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High-dimensional Menger-type curvatures. Part I: Geometric multipoles and multiscale inequalities

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Abstract

We define discrete and continuous Menger-type curvatures. The discrete curvature scales the volume of a $(d+1)$-simplex in a real separable Hilbert space H, whereas the continuous curvature integrates the square of the discrete one according to products of a given measure (or its restriction to balls). The essence of this paper is to establish an upper bound on the continuous Menger-type curvature of an Ahlfors regular measure μ on H in terms of the Jones-type flatness of μ (which adds up scaled errors of approximations of μ by d-planes at different scales and locations). As a consequence of this result we obtain that uniformly rectifiable measures satisfy a Carleson-type estimate in terms of the Menger-type curvature. Our strategy combines discrete and integral multiscale inequalities for the polar sine with the "geometric multipoles" construction, which is a multiway analog of the well-known method of fast multipoles.

1. Introduction

We introduce Menger-type curvatures of measures and show that they satisfy a Carleson-type estimate when the underlying measures are uniformly rectifiable in the sense of David and Semmes [2, 3]. The main development of the paper (implying this Carleson-type estimate) is a careful bound on the Menger-type curvature of an Ahlfors regular measure in terms of the sizes of least squares approximations of that measure at different scales and locations.

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Our setting includes a real separable Hilbert space H with dimension denoted by $\dim(H)$ (possibly infinite), an intrinsic dimension $d \in \mathbb{N}$, where $d < \dim(H)$, and a d-regular (equivalently, d-dimensional Ahlfors regular) measure μ on H. That is, μ is a locally finite Borel measure on H and there exists a constant $C \geq 1$ such that for all $x \in \operatorname{supp}(\mu)$ and $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$:

$$C^{-1} \cdot r^d \leq \mu(B(x,r)) \leq C \cdot r^d. \tag{1.1}$$

We denote the smallest constant C satisfying equation (1.1) by C_μ, and refer to it as the regularity constant of μ. The estimates developed in this paper only depend on the intrinsic dimension d and the regularity constant C_μ, and no other parameter of either μ or H. In particular, they are independent of the dimension of H.

Our d-dimensional discrete curvature is defined on vectors $v_1, \ldots, v_{d+2} \in H$. We denote the diameter of the set $\{v_1, \ldots, v_{d+2}\}$ by $\operatorname{diam}(v_1, \ldots, v_{d+1})$, and the $(d+1)$-dimensional volume of the parallelotope spanned by $v_2 - v_1, \ldots, v_{d+2} - v_1$ by $\operatorname{Vol}_{d+1}(v_1, \ldots, v_{d+2})$. Equivalently, $\operatorname{Vol}_{d+1}(v_1, \ldots, v_{d+2})$ is $(d+1)!$ times the volume of the simplex (i.e., convex polytope) with vertices at $v_1, \ldots, v_{d+2}$. The square of our d-dimensional curvature $c_d(v_1, \ldots, v_{d+2})$ has the form

$$c_d^2(v_1, \ldots, v_{d+2}) = \frac{1}{d+2} \cdot \frac{\operatorname{Vol}_{d+1}^2(v_1, \ldots, v_{d+2})}{\operatorname{diam}(v_1, \ldots, v_{d+2})^{d \cdot (d+1)}} \sum_{i=1}^{d+2} \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^{d+2} \|v_j - v_i\|_2^2}.$$

The one-dimensional curvature $c_1(v_1, v_2, v_3)$ is comparable to the Menger curvature [13, 11], $c_M(v_1, v_2, v_3)$, which is the inverse of the radius of the circle through the points $v_1, v_2, v_3 \in H$. Indeed, we note that

$$c_M^2(v_1, v_2, v_3) = \frac{4 \sin^2(v_2 - v_1, v_3 - v_1)}{\|v_2 - v_3\|^2},$$

$$c_1^2(v_1, v_2, v_3) = \frac{\sin^2(v_2 - v_1, v_3 - v_1) + \sin^2(v_1 - v_2, v_3 - v_2) + \sin^2(v_1 - v_3, v_2 - v_3)}{3 \cdot \operatorname{diam}^2(v_1, v_2, v_3)}$$

and consequently

$$\frac{1}{12} \cdot c_M^2(v_1, v_2, v_3) \leq c_1^2(v_1, v_2, v_3) \leq \frac{1}{4} \cdot c_M^2(v_1, v_2, v_3).$$

We thus view the Menger-type curvature c_d as a higher-dimensional generalization of the Menger curvature c_M. Clearly, one can directly generalize the Menger curvature to the following function of $v_1, \ldots, v_{d+2}$:

$$\frac{\operatorname{Vol}_{d+1}(v_1, \ldots, v_{d+2})}{\prod_{\substack{i,j=1 \\ i \neq j}}^{d+2} \|v_i - v_j\|}. \tag{1.2}$$

However, the methods developed here do not apply to that curvature (see Remark 8.2).

Essentially, this paper shows how the multivariate integrals of the discrete d-dimensional Menger-type curvature can be controlled from above by d-dimensional least squares approximations of μ, which are used to characterize uniform rectifiability [2, 3].

We first exemplify this in the simplest setting of approximating μ by a fixed d-dimensional plane at a given scale and location, indicated by the ball $B = B(x,t)$, for $x \in \text{supp}(\mu)$ and $0 < t \leq \text{diam}(\text{supp}(\mu))$. We denote the scaled least squares error of approximating μ at $B = B(x,t)$ by a d-plane (i.e., d-dimensional affine subspace) by

$$\beta_2^2(x,t) = \beta_2^2(B) = \min_{d\text{−planes } L} \int_B \left(\frac{\text{dist}(x,L)}{\text{diam}(B)} \right)^2 \frac{\mathrm{d}\mu(x)}{\mu(B)} .$$

Fixing $\lambda > 0$ and sampling sufficiently well separated simplices in B^{d+2}, i.e., simplices in the set

$$U_\lambda(B) = \left\{ (v_1, \ldots, v_{d+2}) \in B^{d+2} : \min_{1 \leq i < j \leq d+2} \|v_i - v_j\| \geq \lambda \cdot t \right\}, \tag{1.3}$$

we bound $\beta_2^2(x,t)$ from below by averages of the squared curvature c_d^2 in the following way:

Proposition 1.1. *There exists a constant $C_0 = C_0(d, C_\mu) \geq 1$ such that*

$$\int_{U_\lambda(B(x,t))} c_d^2(X) \, \mathrm{d}\mu^{d+2}(X) \leq \frac{C_0}{\lambda^{d(d+1)+4}} \cdot \beta_2^2(x,t) \cdot \mu(B(x,t)), \tag{1.4}$$

for all $\lambda > 0$, $x \in \text{supp}(\mu)$, and $t \in \mathbb{R}$ with $0 < t \leq \text{diam}(\text{supp}(\mu))$.

An opposite inequality is established in [8, Theorem 1.1]. An extension of Proposition 1.1 to more general measures and to arbitrary simplices (while slightly modifying the curvature) appears in [9].

We next extend the above estimate to multiscale least squares approximations. For this purpose we first define the Jones-type flatness [7, 2, 3] of the measure μ when restricted to a ball $B \subseteq H$ as follows

$$J_d(\mu|_B) = \int_0^{\text{diam}(B)} \int_B \beta_2^2(x,t) \, \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t} . \tag{1.5}$$

This quantity measures total flatness or oscillation of μ around B by combining the errors of approximating it with d-planes at different scales and locations. The actual weighting of the β_2 numbers is designed to capture the uniform rectifiability of μ (see Section 4). We also define the continuous Menger-type curvature of μ when restricted to B, $c_d(\mu|_B)$, as follows

$$c_d(\mu|_B) = \sqrt{\int_{B^{d+2}} c_d^2(v_1, \ldots, v_{d+2}) \, \mathrm{d}\mu(v_1) \ldots \mathrm{d}\mu(v_{d+2})} .$$

The primary result of this paper bounds the local Jones-type flatness from below by the local Menger-type curvature in the following way.

Theorem 1.1. *There exists a constant* $C_1 = C_1(d, C_\mu)$ *such that*

$$c_d^2\,(\mu|_B) \leq C_1 \cdot J_d\,(\mu|_{6 \cdot B}) \quad \textit{for all balls } B \subseteq H.$$

This theorem is relevant for the theory of uniform rectifiability [2, 3] (briefly reviewed in Section 4) and it in fact implies the following result.

Theorem 1.2. *If H is a real separable Hilbert space and μ is a d-dimensional uniformly rectifiable measure then there exists a constant* $C_2 = C_2(d, C_\mu)$ *such that*

$$c_d^2(\mu|_B) \leq C_2 \cdot \mu(B) \quad \textit{for all balls } B \subseteq H. \tag{1.6}$$

In [8] we establish the opposite direction of Theorem 1.1, and thus also conclude the opposite direction of Theorem 1.2, that is, the Carleson-type estimate of Theorem 1.1 implies the uniform rectifiability of μ. As such, we obtain a characterization of uniformly rectifiable measures by the Carleson-type estimate of equation (1.6) (extending a one-dimensional result of [11]).

When $d = 1$ a similar version of Theorem 1.1 was formulated and proved in [14, Theorem 31] following an unpublished work of Peter Jones [6]. The difference is that [14, Theorem 31] uses the larger β_∞ numbers (i.e., the analogs of the β_2 numbers when using the L_∞ norm instead of L_2) and restricts the support of μ to be contained in a rectifiable curve. The latter restriction requires only linear growth of μ, i.e., one can consider Borel measures satisfying only the RHS of equation (1.1).

The proof of Theorem 1.1 when $d > 1$ requires more substantial development than that of [6] and [14, Theorem 31]. This is for a few reasons, a basic one being the greater complexity of higher-dimensional simplices vis á vis the simplicity of triangles. The result is that many more things can go wrong while trying to control the curvature c_d for $d > 1$, and we are forced to invent strategies for obtaining the proper control.

A more subtle reason involves the difference between the L_∞ and L_2 quantities and the way that these interact with some pointwise inequalities. In the case for $d = 1$ and the β_∞ numbers, much of the proof (as recorded in [14]) is driven by the "triangle inequality" for the ordinary sine function, i.e., the subadditivity of the absolute value of the sine function. The "robustness" of this inequality combined with the simplicity of a basic pointwise inequality between the sine function and the β_∞ numbers results in a relatively simple integration procedure. For $d > 1$ and the β_2 numbers this whole framework breaks down. One such breakdown is that the "correct" analog of the triangle inequality holds much more sparsely (see Proposition 3.3). Other reasons will become apparent in the rest of the work.

In principal, there are two kinds of methods in the current work. We refer to the first as geometric multipoles. It decomposes the underlying multivariate integral over a set of well-scaled simplices (i.e., with comparable edge lengths) according to multiscale regions in H, emphasizing in each region approximations of the support of the measure by d-planes. We view it as a d-way analog of the zero-dimensional fast multipoles method [4], which decomposes an integral according to dyadic grids of $\mathbb{R}^n$ and emphasizes near-field interactions. This method, which is rather implicit in [14, Theorem 31], can be used to decompose integrals of many other multivariate functions.

Our second method relies on both discrete and integral multiscale inequalities for the Menger-type curvature (or more precisely the polar sine function defined in Section 3). They allow us to bound $c_d^2(\mu|_B)$ by multivariate integrals restricted to sets of well-scaled simplices, so that geometric multipoles can then be applied. When $d = 1$ both the multiscale inequality and its application are rather trivial (see [14, Lemma 36] and the way it is used in [14, Theorem 31]).

1.1. Organization of the paper

In Section 2 we describe the basic context, notation, definitions and related elementary propositions. In Section 3 we review some geometric properties of simplices as well as of the d-dimensional polar sine, and we decompose simplices and correspondingly the Menger-type curvature according to scales and configurations. Section 4 reviews aspects of the theory of uniform rectifiability relevant to this work. In Section 5 we establish Proposition 1.1, whereas in Section 6 we reduce Theorem 1.1 to three propositions, which we subsequently prove in Sections 7-9. Section 10 contains a brief discussion concluding this work.

2. Basic notation and definitions

We denote the support of the d-regular measure μ on H by $\mathrm{supp}(\mu)$. The inner product and induced norm on H are denoted by $< \cdot, \cdot >$ and $\| \cdot \|$. For $m \in \mathbb{N}$, we denote the Cartesian product of m copies of H by H^m and the corresponding product measure by μ^m.

We summarize some notational conventions as follows. We typically denote scalars larger than 1 by upper-case plain letters, e.g., C; arbitrary integers by lowercase letters, e.g., i, j, and large integers by M and N; real numbers by lower-case Greek or script letters, e.g., α_0, r; subsets of H^m, where $m \in \mathbb{N}$, by upper-case plain letters, e.g., A; families of subsets (e.g., collections of balls) by calligraphic letters, e.g., $\mathcal{B}$; subsets of $\mathbb{N}$ (used for indexing) by capital Greek letters, e.g., Λ; and measures on H by Greek lower-case letters, e.g., μ.

We reserve x, y and z to denote elements of H; X, Y and Z to denote elements of H^m for $m \geq 3$; L for a complete affine subspace of H (possibly a linear subspace); V to denote a complete linear subspace of H. If $x \in \mathbb{R}$, then we denote the corresponding ceiling and floor functions by $\lceil x \rceil$ and $\lfloor x \rfloor$.

If $A \subseteq H$, we denote its diameter by $\operatorname{diam}(A)$, its complement by A^c and the restriction of μ to it by $\mu|_A$.

We denote the closed ball centered at $x \in H$ of radius r by $B(x,r)$, and if both the center and radius are indeterminate, we use the notation B or Q. For a ball $B(x,r)$ and $\gamma > 0$, we denote the corresponding blow up by $\gamma \cdot B(x,r)$, i.e., $\gamma \cdot B(x,r) = B(x, \gamma \cdot r)$. If $\mathcal{B}$ is a family of balls, then we denote the corresponding blow up by $\gamma \cdot \mathcal{B} = \{\gamma \cdot B : B \in \mathcal{B}\}$.

If L is a complete affine subspace of H and $x \in H$, we denote the distance between x and L by $\operatorname{dist}(x,L)$, that is, $\operatorname{dist}(x,L) = \min_{y \in L} \|x - y\|$. If $n \leq \dim(H)$, we at times use the phrase n-plane to refer to an n-dimensional affine subspace of H.

If $\mathcal{B}$ is a family of balls, then we denote the union of its elements by $\bigcup \mathcal{B}$, and we distinguish the latter notation from $\bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$, which is a family of balls formed by the countable union of other families.

We fix the constant

$$C_{\mathrm{p}} = C_{\mathrm{p}}(d, C_\mu) = \begin{cases} \dfrac{\sqrt{5} \cdot \pi^2}{4 \cdot \arcsin\left(2^{-(5/2 \cdot d + 1)} \cdot C_\mu^{-2}\right)}, & \text{if } d > 1; \\ 1, & \text{if } d = 1. \end{cases} \tag{2.1}$$

We also fix the following constant α_0 and use its powers to provide appropriate scales:

$$\alpha_0 = \alpha_0(d, C_\mu) = \min\left\{\frac{1}{2 \cdot C_{\mathrm{p}}^2}, \left(\frac{1}{4 \cdot C_\mu^2}\right)^{1/d}\right\} = \begin{cases} \dfrac{1}{4 \cdot C_\mu^2}, & \text{if } d = 1; \\ \dfrac{1}{2 \cdot C_{\mathrm{p}}^2}, & \text{if } d > 1. \end{cases} \tag{2.2}$$

Finally, for a fixed d-plane L and a ball $B = B(x,t)$, we let

$$\beta_2^2(B, L) = \beta_2(x, t, L) = \int_B \left(\frac{\operatorname{dist}(x, L)}{\operatorname{diam}(B)}\right)^2 \frac{\mathrm{d}\mu(x)}{\mu(B)},$$

where $\beta_2^2(B, L) = 0$ if $\mu(B) = 0$. We note that

$$\beta_2^2(B) = \inf_L \beta_2^2(B, L).$$

2.1. Notation corresponding to elements of H^{n+1}

Throughout this paper, we frequently refer to n-simplices in H for $n \geq 2$ where usually $n = d + 1$. Rather than formulate our work with respect to subsets of H, we work with ordered $(n + 1)$-tuples of the product space, H^{n+1}, representing the set of vertices of the corresponding simplex in H. Fixing $n \geq 2$, we denote an element of H^{n+1} by $X = (x_0, \ldots, x_n)$, and for $0 \leq i \leq n$, we let

$$(X)_i = x_i$$

denote the projection of X onto its i^{th} H-valued *coordinate*. We make a clear distinction between the symbol X_i, denoting an indexed element of the product space H^{n+1}, and the coordinate $(X)_i$ which is a point in H. The zeroth coordinate $(X)_0 = x_0$ is special in many of our calculations. With some abuse of notation we refer to X both as an ordered set of $d+2$ vertices and as a $(d+1)$-simplex.

