)\Big|.$$

Let

$$II_1 = \sup_{l\in\mathbb{Z}} \Big|\Big(\Phi_l * \Big(\sum_{s=-\infty}^{l-1} K_{s,m,j}\Big)\Big)_{b,k} f(x)\Big|$$

and

$$II_2 = \sup_{l\in\mathbb{Z}} \Big|\sum_{u=0}^{k-1} C_k^u G_{l;b,k-u}\Big(\sum_{s=-\infty}^{l-1} T_{s,m,j;b,u} f\Big)(x)\Big|.$$

Therefore, if we can show that for some $0 < \beta < (1-\theta)/2$,

$$\Big\| \Big(\sum_{j=1}^{D_m} II_1^2 \Big)^{1/2} \Big\|_{L^2} \le C m^{-1+\beta} \|f\|_{L^2} \tag{3.2.11}$$

and

$$\Big\| \Big(\sum_{j=1}^{D_m} II_2^2 \Big)^{1/2} \Big\|_{L^2} \le C m^{-1+\beta} \|f\|_{L^2}, \tag{3.2.12}$$

then (3.2.3) holds for II.

For II_1, applying Minkowski inequality we have

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^{D_m} II_1^2 \Big)^{1/2} \Big\|_{L^2} &= \Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \Big(\sup_{l\in\mathbb{Z}} \Big| \sum_{s=-\infty}^{l-1} (\Phi_l * K_{s,m,j})_{b,k} f(x) \Big| \Big)^2 dx \Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \sum_{l\in\mathbb{Z}} \Big| \sum_{s=-\infty}^{l-1} (\Phi_l * K_{s,m,j})_{b,k} f(x) \Big|^2 dx \Big)^{1/2} \\
&= \Big(\int_{\mathbb{R}^n} \sum_{j=1}^{D_m} \sum_{l\in\mathbb{Z}} \Big| \sum_{s=1}^{\infty} (\Phi_l * K_{l-s,m,j})_{b,k} f(x) \Big|^2 dx \Big)^{1/2} \\
&\le \sum_{s=1}^{\infty} \Big(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |(\Phi_l * K_{l-s,m,j})_{b,k} f(x)|^2 dx \Big)^{1/2} \\
&:= W.
\end{aligned}$$

Let

$$U_{s,l,m,j} f = \Phi_l * K_{l-s,m,j} * f.$$

Denote

$$U_{s,l,m,j,b,k} = (\Phi_l * K_{l-s,m,j})_{b,k}.$$

Let $\psi \in C_0^\infty$ be a radial function such that $0 \le \psi \le 1$, $\mathrm{supp}\psi \subset \{\xi : 1/2 \le |\xi| \le 2\}$ and $\sum_{i\in\mathbb{Z}} \psi^2(2^{-i}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_i by $\widehat{S_i f}(\xi) = \psi(2^{-i}\xi)\widehat{f}(\xi)$. For $l \in \mathbb{Z}$, $s \in \mathbb{N}$, $m = 1, 2 \ldots$, and $j = 1, \ldots, D_m$. Set

$$B_{s,l,m,j}(\xi) = \widehat{\Phi}_l(\xi)\widehat{K_{l-s,m,j}}(\xi), \quad B^i_{s,l,m,j}(\xi) = B_{s,l,m,j}(\xi)\psi(2^{l-i}\xi).$$

Define the operator $U^i_{s,l,m,j}$ by $(U^i_{s,l,m,j} f)\widehat{\ }(\xi) = (U_{s,l,m,j} f)\widehat{\ }(\xi)\psi(2^{l-i}\xi)$. Denote by $U^i_{s,l,m,j;b,k}$ the k-th order commutator of $U^i_{s,l,m,j}$. Then

$$U_{s,l,m,j;b,k} f(x) = \sum_{i\in\mathbb{Z}} ((U^i_{s,l,m,j} S_{i-l})_{b,k} f)(x).$$

Applying the equation above and Minkowski inequality, we have

$$\begin{aligned} W &= \sum_{s=1}^{\infty}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|\sum_{i\in\mathbb{Z}}((U^i_{s,l,m,j}S_{i-l})_{b,k}f)(x)|^2dx\Big)^{1/2}\\ &\le \sum_{s=1}^{\infty}\sum_{i\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|((U^i_{s,l,m,j}S_{i-l})_{b,k}f)(x)|^2dx\Big)^{1/2} \qquad (3.2.13)\\ &:= \sum_{s=1}^{\infty}\sum_{i\in\mathbb{Z}}W_{i,s}. \end{aligned}$$

