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A proof of hypoellipticity for Kohn's operator via FBI

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Abstract

A new proof of both analytic and C^∞ hypoellipticity of Kohn's operator is given using FBI techniques introduced by J. Sjöstrand. The same proof allows us to obtain both kind of hypoellipticity at the same time.

1. Introduction

In 2005, J. J. Kohn published a seminal paper [6], mainly concerned with the problem of $C^\infty-$hypoellipticity for sums of squares of complex vector fields.

It turns out that, in contrast with the case of real vector fields satisfying Hörmander's bracket condition, a sum of squares of complex vector fields can be hypoelliptic and, at the same time, lose an arbitrary number of derivatives. As a consequence, the primary tool (the a priori estimate) used to prove hypoellipticity can be rather difficult to obtain. As a matter of fact, J. J. Kohn in [6] produced an example of a sum of squares of complex vector fields with real analytic coefficients having a symplectic characteristic manifold that is $C^\infty-$ hypoelliptic and, as Derridj and Tartakoff prove in the appendix to [6], is also analytic hypoelliptic.

We recall that a sum of squares of complex vector fields is related to a sum of squares of real vector fields with special lower order terms. In this situation, in a transversally non-degenerate case, Treves [14] and Kwon [7, 8] have proved both C^∞ and real analytic hypoellipticity.

The purpose of the present paper is to give an alternate proof of both Kohn's result and that of Derridj and Tartakoff on Kohn's model operator,

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or rather that studied in [2]. We use F.B.I. transform techniques. The advantage of such an approach is that the same proof works in both the C^∞ and C^ω categories at the same time. This is due to the fact that, roughly speaking, the F.B.I. transform has a built in localization that is well adapted to the analytic category. This results in a characterization of both the C^∞ wave front set, WF, and the analytic wave front set, WF_a, with a uniform decay rate (near a certain point) of the F.B.I. transform of the distribution in examination (see, e.g., [10]). The technique we use to derive an a priori estimate on the F.B.I. side is inspired by the work of J. Sjöstrand [11]. A similar technique, not dealing with the spectral degeneracy involved in Kohn's operator, has permitted J. Sjöstrand to give an F.B.I. proof of the analytic hypoellipticity results of Treves [15] and Tartakoff [12, 13].

Let us now state our theorem. Let q be an even positive integer and k a positive integer. Define

$$L = D_x + ix^{q-1}D_t\,,$$

where $D_x = i^{-1}\partial_x$, $x \in \mathbb{R}$. The Kohn's operator is defined as

$$P(x, D_x, D_t) = LL^* + L^*\, x^{2k} L. \tag{1.1}$$

Then we have the

Theorem 1.1. *Kohn's operator* (1.1) *is both C^∞ and C^ω hypoelliptic.*

We point out that the operator P is considered only when q is even and positive. As a matter of fact the case when q is an odd integer is much easier since then P is both C^∞ and C^ω hypoelliptic with a loss of one derivative. This may be explained by remarking that the kernel of L^*_τ is empty in the distribution category, where $L^*_\tau = D_x - ix^{q-1}\tau$, since $L^*_\tau e^{-x^q\tau/q} = 0$.

2. Construction of the left parametrix

Let λ be a positive large parameter. Consider the operator $\lambda^{-2}P$:

$$(\tilde{D}_x + ix^{q-1}\tilde{D}_t)(\tilde{D}_x - ix^{q-1}\tilde{D}_t) + (\tilde{D}_x - ix^{q-1}\tilde{D}_t)x^{2k}(\tilde{D}_x + ix^{q-1}\tilde{D}_t), \tag{2.1}$$

where we used the notation $\tilde{D}_x = \frac{1}{\lambda\sqrt{-1}}\frac{\partial}{\partial x}$ and $\tilde{D}_t = \frac{1}{\lambda\sqrt{-1}}\frac{\partial}{\partial t}$.

The symbol associated with (1.1) via λ-Fourier transform

$$\hat{u}(\xi,\tau) = \int e^{-i\lambda(x\xi+t\tau)}u(x,t)dxdt,$$

$$u(x,t) = \left(\frac{\lambda}{2\pi}\right)^2 \int e^{i\lambda(x\xi+t\tau)}\hat{u}(\xi,\tau)d\xi d\tau,$$

is

$$(1+x^{2k})(\xi^2+x^{2(q-1)}\tau^2)+ \\ +\frac{1}{\lambda}((x^{2k}-1)(q-1)x^{q-2}\tau-2ikx^{2k-1}(\xi+ix^{q-1}\tau)). \tag{2.2}$$

Its characteristic set is

$$\Sigma=\Big\{(x,t,\xi,\tau)\in\mathbb{R}^4\setminus\{0\}:\xi=0=x \text{ with } \tau\neq 0\Big\}; \tag{2.3}$$

we put $\Sigma=\Sigma^+\cup\Sigma^-$ where $\Sigma^\pm=\{\rho\in\Sigma:\tau\gtrless 0\}$. Since the theorem we want to prove is microlocal in essence, we shall argue in a neighborhood of the point $\rho_0=(0,0,0,1)\in\Sigma^+$.

Let us consider the λ-Fourier transform of the operator P with respect to the variable t

$$\tilde{D}_x^2+x^{2(q-1)}\tau^2-\frac{q-1}{\lambda}x^{q-2}\tau+ \\ +x^{2k}(\tilde{D}_x^2+x^{2(q-1)}\tau^2+\frac{q-1}{\lambda}x^{q-2}\tau)-\frac{2ik}{\lambda}x^{2k-1}(\tilde{D}_x+ix^{q-1}\tau) \tag{2.4}$$

It is useful adopt the notation:

1. $P_0=\tilde{D}_x^2+x^{2(q-1)}\tau^2-\frac{q-1}{\lambda}x^{q-2}\tau$,
2. $P_k=x^{2k}(\tilde{D}_x^2+x^{2(q-1)}\tau^2+\frac{q-1}{\lambda}x^{q-2}\tau)-\frac{2ik}{\lambda}x^{2k-1}(\tilde{D}_x+ix^{q-1}\tau)$.

We point out that the above subdivision of P reflects a homogeneity property. More precisely let us consider the dilation $x\mapsto\lambda^{-1/q}x$, $t\mapsto t$ as well as its canonical action on the covariables $\xi\mapsto\lambda^{1/q}\xi$, $\tau\mapsto\tau$. Then if p_0, p_k denote the symbol of P_0 and P_k w.r.t. the usual Fourier transform, we have

$$p_0(\lambda^{-1/q}x,\lambda^{1/q}\xi/\lambda,\tau)=\lambda^{-2+\frac{2}{q}}p_0(x,\xi,\tau) \quad \text{and} \\ p_k(\lambda^{-1/q}x,\lambda^{1/q}\xi/\lambda,\tau)=\lambda^{-2+\frac{2}{q}-\frac{2k}{q}}p_k(x,\xi,\tau).$$

We call this homogeneity property "global homogeneity".