For $0 \leq i \leq n$ and $X = (x_0, \ldots, x_n) \in H^{n+1}$, let $X(i)$ be the following element of H^n:

$$X(i) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

That is, $X(i)$ is the projection of X onto H^n that eliminates its i^{th} coordinate.

If $X \in H^{n+1}$, $y \in H$ and $1 \leq i \leq n$, we form $X(y, i) \in H^{n+1}$ as follows:

$$X(y, i) = (x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$$

That is, $X(y, i)$ is obtained from X by replacing its i^{th} coordinate $(X)_i$ with y.

We say that X is *non-degenerate* if the set $\{x_1 - x_0, \ldots, x_n - x_0\}$ is linearly independent. For $X \in H^{n+1}$ as above, let $L[X]$ denote the affine subspace of H of minimal dimension containing set of vertices of X, $\{x_0, \ldots, x_n\}$, and let $V[X]$ be the linear subspace parallel to $L[X]$.

3. Simplices, polar sines, and Menger-type curvatures

We are only interested in simplices without any coinciding vertices, that is, simplices represented by elements in the set

$$S = \{X \in H^{d+2} : \min(X) > 0\}. \tag{3.1}$$

We note that the restriction to S is natural since $\mu^{d+2}\left(H^{d+2} \setminus S\right) = 0$. We describe some properties and functions of such simplices as follows.

3.1. Height, content and scale

Fixing $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ and $0 \le i \le d+1$, we define the height of the simplex X through the vertex x_i as

$$h_{x_i}(X) = \operatorname{dist}\left(x_i, L[X(i)]\right). \tag{3.2}$$

We denote the minimal height by

$$h(X) = \min_{x_i} \ h_{x_i}(X). \tag{3.3}$$

The *n-content* of X, denoted by $\mathrm{M}_n(X)$, is

$$\mathrm{M}_n(X) = \left(\det\left[\left\{\langle x_i - x_0, x_j - x_0\rangle\right\}_{i,j=1}^{n}\right]\right)^{\frac{1}{2}}. \tag{3.4}$$

Alternatively, the n-content of X is the n-dimensional Lebesgue measure of a parallelotope generated by the images of the vertices of X under any isometric embedding of $L[X]$ in $\mathbb{R}^n$.

For X, we denote its largest edge length by

$$\operatorname{diam}(X) = \max_{0 \le i < j \le n} \|x_i - x_j\|,$$

and its minimal edge length by

$$\min(X) = \min_{0 \le i < j \le n} \|x_i - x_j\|.$$

Given a simplex X, we quantify the disparity between the largest and smallest edges of X at x_0 using the functions

$$\max\nolimits_{x_0}(X) = \max_{1 \le i \le d+1} \|x_i - x_0\| \quad \text{and} \quad \min\nolimits_{x_0}(X) = \min_{1 \le i \le d+1} \|x_i - x_0\|,$$

as well as the function

$$\operatorname{scale}_{x_0}(X) = \frac{\min_{x_0}(X)}{\max_{x_0}(X)}.$$

3.2. Polar sines and elevation sines

We define the *d-dimensional polar sine* of the element $X = (x_0, \ldots, x_{d+1}) \in S$ with respect to the coordinate x_i, $0 \le i \le d+1$, as

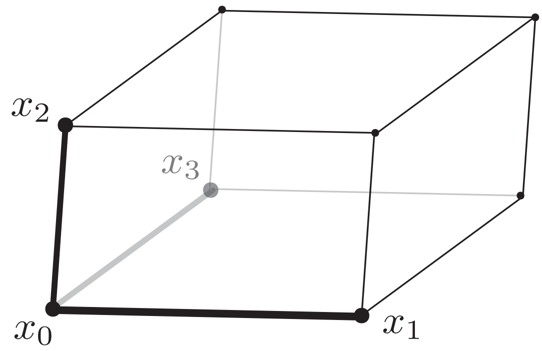
$$\mathrm{p}_d\mathrm{sin}_{x_i}(X) = \frac{\mathrm{M}_{d+1}(X)}{\prod_{\substack{0 \le j \le d+1 \\ j \ne i}} \|x_j - x_i\|}, \tag{3.5}$$

and exemplify it in Figure 1(a). If $X \notin S$, we let $\mathrm{p}_d\mathrm{sin}_{x_i}(X) = 0$. When $d = 1$, the polar sine reduces to the ordinary sine of the angle between two vectors. Unlike [10], our definition here only allows non-negative values of the polar sine. We note that $\mathrm{p}_d\mathrm{sin}_{x_i}(X) = 0$ for some $0 \leq i \leq d+1$ if and only if X is degenerate.

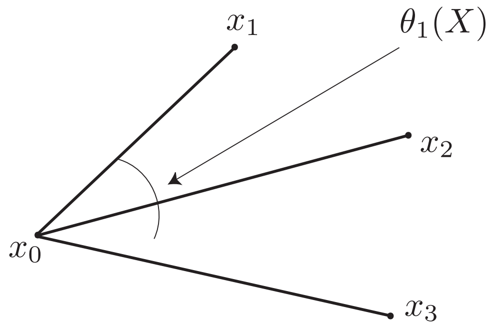
For $n \geq 1$, $X \in S$, and $1 \leq i \leq n+1$, we also define the *elevation angle of* $x_i - x_0$ *with respect to* $V\,[X(i)]$, denoted by $\theta_i(X)$, to be the acute angle such that

$$\sin(\theta_i(X)) = \frac{\operatorname{dist}(x_i, L[X(i)])}{\|x_i - x_0\|}. \tag{3.6}$$

We exemplify it in Figure 1(b).



(a) The induced parallelotope and normalizing edges for computing $\mathrm{p}_2\sin_{x_0}(X)$



(b) The elevation angle $\theta_1(X)$

Figure 1: Exemplifying the concepts of polar sine and elevation angle for tetrahedra of the form $X = (x_0, x_1, x_2, x_3)$. The polar sine $\mathrm{p}_2\sin_{x_0}(X)$ is obtained by dividing the volume of the parallelotope in (a) by the lengths of the corresponding edges through x_0 (indicated in (a) by the thick lines, where the two in the foreground are dark and the one in the background is light). The elevation angle $\theta_1(X)$ in (b) is the angle between the vector $x_1 - x_0$ and its projection onto the face $X(1)$.

The polar sine has the following product formula in terms of elevation angles [10]:

Proposition 3.1. *If* $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ *and* $1 \le i \le d+1$*, then*

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) = \sin\left(\theta_i(X)\right) \cdot \mathrm{p}_{d\text{-}1}\mathrm{sin}_{x_0}\left(X(i)\right).$$

Iterating this product formula we have the estimate

$$0 \le \mathrm{p}_d\mathrm{sin}_{x_i}(X) \le 1, \quad \text{for all } 0 \le i \le d+1. \tag{3.7}$$

3.2.1. Linear deviations and their use in bounding the polar sine

Fixing $X \in H^{d+2}$ and L an affine subspace of H, we define the ℓ_2 deviation of X from L, denoted by $D_2(X, L)$, as follows:

$$D_2(X, L) = \left(\sum_{i=0}^{d+1} \mathrm{dist}^2\left(x_i, L\right)\right)^{1/2}. \tag{3.8}$$

Using this quantity, we get the following upper bound on the polar sine, which we establish in Appendix A.1.

Proposition 3.2. *If* $X \in S$ *and* L *is an arbitrary d-plane of* H*, then*

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \le \frac{\sqrt{2} \cdot (d+1) \cdot (d+2)}{\mathrm{scale}_{x_0}(X)} \, \frac{D_2(X, L)}{\mathrm{diam}(X)}.$$

3.2.2. Concentration inequality for the polar sine

For $X \in [\mathrm{supp}(\mu)]^{d+2}$, $C \ge 1$, and $1 \le i < j \le d+1$, we define

$$U_C(X, i, j) = \tag{3.9}$$
$$= \left\{ y \in \mathrm{supp}(\mu) : \mathrm{p}_d\mathrm{sin}_{x_0}(X) \le C \cdot \left(\mathrm{p}_d\mathrm{sin}_{x_0}(X(y,i)) + \mathrm{p}_d\mathrm{sin}_{x_0}(X(y,j))\right) \right\}.$$

Using C_p of equation (2.1), we have the following concentration inequality proved in [10].

Proposition 3.3. *If* $X \in [\mathrm{supp}(\mu)]^{d+2}$ *with* $x_0 = (X)_0$*, and* $1 \le i < j \le d+1$*, then the following inequality holds for all* $r \in \mathbb{R}$ *such that* $0 < r \le \mathrm{diam}(\mathrm{supp}(\mu))$:

$$\frac{\mu\left(U_{C_\mathrm{p}}(X, i, j) \cap B(x_0, r)\right)}{\mu(B(x_0, r))} \ge \begin{cases} 1, & \text{if } d = 1; \\ 0.75, & \text{if } d > 1. \end{cases}$$

3.3. Categorizing simplices by scales and configurations

We decompose the set S of equation (3.1) according to the size of $\mathrm{scale}_{x_0}(X)$ and the configuration of the vertices of X.

3.3.1. Decomposing the set of simplices S according to scale

Here we categorize simplices according to their scales (defined with respect to their zeroth coordinate x_0) and distinguish between well-scaled and poorly-scaled simplices (with respect to x_0). For $k \in \mathbb{N}_0$ and $p \in \mathbb{N}$ we form the following subset of S

$$S_{k,p} = \left\{ X \in S : \ \alpha_0^{k+p} < \mathrm{scale}_{x_0}(X) \leq \alpha_0^k \right\}. \tag{3.10}$$

We can then decompose S by $\{S_{k,p}\}_{k,p \in \mathbb{N}}$.

Along these lines, we also denote the distinguished set of simplices

$$\widehat{S} = S_{0,3}.$$

We refer to the elements of $\widehat{S}$ as *well-scaled simplices at* x_0, and the elements of $S \setminus \widehat{S}$ as *poorly-scaled simplices at* x_0. Quantifiably, $X \in S$ is well-scaled at x_0 if and only if

$$\frac{\min_{x_0}(X)}{\max_{x_0}(X)} > \alpha_0^3\,. \tag{3.11}$$

A poorly-scaled triangle is exemplified in Figure 2.

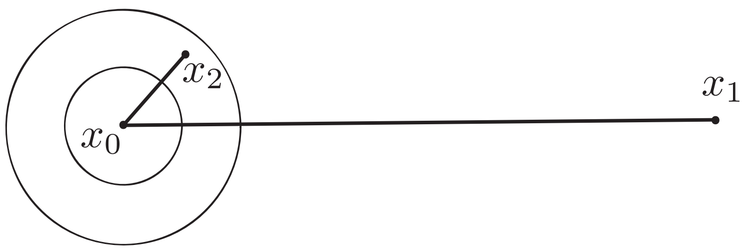


Figure 2: Exemplifying a poorly-scaled triangle (at x_0) for $d = 1$, $k = 3$, and $p = 1$, where the radii of the outer and inner circles are $\alpha_0^3 \cdot \|x_1 - x_0\|$ and $\alpha_0^4 \cdot \|x_1 - x_0\|$ respectively. Note that if x_2 is relocated outside the two circles but still closer to x_0 than x_1, then the modified triangle is well-scaled. Like all poorly-scaled triangles it is a single-handled rake, where the edge connecting x_0 and x_1 is a handle and the edge connecting x_2 and x_1 is a tine.

For a ball Q in H and $p \in \mathbb{N}$, we often use localized versions of the sets S, $\widehat{S}$, and $S_{k,p}$, $k \geq 3$, defined as

$$S(Q) = S \cap Q^{d+2}\,, \quad \widehat{S}(Q) = \widehat{S} \cap Q^{d+2} \;\text{ and }\; S_{k,p}(Q) = S_{k,p} \cap Q^{d+2}\,. \tag{3.12}$$

In most of the paper it will be sufficient to assume $p = 1$, however in Section 9 we will need to consider the case $p = 2$, and we thus formulate some corresponding definitions and propositions in other sections for both $p = 1$ and $p = 2$. We note that if $p = 1$, then the sets $S_{k,1}$, $k \geq 3$, partition $S \setminus \widehat{S}$, whereas if $p > 1$, then they cover it.

3.3.2. Decomposing simplices in $S_{k,p}$ according to configuration

So far we have decomposed the set of simplices S according to the ratio between the shortest and longest lengths of edges at x_0. Next, we further decompose simplices according to ratios of lengths of other edges (at x_0) and the length of the largest edge length (at x_0).

We start with some terminology, while fixing arbitrarily $k \geq 3$, $p \in \{1, 2\}$, and $X = (x_0, \ldots, x_{d+1}) \in S_{k,p}$. We say that an edge connecting x_0 and x_i, $1 \leq i \leq d+1$, is a *handle* if

$$\frac{\|x_0 - x_i\|}{\max_{x_0}(X)} > \alpha_0^k \tag{3.13}$$

and a *tine* otherwise, i.e.,

$$\alpha_0^{k+p} < \frac{\|x_i - x_0\|}{\max_{x_0}(X)} \leq \alpha_0^k.$$

This terminology is intended to evoke the image of the gardening implement known in English as a *rake*. We say that X is a *rake*, or a single-handled rake, if it only has one handle. Similarly, X is an *n-handled rake* if it has n handles for $1 \leq n \leq d$. Clearly $X \in S_{k,p}$ has at least one handle (obtaining the maximal edge length at x_0) and at most d of them (excluding the one of minimal edge length at x_0). We remark that these notions depend on our fixed choice of p which will be clear from the context. A single-handled rake and a double-handled rake are illustrated in Figures 2 and 3 respectively.

We partition $S_{k,p}$, $p = 1, 2$, according to the number of handles in the elements $X = (x_0, \ldots, x_{d+1})$ at x_0. Formally, for $1 \leq n \leq d$, we define the sets:

$$S_{k,p}^n = \left\{ X = (x_0, \ldots, x_{d+1}) \in S_{k,p} : \frac{\|x_i - x_0\|}{\max_{x_0}(X)} > \alpha_0^k \text{ for exactly } n \text{ vertices } x_i \right\}. \tag{3.14}$$

We note that

$$(3.15) \qquad S_{k,p} = \bigcup_{n=1}^{d} S_{k,p}^{n}, \quad \text{and} \quad S_{k,p}^{n} \cap S_{k,p}^{n'} = \emptyset, \quad \text{for} \quad 1 \leq n \neq n' \leq d.$$

In order to reduce unnecessary information regarding the position of the handles at x_0 we concentrate on the following subset of $S_{k,p}^n$.

$$\mathbf{S}_{k,p}^{n} = \left\{ X \in S_{k,p}^{n} : \frac{\|(X)_\ell - x_0\|}{\max_{x_0}(X)} > \alpha_0^k \ \text{ for all } \ 1 \leq \ell \leq n \right\}.$$

That is, $\mathbf{S}_{k,p}^n$ is the subset of $S_{k,p}^n$ whose edges connecting x_0 with $x_1, \ldots, x_n$ are handles and whose edges connecting x_0 with $x_{n+1}, \ldots, x_{d+1}$ are tines. We illustrate an element of $\mathbf{S}_{3,2}^2$ where $d = 4$ in Figure 3.

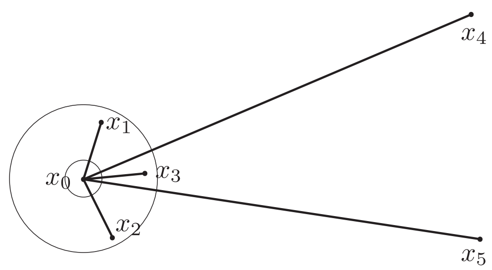


Figure 3: Example of a simplex $X = (x_0, x_1, x_2, x_2, x_4, x_5) \in \mathbf{S}_{3,2}^2$. The radii of the outer and inner circles are $\alpha_0^3\|x_1 - x_0\|$ and $\alpha_0^5\|x_1 - x_0\|$ respectively.

Given a ball Q in H we denote the restrictions of the above sets to Q^{d+2} by $S_{k,p}^n(Q)$ and $\mathbf{S}_{k,p}^n(Q)$ respectively.