Write $(U^i_{s,l,m,j}S_{i-l})_{b,k}f(x) = \sum_{\alpha=0}^{k} C_k^\alpha U^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)$ and

$$U^i_{s,l,m;b,\alpha}f(x) = \Big(\sum_{j=1}^{D_m}|U^i_{s,l,m,j;b,\alpha}f(x)|^2\Big)^{1/2},$$

then

$$\begin{aligned} W_{i,s} &= \Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big|\sum_{\alpha=0}^{k}C_k^\alpha U^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)\Big|^2dx\Big)^{1/2}\\ &\le C\sum_{\alpha=0}^{k}C_k^\alpha\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|U^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)|^2dx\Big)^{1/2}\\ &= C\sum_{\alpha=0}^{k}C_k^\alpha\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}|U^i_{s,l,m;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)|^2dx\Big)^{1/2}. \end{aligned}$$

We claim that there exists $0 < v < 1$ such that

$$\|U^i_{s,l,m;b,\alpha}f\|_{L^2} \le Cm^{-1+\beta}2^{-s}2^{-v|i|}\|f\|_{L^2}. \qquad (3.2.14)$$

If so, then by Lemma 2.3, we have

$$\begin{aligned} W_{i,s} &\le C\sum_{\alpha=0}^{k}C_k^\alpha m^{-1+\beta}2^{-s}2^{-v|i|}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}|(S_{i-l;b,k-\alpha}f)(x)|^2dx\Big)^{1/2}\\ &\le Cm^{-1+\beta}2^{-s}2^{-v|i|}\|f\|_{L^2}. \end{aligned} \qquad (3.2.15)$$

Thus, (3.2.11) will follow from (3.2.13) and (3.2.15). Now we estimate (3.2.14). Define the operator $U^i_{s,l,m,j}$ by $(\tilde{U}^i_{s,l,m,j}f)^\frown(\xi) = B^i_{s,l,m,j}(2^{-l}\xi)\hat{f}(\xi)$. Denote by $\tilde{U}^i_{s,l,m,j;b,\alpha}$ the α-th order commutator of $\tilde{U}^i_{s,l,m,j}$. Let

$$\tilde{U}^i_{s,l,m;b,\alpha}f(\xi) = \Big(\sum_{j=1}^{D_m}|\tilde{U}^i_{s,l,m,j;b,\alpha}f(\xi)|^2\Big)^{1/2}.$$

By the definition of $B_{s,l,m,j}$ and Lemma 2.2, we have

$$|B_{s,l,m,j}(\xi)| \le Cm^{-\lambda-1}2^{-s}\min\{|2^l\xi|, |2^l\xi|^{-1}\}|Y_{m,j}(\xi')|,$$
$$|B_{s,l,m,j}(\xi)| \le Cm^{-\lambda-1}2^{-s}|Y_{m,j}(\xi')|,$$
$$|\nabla B_{s,l,m,j}(\xi)| \le C2^l2^{-s}.$$

Note that $\mathrm{supp}(B^i_{s,l,m,j}(2^{-l}\cdot)) \subset \{\xi : 2^{i-1} \le |\xi| \le 2^{i+1}\}$, we get

$$|B^i_{s,l,m,j}(2^{-l}\xi)| \le Cm^{-\lambda-1}2^{-s}\min\{2^i, 2^{-i}\}|Y_{m,j}(\xi')|,$$
$$|B^i_{s,l,m,j}(2^{-l}\xi)| \le Cm^{-\lambda-1}2^{-s}|Y_{m,j}(\xi')|,$$
$$|\nabla B^i_{s,l,m,j}(2^{-l}\xi)| \le C2^{-s}$$

Using $(3.1.4)'$ with $\delta = 2^i$, for any fixed $0 < v < 1$ and $\alpha \in \mathbb{N}$

$$\|\tilde{U}^i_{s,l,m;b,\alpha}\|_{L^2} \le Cm^{-v}2^{-s}2^{-|i|v}\|f\|_{L^2}.$$

Thus, for $0 < \beta < (1-\theta)/2$ $(0 < \theta < 1)$, we can take $0 < v_0 < 1-\beta$ in the above estimate. Hence we obtain

$$\|\tilde{U}^i_{s,l,m;b,\alpha}\|_{L^2} \le Cm^{-1+\beta}2^{-s}2^{-|i|v_0}\|f\|_{L^2},$$

which implies, by dilation-invariance,

$$\|U^i_{s,l,m;b,\alpha}\|_{L^2} \le Cm^{-1+\beta}2^{-s}2^{-|i|v_0}\|f\|_{L^2}.$$