To construct an approximate parametrix for P_0+P_k, which will be sufficient for our purpose, we shall use an algebraic technique following Sjöstrand [9]. We use the same framework as Boutet de Monvel [1]. In the latter paper classes of symbols are studied via the distance function to a symplectic characteristic manifold. We use basically the same classes, replacing quadratic distance with an adapted anisotropic function, keeping into account the higher vanishing order w.r.t. the x variable of $p(x,\xi,\tau)$. We set

$$m(x,\xi,\lambda)=\left(d_\Sigma^2+\lambda^{-\frac{2(q-1)}{q}}\right)^{\frac{1}{2}} \text{ where } d_\Sigma^2=|\xi|^2+|x|^{2(q-1)},$$

and we say that a C^∞ function a belongs to $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}^2_{\xi,\tau}, \Sigma)$, or briefly $S_q^{m,k}$, if for any positive integers γ, μ, α and β we have

$$\left|\partial_\tau^\gamma \partial_t^\mu \partial_\xi^\beta \partial_x^\alpha a(x,t,\xi,\tau)\right| \lesssim \lambda^{m-\beta-\gamma} m(x,\xi,\lambda)^{k-\beta-\frac{\alpha}{q-1}} ,$$

for $\lambda \geq 1$ and (x,t,ξ,τ) in a neighborhood of $(0,0,0,1) \in \Sigma^+$. In the same way we define the symbol class $\mathscr{H}_q^m = \cap_{j=0}^\infty S_q^{m-j,k-jq/(q-1)}$(see [1] for more details.) We denote by $OP\mathscr{H}_q^m = \cap_{j=0}^\infty OPS_q^{m-j,k-jq/(q-1)}$ the set of pseudodifferential operator corresponding to $\mathscr{H}_q^m$.

We recall the inclusion relation: $S_q^{m,k} \subseteq S_q^{m',k'}$ if and only if $m \leq m'$ and $m - k(q-1)/q \leq m' - k'(q-1)/q$.

We have that

$$P_0(x, \tilde{D}_x, \tilde{D}_t) \in OPS_q^{0,2} \quad \text{ and } \quad P_k(x, \tilde{D}_x, \tilde{D}_t) \in OPS_q^{0,2+\frac{2k}{q-1}}.$$

Let Σ_1=$\Pi_x(\Sigma)$ be the projection of characteristic set Σ on $\mathbb{R}_x$. We define the space $\mathcal{H}_q^{m+1/2q}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, or shortly $\mathcal{H}_q^{m+1/2q}$, as the space of all smooth functions belonging to $\cap_{j=0}^\infty S_q^{m-j+1/2q,-jq/(q-1)}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, where $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ denotes the set of all smooth functions such that

$$|\partial_\tau^\gamma \partial_t^\mu \partial_x^\alpha a(x,t,\tau,\lambda)| \lesssim \lambda^{m-\gamma} \left(|x|^{2(q-1)} + \lambda^{-\frac{2(q-1)}{q}}\right)^{\frac{k}{2}-\frac{\alpha}{2(q-1)}} .$$

The action of a symbol a in $\mathcal{H}_q^{m+1/2q}$ as a map $a(x,t,\tilde{D}_t) \colon C_0^\infty(\mathbb{R}_t) \to C^\infty(\mathbb{R}^2_{x,t})$ is defined by

$$\left(a(x,t,\tilde{D}_t,\lambda)u\right)(x,t) = \frac{\lambda}{2\pi} \int e^{i\lambda(t-t')\tau} a(x,t,\tau) u(t') dt' d\tau.$$

Such an operator, modulo a regularizing operator (w.r.t. the t-variable) is called an Hermite operator of degree m and we denote by $OP\mathcal{H}_q^m$ the corresponding operator class.

Let $a \in \mathcal{H}_q^{m+1/2q}$, we define the adjoint of the Hermite operator a as the map $a^*(x,t,\tilde{D}_t,\lambda) : C_0^\infty(\mathbb{R}^2_{x,t}) \to C^\infty(\mathbb{R}_t)$ defined by

$$\left(a^*(x,t,\tilde{D}_t,\lambda)u\right)(t) = \frac{\lambda}{2\pi} \iint e^{i\lambda(t-t')\tau} \overline{a(x,t,\tau)} u(x,t') dx dt' d\tau.$$

We denote by $OP\mathcal{H}_q^{*m}$ the related space of operators.

$P_0(x,\tilde{D}_x,\tau)$ is a self-adjoint operator on $\mathscr{S}(\mathbb{R}_x)$; moreover, since q is even, it is not injective and actually it has an one dimensional kernel. For more details on this subject see [3].

Let $e_{0,\tau}(x,\lambda)$ be the L^2-normalized null eigenfunction of P_0, $\|e_{0,\tau}\|_{L^2} = 1$,

$$e_{0,\tau}(x,\lambda) = \sqrt{c_q}\,\lambda^{\frac{1}{2q}}\tau^{\frac{1}{2q}}e^{-\frac{\lambda\tau x^q}{q}}$$

where $c_q = \frac{2}{q}\sqrt[q]{\frac{q}{2}}\,\Gamma\left(\frac{1}{q}\right)$. We remark that $e_{0,\tau}$ belongs to the space $\mathcal{H}_q^{1/2q}$ and the operator

$$e_0(x,\tilde{D}_t,\lambda)f(x,t) = \frac{\lambda}{2\pi}\int e^{i\lambda t\tau}e_{0,\tau}(x,\lambda)\hat{f}(\tau)d\tau$$

is an Hermite operator of degree 0, $e_0(x,\tilde{D}_t,\lambda) \in OP\mathcal{H}_q^0$. We recall that, by the results of [4] and [5], we have

$$|\hat{e}_{0,\tau}(\xi,\lambda)| \leq c\lambda^{-\frac{1}{2q}}\tau^{-\frac{1}{2q}}e^{-\varepsilon\lambda\tau^{-\frac{1}{q-1}}\xi^{\frac{q}{q-1}}}$$

where c and ε are suitable positive constants.

We define the operators

$$E : L^2(\mathbb{R}_\tau) \longrightarrow L^2(\mathbb{R}^2_{x,\tau}), \qquad E(f(\tau)) = e_{0,\tau}(x,\lambda)f(\tau)$$

and

$$E^* : L^2(\mathbb{R}^2_{x,\tau}) \longrightarrow L^2(\mathbb{R}_\tau), \qquad E^*(u(x,\tau)) = \int e_{0,\tau}(x,\lambda)u(x,\tau)dx.$$

In order to obtain an approximate parametrix of the operator P we begin by constructing an inverse of the matrix operator

$$\mathfrak{P} = \begin{pmatrix} (P_0+P_k)(x,\tilde{D}_x,\tau) & E \\ E^* & 0 \end{pmatrix}. \tag{2.5}$$

The operator $P_0(x,\tilde{D}_x,\tau)$ on $\langle e_{0,\tau}\rangle^\perp$, the L^2-orthogonal complement of the one dimensional spaces generated by $e_{0,\tau}$, is injective and admits an inverse. We have that the matrix

$$\begin{pmatrix} P_0(x,\tilde{D}_x,\tau) & E \\ E^* & 0 \end{pmatrix}, \tag{2.6}$$

is a bijection on $\mathscr{S}(\mathbb{R}^2_{x,\tau}) \times L^2(\mathbb{R}_\tau)$ and it has an inverse:

$$\begin{pmatrix} F_0(x,\tilde{D}_x,\tau) & E \\ E^* & 0 \end{pmatrix}, \tag{2.7}$$

where $F_0(x, \tilde{D}_x, \tau)$ is a parametrix of $P_0(x, \tilde{D}_x, \tau)$ restricted to the range of $1 - EE^*$:

$$P_0 \# F_0 = 1 - EE^*.$$

Here # denotes the Weyl composition. The operator $F_0(x, \tilde{D}_x, \tilde{D}_t)$ belongs to $OPS_q^{0,-2}$. We remark that $1 - EE^*$ is the projection on $\langle e_{0,\tau} \rangle^\perp$.