3.4. Decomposing the Menger-type curvature

We decompose the continuous Menger-type curvature according to the regions described above (Subsection 3.3). We start by expressing the discrete and continuous d-dimensional Menger-type curvatures of $X \in S$ and $\mu|_Q$ respectively in terms of the polar sine as follows:

$$(3.16) \qquad c_d(X) = \sqrt{\frac{\sum_{i=0}^{d+1} \mathrm{p}_d\mathrm{sin}_{x_i}^2(X)}{(d+2) \cdot \mathrm{diam}(X)^{d(d+1)}}}$$

and

$$(3.17) \qquad c_d^2(\mu|_Q) = \frac{1}{d+2} \sum_{i=0}^{d+1} \int_{Q^{d+2}} \frac{\mathrm{p}_d\mathrm{sin}_{x_i}^2(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X)$$
$$= \int_{Q^{d+2}} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X).$$

In order to control the continuous curvature $c_d^2(\mu|_Q)$ we break it into "smaller" parts. We first decompose it according to the sets $\{S_{k,1}(Q)\}_{k\geq 3}$ and $\widehat{S}(Q)$ of Subsection 3.3.2 in the following way:

$$\text{(3.18)} \quad \int_{Q^{d+2}} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) = \\ = \int_{\widehat{S}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) + \sum_{k\geq 3}\int_{S_{k,1}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X)\,.$$

We further break the terms of the infinite sum in equation (3.18) according to the regions $\mathbf{S}^n_{k,1}(Q)$ of Subsection 3.3.2 in the following way:

$$\text{(3.19)} \quad \int_{S_{k,1}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) = \\ = \sum_{n=1}^{d}\binom{d+1}{n}\int_{\mathbf{S}^n_{k,1}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X).$$

To verify this formula we first decompose $S_{k,1}(Q)$ by $S^n_{k,1}(Q)$, $n=1,\ldots,d$, according to equation (3.15). We then partition each $S^n_{k,1}(Q)$, $n = 1,\ldots,d$, according to the subsets of indices representing the n handles. Clearly this results in $\binom{d+1}{n}$ subsets of $S^n_{k,1}(Q)$, where one of these is associated with $\mathbf{S}^n_{k,p}(Q)$. Now the integral of $\mathrm{p}_d\mathrm{sin}^2_{x_0}(X)/\,\mathrm{diam}(X)^{d(d+1)}$ over these subsets is the same (due to the invariance of the polar sine to permutations fixing x_0 and an immediate change of variables), and consequently equation (3.19) is established.

Therefore, to control the LHS of equation (3.18) we only need to concentrate on the first term of the RHS of equation (3.18) and the terms of the RHS of equation (3.19) for all $k \geq 3$. This is what we do for the rest of the paper.

4. Uniform rectifiability

We review here basic notions in the theory of uniform rectifiability [2, 3]. Even though the original theory is formulated in finite dimensional Euclidean spaces, the part presented here generalizes to any separable real Hilbert space.

4.1. A_1 weights and ω-regular surfaces

Let $\mathcal{L}_d$ denote the d-dimensional Lebesgue measure on $\mathbb{R}^d$. Given a locally integrable function $\omega : \mathbb{R}^d \to [0,\infty)$, we induce a measure on Borel subsets A

of $\mathbb{R}^d$ by defining

$$\omega(A) = \int_A \omega(x)\, \mathrm{d}\mathcal{L}_d(x).$$

We say that ω is an A_1 weight if there exists $C \geq 1$ such that for any ball Q in $\mathbb{R}^d$,

$$\frac{\omega(Q)}{|Q|} \leq C \cdot \omega(x), \quad \text{for } \mathcal{L}_d \text{ a.e. } x \in Q.$$

We note that the measure induced by ω is doubling in the following sense: $\omega(Q) \approx \omega(2 \cdot Q)$ for any ball Q. Consequently, the following function is a quasidistance (i.e., a symmetric positive definite function satisfying a relaxed version of the triangle inequality with controlling constant $C \geq 1$ instead of one):

$$\mathrm{qdist}_\omega(x, y) = \sqrt[d]{\omega\Big(B\Big(\frac{x+y}{2}, \frac{|x-y|}{2}\Big)\Big)}, \quad \text{for all } x, y \in \mathbb{R}^d.$$

Given an A_1 weight ω, we define ω-regular surfaces as follows.

Definition 4.1. *Let ω be an A_1 weight on $\mathbb{R}^d$. A subset Γ of H is called an ω-*regular surface *if there exists a function $f\colon \mathbb{R}^d \to H$ and constants L and C such that $\Gamma = f(\mathbb{R}^d)$,*

$$\|f(x) - f(y)\| \leq L \cdot \mathrm{qdist}_\omega(x, y), \qquad \text{for all } x, y \in \mathbb{R}^d, \tag{4.1}$$

and

$$\omega\left(f^{-1}(B(x, r))\right) \leq C \cdot r^d, \qquad \text{for all } x \in H \text{ and } r > 0. \tag{4.2}$$

4.2. Two equivalent definitions of uniform rectifiability

We provide here two equivalent definitions of uniform rectifiability. Many other definitions appear in [2, 3, 12, 16].

Given a Borel measure μ on H, we let

$$\widehat{H} = \begin{cases} H \times \mathbb{R}, & \text{if } \dim(H) < 2 \cdot d; \\ H, & \text{otherwise,} \end{cases}$$

and we define the induced measure $\widehat{\mu}$ on $\widehat{H}$ to be $\widehat{\mu}(A) = \mu(A \cap H)$, for all Borel sets $A \subseteq \widehat{H}$.

We define d-dimensional uniformly rectifiable measures as follows:

Definition 4.2. *A Borel measure μ on H is said to be d-dimensional uniformly rectifiable if it is d-regular and there exist an A_1 weight ω on $\mathbb{R}^d$ along with an ω-regular surface $\Gamma \subseteq \widehat{H}$ such that $\widehat{\mu}\big(\widehat{H} \setminus \Gamma\big) = 0$.*

David and Semmes [2, 3] have shown that the Jones-type flatness of equation (1.5) can be used to quantify and thus redefine uniform rectifiability as follows.

Theorem 4.1. *A d-regular measure μ on H is uniformly rectifiable if and only if there exists a constant $C = C(d, C_\mu)$ such that*

$$J_d(\mu|_Q) \leq C \cdot \mu(Q) \quad \textit{for any ball } Q \textit{ in } H.$$

We note that Theorem 1.2 is an immediate consequence of Theorems 1.1 and 4.1.

4.3. Multiscale resolutions and modified Jones-type flatness

Multiscale decomposition of $\operatorname{supp}(\mu)$ are common in the theory of uniform rectifiability [1, 15], and they are often used to compress information from all scales and locations. Here we also use them to construct covers and partitions of $S \cap [\operatorname{supp}(\mu)]^{d+2}$ and consequently control the Menger-type curvature by a discretized Jones-type flatness.

Following the spirit of [1, 15] we cover $\operatorname{supp}(\mu)$ by balls $\{\mathcal{B}_n\}_{n\in\mathbb{Z}}$ which correspond to the length scales $\{\alpha_0^n\}_{n\in\mathbb{Z}}$. We then use them to construct a corresponding sequence of partitions, $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ of $\operatorname{supp}(\mu)$.

We say that a collection of points $E_n \subseteq \operatorname{supp}(\mu)$ is an *n-net* for $\operatorname{supp}(\mu)$ if

1. $\|x - y\| > \alpha_0^n$, for all x and y in E_n.
2. $\operatorname{supp}(\mu) \subseteq \bigcup_{x\in E_n} B(x, \alpha_0^n)$.

For each $n \in \mathbb{Z}$, we arbitrarily form an n-net, E_n, and among all balls in the family $\{B(x, 4\cdot\alpha_0^n)\}_{x\in E_n}$ we fix a subfamily $\mathcal{B}_n$ such that $\frac{1}{4}\cdot\mathcal{B}_n$ is maximally mutually disjoint. Since H is separable, $\operatorname{supp}(\mu)$ is separable and $\mathcal{B}_n$ is countable. Furthermore, we note that $\mathcal{B}_n$ covers $\operatorname{supp}(\mu)$.

We index the elements of $\mathcal{B}_n$ by $\Lambda_n = \{1, 2, \ldots\}$, which is either finite or $\mathbb{N}$, so that

$$\mathcal{B}_n = \{B_{n,j}\}_{j\in\Lambda_n}. \tag{4.3}$$

We define the corresponding *multiresolution family* for $\operatorname{supp}(\mu)$ to be

$$\mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n. \tag{4.4}$$

These resolutions can be replaced by multiscale partitions of $\operatorname{supp}(\mu)$ in the following manner (see proof in Appendix A.2).

Lemma 4.3.1. *For any $n \in \mathbb{Z}$ there exists a partition of* $\operatorname{supp}(\mu)$, $\mathcal{P}_n = \{P_{n,j}\}_{j \in \Lambda_n}$, *such that for any $j \in \Lambda_n$ there exists a unique $B_{n,j} \in \mathcal{B}_n$ with*

$$\operatorname{supp}(\mu) \bigcap \frac{1}{4} \cdot B_{n,j} \subseteq P_{n,j} \subseteq \operatorname{supp}(\mu) \bigcap \frac{3}{4} \cdot B_{n,j} .$$

We typically work with localized resolutions which we define as follows. For Q a ball in H, we let $m(Q)$ be the smallest integer m such that $\alpha_0^m \leq \operatorname{diam}(Q)$, i.e.,

$$m(Q) = \left\lceil \frac{\ln(\operatorname{diam}(Q))}{\ln(\alpha_0)} \right\rceil , \tag{4.5}$$

For $n \geq m(Q)$ we define

$$\mathcal{B}_n(Q) = \{B_{n,j} \in \mathcal{B}_n : B_{n,j} \cap Q \neq \emptyset\} , \tag{4.6}$$

and form the *local multiresolution family* as follows

$$\mathcal{D}(Q) = \bigcup_{n \geq m(Q)} \mathcal{B}_n(Q) . \tag{4.7}$$

We also define the set of indices (possibly empty)

$$\Lambda_n(Q) = \{j \in \Lambda_n : P_{n,j} \cap Q \neq \emptyset\}.$$

4.3.1. Jones-type flatness via multiscale resolutions

For a ball Q in H, and the local multiresolution $\mathcal{D}(Q)$, we define the corresponding local Jones-type d-flatness as follows:

$$J_d^{\mathcal{D}}(\mu|_Q) = \sum_{B \in \mathcal{D}(Q)} \beta_2^2(B) \cdot \mu(B).$$

Both quantities of Jones-type flatness, J_d and $J_d^{\mathcal{D}}$, are comparable. We will only use the following part of the comparability whose proof practically follows the same arguments of [15, Lemma 3.2]:

Proposition 4.1. *There exists a constant $C_3 = C_3(d, C_\mu)$ such that for any multiresolution family $\mathcal{D}$ on* $\operatorname{supp}(\mu)$*:*

$$J_d^{\mathcal{D}}(\mu|_Q) \leq C_3 \cdot J_d(\mu|_{6 \cdot Q}) \text{ for any ball } Q \text{ in } H . \tag{4.8}$$

5. Proof of Proposition 1.1

We first note that $U_\lambda(B(x,t))$ is invariant under any permutation of the coordinates. Thus, by the same argument producing equation (3.17) we have the equality

$$\int_{U_\lambda(B(x,t))} c_d^2(X)\,\mathrm{d}\mu^{d+2}(X) = \int_{U_\lambda(B(x,t))} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X). \tag{5.1}$$

Furthermore, $\operatorname{diam}(X) \geq \lambda \cdot t$ and $\operatorname{scale}_{x_0}(X) \geq \lambda$ for all $X \in U_\lambda(B(x,t))$. Hence, applying Proposition 3.2 to the RHS of equation (5.1) we get that for any d-plane L

$$\begin{aligned} \int_{U_\lambda(B(x,t))} c_d^2(X)\,\mathrm{d}\mu^{d+2}(X) \leq \\ \leq \frac{2\cdot(d+1)^2\cdot(d+2)^2}{\lambda^{d(d+1)+4}} \int_{U_\lambda(B(x,t))} \frac{D_2^2(X,L)}{t^2}\,\frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}} \\ = \frac{8\cdot(d+1)^2\cdot(d+2)^2}{\lambda^{d(d+1)+4}} \sum_{i=0}^{d+1} \int_{U_\lambda(B(x,t))} \left(\frac{\operatorname{dist}(x_i,L)}{2\cdot t}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}}. \end{aligned} \tag{5.2}$$

For P_i, $0 \leq i \leq d+1$, the projection of H^{d+2} onto its i^{th} coordinate, we have the inclusion

$$P_i(U_\lambda(B(x,t))) \subseteq B(x,t). \tag{5.3}$$

Hence, fixing $0 \leq i \leq d+1$ and applying equation (5.3) and Fubini's Theorem to the corresponding term on the RHS of equation (5.2), we get the inequality

$$\begin{aligned} \int_{U_\lambda(B(x,t))} \left(\frac{\operatorname{dist}(x_i,L)}{2\cdot t}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{t^{d(d+1)}} \leq \\ \leq \left(\frac{\mu(B(x,t))}{t^d}\right)^{d+1} \int_{B(x,t)} \left(\frac{\operatorname{dist}(x_i,L)}{2\cdot t}\right)^2 \mathrm{d}\mu(x_i) \\ = \left(\frac{\mu(B(x,t))}{t^d}\right)^{d+1} \cdot \beta_2^2(x,t,L)\cdot\mu(B(x,t)). \end{aligned} \tag{5.4}$$

Combining equations (5.2) and (5.4), while summing the RHS of equation (5.4) over $0 \leq i \leq d+1$ as well as applying the d-regularity of μ, we obtain the following bound on the LHS of equation (1.4):

$$\begin{aligned} \int_{U_\lambda(B(x,t))} c_d^2(X)\,\mathrm{d}\mu^{d+2}(X) \leq \\ \leq \frac{8\cdot(d+1)^2\cdot(d+2)^3}{\lambda^{d(d+1)+4}} \cdot C_\mu^{d+1}\cdot\beta_2^2(x,t,L)\cdot\mu(B(x,t)). \end{aligned} \tag{5.5}$$

Since L is arbitrary, taking the infimum over all L on the RHS of equation (5.5) proves the proposition. ■

6. Reduction of Theorem 1.1

We reduce Theorem 1.1 by applying the following decomposition of the multivariate integral $c_d^2(\mu|_Q)$, obtained by combining equations (3.18) and (3.19):

$$c_d^2(\mu|_Q) = \int_{\widehat{S}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) + \tag{6.1}$$
$$+ \sum_{k\geq 3}\sum_{n=1}^{d}\binom{d+1}{n}\int_{\mathbf{S}_{k,1}^{n}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X).$$

In view of Proposition 4.1 and equation (6.1), we prove Theorem 1.1 by establishing the following three propositions:

Proposition 6.1. *There exists a constant $C_4 = C_4(d, C_\mu)$ such that*

$$\int_{\widehat{S}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2}(X) \leq \frac{C_4}{\alpha_0^6}\cdot J_d^{\mathcal{D}}(\mu|_Q)\,. \tag{6.2}$$

Proposition 6.2. *There exists a constant $C_5 = C_5(d, C_\mu)$ such that for any ball Q in H*

$$\int_{\mathbf{S}_{k,1}^{1}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2} \leq C_5\cdot(k\cdot d+1)\cdot\left(\alpha_0^d\cdot C_{\mathrm{p}}^2\right)^{k\cdot d}\cdot J_d^{\mathcal{D}}(\mu|_Q)\,.$$

Proposition 6.3. *If $1 < n \leq d$, then there exists a constant $C_6 = C_6(d, C_\mu)$ such that for any ball Q in H*

$$\int_{\mathbf{S}_{k,1}^{n}(Q)} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}^2(X)}{\mathrm{diam}(X)^{d(d+1)}}\,\mathrm{d}\mu^{d+2} \leq C_6\cdot(k\cdot d+1)^3\cdot\left(\alpha_0\cdot C_{\mathrm{p}}^2\right)^{k\cdot d}\cdot J_d^{\mathcal{D}}(\mu|_Q)\,.$$

Proposition 6.1 reduces the integration of the Menger-type curvature in Theorem 1.1 to well-scaled simplices. We prove it in Section 7 by straightforward decomposition of the multivariate integral which we refer to as geometric multipoles.

Propositions 6.2 and 6.3 reduce the integration of the Menger-type curvature in Theorem 1.1 to poorly-scaled simplices, which are single-handled rakes in the former proposition and multi-handled rakes in the latter one. We prove those propositions in Sections 8 and 9 respectively. Unlike Proposition 6.1, the method of geometric multipoles is not sufficient for their proof. Our basic idea is in the spirit of the proof of [6] (i.e., [14, Theorem 31]), and trades any poorly-scaled simplex for a predictable sequence of well-scaled simplices satisfying an extended version of the "triangle inequality" for the polar sine (Proposition 3.3). We then apply the method of geometric multipoles to the well-scaled simplices of the sequence.