So we proved (3.2.14). Now let us turn to (3.2.12). Write

$$\begin{aligned} II_2 &= \sup_{l\in\mathbb{Z}}\left|\sum_{u=0}^{k-1} C^u_k G_{l;b,k-u}\left(\sum_{s=-\infty}^{l-1} T_{s,m,j;b,u}f\right)(x)\right| \\ &= \sup_{l\in\mathbb{Z}}\left|\sum_{u=0}^{k-1} C^u_k G_{l;b,k-u}\left(T_{m,j,b,u}f - \sum_{s=l}^{\infty} T_{s,m,j;b,u}f\right)(x)\right| \\ &\le \sup_{l\in\mathbb{Z}}\left|\sum_{u=0}^{k-1} C^u_k G_{l;b,k-u}(T_{m,j,b,u}f)(x)\right| \\ &\quad + \sup_{l\in\mathbb{Z}}\left|\sum_{u=0}^{k-1} C^u_k G_{l;b,k-u}\left(\sum_{s=l}^{\infty} T_{s,m,j;b,u}f\right)(x)\right| \\ &:= II_{21} + II_{22}. \end{aligned}$$

Thus, to prove (3.2.12), it is sufficient to show that for some $0 < \beta < (1-\theta)/2$,

$$\left\|\left(\sum_{j=1}^{D_m} II^2_{21}\right)^{1/2}\right\|_{L^2} \le Cm^{-1+\beta}\|f\|_{L^2} \tag{3.2.17}$$

and

$$\left\|\left(\sum_{j=1}^{D_m} II_{22}^2\right)^{1/2}\right\|_{L^2} \le Cm^{-1+\beta}\|f\|_{L^2}. \qquad (3.2.18)$$

However, (3.2.17) is just a consequence of Lemma 3.1.2 and (2.7). In fact, for $0<\beta<(1-\theta/2)$,

$$\begin{aligned}\left\|\left(\sum_{j=1}^{D_m} II_{21}^2\right)^{1/2}\right\|_{L^2}^2 &\le \sum_{j=1}^{D_m}\sum_{u=0}^{k-1} C_k^u \sup_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}|G_{l;b,k-u}(T_{m,j,b,u}f)(x)|^2dx\\ &\le \sum_{j=1}^{D_m}\sum_{u=0}^{k-1} C_k^u\|M_{b,k-u}(T_{m,j,b,u}f)\|_{L^2}^2\\ &\le C\sum_{j=1}^{D_m}\|T_{m,j,b,u}f\|_{L^2}^2\\ &\le Cm^{-2+2\beta}\|f\|_{L^2}^2.\end{aligned}$$

On the other hand, to obtain (3.2.18), applying (2.7) and the induction assumptions for $0\le u\le k-1$, we have

$$\begin{aligned}\left\|\left(\sum_{j=1}^{D_m} II_{22}^2\right)^{1/2}\right\|_{L^2}^2 &= \left\|\left(\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}|\sum_{u=0}^{k-1}C_k^u G_{l;b,k-u}\left(\sum_{s=l}^{\infty}T_{s,m,j;b,u}f\right)(x)|^2\right)^{1/2}\right\|_{L^2}^2\\ &\le C\sum_{u=0}^{k-1}\left\|\left(\sum_{j=1}^{D_m}\sup_{l\in\mathbb{Z}}\left|G_{l;b,k-u}\left(\sum_{s=l}^{\infty}T_{s,m,j;b,u}f(x)\right)\right|^2\right)^{1/2}\right\|_{L^2}^2\\ &\le C\sum_{u=0}^{k-1}\left\|\left(\sum_{j=1}^{D_m}\left|M_{b,k-u}\left(\sup_{l\in\mathbb{Z}}|\sum_{s=l}^{\infty}T_{s,m,j;b,u}f(x)|\right)\right|^2\right)^{1/2}\right\|_{L^2}^2\\ &\le C\sum_{u=0}^{k-1}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\left(\sup_{l\in\mathbb{Z}}|\sum_{s=l}^{\infty}T_{s,m,j;b,u}f(x)|\right)^2dx\\ &\le Cm^{-2+2\beta}\|f\|_{L^2}^2.\end{aligned}$$