We have

$$\begin{pmatrix} F_0(x, \tilde{D}_x, \tau) & E \\ E^* & 0 \end{pmatrix} \begin{pmatrix} (P_0 + P_k)(x, \tilde{D}_x, \tau) & E \\ E^* & 0 \end{pmatrix} = \tag{2.8}$$
$$= \begin{pmatrix} 1 + F_0 P_k(x, \tilde{D}_x, \tau) & 0 \\ E^* P_k & 1 \end{pmatrix}.$$

Here $F_0 P_k(x, \tilde{D}_x, \tilde{D}_t) \in OPS_q^{0, \frac{2k}{q-1}}$; a direct computation shows that $P_k e_{0,\tau}$ is equal to $\lambda^{-1} 2(2k + q - 1)\tau x^{2k+q-2} e_{0,\tau}$, then we have that $E^* P_k$ belongs to $OP\mathcal{H}_q^{* - \frac{2k}{q} - \frac{2(q-1)}{q}}$. We recall that if $A \in OPS_q^{m,k}$ and $H \in OP\mathscr{H}_q^{m'}$ then $AH \in OP\mathscr{H}_q^{m+m'-k(q-1)/q}$. We remark that the operator $1 + F_0 P_k$ satisfies the second condition of the Proposition 6.1 of [1] so it admits a left parametrix. ([1])

The inverse of the matrix in the right hand side of (2.8) is

$$\begin{pmatrix} (1 + F_0 P_k)^{-1} & 0 \\ -E^* P_k (1 + F_0 P_k)^{-1} & 1 \end{pmatrix}.$$

Formally we have $(1 + F_0 P_k)^{-1} = 1 - \sum_{j \geq 0} (-1)^j (F_0 P_k)^{j+1}$, for every j we have $\sigma(F_0 P_k)^{j+1} \in S_q^{0, \frac{2k}{q-1} + j\frac{2k}{q-1}}$. By the Proposition 1.11, of [1], exists a symbol $r \in S_q^{0, \frac{2k}{q-1}}$ such that for all N, $r - \sum_{j \leq N} (-1)^j \sigma(F_0 P_k)^{j+1}$ belongs to $S_q^{0, \frac{2k}{q-1} + \frac{N}{q-1}}$. Let $R \sim \sum (-1)^{j+1} (F_0 P_k)^j$ modulo $OPS_q^{0,\infty}$. We can write the inverse of the matrix $\mathfrak{P}$

$$\mathfrak{Q} = \begin{pmatrix} (1 + F_0 P_k)^{-1} & 0 \\ -E^* P_k (1 + F_0 P_k)^{-1} & 1 \end{pmatrix} \begin{pmatrix} F_0 & E \\ E^* & 0 \end{pmatrix} \tag{2.9}$$
$$= \begin{pmatrix} (1 + F_0 P_k)^{-1} F_0 & (1 + F_0 P_k)^{-1} E \\ E^* - E^* P_k (1 + F_0 P_k)^{-1} F_0 & -E^* P_k (1 + F_0 P_k)^{-1} E \end{pmatrix}$$

[1] Actually if an operator $A \in OPS_q^{m,k}$ satisfies condition (ii) of the Proposition 6.1 then the operator $A + B$, with $B \in OPS_q^{m,k+k'}$ and $k' \geq 1$, satisfies the same condition.

We have that $\mathfrak{Q}\mathfrak{P} \sim Id$, this relation gives

$$\text{(2.10)} \qquad \begin{cases} ((1+F_0P_k)^{-1}F_0)(P_0+P_k)+(1+F_0P_k)^{-1}EE^*=1\,; \\ (E^*-E^*P_k(1+F_0P_k)^{-1}F_0)(P_0+P_k)-E^*P_k(1+F_0P_k)^{-1}EE^*=0. \end{cases}$$

Using the second equation in above system we want to show that $E^* = S(P_0+P_k) + OP\mathcal{H}_q^{*-\frac{2k}{q}N-\frac{1}{q}}$ for a suitable operator S. For this purpose we give an expression of $E^*P_k(1+F_0P_k)^{-1}E$ (as an operator in λ and τ.) We have

$$\text{(2.11)} \qquad P_k\,(1+F_0P_k)^{-1}\,E = P_kE + \sum_{j\geq 0}(-1)^{j+1}P_k\,(F_0P_k)^{j+1}\,E\ .$$

For any j, the symbol associated to the operator $P_k\,(F_0P_k)^{j+1}\,E$ belongs to the class $S_q^{-\frac{4k}{q}-\frac{2(q-1)}{q}-\frac{j}{q},-\frac{j}{q-1}}$. By [1, Proposition 1.11], there exists an $r_1 \in S_q^{-\frac{4k}{q}-\frac{2(q-1)}{q},0}$ such that, for all N,

$$r_1 - \sum_{j\geq 0}^{N}\sigma\left(P_k\,(F_0P_k)^{j+1}\,E\right) \in S_q^{-\frac{4k}{q}-\frac{2(q-1)}{q}-\frac{N}{q},-\frac{N}{q-1}}$$

(r_1 is defined modulo an element in $\mathscr{H}_q^{-\frac{4k}{q}-\frac{2(q-1)}{q}}$.)

We have that

$$\begin{aligned} E^*P_k(1+F_0P_k)^{-1}E &= E^*P_kE+ \\ &+\sum_{j=0}^{N-1}(-1)^{j+1}P_k\,(F_0P_k)^{j+1}\,E + \sum_{l\geq 0}(-1)^{l+N}P_k\,(F_0P_k)^{l+N}\,E\ . \end{aligned}$$

E^*P_kE is the leading symbol, $\sigma\,(E^*P_kE) \sim \tau^{-\frac{2k}{q}+\frac{2}{q}}\lambda^{-\frac{2k}{q}-\frac{2(q-1)}{q}}$.

We point out that the number $\frac{2k}{q}+\frac{2(q-1)}{q}$, appearing as the exponent of λ, is the loss of derivatives of P.

Furthermore

$$E^*\sum_{j\geq 0}^{N-1}(-1)^{j+1}P_k\,(F_0P_k)^{j+1}\,E$$

is a symbol in τ and λ of order $-\frac{4k}{q}-\frac{2(q-1)}{q}$ and

$$E^*\sum_{l\geq 0}P_k(F_0P_k)^{l+N}E$$

is a symbol in τ and λ of order $-\frac{2k}{q}(N+1)-\frac{2(q-1)}{q}$.