7. Proof of Proposition 6.1 via geometric multipoles

Our proof of Proposition 6.1 generates approximate decompositions of the Menger-type curvature of μ according to goodness of approximations by d-planes at different scales and locations. We refer to this strategy as *geometric multipoles* and see it as a geometric analog of the decomposition of special potentials by near-field interactions at different locations, as applied in the fast multipoles algorithm [4]. Unlike fast multipoles, which considers interactions between pairs of points, geometric multipoles takes into account simultaneous interactions between $d+2$ points. While fast multipoles neglects terms of distant interactions, d-dimensional geometric multipoles may neglect locations and scales well-approximated by d-planes.

We first break the integral on the LHS of equation (6.2) into a sum of integrals reflecting different scales and locations in $Q \cap \operatorname{supp}(\mu)$. Then we control each such integral by $\beta_2^2(B) \cdot \mu(B)$ for a unique $B \in \mathcal{D}(Q)$.

For fixed $m \geq m(Q)$ (see equation (4.5)), we define

$$\widehat{S}(m) = \left\{ X \in \widehat{S} : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m] \right\}, \tag{7.1}$$

and we let $\widehat{S}(m)(Q)$ denote the restriction of $\widehat{S}(m)$ to the set Q^{d+2}. The argument m indicates the overall length scale of the elements and the subscript indicates the relative scaling between the edges at x_0. We note that the family $\{\widehat{S}(m)(Q)\}_{m\in\mathbb{Z}}$ partitions $\widehat{S}(Q)$, and thus

$$\int_{\widehat{S}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) = \tag{7.2}$$
$$= \sum_{m \geq m(Q)} \int_{\widehat{S}(m)(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X).$$

Next, fixing the length scale determined by m, we partition each $\widehat{S}(m)(Q)$ according to location in $\operatorname{supp}(\mu)$ determined by the partition $\mathcal{P}_m$. For fixed $m \geq m(Q)$, $j \in \Lambda_m$, and $P_{m,j}$ as in Lemma (4.3.1), let

$$\widehat{P}_{m,j} = \left\{ X \in \widehat{S}(m)(Q) : x_0 \in P_{m,j} \right\}. \tag{7.3}$$

We thus obtain the inequality

$$\int_{\widehat{S}(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) \leq$$
$$\leq \sum_{m \geq m(Q)} \sum_{j \in \Lambda_m(Q)} \int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X).$$

Fixing m , $j \in \Lambda_m(Q)$, and $B_{m,j} \in \mathcal{B}_m(Q)$ such that

$$\frac{1}{4} \cdot B_{m,j} \cap \operatorname{supp}(\mu) \subseteq P_{m,j} \subseteq \frac{3}{4} \cdot B_{m,j}$$

(see Lemma 4.3.1) we will show that

$$\int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) \leq \frac{C_4}{\alpha_0^6} \cdot \beta_2^2(B_{m,j}) \cdot \mu(B_{m,j}) \, . \tag{7.4}$$

Summing over $j \in \Lambda_m(Q)$ and $m \geq m(Q)$ will then conclude the proposition.

We establish equation (7.4) by following the basic argument behind Proposition 1.1. We first note that for all $X \in \widehat{P}_{m,j} \subseteq \widehat{S}(m)(Q)$

$$\frac{\alpha_0}{8} \cdot \operatorname{diam}\left(B_{m,j}\right) = \alpha_0^{m+1} < \max_{x_0}(X) \leq \operatorname{diam}(X) \, . \tag{7.5}$$

Then, fixing an arbitrary d-plane L in H, and combining Proposition 3.2 with equation (7.5), we obtain the following inequality for all $X \in \widehat{P}_{m,j}$

$$\begin{aligned} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} &\leq \\ &\leq \frac{2 \cdot 8^2 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha^6 \cdot \alpha_0^2} \cdot \frac{D_2^2\left(X, L\right)}{\operatorname{diam}^2\left(B_{m,j}\right)} \cdot \frac{1}{\left(\alpha_0^{d(m+1)}\right)^{d+1}}. \end{aligned} \tag{7.6}$$

Furthermore, if $X = (x_0, \dots, x_{d+1}) \in \widehat{P}_{m,j}$, then $x_0 \in \frac{3}{4} \cdot B_{m,j}$, and $\|x_i - x_0\| \leq \alpha_0^m = \operatorname{diam}(B_{m,j})/4$ for all $1 \leq i \leq d+1$. Consequently, $x_i \in B_{m,j}$ for all $0 \leq i \leq d+1$, and thus

$$\widehat{P}_{m,j} \subseteq \left(B_{m,j}\right)^{d+2} . \tag{7.7}$$

Combining equations (7.6) and (7.7) we obtain the bound

$$\begin{aligned} \int_{\widehat{P}_{m,j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \, \mathrm{d}\mu^{d+2}(X) &\leq \\ \leq \frac{2 \cdot 8^2 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha^6 \cdot \alpha_0^2} &\int_{(B_{m,j})^{d+2}} \frac{D_2^2\left(X, L\right)}{\operatorname{diam}^2\left(B_{m,j}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\left(\alpha_0^{d(m+1)}\right)^{d+1}}. \end{aligned} \tag{7.8}$$

Finally, applying the same types of computations to the RHS of equation (7.8) that led to equations (5.4) and (5.5) we obtain equation (7.4) and hence the proposition. ■

8. Proof of Proposition 6.2 via multiscale inequalities

Here we develop a multiscale inequality for the polar sine that results in a proof of Proposition 6.2. The basic idea is to take an $X \in \mathbf{S}_{k,1}$ and carefully construct a sequence of simplices, $\{X_n\}_{n=1}^{n(X)}$, well-scaled at x_0 and depending on X such that

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \lesssim \sum_{j=1}^{n(X)} \mathrm{p}_d\mathrm{sin}_{x_0}(X_j), \tag{8.1}$$

where neither the length of the sequence, $n(X)$, nor the size of the comparability constant in the above inequality (also depending on $n(X)$) are "too large".

Such a sequence will have zeroth coordinate $x_0 = (X)_0$, and overall length scales progressing geometrically from $\min_{x_0}(X)$ to $\max_{x_0}(X)$. In this way the length of the sequence, $n(X)$, will be such that

$$n(X) \lesssim \log\left(\frac{\max_{x_0}(X)}{\min_{x_0}(X)}\right) \lesssim k.$$

As mentioned previously, a similar approach used in the one-dimensional case to control $c_M^2(\mu)$ by the β_∞ numbers [6, 14] was the inspiration for our approach. However, generalizing it to higher-dimensions, as well as the β_2 numbers, required us to go substantially beyond what was present for $d = 1$ in a few ways as shown throughout the arguments of the proof.

In order to construct these sequences and clearly formulate the inequalities we require some development in terms of ideas and notation. Subsection 8.1 develops the notion of well-scaled sequences and a related discrete multiscale inequality for the polar sine. Subsection 8.2 turns this inequality into a multiscale integral inequality. Finally, in Subsection 8.4 we prove Proposition 6.2. Throughout this section we arbitrarily fix $k \geq 3$ and take $p = 1$ or $p = 2$ (depending on the context), where $p = 2$ is only needed for definitions or calculations that will be used in Section 9.

8.1. From rakes to well-scaled sequences

We recall that $\mathbf{S}_{k,p}^1$ is the set of rakes whose single handles are obtained at their first coordinate, such that

$$\alpha_0^{k+p} < \mathrm{scale}_{x_0}(X) \leq \alpha_0^k.$$

We explain here how to "decompose" elements of $\mathbf{S}_{k,p}^1$ into a sequence of well-scaled simplices satisfying an inequality of the form in equation (8.1).

That is, we take an element with one very large edge at x_0 and swap it for a predictable sequence of elements, each with all edges at x_0 comparable, such that the sum of the polar sines controls that of the original simplex. This involves (regrettably) some technical definitions and corresponding notation, the first of which is the following shorthand notation for annuli.

If $n \in \mathbb{Z}$ and α_0 is the fixed constant of equation (2.2), we use the notation

$$A_n(x,r) = B(x, \alpha_0^n \cdot r) \setminus B(x, \alpha_0^{n+1} \cdot r)\,. \tag{8.2}$$

8.1.1. Well-scaled pieces and augmented elements

We define a *well-scaled piece* for $X \in \mathbf{S}^1_{k,p}$ to be a $(k \cdot d)$-tuple of the form

$$Y_X = (y_1, \dots, y_{k \cdot d}) \in \prod_{q=1}^{k \cdot d} A_{k - \lceil \frac{q}{d} \rceil}(x_0, \max_{x_0}(X))\,. \tag{8.3}$$

The coordinates of Y_X are grouped into k distinct clusters of d points, with each individual cluster lying in a distinct annulus centered at x_0 (see Figure 4 (a)).

For $X \in \mathbf{S}^1_{k,p}$ and a well-scaled piece, Y_X, we define the *augmentation of X by Y_X* as

$$\underline{X} = X \times Y_X = (x_0, \dots, x_{d+1}, y_1, \dots, y_{k \cdot d}) \in \mathbf{S}^1_{k,p} \times H^{k \cdot d}. \tag{8.4}$$

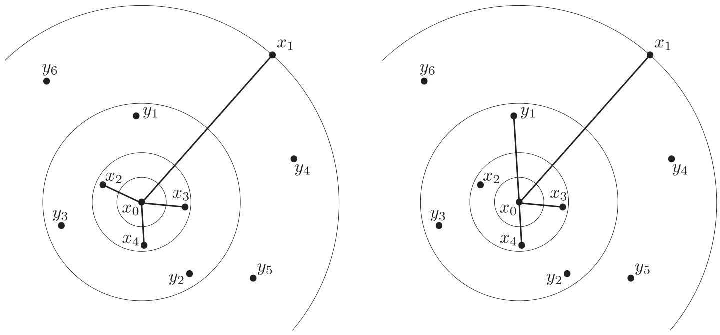
For a fixed augmented element $\underline{X}$, we define two sequences in H^{d+2}, the *auxiliary sequence*,

$$\widetilde{\Phi}_k(\underline{X}) = \left\{\widetilde{X}_q\right\}_{q=0}^{k \cdot d},$$

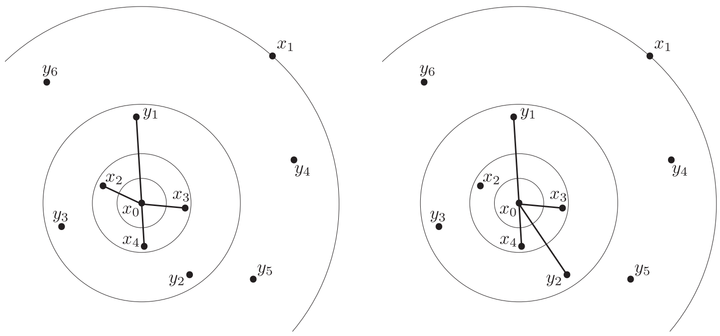
and the *well-scaled sequence*

$$\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}\,.$$

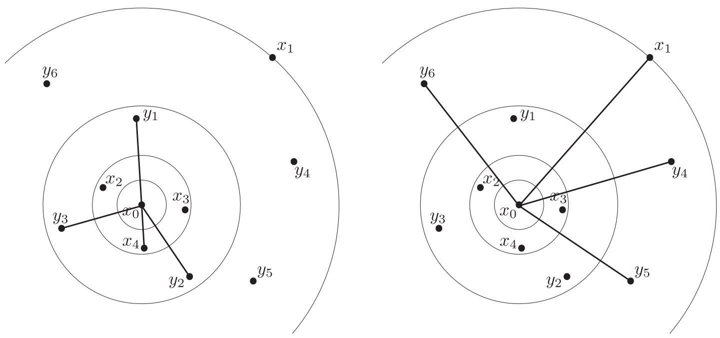
They will be used to formulate a multiscale inequality for the polar sine function. We remark that the auxiliary sequence is auxiliary in the sense that it is only used to establish the inequality of equation (8.1) for the well-scaled sequence $\Phi(\underline{X})$.



(a) simplex X and Y_X (b) auxiliary simplex $\widetilde{X}_1$



(c) well-scaled simplex X_1 (d) well-scaled simplex X_2



(e) well-scaled simplex X_4 (f) well-scaled simplex X_7

Figure 4: Illustration of the well-scaled piece and induced sequences. We assume $d = 3$, $k = 2$, and $p = 1$ and a simplex $X = (x_0, \dots, x_5)$ (we use the case $k = 2$ for convenience of drawing even though X itself is well-scaled, but we notice that the construction produces simplices of smaller scales). The radii of the circles (ordered from outer to inner) are $\|x_1 - x_0\|$, $\alpha_0\|x_1 - x_0\|$, $\alpha_0^2\|x_1 - x_0\|$, and $\alpha_0^3\|x_1 - x_0\|$. The simplex X and the well-scaled piece Y_X are shown in (a). The auxiliary simplex, $\widetilde{X}_1$, is exemplified in (b) as an element mediating between the well-scaled simplex X_1 shown next and the original simplex X. The elements of the well-scaled sequence X_1, X_2, X_4 and X_7 are shown in (c)-(f) respectively (note that X_7 is not listed in the example below).

8.1.2. The auxiliary sequence

If $a \in \mathbb{Z}$, then let $\overline{a} \in \{2, \ldots, d+1\}$ denote the unique integer such that $\overline{a} = a \bmod d$. We form the *auxiliary sequence* $\{\widetilde{X}_q\}_{q=0}^{k \cdot d}$ recursively as follows.

Definition 8.1. *If* $X = (x_0, \ldots, x_{d+1}) \in \mathbf{S}^1_{k,p}$ *and*

$$\underline{X} = X \times Y_X = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d}),$$

then let $\widetilde{\Phi}_k(\underline{X}) = \{\widetilde{X}_q\}_{q=0}^{k \cdot d}$ *be the sequence of elements in* H^{d+2} *defined recursively as follows:*

$$\widetilde{X}_0 = X,$$

and

$$\widetilde{X}_q = \widetilde{X}_{q-1}\left(y_q, \overline{q+1}\right) \quad \text{for} \quad 1 \leq q \leq k \cdot d. \tag{8.5}$$

For example, if $d = 3$, then

$$\widetilde{X}_1 = (x_0, x_1, y_1, x_3, x_4), \quad \widetilde{X}_2 = (x_0, x_1, y_1, y_2, x_4), \quad \widetilde{X}_3 = (x_0, x_1, y_1, y_2, y_3);$$
$$\widetilde{X}_4 = (x_0, x_1, y_4, y_2, y_3), \quad \widetilde{X}_5 = (x_0, x_1, y_4, y_5, y_3), \quad \widetilde{X}_6 = (x_0, x_1, y_4, y_5, y_6).$$

where $\widetilde{X}_1$ is illustrated in Figure 4 (b).

In general, we note that the elements $\widetilde{X}_q$ have the following form:

$$\widetilde{X}_q = \begin{cases} (x_0, x_1, y_1, \ldots, y_q, x_{q+2}, \ldots, x_{d+1}), & \text{if } 1 \leq q \leq d-1; \\ (x_0, x_1, y_{j \cdot d+1}, \ldots, y_q, y_{q-d+1}, \ldots, y_{j \cdot d}), & \text{if } j \cdot d < q < (j+1) \cdot d \\ & \quad \text{for } 1 \leq j \leq k-1; \\ (x_0, x_1, y_{(j-1) \cdot d+1}, \ldots, y_{j \cdot d}), & \text{if } q = j \cdot d \text{ for } 1 \leq j \leq k. \end{cases} \tag{8.6}$$

In the special case where $d = 1$, the first two cases of equation (8.6) are meaningless and $\widetilde{X}_q = (x_0, x_1, y_q)$ for all $1 \leq q \leq k$.

8.1.3. The well-scaled sequence

We derive the *well-scaled* sequence $\Phi_k(\underline{X})$ from the auxiliary sequence $\widetilde{\Phi}_k(\underline{X})$ as follows.

Definition 8.2. *If* $X \in \mathbf{S}^1_{k,p}$ *and* $\underline{X} = X \times Y_X = (x_0, \ldots, x_{d+1}, y_1, \ldots, y_{k \cdot d})$, *then let* $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ *be the sequence of elements in* H^{d+2} *such that*

$$X_q = \begin{cases} \widetilde{X}_{q-1}(y_q, 1), & \text{if } 1 \leq q \leq k \cdot d; \\ \widetilde{X}_{k \cdot d}, & \text{if } q = k \cdot d + 1. \end{cases} \tag{8.7}$$

For example, if $d = 3$, then the first six elements of the sequence are

$$X_1 = (x_0, y_1, x_2, x_3, x_4), \quad X_2 = (x_0, y_2, y_1, x_3, x_4), \quad X_3 = (x_0, y_3, y_1, y_2, x_4);$$
$$X_4 = (x_0, y_4, y_1, y_2, y_3), \quad X_5 = (x_0, y_5, y_4, y_2, y_3), \quad X_6 = (x_0, y_6, y_4, y_5, y_3).$$

We illustrate X_1, X_2, X_4 and even X_7 in Figures 4 (c)–4 (f).