Thus we prove (3.2.3) for II. Finally, we show (3.2.3) is true for III. It is easy to check that

$$\begin{aligned}III &= \left|\sum_{s=l}^{\infty}T_{s,m,j;b,k}f(x)-\left(\Phi_l*\sum_{s=l}^{\infty}K_{s,m,j}\right)_{b,k}f(x)\right.\\ &\qquad\left.-\sum_{u=0}^{k-1}C_k^u G_{l;b,k-u}\left(\sum_{s=l}^{\infty}T_{s,m,j;b,u}f\right)(x)\right|\\ &\le III_1+II_{22},\end{aligned}$$

where $III_1=\sup_{l\in\mathbb{Z}}\left|\left(\sum_{s=l}^{\infty}(\delta-\Phi_l)*K_{s,m,j}\right)_{b,k}f(x)\right|$.

Thus, by (3.2.18), we need only to verify that

$$\Big\|\Big(\sum_{j=1}^{D_m} III_1^2\Big)^{1/2}\Big\|_{L^2} \le Cm^{-1+\beta}\|f\|_{L^2}. \tag{3.2.19}$$

Applying Minkowski inequality, we get

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{D_m} III_1^2\Big)^{1/2}\Big\|_{L^2} = \\
&= \Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big(\sup_{l\in\mathbb{Z}}\Big|\sum_{s=l}^{\infty}((\delta-\Phi_l)*K_{s,m,j})_{b,k}f(x)\Big|\Big)^2 dx\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\sum_{l\in\mathbb{Z}}\Big|\sum_{s=0}^{\infty}((\delta-\Phi_l)*K_{s+l,m,j})_{b,k}f(x)\Big|^2 dx\Big)^{1/2} \qquad (3.2.20)\\
&\le \sum_{s=0}^{\infty}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|((\delta-\Phi_l)*K_{s+l,m,j})_{b,k}f(x)|^2 dx\Big)^{1/2} \\
&:= Q.
\end{aligned}$$

Let $V_{s,l,m,j}f = (\delta-\Phi_l)*K_{s+l,m,j}*f$ and $V_{s,l,m,j;b,k} = ((\delta-\Phi_l)*K_{s+l,m,j})_{b,k}$. Let $\psi \in C_0^\infty$ be a radial function such that $0 \le \psi \le 1$, supp$\psi \subset \{\xi : 1/2 \le |\xi| \le 2\}$ and $\sum_{i\in\mathbb{Z}}\psi^2(2^{-i}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_i by $\widehat{S_i f}(\xi) = \psi(2^{-i}\xi)\widehat{f}(\xi)$. For $l \in \mathbb{Z}$, $s \in \mathbb{Z}_+$, $m \in \mathbb{N}$ and $j = 1,\dots,D_m$, set

$$D_{s,l,m,j}(\xi) = (1-\widehat{\Phi}_l(\xi))\widehat{K_{s+l,m,j}}(\xi)$$

and

$$D^i_{s,l,m,j}(\xi) = D_{s,l,m,j}(\xi)\psi(2^{l-i}\xi).$$

Define the operator $V^i_{s,l,m,j}$ by $(V^i_{s,l,m,j}f)\widehat{\ }(\xi) = (V_{s,l,m,j}f)\widehat{\ }(\xi)\psi(2^{l-i}\xi)$ and denote by $V^i_{s,l,m,j;b,k}$ the k-th order commutator of $V^i_{s,l,m,j}$. Then it is clear that

$$V_{s,l,m,j;b,k}f(x) = \sum_{i\in\mathbb{Z}}((V^i_{s,l,m,j}S_{i-l})_{b,k}f)(x).$$

By Minkowski inequality, we have

$$\begin{aligned}
Q &= \sum_{s=0}^{\infty}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}\Big|\sum_{i\in\mathbb{Z}}((V^i_{s,l,m,j}S_{i-l})_{b,k}f)(x)\Big|^2 dx\Big)^{1/2} \\
&\le \sum_{s=0}^{\infty}\sum_{i\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|((V^i_{s,l,m,j}S_{i-l})_{b,k}f)(x)|^2 dx\Big)^{1/2} \qquad (3.2.21)\\
&:= \sum_{s=0}^{\infty}\sum_{i\in\mathbb{Z}} Q_{i,s}.
\end{aligned}$$