Then we have

$$\sigma\left(E^* P_k \left(1+F_0P_k\right)^{-1} E\right) \sim A+B$$

here

$$A=\lambda^{-\frac{2k}{q}-\frac{2(q-1)}{q}}\tau^{-\frac{2k}{q}+\frac{2}{q}}\left(1-\sum_{j=1}^{N-1} c_j\tau^{-\frac{2k}{q}j}\lambda^{-\frac{2k}{q}j}\right)$$

and

$$B=c_N\tau^{-\frac{2k}{q}(N+1)+\frac{2}{q}}\lambda^{-\frac{2k}{q}(N+1)-\frac{2(q-1)}{q}}\left(1+R(t,\tau,\lambda)\right)$$

where R is a symbol of order $-\frac{2k}{q}$. Using the second equation in (2.10) we deduce that

$$E^*=A^{-1}BE^*+A^{-1}\left(E^*-E^*P_k(1-F_0P_k)^{-1}F_0\right)(P_0+P_k)$$

or more explicitly

$$E^*=\frac{\tau^{\frac{2k}{q}-\frac{2}{q}}\lambda^{\frac{2k}{q}+\frac{2(q-1)}{q}}}{\left(1-\sum_{j=1}^{N-1}c_j\tau^{-\frac{2k}{q}j}\lambda^{-\frac{2k}{q}j}\right)}\left(E^*-E^*P_k(1-F_0P_k)^{-1}F_0\right)(P_0+P_k)$$
$$+c_N\tau^{-\frac{2k}{q}N}\lambda^{-\frac{2k}{q}N}E^* \qquad \operatorname{mod}(\mathcal{H}_q^{*-\frac{2k}{q}N-\frac{1}{q}}).$$

Hence

$$E^*=\tau^{\frac{2k}{q}-\frac{2}{q}}\lambda^{\frac{2k}{q}+\frac{2(q-1)}{q}}\sum_{s<N+1+\frac{q-1}{k}}\tilde{c}_s(\tau\lambda)^{-\frac{2k}{q}s}\left(E^*-E^*P_k(1-F_0P_k)^{-1}F_0\right)(P_0+P_k)$$
$$+c_N\tau^{-\frac{2k}{q}N}\lambda^{-\frac{2k}{q}N}E^* \qquad \operatorname{mod}(\mathcal{H}_q^{*-\frac{2k}{q}N-\frac{1}{q}}).$$

where $\tilde{c}_s$ are real constants. Using the above representation of E^* and the first equation in (2.10) we get

$$\left(\tau^{\frac{2k}{q}-\frac{2}{q}}\lambda^{\frac{2k}{q}+\frac{2(q-1)}{q}}\sum\tilde{c}_s(\tau\lambda)^{-\frac{2k}{q}s}EE^*(1+P_k(1+F_0P_k)^{-1}F_0)+\right.$$
$$\left.+(1+F_0P_k)^{-1}F_0\right)(P_0+P_k)=1-\tau^{-\frac{2k}{q}N}\lambda^{-\frac{2k}{q}N}EE^* \quad \operatorname{mod}(\mathscr{H}_q^{-\frac{2k}{q}N-\frac{1}{q}}).$$

The term

$$\tau^{-\frac{2k}{q}N}\lambda^{-\frac{2k}{q}N}EE^*\in OPS_q^{-\frac{2k}{q}N,0}\subset OPS_q^{-\frac{1}{q},-\frac{1}{q-1}}.$$

We do not give a direct construction of the parametrix but, from the condition (i) of the Proposition 6.1 ([1]), it is easy to see that the operator $\lambda^{-2}P$ admits a left parametrix. For future purposes we state here what has been done until now.

Proposition 2.1. *There exists a left approximate parametrix for the operator P in (1.1), i.e. there exists a symbol $q(x,\xi,\tau,\lambda)$ such that, for $N \in \mathbb{N}$, a large integer,*

a) $q(x,\xi,\tau,\lambda)=\sum_{j=1}^{3} q_j(x,\xi,\tau,\lambda)$ *with*

1. $q_1 = \tau^{\frac{2k}{q}-\frac{2}{q}}\lambda^{\frac{2k}{q}+\frac{2(q-1)}{q}}\sum_s \tilde{c}_s(\tau\lambda)^{-\frac{2k}{q}s}\sigma(EE^*)$ *where* $s < N+1+\frac{q-1}{k}$ *and* $\sigma(EE^*) \in \mathscr{H}_q^{0}$;
2. $q_2 = \tau^{\frac{2k}{q}-\frac{2}{q}}\lambda^{\frac{2k}{q}+\frac{2(q-1)}{q}}\sum \tilde{c}_s(\tau\lambda)^{-\frac{2k}{q}s}\sigma(EE^*P_k(1+F_0P_k)^{-1}F_0)$ *where* $\sigma(EE^*P_k(1+F_0P_k)^{-1}F_0) \in \mathscr{H}_q^{-\frac{2k}{q}}$;
3. $q_3 = \sigma((1+F_0P_k)^{-1}F_0)$ *belongs to* $S_q^{0,-2}$.

b) we have

$$(2.12)\quad (Q\circ P)\,u(x,t) = u(x,t)+ + \left(\frac{\lambda}{2\pi}\right)^2 \iint e^{i\lambda(x-x')\eta+i\lambda(t-t')\tau}\, a(x,\xi,\tau,\lambda)u(x',t')dx'dt'd\xi d\tau$$

modulo terms in $OP\mathscr{H}_q^{-\frac{2k}{q}N-\frac{1}{q}}$, *where* $a=\tau^{-\frac{2k}{q}N}\lambda^{-\frac{2k}{q}N}\sigma(EE^*)$.

3. The local a priori estimate via FBI

We recall some basic notions related to the *Fourier-Bros-Iagolnitzer* transformation, fore more details see [10]. We consider the F.B.I.-transformation with the classical phase function

$$Tu(z,\lambda) = \int_{\mathbb{R}^2} e^{-\frac{\lambda}{2}(z-y)^2}u(y)dy \tag{3.1}$$

where $\lambda \geq 1$, $z=(z_1,z_2) \in \mathbb{C}^2$ and $y=(x,t)\in\mathbb{R}^2$. Let

$$\varphi_0(z) = \sup_{y\in\mathbb{R}^2}\left(-\operatorname{Im}\left(\frac{i}{2}(z-y)^2\right)\right) = \frac{(\operatorname{Im} z_1)^2}{2} + \frac{(\operatorname{Im} z_2)^2}{2}$$

be the plurisubharmonic weight function associated to the classical phase function. We put

$$\varphi_{0,1}(z_1) = \frac{(\operatorname{Im} z_1)^2}{2} \quad \text{and} \quad \varphi_{0,2}(z_2) = \frac{(\operatorname{Im} z_2)^2}{2}\ .$$

We also recall that T is associated to a canonical transformation $\mathcal{H}_T$ from $\mathbb{C}^4$ in itself:

$$(w, i(z-w)) \longmapsto (z, i(z-w))$$

such that

$$\mathcal{H}_T(T^*\mathbb{R}^2) = \Lambda_{\varphi_0} = \{(z, -2i\partial_z\varphi_0(z))|z \in \mathbb{C}^2\},$$

$\mathcal{H}_T(x,t,\xi,\tau) = (x - i\xi, t - i\tau, \xi, \tau)$. Λ_{φ_0} is an I-Lagrangian, $\mathbb{R}$-symplectic (totally real) subspace in $\mathbb{C}^4$. If $u \in \mathscr{S}'(\mathbb{R}^2)$ then Tu is an holomorphic function of $z \in \mathbb{C}^2$ such that $|Tu(z,\lambda)| \leq \mathcal{O}_\lambda(1) \langle z\rangle^N e^{\lambda\varphi_0(z)}$ for some N depending on u where $\langle z\rangle = (1+|z|^2)^{1/2}$ and $\mathcal{O}_\lambda(1)$ denotes a uniformly bounded quantity when $\lambda \to +\infty$; moreover if u belongs to $L^2(\mathbb{R}^2)$ then $Tu \in L^2(\mathbb{C}^2; e^{-\lambda\varphi_0(z)}L(dz))$ where $L(dz)$ is the Lebesgue measure in $\mathbb{R}^4$, $L(dz) = \pm(2i)^{-2}dz \wedge d\bar{z}$.