We also note that in the very special case where $d = 1$, then $X_1 = (x_0, y_1, x_2)$, $X_q = (x_0, y_q, y_{q-1})$, $1 < q \leq k \cdot d + 1$ and $X_{k \cdot d+1} = (x_0, x_1, y_{k \cdot d})$.

The following lemma shows that the elements $X_q \in \Phi_k(\underline{X})$, $1 \leq q \leq k \cdot d + 1$, are indeed well-scaled at the vertex x_0. It follows directly from the definition of the well-scaled sequence by checking that each of the coordinates of the simplices X_q, $q = 1, \ldots, k \cdot d + 1$, are in the correct annulus centered at x_0.

Lemma 8.1.1. *If $X \in \mathbf{S}^1_{k,p}$ and $\underline{X} = X \times Y_X$, then each term of the sequence $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ is well-scaled at x_0 and we have the following estimates:*

$$\alpha_0^{k+1-\lceil \frac{q}{d} \rceil} \cdot \max\nolimits_{x_0}(X) < \max\nolimits_{x_0}(X_q) \leq \alpha_0^{k-\lceil \frac{q}{d} \rceil} \cdot \max\nolimits_{x_0}(X), \tag{8.8}$$

if $1 \leq q \leq k \cdot d$, *and*

$$\alpha_0 \cdot \max\nolimits_{x_0}(X) < \min\nolimits_{x_0}(X_q) \leq \max\nolimits_{x_0}(X_q) = \max\nolimits_{x_0}(X), \tag{8.9}$$

if $q = k \cdot d + 1$.

8.1.4. Augmented sets and a discrete multiscale inequality

Lemma 8.1.1 assures us that the elements X_q have the correct structure in terms of relative scale, but we still have to assure ourselves that we can pick the sequence $\Phi(\underline{X}) = \{X_q\}_{q=1}^{k \cdot d+1}$ so that it satisfies the inequality of equation (8.1). This is easy to accomplish as long as we impose some conditions on this sequence as well as the auxiliary sequence as we are choosing them. We clarify this as follows.

Using the constant C_{p} of Proposition 3.3, we form the *set augmentation* of the set $\mathbf{S}^1_{k,p}$, denoted by $\underline{\mathbf{S}^1_{k,p}}$, as follows:

(8.10)

$$\underline{\mathbf{S}^1_{k,p}} = \Big\{ \underline{X} \in \mathbf{S}^1_{k,p} \times [\mathrm{supp}(\mu)]^{k \cdot d} : \ \widetilde{\Phi}_k(\underline{X}) \text{ and } \Phi_k(\underline{X}) \text{ satisfy the inequality}$$
$$\mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_q) \leq C_{\mathrm{p}} \cdot \big(\mathrm{p}_d\mathrm{sin}_{x_0}(X_{q+1}) + \mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_{q+1}) \big) \ \ \forall\, 0 \leq q < k \cdot d \Big\}.$$

We note that this set has some complicated structure and that it is not simply a product set. Essentially it is a set of augmented elements such that each coordinate is conditioned on the previous coordinates in such a way as to satisfy a sequence of two-term inequalities. In this way the well-scaled sequence $\Phi(\underline{X})$ satisfies an inequality of the form in equation (8.1) by simply iterating the two-term inequality of equation (8.10).

In fact, the sets $\underline{\mathbf{S}}^1_{k,p}$ give rise to the following multiscale inequality, whose direct proof (which we omit) is based on a simple iterative argument followed by an application of the Cauchy-Schwartz inequality.

Lemma 8.1.2. *If $\underline{X} \in \underline{\mathbf{S}}^1_{k,p}$, then the elements of the corresponding well-scaled sequence $\Phi_k(\underline{X}) = \{X_q\}_{q=1}^{k\cdot d+1}$ satisfy the inequality*

$$\mathrm{p}_d\mathrm{sin}^2_{x_0}(X) \leq (k \cdot d + 1) \cdot C_{\mathrm{p}}^{2\cdot k\cdot d} \sum_{q=1}^{k\cdot d+1} \mathrm{p}_d\mathrm{sin}^2_{x_0}(X_q)\,.$$

At this point, the motivation behind our construction of the well-scaled and auxiliary sequences may become somewhat more apparent. Our construction was governed by the iteration of the two-term inequality of equation (8.10), with the purpose of iteratively swapping out the simplices with bad scaling, $\widetilde{X}_q$, for a well-scaled simplex, X_q, and a simplex $\widetilde{X}_{q+1}$ whose scaling is slightly better than that of $\widetilde{X}_q$ because *fewer* of its edges are grossly disproportionate. In this way, we gradually move from bad to better, with each stage leaving us with an acceptable "remainder", i.e., X_q. We must be somewhat careful in this process, because if we move too quickly, somewhere down the line we will generate simplices with worse structure than what we want. This is the basic reason that we have d interpolating coordinates, y_i, in a given annulus. We must make sure that while we are "growing" the interpolating coordinates out from x_0, they still remain concordant with the length scales of the previous step.

8.2. Estimating the size of $\underline{\mathbf{S}}^1_{k,p}$

We show here that for any $X \in \mathbf{S}^1_{k,p}$, the corresponding "slice" in $\underline{\mathbf{S}}^1_{k,p}$ is uniformly "quite large" for each such X, and thus Lemma 8.1.2 can be applied somewhat indiscriminately. Later in Section 8.3 we will use that fact to show that the integral over $\mathbf{S}^1_{k,1}$, can be bounded in a meaningful way by a corresponding integral over the set $\underline{\mathbf{S}}^1_{k,1}$.

Once again, for the sake of clarity we need to rely on a bit of technical notation to account for the structure of the set $\underline{\mathbf{S}}^1_{k,p}$. While the notation is a bit cumbersome, we believe that the idea is simple enough.

8.2.1. Truncations and projections of $\underline{\mathbf{S}^1_{k,p}}$

We fix $0 \le q \le k \cdot d$. If $\underline{X} = (x_0, \dots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, then we define the q^{th} *truncation of* $\underline{X}$ to be the function $T_q : \underline{\mathbf{S}^1_{k,p}} \to H^{d+2+q}$, where

$$T_q\left(\underline{X}\right) = \begin{cases} X, & \text{if } q = 0; \\ (x_0, \dots, x_{d+1}, y_1, \dots, y_q), & \text{if } 1 \le q \le k \cdot d. \end{cases} \tag{8.11}$$

The $(d+2+q)$-tuple $T_q\left(\underline{X}\right)$ is not to be confused with the projection $(\underline{X})_q \in H$. If $A \subseteq \underline{\mathbf{S}^1_{k,p}}$, then we denote the image of A by $T_q(A)$.

For $\underline{X} = (x_0, \dots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, we denote the pre-image of $T_q\left(\underline{X}\right) = (x_0, \dots, y_q)$ by

$$T_q^{-1}(x_0, \dots, y_q) = \left\{ \underline{X}' \in \underline{\mathbf{S}^1_{k,p}} : \ T_q\left(\underline{X}'\right) = T_q\left(\underline{X}\right) = (x_0, \dots, y_q) \right\}, \tag{8.12}$$

where $(x_0, \dots, y_q)$ is taken to mean X if $q = 0$.

Now, fixing $1 \le q \le k \cdot d$, we define the q^{th} projection of $\underline{X}$ onto H. For $\underline{X} = (x_0, \dots, y_{k \cdot d}) \in \underline{\mathbf{S}^1_{k,p}}$, let

$$\pi_q(\underline{X}) = y_q = \left(\underline{X}\right)_{d+1+q}.$$

The set $\pi_q\left(T_{q-1}^{-1}\left(x_0, \dots, y_{q-1}\right)\right)$ is composed of all possible q^{th} coordinates of the well-scaled pieces $Y_X = (y_1, \dots, y_{k \cdot d})$ such that $\underline{X}' = X \times Y_X \in T_{q-1}^{-1}(x_0, \dots, y_{q-1})$.

We note that in this way we are giving another clear indication of how the coordinates y_q of $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ are "conditioned" on the previous coordinates $(x_0, \dots, x_{d+1}, \dots, y_{q-1})$.

8.2.2. The "size" of $\underline{\mathbf{S}^1_{k,p}}$

For any $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and all $1 \le q \le k \cdot d$, we define the functions

$$g^1_{k,q}\left(\underline{X}\right) = \mu\left(\pi_q\left(T_{q-1}^{-1}\left(T_{q-1}\left(\underline{X}\right)\right)\right)\right). \tag{8.13}$$

The double subscript in this case is not intended to evoke the index $p = 1, 2$. It is important to note that the functions $g^1_{k,q}(\underline{X})$ are independent of the values of the coordinates y_ℓ for $\ell > q$, and as such, they are well defined on the truncations $T_q(\underline{X})$. That is, we always have the equality (with a slight abuse of notation)

$$g^1_{k,q}(\underline{X}) = g^1_{k,q}(T_q(\underline{X}))$$

For $1 \le q \le k \cdot d$, the following proposition (which is proved in Appendix A.3) estimates the sizes of the functions $g^1_{k,q}$ (defined on $\underline{\mathbf{S}^1_{k,p}}$):

Proposition 8.1. *If $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and $1 \leq q \leq k \cdot d$, then*

(8.14)

$$\mu\Big(B(x_0, \alpha_0^{k-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X))\Big) \geq g^1_{k,q}(\underline{X}) \geq \frac{1}{2} \cdot \mu\Big(B(x_0, \alpha_0^{k-\lceil \frac{q}{d} \rceil} \cdot \max_{x_0}(X))\Big).$$

8.3. Multiscale integral inequality

Taking the product of the functions $g^1_{k,q}$ over q we have the strictly positive function

$$f^1_k(\underline{X}) = \prod_{q=1}^{k \cdot d} g^1_{k,q}(\underline{X}). \tag{8.15}$$

Thus, if we take the measure induced by

$$\left.\frac{\mathrm{d}\mu^{d+2+k\cdot d}(\underline{X})}{f^1_k(\underline{X})}\right|_{\underline{X} \in \underline{\mathbf{S}^1_{k,p}}}, \tag{8.16}$$

then we essentially obtain the measure $\mathrm{d}\mu^{d+2}(X)$ over $\mathbf{S}^1_{k,p}$ modified by a set of conditional probability distributions on the coordinates y_q for $1 \leq q \leq k \cdot d$.

As such, integrating a function of X according to the measure of equation (8.16) reduces (after Fubini's Theorem) to integrating the function with respect to μ^{d+2}. In this way we can turn Lemma 8.1.2 into a meaningful integral inequality. We formulate such an inequality as follows, while using the notation

$$N_k = (k+1) \cdot d + 2.$$

Proposition 8.2. *If Q is a ball in H, $k \geq 3$, and $p = 1, 2$, then*

$$\int_{\mathbf{S}^1_{k,p}(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2} \leq \tag{8.17}$$

$$\leq (k \cdot d + 1) \cdot C_{\mathrm{p}}^{2 \cdot k \cdot d} \sum_{q=1}^{k \cdot d+1} \int_{\underline{\mathbf{S}^1_{k,p}(Q)}} \frac{\mathrm{p}_d\sin^2_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})}.$$

Proof. This proposition is a direct consequence of Lemma 8.1.2 and the following equation

$$\int_{\mathbf{S}^1_{k,p}(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X) = \tag{8.18}$$

$$= \int_{\underline{\mathbf{S}^1_{k,p}(Q)}} \frac{\mathrm{p}_d\sin^2_{x_0}(X)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})}, \ \forall \text{ ball } Q \subseteq H.$$

For simplification, we prove equation (8.18) for $Q = H$, however, the idea applies to any ball Q in H. First, using Fubini's Theorem we obtain

$$\int_{\underline{\mathbf{S}^1_{k,p}}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})} = \tag{8.19}$$

$$= \int_{\mathbf{S}^1_{k,p}} \frac{\mathrm{p}_d \sin^2_{x_0}(X)}{\operatorname{diam}(X)^{d(d+1)}} \left(\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} \right) \mathrm{d}\mu^{d+2}(X).$$

Then, we iterate the inner integral on the RHS of equation (8.19) and get

$$\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} \tag{8.20}$$

$$= \int_{\pi_1\left(T_0^{-1}(X)\right)} \cdots \int_{\pi_q\left(T_{q-1}^{-1}(x_0,\ldots,y_{q-1})\right)} \cdots \int_{\pi_{k\cdot d}\left(T_{k\cdot d-1}^{-1}(x_0,\ldots,y_{k\cdot d-1})\right)} \frac{\mathrm{d}\mu\,(y_{k\cdot d}) \cdots \mathrm{d}\mu(y_1)}{\prod_{q=1}^{k\cdot d} g^1_{k,q}\,(\underline{X})},$$

for the sets $\pi_q\left(T_{q-1}^{-1}(x_0,\ldots,y_{q-1})\right)$ conditionally defined given

$$y_{q-1} \in \pi_{q-1}\left(T_{q-2}^{-1}(x_0,\ldots,y_{q-2})\right), \qquad \text{for } 2 \le q \le k\cdot d.$$

For any fixed $(x_0,\ldots,y_{q-1}) \in H^{(d+1+q)}$, by the definition of $g^1_{k,q}$ we have the equality

$$\int_{\pi_q\left(T_{q-1}^{-1}(x_0,\ldots,y_{q-1})\right)} \frac{\mathrm{d}\mu(y_q)}{g^1_{k,q}\,(\underline{X})} = \tag{8.21}$$

$$= \int_{\pi_q\left(T_{q-1}^{-1}(x_0,\ldots,y_{q-1})\right)} \frac{\mathrm{d}\mu(y_q)}{\mu\left(\pi_q\left(T_{q-1}^{-1}(x_0,\ldots,y_{q-1})\right)\right)} = 1.$$

Applying this to the iterated integral on the RHS of equation (8.20) we obtain

$$\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} = 1, \text{ for all } X \in \mathbf{S}^1_{k,p}. \tag{8.22}$$

Combining equations (8.19) and (8.22), we conclude equation (8.18) and the proposition. ■

Remark 8.1. *Since we do not take the time to establish the measurability of* f^1_k*, one can instead follow an alternative strategy: Proposition* 8.1 *implies that*

$$f^1_k \approx \prod_{q=1}^{k\cdot d} \mu\left(B(x_0, \alpha_0^{k-\lceil \frac{q}{d} \rceil} \cdot \max\nolimits_{x_0}(X))\right),$$

where the constant of comparability is at worst $2^{k\cdot d}$*. The latter function is clearly measurable in* $\underline{X}$*, and one can thus use it instead of* f_k^1 *for Proposition* 8.2*. Nevertheless such a strategy will increase the estimate on the constant* C_1 *of Theorem* 1.1*, and requires a slight change in the choice of* α_0 *in equation* (2.2)*.*

8.4. Concluding the proof of Proposition 6.2

The rest of our analysis for Proposition 6.2 consists of taking each term from the RHS of equation (8.17) and adapting the argument of Proposition 6.1. The basic idea is to chop the set $\underline{\mathbf{S}^1_{k,1}(Q)}$ in a multiscale fashion depending on the relative sizes of X_q and X. This requires some careful bookkeeping, but it allows us to involve the very small (relative to $\operatorname{diam}(X)$) length scales in the integration, and this will produce a constant (uniform in q) that decays rapidly enough with k.

More specifically, we verify the following proposition, whose combination with Proposition 8.2 establishes Proposition 6.2.