Write $(V^i_{s,l,m,j}S_{i-l})_{b,k}f(x) = \sum_{\alpha=0}^{k} C^\alpha_k V^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)$ and let

$$V^i_{s,l,m;b,\alpha}f(x) = \left(\sum_{j=1}^{D_m} |V^i_{s,l,m,j;b,\alpha}f(x)|^2\right)^{1/2},$$

then

$$\begin{aligned} Q_{i,s} &= \left(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |\sum_{\alpha=0}^{k} C^\alpha_k V^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)|^2 dx\right)^{1/2} \\ &\le C\sum_{\alpha=0}^{k} C^\alpha_k \left(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{j=1}^{D_m} |V^i_{s,l,m,j;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)|^2 dx\right)^{1/2} \\ &= C\sum_{\alpha=0}^{k} C^\alpha_k \left(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} |V^i_{s,l,m;b,\alpha}(S_{i-l;b,k-\alpha}f)(x)|^2 dx\right)^{1/2}. \end{aligned}$$

Obviously, if we can prove that there exists $0 < v < 1$ such that,

$$\|V^i_{s,l,m;b,\alpha}f\|_{L^2} \le Cm^{-1+\beta}2^{-\beta s/2}2^{-v|i|\beta/2}\|f\|_{L^2}, \tag{3.2.22}$$

then applying Lemma 2.3, we get

$$\begin{aligned} Q_{i,s} &\le C\sum_{\alpha=0}^{k} C^\alpha_k m^{-1+\beta}2^{-\beta s/2}2^{-v\beta|i|/2} \cdot \left(\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} |(S_{i-l;b,k-\alpha}f)(x)|^2 dx\right)^{1/2} \\ &\le Cm^{-1+\beta}2^{-\beta s/2}2^{-v\beta|i|/2}\|f\|_{L^2}. \end{aligned} \tag{3.2.23}$$

Thus, (3.2.19) follows by (3.2.20), (3.2.21) and (3.2.23). Hence, it remains to show (3.2.22). To this end, define the multiplier $\tilde{V}^i_{s,l,m,j}$ by

$$\widehat{\tilde{V}^i_{s,l,m,j}f}(\xi) = D^i_{s,l,m,j}(2^{-l}\xi)\hat{f}(\xi)$$

and denote by $\tilde{V}^i_{s,l,m,j;b,\alpha}$ the α-th order commutator of $\tilde{V}^i_{s,l,m,j}$. Let

$$\tilde{V}^i_{s,l,m;b,\alpha}f(\xi) = \left(\sum_{j=1}^{D_m} |\tilde{V}^i_{s,l,m,j;b,\alpha}f(\xi)|^2\right)^{1/2}.$$

By Lemma 2.2, we have

$$|D_{s,l,m,j}(\xi)| \le Cm^{-\lambda-1+\beta/2}2^{-\beta s/2}\min\{|2^l\xi|, |2^l\xi|^{-\beta/2}\}|Y_{m,j}(\xi')|,$$

$$|D_{s,l,m,j}(\xi)| \le Cm^{-\lambda-1}|Y_{m,j}(\xi')|,$$
$$|\nabla D_{s,l,m,j}(\xi)| \le C2^l2^s.$$

Since $\mathrm{supp}(D^i_{s,l,m,j}(2^{-l}\cdot)) \subset \{\xi : 2^{i-1} \le |\xi| \le 2^{i+1}\}$, we have the following estimates:

$$|D^i_{s,l,m,j}(2^{-l}\xi)| \le Cm^{-\lambda-1+\beta/2}2^{-\beta s/2}\min\{2^i, 2^{-i\beta/2}\}|Y_{m,j}(\xi')|,$$

$$|D^i_{s,l,m,j}(2^{-l}\xi)| \le Cm^{-\lambda-1}|Y_{m,j}(\xi')|,$$

$$|\nabla D^i_{s,l,m,j}(2^{-l}\xi)| \le C2^s.$$

Applying Lemma 3.1.1 with $\delta = 2^i$, we know for any fixed $0 < v < 1$ and nonnegative integer u

$$\|\tilde{V}^i_{s,l,m;b,\alpha}f\|_{L^2} \le C2^{-\beta sv/2}m^{(-1+\beta/2)}2^{-\beta|i|v/2}\|f\|_{L^2},$$

which implies

$$\|V^i_{s,l,m;b,\alpha}f\|_{L^2} \le C2^{-\beta sv/2}m^{(-1+\beta/2)}2^{-\beta|i|v/2}\|f\|_{L^2}$$

by dilation-invariance. This is (3.2.22) and (3.2.3) holds for III. This completes the proof of Theorem 1.

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