We recall briefly the characterization of analytic and C^∞ wave front set in the F.B.I. setting (see Sjöstrand [10]): a point $(y_0, \eta_0) \in \mathbb{R}^4$ does not belong to $WF_a(u)$ if and only if there exist a positive constant ε, a neighborhood Ω of $y_0 - i\eta_0$ in $\mathbb{C}^2$ and a positive constant C_Ω, depending on Ω, such that

$$|Tu(z,\lambda)|\, e^{-\frac{\lambda}{2}\varphi_0(z)} \leq C_\Omega\, e^{-\varepsilon\lambda} \qquad \forall z \in \Omega;$$

analogously we say that $(y_0, \eta_0) \notin WF(u)$ if there exists a neighborhood Ω of $y_0 - i\eta_0$ in $\mathbb{C}^2$ such that

$$|Tu(z,\lambda)|\, e^{-\frac{\lambda}{2}\varphi_0(z)} \leq C_\Omega\, \lambda^{-N} \qquad \forall N \in \mathbb{N} \text{ and } \forall z \in \Omega.$$

A direct computation gives $T(D_x u) = D_{z_1}Tu$, $T(D_t u) = D_{z_2}Tu$ and $T(xu) = (z_1 + i\lambda^{-1}D_{z_1})Tu$; it is not more difficult to compute TP directly.

We denote by P the zero order *Kohn* operator, $\lambda^{-2}P$, after the F.B.I. and we put $\tilde{\Sigma} = \mathcal{H}_T(\Sigma) = \Sigma^{\mathbb{C}} \cap \Lambda_{\varphi_0}$, where Σ is the characteristic set of P before the F.B.I. transformation and $\Sigma^{\mathbb{C}}$ denotes the complexification of Σ.

In order to study the wave front set we need an a priori estimate on the F.B.I. side. This estimate is obtained by a technique inspired by [11]. We have an additional difficulty do to the fact that we need to work with Hermite operators on the F.B.I. side.

Let $(z_0, \zeta_0) \in \mathcal{H}_T(\Sigma)$ and W a neighborhood of (z_0, ζ_0) such that $W \cap \Lambda_{\varphi_0}$ is a suitably small neighborhood of (z_0, ζ_0) in Λ_{φ_0}. Let F be a C^ω map

$$F : W \longrightarrow \mathbb{C}^4$$

such that

1. F is close to the identity map in the C^1 norm, $\|F - I\|_{C^1} = \mathcal{O}(\varepsilon)$, where ε is a small positive parameter; we want that $F(W \cap \Lambda_{\varphi_0})$ has a injective projection onto $\mathbb{C}^2_z$. Thus it is a graph.
2. There exists a real valued non negative plush function φ such that $\varphi(z) = \varphi_1(z_1) + \varphi_2(z_2)$ and

$$F(W \cap \Lambda_{\varphi_0}) = \Lambda_\varphi \cap F(W)$$

where $\Lambda_\varphi = \{(z, -2i\partial_z\varphi(z))|z \in \mathbb{C}^2\}$.

Moreover we put $\varphi(z_0) = \varphi_0(z_0)$. Then φ is as close as we want to φ_0 if F is close to the identity (in a suitable small neighborhood of (z_0, ζ_0).)

We remark that Λ_φ is in a small tubular neighborhood of Λ_{φ_0}.

We define $L^2_\varphi(\Omega)$ as the set of all locally square integrable functions defined on Ω equipped with the norm

$$\|u\|^2_{\varphi,\Omega} = \int_\Omega |u(z)|^2 \, e^{-2\lambda\varphi(z)} \, L(dz)$$

and $L^{2,2}_{\varphi,\Omega}$ as the set of all locally square integrable functions defined on Ω equipped with the norm

$$|||u|||^2_{\varphi,\Omega} = \int_\Omega |u(z)|^2 \, (d^2(z) + \lambda^{-2\frac{q-1}{q}})^2 \, e^{-2\lambda\varphi(z)} L(dz),$$

where $d = d_{\Sigma^{\mathbb{C}}\cap\Lambda_\varphi}$ is $(d \circ \mathcal{H}_T)_{|\Lambda_\varphi}$ (d, in the last formula, is the distance function defined in the previous section.) Since F is close to the identity we have that $p_{2|\Lambda_\varphi} \in S^{0,2}_q$ and $p_{k|\Lambda_\varphi} \in S^{0,2+\frac{2k}{q-1}}_q$, i.e.

$$|p_{2|\Lambda_\varphi}| \lesssim (d^2 + \lambda^{-2(q-1)/q}) \qquad \text{and} \qquad |p_{k|\Lambda_\varphi}| \lesssim (d^2 + \lambda^{-2(q-1)/q})^{\frac{k}{q-1}+1}.$$

Let $(z^0, \zeta^0) = (0, -i, 0, 1) = \mathcal{H}_T(\rho)$, where $\rho = (0, 0, 0, 1) \in \Sigma^+$, let $(z^1, \zeta^1) = F(z^0, \zeta^0) \in \Sigma^{\mathbb{C}} \cap \Lambda_\varphi$, and let Ω, Ω_1 be open suitably small neighborhoods of $z^0 (= z^1)$ in $\mathbb{C}^2$ with $\Omega_1 \subset\subset \Omega$. The open set Ω is strictly contained in $\Pi_{\mathbb{C}^2_z}(W)$, where $\Pi_{\mathbb{C}^2_z}$ is the projection on $\mathbb{C}^2_z$ and W is the domain of F in $\mathbb{C}^4_{z,\zeta}$.

For an analytic symbol $q(z, \zeta, \lambda)$ we define the corresponding pseudodifferential operator acting on holomorphic functions as

$$Qu(z) = \left(\frac{\lambda}{2\pi}\right)^2 \iint e^{i\lambda(z-w)\zeta} \, q((z+w)/2, \zeta, \lambda) \, u(w) \, dw \wedge d\zeta \; . \tag{3.2}$$

The integral is taken along an integration contour of the form

$$\Gamma := \zeta = \frac{2}{i}\frac{\partial\varphi}{\partial z}\left(\frac{z+w}{2}\right) + iC\overline{(z-w)} \quad \text{with} \;\; |z-w| \le r \tag{3.3}$$

where φ is the phase function given above, r is a small positive constant such that $\operatorname{dist}(\Omega_1; \complement\Omega) > r$ and C is a positive suitable constant such that

$$\left|e^{i\lambda(z-w)\zeta}\right| \, e^{-\lambda\varphi(z)} \, e^{\lambda\varphi(w)} \le e^{-\delta|z-w|^2},$$

for some $\delta > 0$.

Since we need to work with Hermite operators on the F.B.I. side, we must introduce an adapted integration path for this type of operators, taking into account the anisotropic behavior of the operator P w.r.t. the x variable. We introduce the integration contour

$$\begin{cases} \zeta_1 = \dfrac{2}{i}\dfrac{\partial \varphi_1}{\partial z}\left(\dfrac{z_1+w_1}{2}\right) + iC((z_1'-w_1')^{q-1} - i(z_1''-w_1'')^{\frac{1}{q-1}}) \\ \zeta_2 = \dfrac{2}{i}\dfrac{\partial \varphi_2}{\partial z}\left(\dfrac{z_2+w_2}{2}\right) + iC\overline{(z_2-w_2)} \end{cases} \tag{3.4}$$

where $z_j = z_j' + iz_j''$, where z_j', z_j'' are real, and C is a suitably small constant. We remark, that, in the case $q=2$, the above integration path is the usual one and we only need to chose C suitably small.