Proposition 8.3. *If* $1 \leq q \leq k \cdot d + 1$*, then there exists a constant* $C_7 = C_7(d, C_\mu)$ *such that*

$$\int_{\underline{\mathbf{S}^1_{k,1}(Q)}} \frac{\mathrm{p}_d\mathrm{sin}_{x_0}(X_q)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})} \leq C_7 \cdot \alpha_0^{k\cdot d^2} \cdot J_d^{\mathcal{D}}(\mu|_Q) \tag{8.23}$$

for any ball $Q \subseteq H$*.*

Proof of Proposition 8.3. We partition the sets $\underline{\mathbf{S}^1_{k,1}}(Q)$ by the size of $\max_{x_0}(X)$. If $m \geq m(Q)$, then let

$$\underline{\mathbf{S}^1_{k,1}(m)(Q)} = \left\{ \underline{X} \in \underline{S_1^{k,1}(\widetilde{\eta})(Q)} : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m] \right\}. \tag{8.24}$$

Throughout the rest of the proof we fix $m \geq m(Q)$ and $1 \leq q \leq k\cdot d+1$, and we further partition the set $\underline{\mathbf{S}^1_{k,p}(m)(Q)}$ according to location in $\operatorname{supp}(\mu)$ in order to reflect the quantity $\max_{x_0}(X_q)$. Specifically, we define the *scale exponent of* m *and* q

$$\operatorname{sc}(m,q) = \begin{cases} m + k - \left\lceil \frac{q}{d} \right\rceil, & \text{if } 1 \leq q \leq k \cdot d; \\ m, & \text{if } q = k \cdot d + 1. \end{cases} \tag{8.25}$$

The exponent $\operatorname{sc}(m,q)$ indicates the correct length scale for the decomposition of the set $\underline{\mathbf{S}^1_{k,1}(m)(Q)}$. Specifically, according to the estimates of Lemma 8.1.1, we have that

$$\max_{x_0}(X_q) \in \left(\alpha_0^{\operatorname{sc}(m,q)+2}, \alpha_0^{\operatorname{sc}(m,q)}\right].$$

Thus, for $j \in \Lambda_{\mathrm{sc}(m,q)}(Q)$, we let

$$\underline{P_{\mathrm{sc}(m,q),j}} = \left\{ \underline{X} \in \underline{\mathbf{S}^1_{k,1}(m)} : x_0 \in P_{\mathrm{sc}(m,q),j} \right\}, \tag{8.26}$$

and obtain the following cover of $\underline{\mathbf{S}^1_{k,1}(m)(Q)}$:

$$\underline{\mathcal{P}_{\mathrm{sc}(m,q)}}(Q) = \left\{ \underline{P_{\mathrm{sc}(m,q),j}} \right\}_{j \in \Lambda_{\mathrm{sc}(m,q)}(Q)}. \tag{8.27}$$

Letting $m \geq m(Q)$ and j vary, the LHS of equation (8.23) satisfies the following inequality:

$$\begin{aligned}\int_{\underline{\mathbf{S}^1_{k,1}}(Q)} \frac{\mathrm{p}_d \sin_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})} \leq \\ \leq \sum_{m \geq m(Q)} \left[\sum_{j \in \Lambda_{\mathrm{sc}(m,q)}(Q)} \int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{\mathrm{p}_d \sin_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})} \right].\end{aligned} \tag{8.28}$$

Fixing $j \in \Lambda_{\mathrm{sc}(m,q)}(Q)$, in addition to the fixed k, m and q, we will establish that

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{\mathrm{p}_d \sin_{x_0}(X_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f^1_k(\underline{X})} \leq C_1 \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2 \left(B_{\mathrm{sc}(m,q),j} \right) \cdot \mu \left(B_{\mathrm{sc}(m,q),j} \right). \tag{8.29}$$

Equations (8.28) and (8.29) will directly imply Proposition 8.3.

We prove equation (8.29) as follows. Fix an arbitrary d-plane L. Since the elements $\{X_q\}_{q=1}^{k \cdot d+1}$ are well-scaled at x_0, by Proposition 3.2 and equation (3.11) we have the following bound on the LHS of equation (8.29):

$$\begin{aligned}\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \mathrm{p}_d \sin^2_{x_0}(X_q) \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{\mathrm{diam}(X)^{d(d+1)} \cdot f^1_k(\underline{X})} \leq \\ \leq \frac{2 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha_0^6} \int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2^2(X_q, L)}{\mathrm{diam}(X_q)^2} \cdot \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{\mathrm{diam}(X)^{d(d+1)} \cdot f^1_k(\underline{X})}.\end{aligned} \tag{8.30}$$

To bound the RHS of equation (8.30) we focus on the individual terms of

$$\frac{D_2^2(X_q, L)}{\mathrm{diam}^2(X_q)} = \sum_{s=0}^{d+1} \frac{\mathrm{dist}^2((X_q)_s, L)}{\mathrm{diam}^2(X_q)}.$$

We arbitrarily fix $0 \leq s \leq d+1$ and note the following cases of possible values of $(X_q)_s$:

Case 1: $(X_q)_s = x_0$. In this case q has no restriction, that is, $1 \le q \le k\cdot d+1$.

Case 2: $(X_q)_s = x_1$. In this case $q = k \cdot d + 1$ (see equations (8.6)-(8.7)) and thus $\mathrm{sc}(m,q) = m$.

Case 3: $(X_q)_s = x_i$, where $2 \le i \le d+1$. In this case $1 \le q \le d$ (see equations (8.6)-(8.7)) and thus $\mathrm{sc}(m,q) = m + k - 1$.

Case 4: $(X_q)_s = y_\ell$, where $1 \le \ell \le k\cdot d$. In this case for each $1 \le q \le k\cdot d+1$, we have the following restriction on ℓ: $\max\{1, q-d\} \le \ell \le q$ and thus

$$\max\left\{1, \left\lceil\frac{q}{d}\right\rceil - 1\right\} \le \left\lceil\frac{\ell}{d}\right\rceil \le \left\lceil\frac{q}{d}\right\rceil . \tag{8.31}$$

The calculation of the upper bound varies slightly according to which case we consider.

Considering the first three cases simultaneously, we let $0 \le i \le d+1$ and examine the integrals decomposing the RHS of equation (8.30), each of the form

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot \mathrm{diam}(X)^{d(d+1)}}.$$

Per Fubini's Theorem we obtain the equality

$$\begin{aligned}\int_{\underline{P_{\mathrm{sc}(m,q),j}}} &\left(\frac{\mathrm{dist}\,(x_i, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot \mathrm{diam}(X)^{d(d+1)}} = \\ &= \int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \left(\int_{\left\{Y_X : X\times Y_X \in \underline{P_{\mathrm{sc}(m,q),j}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{\mathrm{diam}^2(X_q)\cdot f_k^1(\underline{X})}\right) \\ &\qquad\qquad \cdot \mathrm{dist}^2(x_i, L)\, \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}} .\end{aligned} \tag{8.32}$$

To bound the inner integral on the RHS of equation (8.32) we first note that

$$\mathrm{diam}(X) \ge \max\nolimits_{x_0}(X) = \|x_0 - x_1\| > \alpha_0^{m+1} \quad \text{for all } X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right). \tag{8.33}$$

Combining Lemma 8.1.1 with equations (8.25) and (8.33) we see that

$$\mathrm{diam}(X_q) \ge \max\nolimits_{x_0}(X_q) > \alpha_0^{\mathrm{sc}(m,q)+1}. \tag{8.34}$$

Since $\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right) = 8\cdot \alpha_0^{\mathrm{sc}(m,q)}$, we rewrite equation (8.34) as follows:

$$\mathrm{diam}(X_q) \ge \frac{\alpha_0}{8}\cdot \mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right). \tag{8.35}$$

Combining equations (8.22) and (8.35), we obtain that if $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$, then

$$\int_{\left\{Y_X : X\times Y_X \in \underline{P_{\mathrm{sc}(m,q),j}}\right\}} \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f_k^1(\underline{X})\cdot \mathrm{diam}^2(X_q)} \leq \frac{64}{\alpha_0^2}\cdot \frac{1}{\mathrm{diam}^2(B_{\mathrm{sc}(m,q),j})}. \tag{8.36}$$

By the definition of $\underline{P_{\mathrm{sc}(m,q),j}}$, $\underline{\mathbf{S}^1_{k,1}(m)}$ and $\mathbf{S}^1_{k,1}(m)$ we have the equality

$$T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right) = \bigcup_{x_0\in P_{\mathrm{sc}(m,q),j}} \left[\bigcup_{x_1\in A_m(x_0,1)} \{(x_0,x_1)\}\times [A_k(x_0,\|x_1-x_0\|)]^d\right].$$

From this we trivially obtain the inclusion

$$T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right) \subseteq \bigcup_{x_0\in P_{\mathrm{sc}(m,q),j}} \{x_0\}\times B(x_0,\alpha_0^m)\times \left[B(x_0,\alpha_0^{m+k})\right]^d.$$

Applying this together with the inequalities of equations (8.33) and (8.36) to the RHS of equation (8.32) gives the following inequality for all $0\leq i\leq d{+}1$:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(x_i,L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot \mathrm{diam}(X)^{d(d+1)}} \leq \tag{8.37}$$
$$\leq \frac{64}{\alpha_0^{d(d+1)+2}} \int_{P_{\mathrm{sc}(m,q),j}} \int_{B(x_0,\alpha_0^m)} \int_{\left[B(x_0,\alpha_0^{m+k})\right]^d} \left(\frac{\mathrm{dist}\,(x_i,L)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{d+2}(X)}{\left[\alpha_0^m\right]^{d(d+1)}}.$$

Assume Case 1, that is, $i=0$. Then, after iterating the integral on the RHS of equation (8.37), applying the defining property of d-regular measure and the inclusion $P_{\mathrm{sc}(m,q),j}\subseteq B_{\mathrm{sc}(m,q),j}$, we see that the term on the RHS of equation (8.37) has the bound

$$\frac{64\cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+2}}\cdot \alpha_0^{k\cdot d^2}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j},L\right)\cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \tag{8.38}$$

Assume Case 2, that is, $i=1$, and recall that in this case $q=k\cdot d+1$ and $\mathrm{sc}(m,q)=m$. Thus we have the inclusion

$$B(x_0,\alpha_0^m)\subseteq B_{\mathrm{sc}(m,q),j}, \text{ for all } x_0\in P_{\mathrm{sc}(m,q),j}.$$

Hence, iterating the integral on the RHS of equation (8.37) and then applying similar arguments to Case 1, we obtain the following bound for the LHS of equation (8.37):

$$\frac{64\cdot 4^d\cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+2}}\cdot \alpha_0^{k\cdot d^2}\cdot \beta_2^2\left(B_{m,j},L\right)\cdot \mu\left(B_{m,j}\right). \tag{8.39}$$

Next, assume Case 3, that is, $2 \leq i \leq d+1$ and recall that in this case $1 \leq q \leq d$ and $\mathrm{sc}(m,q) = m+k-1$. Using the fact that $P_{\mathrm{sc}(m,q),j} \subseteq \frac{3}{4} \cdot B_{\mathrm{sc}(m,q),j}$, and the defining property of d-regular measures we have the inequality

$$\mu\left(P_{\mathrm{sc}(m,q),j}\right) \leq \mu\Big(\frac{3}{4} \cdot B_{\mathrm{sc}(m,q),j}\Big) \leq C_\mu \cdot \left(3 \cdot \alpha_0^{m+k-1}\right)^d .$$

Furthermore, we have the inclusion

$$B\left(x_0, \alpha_0^{m+k}\right) \subseteq B_{\mathrm{sc}(m,q),j}, \text{ for all } x_0 \in P_{\mathrm{sc}(m,q),j}.$$

Iterating the integral as in the previous calculations, the LHS of equation (8.37) is bounded by

$$\frac{64 \cdot 3^d \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+d+2}} \cdot \alpha_0^{k \cdot d^2} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right) . \tag{8.40}$$

Therefore, taking the maximal coefficient from equations (8.38), (8.39) and (8.40), the LHS of equation (8.37) has the following uniform bound for all $0 \leq i \leq d+1$:

$$\frac{3^d \cdot 2^7 \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+d+2}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right) . \tag{8.41}$$

At last we consider Case 4, where the terms in the sum comprising the RHS of equation (8.30) are of the form:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \operatorname{diam}(X)^{d(d+1)}} ,$$

where $1 \leq l \leq k \cdot d$.

Iterating the integral and applying equation (8.35), we obtain

$$\begin{aligned} &\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X}) \cdot \operatorname{diam}(X)^{d(d+1)}} \leq \\ &\leq \frac{64}{\alpha_0^2} \int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \left(\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}_{k,p}^1}\right\}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k \cdot d}(Y_X)}{f_k^1(\underline{X})}\right) \\ &\qquad \cdot \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}} . \end{aligned} \tag{8.42}$$

In order to bound the RHS of equation (8.42), we first calculate a uniform bound in $1 \leq l \leq k \cdot d$ for the interior integral. Then, completing the integration with respect to $X \in T_0\big(\underline{P_{\mathrm{sc}(m,q),j}}\big)$ will give the desired bound in terms of the corresponding β_2 number.

For fixed $X \in T_0\big(\underline{P_{\mathrm{sc}(m,q),j}}\big)$ and $1 \le \ell \le k \cdot d$, after iterating the interior integral on the RHS of equation (8.42) and applying equation (8.21) we have that

$$\int_{\left\{Y_X : X \times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f^1_k(\underline{X})} = \tag{8.43}$$

$$= \int_{\pi_1\left(T_0^{-1}(X)\right)} \cdots \int_{\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\dots,y_{\ell-1})\right)} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu\,(y_\ell)\cdots \mathrm{d}\mu(y_1)}{\prod_{s=1}^{\ell} g^1_{k,s}},$$

where we used the notation $g^1_{k,s}$ defined in equation (8.13). Given $\underline{X} \in \underline{P_{\mathrm{sc}(m,q),j}}$ we fix $(x_0,\dots,y_{\ell-1}) = T_{\ell-1}(\underline{X})$ and calculate a bound for the integral

$$\int_{\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\dots,y_{\ell-1})\right)} \left(\frac{\operatorname{dist}(y_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu\,(y_\ell)}{g^1_{k,\ell}}.$$

We first obtain an upper bound for

$$\frac{1}{g^1_{k,\ell}} = \frac{1}{\mu\left(\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\dots,y_{\ell-1})\right)\right)},$$

and then complete the integration.

To obtain that bound, we apply Proposition 8.1 to get that for all $1 \le \ell \le k \cdot d$:

$$\mu\left(\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\dots,y_{\ell-1})\right)\right) \ge \frac{1}{2}\cdot \mu\left(B\left(x_0, \alpha_0^{k-\lceil \frac{\ell}{d}\rceil}\cdot \max_{x_0}(X)\right)\right). \tag{8.44}$$

Next, applying equations (8.25), (8.31) and (8.33) as well as the fact that $\alpha_0 < 1$, we note that

$$\alpha_0^{k-\lceil \frac{\ell}{d}\rceil}\cdot \max_{x_0}(X) \ge \alpha_0^{k-\lceil \frac{\ell}{d}\rceil+m+1} \ge \alpha_0^{k-\lceil \frac{q}{d}\rceil+m+2} = \alpha_0^{\mathrm{sc}(m,q)+2}. \tag{8.45}$$

We note that the RHS of equation (1.1) extends to all $r > 0$ and apply it to obtain the following bound

$$\mu\left(B\left(x_0, \alpha_0^{\mathrm{sc}(m,q)+2}\right)\right) \ge \frac{1}{C_\mu^2}\cdot\left(\frac{\alpha_0^2}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \tag{8.46}$$

Finally, combining equations (8.44), (8.45) and (8.46) we conclude that

$$\mu\left(\pi_\ell\left(T^{-1}_{\ell-1}(x_0,\dots,y_{\ell-1})\right)\right) \ge \frac{1}{2\cdot C_\mu^2}\cdot\left(\frac{\alpha_0^2}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right). \tag{8.47}$$

Noting that $\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right) \subseteq B_{\mathrm{sc}(m,q),j}$ and applying equation (8.47), we have the inequality

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)} \left(\frac{\mathrm{dist}\,(y_\ell, L)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \cdot \frac{\mathrm{d}\mu(y_\ell)}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,y_{\ell-1})\right)\right)} \leq$$

$$\leq \frac{2\cdot 4^d \cdot C_\mu^2}{\alpha_0^{2\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}\right). \tag{8.48}$$

Then, applying this and equation (8.21) to the RHS of equation (8.43), we have the following inequality for all $X \in T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)$:

(8.49)

$$\int_{\left\{Y_X : X\times Y_X \in \underline{\mathbf{S}^1_{k,p}}\right\}} \left(\frac{\mathrm{dist}\,(y_\ell, L)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k\cdot d}(Y_X)}{f_k^1(\underline{X})} \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{2\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,q),j}\right).$$

Furthermore, noting that

$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,q),j}}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}} \leq \frac{C_\mu^{d+1}}{\alpha_0^{d(d+1)}} \cdot \alpha_0^{k\cdot d^2} \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right),$$

per equations (8.42) and (8.49), we have the following uniform bound for all $1 \leq \ell \leq k\cdot d$:

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \left(\frac{\mathrm{dist}\,(y_\ell, L)}{\mathrm{diam}(X_q)}\right)^2 \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot \mathrm{diam}(X)^{d(d+1)}} \leq \tag{8.50}$$

$$\leq \frac{128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2} \cdot \beta_2^2(B_{\mathrm{sc}(m,q),j}) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right).$$

Finally, taking largest coefficient from equations (8.41) and (8.50), we have the bound

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2\,(X_q, L)}{\mathrm{diam}^2(X_q)} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot\mathrm{diam}(X)^{d(d+1)}} \leq$$

$$\leq \frac{(d+2)\cdot 128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2} \cdot \beta_2^2(B_{\mathrm{sc}(m,q),j}, L) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right).$$

Therefore, taking the infimum over all such d-planes L we obtain the bound

$$\int_{\underline{P_{\mathrm{sc}(m,q),j}}} \frac{D_2\,(X_q, L)}{\mathrm{diam}^2(X_q)} \frac{\mathrm{d}\mu^{N_k}(\underline{X})}{f_k^1(\underline{X})\cdot\mathrm{diam}(X)^{d(d+1)}} \leq$$

$$\leq \frac{(d+2)\cdot 128\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d^2+3\cdot d+2}} \cdot \alpha_0^{k\cdot d^2} \cdot \beta_2^2(B_{\mathrm{sc}(m,q),j}) \cdot \mu\left(B_{\mathrm{sc}(m,q),j}\right).$$

Combining this with equation (8.30) establishes the conclusion of equation (8.29). ■

Remark 8.2. *The interpolation procedure applied above verifies the intuitive idea that the small length scales ought not to contribute much information when the overall length scale of a simplex is large. One reason that our argument will not work for the curvature specified in equation* (1.2) *is that such a curvature is "resistant" to our current interpolation procedure. The gains of small length scales made via the interpolation is nullified by the very singular behavior of this curvature on simplices with bad scaling. In a sense, this curvature incorporates too much information for our current method to handle, and if the continuous version of this curvature can be bounded by the Jones-type flatness (in the spirit of Theorem* 1.1)*, then we must use a different method to show it.*

9. Proof of Proposition 6.3

Due to the similarity of various parts of the proof of Proposition 6.3 with the ideas and computations of Section 8, this current section focuses mostly on the new ideas required to prove the proposition, including some technical notation and statements. Computations and ideas presented previously are referenced to as needed.

We define the constant

$$N_n = 2^{n-1} - 1,$$

and we remark that N_n and N_k (defined in Subsection 8.3) are two different constants. We also define the constant

$$M_n = d + 2 + N_n = d + 1 + 2^{n-1}.$$

We recall that $\mathbf{S}^n_{k,1}$ is the set of multi-handled rakes whose handles occur at their first n coordinates and whose tines occur at their last $d+1-n$ coordinates.

Here we adapt the methods of Section 8 to deal with the problems present in integrating the curvature over the regions $\mathbf{S}^n_{k,1}$. Our adaptation consists of two stages. First we split the simplex $X \in \mathbf{S}^n_{k,1}$ into a sequence of *single-handled* rakes using an interpolation procedure similar to that of section 8. The basic idea is to use such a procedure to "break off" each of the n handles from the simplex X, thereby forming a sequence of single-handled rakes with elements denoted by X^s. This procedure generates an "augmentation" of $\mathrm{d}\mu^{d+2}(X)$, as well as an integral inequality along the lines of Proposition 8.2 of Section 8.3.

Once we have made this exchange, we can again apply the methods of Section 8.1 to the single-handled rakes obtained from the first step and obtain the proper control in terms of the β_2 numbers.

The result is that we exchange integrals over $\mathrm{d}\mu^{d+2}(X)$ for integrals over a "doubly augmented" measure (depending on a much larger "variable") which allows us to incorporate the necessary information from small scales. Simply put, we iterate the methods of Section 8.1 in an highly adaptive way. We remark that the type of analysis done in this section was unnecessary in the case $d = 1$ due to the extreme simplicity of the "combinatorial structure" of triangles. Much of the work that we had to do revolved around dealing with the problems of disparities of scale that arise from the more complicated structure of $(d + 1)$-simplices for $d \geq 2$.

9.1. Rake sequences and pre-multiscale inequality

We define a *short-scale piece* for $X \in \mathbf{S}^n_{k,1}$ to be an N_n-tuple of the form

$$Z_X = (z_1, \ldots, z_{N_n}) \in [A_k(x_0, \max_{x_0}(X))]^{N_n}\,, \tag{9.1}$$

and illustrate it in Figure 5 (a).

For $X \in \mathbf{S}^n_{k,1}$ and Z_X we define an augmentation of X by Z_X as

$$\overline{X} = X \times Z_X = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}) \in \mathbf{S}^n_{k,1} \times H^{N_n}. \tag{9.2}$$

We note that $\overline{X} \in H^{M_n}$ and that all coordinates of Z_X are in the annulus centered at x_0 and determined by $\max_{x_0}(X)$.

For $\overline{X}$ we construct a sequence of *single-handled rakes*,

$$\Psi(\overline{X}) = \{X^s\}_{s=1}^{2^{n-1}}\,,$$

in H^{d+2} (à la the construction in Section 8.1.1) and use them to formulate an inequality for the polar sine on $\mathbf{S}^n_{k,1}$ á la equation (8.1). The major difference is that the X^s are not well-scaled, and the length of the sequence is determined by $1 < n \leq d$, not k. Despite the fact that they are not necessarily well-scaled, we can construct them so that their scaling is better than the original simplex X, i.e.,

$$\mathrm{scale}_{x_0}(X^s) \geq \mathrm{scale}_{x_0}(X).$$

Just as before, in order to get the type of sequence we want, we must first define an auxiliary sequence which will only be used to construct the desired type of sequence.

Definition 9.1. *If $X \in \mathbf{S}^n_{k,1}$ and $\overline{X} = X \times Z_X = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n})$, then let Z^j_m, $j = 0, \ldots, n-1$, $m = 1, \ldots, 2^j$, be the doubly indexed sequence of elements of H^{d+2} defined recursively as follows:*

$$Z^0_1 = X,$$

and for $0 \leq j < n-1$ and σ_j denoting the transposition of $n-j-1$ and $n-j$

(acting on Z_m^j by replacing its coordinates at those indices)

$$Z_{2m-1}^{j+1} = Z_m^j\left(n-j, z_{2^j+(m-1)}\right), \tag{9.3}$$

and

$$Z_{2m}^{j+1} = \sigma_j\left(Z_m^j\left(n-j-1, z_{2^j+(m-1)}\right)\right). \tag{9.4}$$

Realizing that this is a fairly technical definition, we remark that its purpose is only to give a sensible and formal framework for isolating the individual handles of the original simplex X. Furthermore, any such method must work under the restrictions imposed by the two-term inequality for the polar sine. As such, we must have some sort of iterative scheme which allows us to swap out one at a time. For this reason, we construct the different generations of the simplices Z_m^j, with the simplices of each successive generation having one less handle than the previous generation. The final generation will have only one handle, and this is the sequence of simplices that we really want to work with.

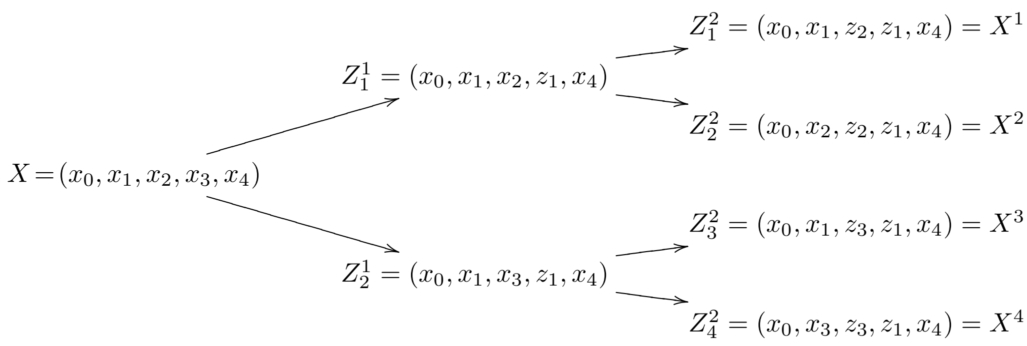
Using the $(n-1)^{\text{th}}$-generation of elements of the auxiliary sequence above we define the *rake sequence* $\Psi_k\left(\overline{X}\right)$ as follows.

Definition 9.2. *If $X \in \mathbf{S}_{k,1}^n$, $\overline{X} = X \times Z_X$, and Z_m^j as above, then let $\Psi_k\left(\overline{X}\right) = \{X^s\}_{s=1}^{2^{n-1}}$ be the sequence of elements in H^{d+2} such that*

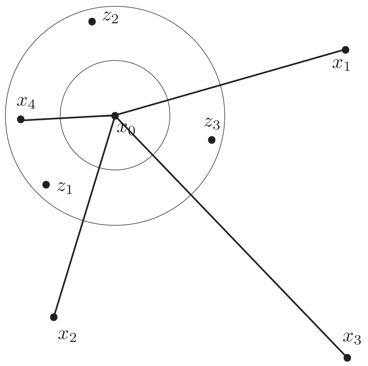
$$X^s = Z_s^{n-1}, \text{ for } 1 \le s \le 2^{n-1}. \tag{9.5}$$

We note that the simplex X^s has exactly three coordinates taken from the original simplex X: the "base" vertex x_0, the handle vertex x_{i_s} for some index $1 \le i_s \le n$, and the tine x_i for some $n+1 \le i \le d+1$. The rest of the coordinates are taken from the short scale piece Z_X. This fact is apparent from following the recursive definitions, as well as the fact that X^s can have only one possible handle, this handled being inherited from one of the handles for X.

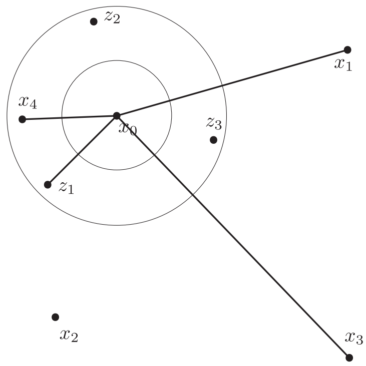
For example, if $d=3$ and $n=3$, then the simplex X has three handles located at the first 3 coordinates, with the 4^{th} coordinate being the tine. Hence the sequences $\widetilde{\Psi}_k\left(\overline{X}\right)$ and $\Psi_k\left(\overline{X}\right)$ fit into the following tree:



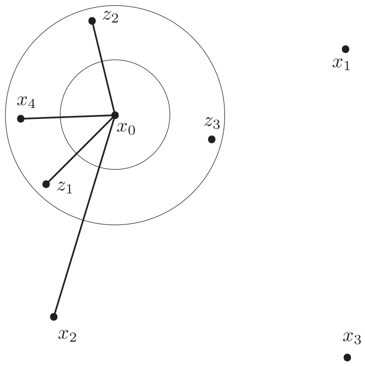
We illustrate the elements Z_2^1 and Z_2^2 of this tree in Figures 5 (b) and 5 (c) respectively.



(a) simplex X and Z_X



(b) auxiliary simplex Z_2^1



(c) rake simplex $X^2 = Z_2^2$

FIGURE 5: Illustration of the short-scale piece and elements of the auxiliary and rake sequences. Here X is a 3-handled rake. Note that the coordinates of Z_X are contained in a single annulus rather than many annuli as was the case for Y_X. The element $Z_2^1 = (x_0, x_1, x_3, z_1, x_4)$ of the auxiliary sequence has two handles rather than three (as X). The element $Z_2^2 = (x_0, x_2, z_2, z_1, x_4)$ has one handle and indeed it is in the rake sequence (denoted by X^2).

We note that the information provided by the original simplex $X = (x_0, x_1, x_2, x_3, x_4)$ has been "distributed" over the new sequence of simplices, and in a very real sense has been "decoupled". As such, this will allow us to pursue our analysis of these simplices more or less independently.

The following lemma establishes that all elements of $\Psi_k\left(\overline{X}\right)$ are single-handled rakes, and follows directly from the definition of the rake sequence.

Lemma 9.1.1. *If* $X \in \mathbf{S}_{k,1}^n$, $\overline{X} = X \times Z_X$, *and* $X^s \in \Psi_k\left(\overline{X}\right)$ *with* $1 \le s \le 2^{n-1}$, *then each* X^s *is a rake, and for* $0 \le k' \le k-1$, *we have that*

$$X^s \in S_{k',2}^1. \tag{9.6}$$

We remark that we take the tolerance for the subscript index to be $p = 2$ in Lemma 9.1.1, and this is simply because in our construction of the rakes, X^s, we lose a tiny bit of accuracy in determining the relative lengths of the smallest and largest edges at x_0. This slight change in accuracy is accounted for by the extra power of α_0 in determining the length of the interval that $\text{scale}_{x_0}(X^s)$ sits in.

Using the ideas of Section 8.2 we construct a set augmentation of $\mathbf{S}^n_{k,1}$, denoted by $\overline{\mathbf{S}^n_{k,1}}$, which has a uniformly large size for a given $X \in \mathbf{S}^n_{k,1}$ in the sense given by Proposition 8.1, and is such that we have a version of equation (8.1). Using the constant C_{p} of Proposition 3.3, we define the set augmentation $\overline{\mathbf{S}^n_{k,1}}$ as

$$\overline{\mathbf{S}^n_{k,1}} = \Big\{ \overline{X} \in \mathbf{S}^n_{k,1} \times [\text{supp}(\mu)]^{N_n} : \text{ for all } 0 \leq j < n-1 \text{ and } 1 \leq m \leq 2^j,$$

$$\text{the sequence } Z^j_m \text{ satisfies the inequalities:}$$

$$\mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^j_m\right) \leq C_{\mathrm{p}}\left[\mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m-1}\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(Z^{j+1}_{2m}\right)\right]\Big\}. \tag{9.7}$$

The sequence $\Psi_k\left(\overline{X}\right)$ gives rise to the following pre-multiscale inequality for the polar sine, which is analogous to Lemma 8.1.2 and similarly can be immediately proved by iterative application of the defining inequality followed by the Cauchy-Schwartz inequality.

Lemma 9.1.2. *If $\overline{X} \in \overline{\mathbf{S}^n_{k,1}}$, then the sequence of single-handled rakes $\Psi_k(\overline{X}) = \{X^s\}_{s=1}^{2^{n-1}}$ satisfy the inequality*

$$\mathrm{p}_d\mathrm{sin}^2_{x_0}(X) \leq 2^{n-1} \cdot C_{\mathrm{p}}^{2\cdot(n-1)} \sum_{s=1}^{2^{n-1}} \mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s).$$

Just as in Subsection 8.2, for $0 \leq s \leq N_n$ we define the truncations, T_s, the projections π_s, and the functions

$$g^n_{k,s}\left(\overline{X}\right) = \mu\left(\pi_s\left(T^{-1}_{s-1}\left(T_{s-1}(\overline{X})\right)\right)\right), \text{ for all } 1 \leq s \leq N_n. \tag{9.8}$$

These functions are again positive and satisfy similar estimates as before. Specifically, adapting the proof of Proposition 8.1 to our current purposes we can demonstrate the following.

Proposition 9.1. *If $\overline{X} \in \overline{\mathbf{S}^n_{k,1}}$ and $1 \leq s \leq N_n$, then*

$$\mu\left(B(x_0, \alpha_0^k \cdot \max_{x_0}(X))\right) \geq g^n_{k,s}\left(\overline{X}\right) \geq \frac{1}{2} \cdot \mu\left(B(x_0, \alpha_0^k \cdot \max_{x_0}(X))\right). \tag{9.9}$$

Analogous to the normalizing function f_k^1 of Section 8.3, we then define

$$f_k^n\left(\overline{X}\right) = \prod_{s=1}^{N_n} g_{k,s}^n\left(\overline{X}\right), \tag{9.10}$$

and the corresponding "augmentation" of the measure $\mathrm{d}\mu^{d+2}(X)$ (restricted to $X \in \mathbf{S}_{k,1}^n$)

$$\left.\frac{\mathrm{d}\mu^{M_n}(\overline{X})}{f_k^n(\overline{X})}\right|_{\overline{\mathbf{S}_{k,1}^n}}. \tag{9.11}$$

The following proposition is then established in parallel to Proposition 8.2 (i.e., applying the inequality of Lemma 9.1.2 together with a direct relation between plain and augmented integration as in equation (8.18)).

Proposition 9.2. *If Q is a ball in H, then*

$$\int_{\mathbf{S}_{k,1}^n(Q)} \frac{\mathrm{p}_d \sin_{x_0}^2(X)}{\operatorname{diam}(X)^{d(d+1)}} \,\mathrm{d}\mu^{d+2}(X) \leq$$
$$\leq 2^{n-1} \cdot C_{\mathrm{p}}^{2\cdot(n-1)} \sum_{s=1}^{N_n} \int_{\overline{\mathbf{S}_{k,1}^n(Q)}} \frac{\mathrm{p}_d \sin_{x_0}^2(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}.$$

We now focus on applying the methods of Section 8 to the individual terms on the RHS of the inequality above.