Denote by $\tilde{Q}$ the approximate parametrix of P after the F.B.I. transformation

$$\tilde{Q}u(z) = \left(\frac{\lambda}{2\pi}\right)^2 \sum_{j=1}^{3} \iint_{\Gamma_j} e^{i\lambda(z-w)\zeta}\, \tilde{q}_j((z_1+w_1)/2, \zeta, \lambda)\, u(w) dw \wedge d\zeta$$

where $\tilde{q}_j \circ \mathcal{H}_T = q_j$ and the integration contour Γ_j is of the form (3.4) for $j=1,2$ and of the form (3.3) for $j=3$.

Let Ω_2 be an open neighborhood of z_0 such that

$$\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega \quad \text{and} \quad \operatorname{dist}(\Omega_2, \complement\Omega_1) > r.$$

Our purpose is to obtain an estimate of the form

$$|||u|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}} \|Pu\|_{\varphi,\Omega_1} + |||u|||_{\varphi,\Omega\setminus\Omega_2} .$$

Let v be an holomorphic function on Ω, we want to show that

$$|||\tilde{Q}v|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}} \|v\|_{\varphi,\Omega_1} \;, . \tag{3.5}$$

We recall that if K is an operator defined by an integral kernel k:

$$Ku(z) = \int_O k(z,w)\, u(w)\, L(dw)$$

where O is a suitable complex domain, we have

$$|||Ku|||^2_{\varphi,\Omega} = \| \left(d^2(\cdot) + \lambda^{-2\frac{q-1}{q}}\right) Ku\|^2_{\varphi,\Omega} \le$$
$$\le \|u\|^2_{\varphi,O} \iint_{\Omega\times O} e^{-2\lambda(\varphi(z)-\varphi(w))} \left|\left(d^2(z) + \lambda^{-2\frac{q-1}{q}}\right) k(z,w)\right|^2 L(dz)\, L(dw).$$

In order to obtain (3.5) we need to estimate kernels of the form

$$e^{-\lambda(\varphi(z)-\varphi(w))}(d^2(z)+\lambda^{-2(q-1)/q})\,k(z,w).$$

We start with $\tilde{Q}_3$ and we want to show that

$$|||\tilde{Q}_3 v|||_{\varphi,\Omega_2} \lesssim \|v\|_{\varphi,\Omega_1}\ .$$

In this situation we work with the classical integration contour. Its kernel k has the form

$$e^{i\lambda(z-w)\zeta}\ \tilde{q}_3\left(\frac{z_1+w_1}{2},\zeta,\lambda\right)\frac{\partial\zeta}{\partial\overline{w}}\ .$$

We have that $|\partial_{\overline{w}}\zeta| \leq$ const. By the Taylor expansion we can replace $\tilde{q}_3$ in the above formula with

$$\tilde{q}_3\Big(z_1,\frac{2}{i}\frac{\partial\varphi}{\partial z}(z),\lambda\Big)+\frac{\partial\tilde{q}_3}{\partial\zeta_1}\Big(z_1,\frac{2}{i}\frac{\partial\varphi}{\partial z},\lambda\Big)\Big(\zeta_1-\frac{2}{i}\frac{\partial\varphi_1}{\partial z_1}(z)\Big)+$$
$$+\frac{\partial\tilde{q}_3}{\partial\zeta_2}\Big(z_1,\frac{2}{i}\frac{\partial\varphi}{\partial z}(z),\lambda\Big)\Big(\zeta_2-\frac{2}{i}\frac{\partial\varphi_2}{\partial z_2}(z)\Big)+\frac{\partial\tilde{q}_3}{\partial z_1}\Big(z_1,\frac{2}{i}\frac{\partial\varphi}{\partial z}(z),\lambda\Big)\Big(\frac{z_1-w_1}{2}\Big)+$$
$$+\mathcal{R}_{1,\lambda}\Big(|z-w|^2\Big)+\mathcal{R}_{2,\lambda}\Big(\Big|\zeta-\frac{2}{i}\frac{\partial\varphi}{\partial z}(z)\Big|^2\Big),$$

where $R_{1,\lambda}$, $R_{2,\lambda}$ denote holomorphic symbols which are uniformly $\mathcal{O}(\lambda^{\frac{2}{q}-1})$. Since F is close to identity and the Lipschitz norm of $F-I$ is bounded by $\mathcal{O}(\varepsilon)$ we have $\nabla\varphi(z+h)-\nabla\varphi(z)=[\nabla^2\varphi_0+\mathcal{O}(\varepsilon)](z-h)$; hence $\Big|\zeta_j+2i\partial_{z_j}\varphi_j(z)\Big|\lesssim|z_j-w_j|$ for $j=1,2$. We have that $\tilde{q}_{3|_{\Lambda_\varphi}}$ is in the class $S_q^{0,-2}$ with respect to the distance function defined above; so we conclude that $|\tilde{q}_{3|_{\Lambda_\varphi}}|\lesssim(d^2+\lambda^{-2(q-1)/q})^{-1}$. In view of the canonical transformation associated to the F.B.I. transformation we have that

$$|\partial_{\zeta_1}\tilde{q}_{3|_{\Lambda_\varphi}}|\lesssim\lambda^{\frac{1}{q}}(d^2+\lambda^{-2(q-1)/q})^{-1}.$$

On the other hand we have

$$|\partial_{z_1}\tilde{q}_{3|_{\Lambda_\varphi}}|\lesssim\lambda^{\frac{1}{q}}(d^2+\lambda^{-2(q-1)/q})^{-1}\quad\text{and}\quad|\partial_{\zeta_2}\tilde{q}_{3|_{\Lambda_\varphi}}|\lesssim\lambda^{-1}(d^2+\lambda^{-2(q-1)/q})^{-1}.$$

The remainder terms give rise to an operator that is uniformly $\mathcal{O}(\lambda^{\frac{2}{q}-1})$ acting from $L^{2,2}_{\varphi,\Omega_2}$ to L^2_{φ,Ω_1}. We can estimate $|||\tilde{Q}_3 v|||_{\varphi,\Omega_2}$ with

$$\|v\|_{\varphi,\Omega_1}\left(\frac{\lambda}{2\pi}\right)^2\iint e^{-\delta\lambda|z-w|^2}(1+\frac{1}{\lambda}|z_2-w_2|+\lambda^{\frac{1}{q}}|z_1-w_1|)^2L(dz')L(dz)$$
$$+\mathcal{O}(\lambda^{\frac{2}{q}-1})\,\|v\|_{\varphi,\Omega_1}\ .$$

Since q is an even integer and in this way $\frac{1}{q}-\frac{1}{2}$ is smaller or equal to zero.

Next we want to estimate $|||\tilde{Q}_1 v|||_{\varphi,\Omega_2}$ where $\tilde{Q}_1$ is realized on the integration path (3.4). Its kernel is

$$e^{i\lambda(z-w)\zeta} e^{-\frac{\lambda\zeta_2}{q}(z_1+i\zeta_1)^q - i\lambda(z_1+i\zeta_1)\zeta_1} \sum_s \tilde{c}_s(\zeta_2\lambda)^{-\frac{2k}{q}s} \times \\ \times \int_{\mathbb{R}} e^{-i\lambda y\zeta_1} e^{-\frac{\lambda\zeta_2}{q}y^q} dy \, \frac{\partial \zeta_1}{\partial \overline{w}_1} \frac{\partial \zeta_2}{\partial \overline{w}_2}$$

where $s < N + 1 + \frac{q-1}{k}$. We have that

$$\left| \frac{\partial \zeta_1}{\partial \overline{w}_1} \right| \leq C_1 \left(1 + (z_1' - w_1')^{q-2} + (z_1'' - w_1'')^{\frac{1}{q-1}-1} \right)$$

and $|\partial_{\overline{w}_2}\zeta_2| \leq$ cost. On Γ, using the same technique of [4] and [5], we have

$$\left| e^{i\lambda(z-w)\zeta - \lambda(\varphi(z)-\varphi(w))} e^{-\frac{\lambda\zeta_2}{q}(z_1+i\zeta_1)^q - i\lambda(z_1+i\zeta_1)\zeta_1} \int_{\mathbb{R}} e^{-i\lambda y\zeta_1} e^{-\frac{\lambda\zeta_2}{q}y^q} dy \right| \lesssim \\ \lesssim e^{-\lambda\delta_2|z_2-w_2|^2} e^{-\lambda\varepsilon_1\left|z_1'-w_1'\right|^q - \lambda\varepsilon_2\left|z_1''-w_1''\right|^{\frac{q}{q-1}}}$$

where ε_1 and ε_2 are suitable constants depending on the constant C in (3.4) and q. We can conclude that

$$|||\tilde{Q}_1 v|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q} + \frac{3(q-1)}{q} - \frac{1}{q}} \|v\|_{\varphi,\Omega_1}.$$

It only remains to estimate $\tilde{Q}_2$. We realize it with the integration path (3.4). Its kernel is

$$e^{i\lambda(z-w)\zeta} e^{-\frac{\lambda\zeta_2}{q}(z_1+i\zeta_1)^q - i\lambda(z_1+i\zeta_1)\zeta_1} \sum_s \tilde{c}_s(\zeta_2\lambda)^{-\frac{2k}{q}s} \times \\ \times \int_{\mathbb{R}} e^{-i\lambda y\zeta_1} p(y,\zeta_2) e^{-\frac{\lambda\zeta_2}{q}y^q} dy \, \frac{\partial \zeta_1}{\partial \overline{w}_1} \frac{\partial \zeta_2}{\partial \overline{w}_2}.$$

where $p(y,\zeta_2)$ is a polynomial of order $2k$ w.r.t. the variable y. Actually $p_2(y,\zeta_2) \sim y^{2k}\tilde{p}_2(y,\zeta_2)$. On the integral contour Γ, using the same technique of [4] and [5], we have

$$\left| e^{i\lambda(z-w)\zeta - \lambda(\varphi(z)-\varphi(w))} e^{-\frac{\lambda\zeta_2}{q}(z_1+i\zeta_1)^q - i\lambda(z_1+i\zeta_1)\zeta_1} \int_{\mathbb{R}} e^{-i\lambda y\zeta_1} p(y,\zeta_2) e^{-\frac{\lambda\zeta_2}{q}y^q} dy \right| \lesssim \\ \lesssim \lambda^{-\frac{2k}{q}} e^{-\lambda\delta_2|z_2-w_2|^2} e^{-\lambda\varepsilon_1\left|z_1'-w_1'\right|^q - \lambda\varepsilon_2\left|z_1''-w_1''\right|^{\frac{q}{q-1}}}.$$

We can conclude that

$$|||\tilde{Q}_2 v|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{3(q-1)}{q} - \frac{1}{q}} \|v\|_{\varphi,\Omega_1}.$$

Then (3.5) follows.

We replace v by Pu in (3.5) and we obtain

$$\lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}}\|Pu\|_{\varphi,\Omega_1} \gtrsim |||\tilde{Q}Pu|||_{\varphi,\Omega_2} \gtrsim |||u|||_{\varphi,\Omega_2} - |||\tilde{Q}Pu-u|||_{\varphi,\Omega_2}.$$

Then

$$|||u|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}}\|Pu\|_{\varphi,\Omega_1} + |||\tilde{Q}Pu-u|||_{\varphi,\Omega_2}; \tag{3.6}$$

Since the Weyl composition and the linear canonical transformation commute, using (2.12), we have

$$\tilde{Q}\#P = 1+\tilde{A} \qquad \text{mod } (\lambda^{-\frac{2k}{q}N-\frac{1}{q}})\ .$$

The second term in the r. h. s. of (3.6) can be estimated as follows

$$|||\tilde{Q}Pu-u|||_{\varphi,\Omega_2} \lesssim |||Lu|||_{\varphi,\Omega_2} + |||\tilde{A}u|||_{\varphi,\Omega_2}, \tag{3.7}$$

where, for a holomorphic function v defined in Ω, we set

$$Lv(z) = v(z) - \left(\frac{\lambda}{2\pi}\right)^2 \iint e^{i\lambda(z-w)\zeta} v(w)dwd\zeta,$$

the integral is performed along the contour $\zeta = -2i\partial_z\varphi(z) - iC\overline{(z-w)}$. Let us estimate now the first term in the r. h. s. of (3.7). We recall that

$$|||Lu|||_{\varphi,\Omega_2} \lesssim e^{-\lambda/C}\|u\|_{\varphi,\Omega}\,.$$

(See Sjöstrand [11] and [10].)

Using the Stokes theorem and the deformation argument

$$-2i\partial_z\varphi(zt+(1-t)z') - iC\overline{(z-w)},$$

with $t \in \left[\frac{1}{2}, 1\right]$, we can replace the above integration path

$$(-2i\partial_z\varphi(z) - iC\overline{(z-w)})$$

with the contour

$$\zeta = -2i\partial_z\varphi(\frac{z+z'}{2}) - iC\overline{(z-w)}.$$

The operator $\tilde{A}$ is realized on the integration path (3.4) and its symbol is $\lambda^{-2kN/q}\zeta_2\sigma(EE^*)(z+i\zeta,\zeta)$. The same technique used to estimate $\tilde{Q}_1$ allows us to estimate the second term in (3.7):

$$|||\tilde{A}u|||_{\varphi,\Omega_2} \lesssim \lambda^{-\frac{2k}{q}N+\frac{3(q-1)}{q}-\frac{1}{q}}|||u|||_{\varphi,\Omega}\ .$$

We choose N such that

$$2kN + 4 > 3q$$

e.g. $2N = 3q$. Then (3.6) can be rewritten as

$$|||u|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}} \|Pu\|_{\varphi,\Omega_1} + \left(\frac{1}{\lambda^\alpha} + e^{-\lambda/C'}\right) |||u|||_{\varphi,\Omega};$$

where $\alpha = -\frac{2k}{q}N + \frac{3(q-1)}{q} - \frac{1}{q}$.

Let λ_0 be such that $\lambda_0^{-\alpha} + e^{-\lambda_0/C'} < 1$. Then for every $\lambda \geq \lambda_0$ we have the

Proposition 3.1. *Let P be as in* (1.1)*. Then on the F.B.I. side we have the a priori estimate*

$$|||u|||_{\varphi,\Omega_2} \lesssim \lambda^{\frac{2k}{q}+\frac{3(q-1)}{q}-\frac{1}{q}} \|Pu\|_{\varphi,\Omega_1} + |||u|||_{\varphi,\Omega\setminus\Omega_2} \quad . \tag{3.8}$$

where φ has been defined at the beginning of Section 2.

4. The construction of the phase function

In the next section we use a deformation argument of Holmgren type due to J. Sjöstrand [11] and [10].

First we construct a weight function φ by solving a Hamilton-Jacobi equation for small times.

Let $r\colon W \to \mathbb{C}$ be a C^∞ function (W is a neighborhood of $(z_0; \zeta_0)$ whose space projection contains Ω.) Consider

$$\begin{cases} \dfrac{\partial\varphi}{\partial s}(s,x) & = (\operatorname{Re} r)\left(x, \dfrac{2}{i}\dfrac{\partial\varphi}{\partial x}(s,x)\right) \\ \varphi(0,x) & = \varphi_0(x), \end{cases} \tag{4.1}$$

for $0 \leq s \leq \varepsilon_0$.

The solution of the above problem is constructed using the standard Hamilton-Jacobi theory with respect to the symplectic form

$$\operatorname{Im}\sigma = \operatorname{Im}(d\xi \wedge dx).$$

Actually, setting $\varphi_s(x) = \varphi(s,x)$, we have

$$\Lambda_{\varphi_s} = \exp\left(sH^{\operatorname{Im}\sigma}_{\operatorname{Re} r}\right)\Lambda_{\varphi_0}.$$

The map $\exp\left(sH^{\operatorname{Im}\sigma}_{\operatorname{Re} r}\right)$ is the function F of the previous section.

We recall that, if r is a holomorphic function on W then we have

$$H_{\operatorname{Re} r}^{\operatorname{Im}\sigma} = \widehat{H_{ir}},$$

where H_{ir} is the usual complex standard Hamilton field of ir and $\widehat{H_{ir}}$ denotes the real part of H_{ir}, i.e. the real field that gives the same result as H_{ir} when acting on holomorphic functions.

If r is holomorphic in W and real valued on Λ_{φ_0} the solution of the above Hamilton-Jacobi problem is obtained as the restriction to the positive s-axis of the solution of the complex equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial s}(s,x) & = r\Big(x, \dfrac{2}{i}\dfrac{\partial \varphi}{\partial x}(s,x)\Big) \\ \varphi(0,x) & = \varphi_0(x), \end{cases}$$

for $|s| < \varepsilon_0$.

Since $\mathbb{R}^4$ and Λ_{φ_0} are isomorphic it is easier to construct the function r in $\mathbb{R}^4$ near the characteristic point $(0,0,0,1) \in \operatorname{Char} P$.

Let us choose

$$r(x,t,\xi,\tau) = t^2 + (\tau-1)^2 + C\Big(\frac{x^q}{q} + \xi^2\Big),$$

where C is a positive constant that makes r as positive as we desire outside Σ. We point out that

$$H_r(\rho) \in T\Sigma, \qquad \text{for } \rho \in \Sigma.$$

Then, on Λ_{φ_0},

$$r\left(z, \frac{2}{i}\frac{\partial \varphi_0}{\partial z}(z)\right) \sim |z_2 - z_{2,0}|^2 + \|z_1\|^2,$$

for every $z \in \Pi_{\mathbb{C}^2_z}(W \cap \Lambda_{\varphi_0})$. Here $z_{2,0}$ is equal to $-i$ and the norm $\|\cdot\|$ is defined by

$$\|z_1\|^2 = z_1'^{\,q} + z_1''^{\,2} \quad .$$

We remark that

$$\varphi_s(z) = \varphi_0(z) + r\left(z, \frac{2}{i}\frac{\partial \varphi}{\partial z}(0,z)\right) s + \mathscr{O}(s^2).$$

5. Proof of Theorem 1.1

We start with the assertion on the analytic wave front set. Denote by (x_0, ξ_0) the point $(0, 0, 0, 1) \in \operatorname{Char} P$.

Our purpose is to show that if $(x_0, \xi_0) \notin WF_a(Pu)$ then $(x_0, \xi_0) \notin WF_a(u)$. We recall estimate (3.8):

$$\text{(5.1)} \qquad |||u|||_{\varphi,\Omega_2} \leq C \left(\lambda^{\frac{2k}{q} + \frac{3(q-1)}{q} - \frac{1}{q}} \|Pu\|_{\varphi,\Omega_1} + |||u|||_{\varphi,\Omega\setminus\Omega_2} \right)$$

where $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega \subset W$ and $z_0 \in \Omega_2$, $z_0 = x_0 - i\xi_0$.

Since u is a tempered distribution before the FBI transform, we have that

$$\|u\|_{\varphi_0,\Omega} \leq C\lambda^{N_0},$$

for a certain $N_0 \in \mathbb{N}$.

Since Pu is real analytic at the real point (x_0, ξ_0) before the FBI transform, we have that

$$\|Pu\|_{\varphi_0,\Omega_3} \leq C_1 e^{-\lambda/C_1},$$

for a positive constant C_1; here Ω_3 is a suitable neighborhood of z_0. Recalling that

$$\varphi_s(z) - \varphi_0(z) \sim s(|z_2 - z_{2,0}|^2 + \|z_1\|^2),$$

we obtain

$$\|Pu\|_{\varphi_s,\Omega_1} \leq \tilde{C} e^{-\lambda/\tilde{C}},$$

for a positive constant $\tilde{C}$.

Since, on $\Omega \setminus \Omega_2$, $r \geq \alpha > 0$, we have

$$\varphi_{s_{|\Omega\setminus\Omega_2}} \geq \varphi_0 + \alpha_1 s, \qquad \alpha_1 > 0,$$

so then

$$|||u|||_{\varphi_t,\Omega\setminus\Omega_2} \leq C e^{-\lambda/C'_s} \|u\|_{\varphi_0,\Omega_1} \leq C_s e^{-\lambda/C_s} \; .$$

Hence the *a priori* estimate (5.1) implies that

$$|||u|||_{\varphi_s,\Omega_2} \leq C_2 e^{-\lambda/C_2}.$$

Let now Ω_4 be a sufficiently small neighborhood of z_0 such that

$$\varphi_s(z) < \varphi_0(z) + \frac{1}{C_4}$$

on Ω_4. Then

$$\|u\|_{\varphi_0,\Omega_4} \leq C_3 e^{-\lambda/C_3},$$

which means that u is real analytic at (x_0, ξ_0) before the FBI transform. This proves the assertion.

We point out that the regularity at the characteristic points in Σ^- is much easier and well known since the operator P loses microlocally only one derivative (i.e. it is maximally hypoelliptic.)

Finally we remark that estimate (3.8) implies C^∞ hypoellipticity as well. In fact the only action we must take is to replace the exponential decay of the (F.B.I. transform of) u with a rapidly decrease decay, i.e. $Pu = \mathcal{O}(\lambda^{-\infty})$ uniformly in a neighborhood of $z_0 \in \Omega_2$. Using the same "canonical deformation" argument yield that the error term, i.e. the second term on the right hand side of (3.8) is exponentially decreasing. This ends the proof of Theorem 1.1.

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