9.2. Generating multiscale discrete and integral inequalities

The individual integrals on the RHS of Proposition 9.2 are similar to the integral on the LHS of Proposition 6.2, mainly because the argument X^s, $1 \leq s \leq N_n$, is a rake. In principle, one would like to change variables in order to directly apply Proposition 6.2 to these integrals. We avoid this for various reasons. The most immediate is because the region $\overline{\mathbf{S}_{k,1}^n}$, $1 < n \leq d-1$, is relatively complicated, and any change of variables would be further obstructed by the normalization of the function f_k^n. Furthermore, for a rake X^s that is poorly-scaled, we must somehow use the small length scales present in X^s, and these may not be accessible via such a change of variables. As such, we find it more straightforward to adapt the methods of Section 8 to the individual terms on the RHS of Proposition 9.2.

Throughout the rest of this section, we fix $1 < n \leq d-1$, $k \geq 3$, and $1 \leq s \leq N_n$.

9.2.1. The decomposition of $\overline{\mathbf{S}^n_{k,1}}$ according to $\mathrm{scale}_{x_0}(X^s)$

We first decompose $\overline{\mathbf{S}^n_{k,1}}$ according to the variety of the rakes $\{X^s\}_{s=1}^{N_n}$, the only meaningful variation being the length of the handle and size of the function $\mathrm{scale}_{x_0}(X^s)$. Furthermore, following the argument leading to equation (3.19) we may assume that the handle occurs at the first coordinate of X^s.

The initial task is accounting for the discrepancies of scale for X^s, that is we must decompose the set $\overline{\mathbf{S}^n_{k,1}}$ according to $\mathrm{scale}_{x_0}(X^s)$. Let

$$\widehat{R}^s = \left\{\overline{X} \in \overline{\mathbf{S}^n_{k,1}} : X^s \text{ is well-scaled at } x_0\right\}.$$

By construction, the scaling of the simplices X^s is no worse (actually slightly better) than that of the original simplex X, and thus for $2 \le k' \le k-1$ we define the sets

$$R^s_{k'} = \left\{\overline{X} \in \overline{\mathbf{S}^n_{k,1}} : X^s \in \mathbf{S}^1_{k',2}\right\}.$$

Furthermore, if Q is a ball in H, then we restrict those sets to Q^{d+2} as before to obtain $\widehat{R}^s(Q)$ and $R^s_{k'}(Q)$, and we note the following set equality:

$$\overline{\mathbf{S}^n_{k,1}(Q)} = \widehat{R}^s(Q) \bigcup \bigcup_{k'=2}^{k-1} R^s_{k'}(Q). \tag{9.12}$$

This decomposition (which is not a partition because the sets $R^s_{k'}(Q)$ may overlap) yields the following inequality

$$\int_{\overline{\mathbf{S}^n_{k,1}(Q)}} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} \le \tag{9.13}$$

$$\le \int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} + \sum_{k'=2}^{k-1} \int_{R^s_{k'}(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)}.$$

Note that the inequality of equation (9.13) is analogous to the equality of equation (6.1), with one of the differences being the fact that we now only need to control a finite sum, rather than an infinite one. The rest of our efforts focus on showing that each term on the RHS above is "small" with respect to the quantity $J^{\mathcal{D}}_d(\mu|_Q)$, that is, we can control them by something that looks basically like $\alpha_0^{k\cdot d} \cdot J^{\mathcal{D}}_d(\mu|_Q)$.

The first term on the RHS of equation (9.13) can be controlled via geometric multipoles. In fact, via the well-scaling of X^s and the small length scales produced by the interpolation procedure, we get such control by chopping it according to the length scales of X^s, and then following the computations of Section 8.4 on these pieces. The result is the following proposition, whose proof appears in Appendix A.4

Proposition 9.3. *If Q is a ball in H, then there exists a constant $C_8 = C_8(d, C_\mu)$ such that*

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq C_8 \cdot \alpha_0^{k\cdot d\cdot(d-n+2)} \cdot J_d^{\mathcal{D}}(\mu|_Q)\,. \tag{9.14}$$

We note that the coefficient $\alpha_0^{k\cdot d\cdot(d-n+2)}$ is larger than the coefficient $\alpha_0^{k\cdot d^2}$ of Proposition 8.3, and this is because the n-handled simplex X has fewer small length scales to work with. In the previous case we had d "small" edges helping us obtain a sufficiently small coefficient in k. In the current case we have less help because now there are only $d+1-n$ small edges, and the coefficient is hence slightly larger.

The terms of the finite sum on the RHS of equation (9.13) require further analysis before we can establish the appropriate bounds, and we develop this in the rest of the section.

9.3. Doubly-augmented elements and a multiscale inequality

We fix $2 \leq k' \leq k-1$ and concentrate on the set $R^s_{k'}$. The integral over the augmented region $R^s_{k'}$ can be exchanged for yet another augmented integral, but we must perform two different types of augmentations. The first element is defined as follows. If $\overline{X} \in R^s_{k'}$, then we take a well-scaled piece for the rake $X^s \in \mathbf{S}^1_{k',2}$,

$$Y_{X^s} = (y_1, \ldots, y_{k'\cdot d}) \in \prod_{q=1}^{k'\cdot d} A_{k'-\lceil \frac{q}{d}\rceil}(x_0, \max\nolimits_{x_0}(X^s)),$$

and we form the "doubly-augmented" element

$$\overline{X} \times Y_{X^s} = (x_0, \ldots, x_{d+1}, z_1, \ldots, z_{N_n}, y_1, \ldots, y_{k'\cdot d}) \in R^s_{k'} \times H^{k'\cdot d}.$$

The "variable" $\overline{X} \times Y_{X^s}$ is the underlying piece of information that controls our process, but the actual simplex driving our decisions is X^s. As such, we introduce the symbol $\overline{X} \times Y_{X^s}$ for clarity, and we focus our development on another type of augmentation. We clarify this as follows.

If $\overline{X} \in R^s_{k'}$ and Y_{X^s} is a well-scaled piece for X^s, then we form the augmented element

$$\underline{X^s} = X^s \times Y_{X^s},$$

which is an augmentation of the type introduced in Subsection 8.1. We then form the sequences

$$\widetilde{\Phi}_{k'}(\underline{X^s}) = \left\{\widetilde{X}^s_q\right\}_{q=0}^{k'\cdot d} \quad \text{and} \quad \Phi_{k'}(\underline{X^s}) = \left\{X^s_q\right\}_{q=1}^{k'\cdot d+1}$$

as given in Definitions 8.1 and 8.2 of Subsection 8.1, and we note that these depend only on $\underline{X^s}$. Following this line of reasoning we define the set augmentation

$$\underline{R^s_{k'}} := \Big\{ \overline{X} \times Y_{X^s} : \overline{X} \in R^s_{k'} \text{ and the sequences } \widetilde{\Phi}_{k'}(\underline{X^s}) \text{ and } \Phi_{k'}(\underline{X^s}) \text{ satisfy, for all } 0 \leq q < k' \cdot d, \text{ the inequality}$$

$$\mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}^s_q\right) \leq C_{\mathrm{p}} \cdot \left[\mathrm{p}_d\mathrm{sin}_{x_0}\left(X^s_{q+1}\right) + \mathrm{p}_d\mathrm{sin}_{x_0}\left(\widetilde{X}^s_{q+1}\right)\right]\Big\}. \tag{9.15}$$

We have the following inequality which is a direct application of Lemma 8.1.2.

Lemma 9.3.1. *If* $\overline{X} \times Y_{X^s} \in \underline{R^s_{k'}}$, *then the well-scaled sequence* $\Phi_{k'}(\underline{X^s}) = \left\{X^s_q\right\}_{q=1}^{k'\cdot d+1}$ *satisfies the inequality*

$$\mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s) \leq (k' \cdot d + 1) \cdot C_{\mathrm{p}}^{2\cdot k' \cdot d} \sum_{q=1}^{k'\cdot d+1} \mathrm{p}_d\mathrm{sin}^2_{x_0}\left(X^s_q\right).$$

With these definitions, we formulate our ultimate multiscale integral inequality whose proof is hardly different from the proof of Proposition 9.2.

Lemma 9.3.2. *If* Q *is a ball in* H *then*

$$\int_{R^s_{k'}(Q)} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f^n_k\left(\overline{X}\right)} \leq$$

$$\leq C_{\mathrm{p}}^{2k'\cdot d} \cdot (k' \cdot d + 1) \sum_{q=1}^{k'\cdot d+1} \int_{\underline{R^s_{k'}(Q)}} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f^n_k\left(\overline{X}\right) \cdot f^1_{k'}\left(\underline{X^s}\right)}.$$

The terms on the RHS of Lemma 9.3.2 can be controlled by geometric multipoles as in the proofs of Propositions 6.2 and 9.3. However, the computations here have more information to take into account because we are dealing with various length scales at the same time, i.e., those of the doubly-indexed well-scaled element X^s_q as well as the original simplex X. We perform these computations in Appendix A.5 and conclude the following bounds on the terms on the RHS of Lemma 9.3.2 and thus also Proposition 6.3.

Proposition 9.4. *If* Q *is a ball in* H *and* $2 \leq k' \leq k - 1$, *then there exists a constant* $C_9 = C_9(d, C_\mu)$ *such that*

$$\int_{\underline{R^s_{k'}(Q)}} \frac{\mathrm{p}_d\mathrm{sin}^2_{x_0}(X^s_q)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f^n_k\left(\overline{X}\right) \cdot f^d_{k'}\left(\underline{X^s}\right)} \leq C_9 \cdot \alpha_0^{k\cdot d\cdot(d-n+1)} \cdot J^{\mathcal{D}}_d(\mu|_Q). \tag{9.16}$$

We remark that the coefficient $\alpha_0^{k\cdot d(d-n+1)}$ is slightly worse than the coefficient produced in analyzing the set $\widehat{R}^s(Q)$, the reason being that the poor scaling of $X^s \in R^s_{k'}(Q)$ deprives us (in a small way) of that extra factor.

10. Discussion

We defined the Menger-type curvature of the restriction of μ to the ball B by integrating a special real-valued function of H^{d+2} over $\mu|_B^{d+2}$. This paper and [8] show that this Menger-type curvature measures a cumulative flatness or oscillation of μ around the ball B. More precisely, these papers compare the Menger-type curvature with the Jones-type flatness, where the comparison depends only on the regularity of μ and the dimension d.

Broader perspectives and open directions for this and [8] are provided in [8]. In particular, [8, Section 6] shows that it is possible to suggest many other functions on H^{d+2}, or discrete curvatures as we refer to them, whose integral over $\mu|_B^{d+2}$ is comparable to the Jones-type flatness of μ around B.

Some of those curvatures are more easily bounded in terms of the Jones-type flatness since they are less singular than the curvature used in this paper. For example, one possible discrete curvature (though not directly exemplified in [8]) is $c_h(X)$ for $X \in H^{d+2}$, whose square is expressed in terms of the minimal height $h(X)$ of equation (3.3):

$$c_h^2(X) = \frac{h^2(X)}{\operatorname{diam}(v_1, \ldots, v_{d+2})^{d\cdot(d+1)+2}}\,. \tag{10.1}$$

For such curvatures, the technically involved application of the interpolating multiscale inequalities (pursued in Sections 8 and 9) is unnecessary. Indeed, the method of Section 7 is sufficient to control the integral of $c_h^2(X)$ over $\mu|_B^{d+2}$ by $J_d^{\mathcal{D}}(\mu|_B)$, and it works as well for a wide variety of such curvatures.

However, we were interested here in controlling the more singular curvature c_d by the Jones-type flatness. The reason for this is the comparability of c_d with the Menger curvature when $d = 1$. On the other hand, the curvature of equation (10.1) is not comparable to the Menger curvature when $d = 1$ and the simpler interpolation technique of Jones [6] is unnecessary as well then.

A. Appendix

A.1. Proof of Proposition 3.2

Proposition 3.2 follows from the two observations:

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq \frac{2\cdot(d+1)}{\mathrm{scale}_{x_0}(X)} \cdot \frac{h(X)}{\operatorname{diam}(X)}\,, \tag{A.1}$$

and

$$h(X) \leq \sqrt{2}\cdot\left\lceil\frac{d+1}{2}\right\rceil\cdot D_2(X, L), \quad \text{for any } d\text{-plane } L. \tag{A.2}$$

The two equations follow from elementary geometric estimates as follows.

A.1.1. Proof of Equation (A.1)

We first note that

$$\max_{0\leq i\leq d+1} \mathrm{M}_d(X(i)) \leq\ (d+1)\cdot \max_{1\leq i\leq d+1} \mathrm{M}_d(X(i)).$$

Indeed, if the maximum on the LHS of the above equation is obtained at $1 \leq i \leq d+1$, then the above inequality is trivial. If on the other hand this maximum is obtained at $i = 0$, then the inequality follows from the fact that the d-content of a face of a $(d+1)$-simplex is less than the sum of the d-contents of the other faces (this is since the d-content does not increase under projections and is subadditive on $\mathbb{R}^d$).

Then, using the fact that the product of any height of a $(d+1)$-simplex with the d-content of the opposite side is a constant (proportional to the $(d+1)$-content of the simplex), we obtain that

$$\min_{1\leq i\leq d+1} h_{x_i}(X)\cdot \max_{1\leq i\leq d+1} \mathrm{M}_d(X(i)) = h(X)\cdot \max_{0\leq i\leq d+1} \mathrm{M}_d(X(i)).$$

Combining the last two equations we deduce the inequality

$$\min_{1\leq i\leq d+1} h_{x_i}(X) \leq (d+1)\cdot h(X)\,. \tag{A.3}$$

Next, by equation (3.6), Proposition 3.1, and also equation (3.7) we obtain that

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq \min_{1\leq i\leq d+1} \frac{h_{x_i}(X)}{\|x_i - x_0\|}\ \leq \frac{\min_{1\leq i\leq d+1} h_{x_i}(X)}{\min_{x_0}(X)}.$$

Applying equation (A.3) to the RHS above, we have that

$$\mathrm{p}_d\mathrm{sin}_{x_0}(X) \leq (d+1)\cdot \frac{h(X)}{\min_{x_0}(X)}.$$

Finally, applying the definition of $\mathrm{scale}_{x_0}(X)$ as well as the bound: $\mathrm{diam}(X) \leq 2\cdot \max_{x_0}(X)$ to the latter equation establishes equation (A.1), and consequently the current proposition.

A.1.2. Proof of Equation (A.2)

We may assume that X is non-degenerate, because otherwise $h(X) = 0$ and the bound holds trivially. Furthermore, since orthogonal projection decreases distances and reduces dimension of subspaces, we may assume that $\dim(H) = d+1$, in particular, $H = \mathbb{R}^{d+1}$.

Our proof utilizes the comparability of the height of the simplex represented by X and its width $w(X)$, which is given by the following infimum over all d-planes L:

$$w(X) = 2 \min_{L} \max_{x \in \text{ the convex hull of } X} \operatorname{dist}(x, L). \tag{A.4}$$

Equivalently, the width $w(X)$ is the shortest distance between any two parallel d-planes supporting the convex hull of X, i.e., the convex hull is trapped between them and its boundary touches them. Gritzman and Lassak [5, Lemma 3] established the following bound on $h(X)$ in terms of $w(X)$:

$$h(X) \leq \left\lceil \frac{d+1}{2} \right\rceil \cdot w(X). \tag{A.5}$$

Equation (A.2) thus follows from combining equation (A.5) with the following bound on $w(X)$, which we verify below.

$$w(X) \leq \sqrt{2} \cdot D_2(X, L) \quad \text{for an arbitrary } d\text{-plane } L\,, \tag{A.6}$$

We verify equation (A.6), and thus conclude equation (A.2), as follows. For a given d-plane L, let L_1 and L_2 be the two unique translates of L supporting the simplex represented by X and let $w_L(X)$ denote the distance between L_1 and L_2. Furthermore, let x_{L_1} be a vertex of X contained in L_1 and x_{L_2} be a vertex contained in L_2. The d-planes L_1 and L_2 separate $\mathbb{R}^{d+1}$ into three regions, being the two disjoint half spaces of $\mathbb{R}^{d+1}$ and the intermediate region bounded by the d-planes whose closure contains the simplex represented by X.

If L is contained in one of the disjoint half spaces described above, then we may assume that L_1 sits between L and L_2. We establish equation (A.6) in this case as follows:

$$w(X) \leq w_L(X) = \operatorname{dist}(x_{L_2}, L_1) \leq \operatorname{dist}(x_{L_2}, L) \leq D_2(X, L).$$

In the second case, where the plane L is contained in the intermediate region, we obtain equation (A.6) in the following way.

$$\begin{aligned} w(X) \leq w_L(X) &= \operatorname{dist}(x_{L_1}, L) + \operatorname{dist}(x_{L_2}, L) \\ &\leq \sqrt{2} \cdot \left(\operatorname{dist}^2(x_{L_1}, L) + \operatorname{dist}^2(x_{L_2}, L)\right)^{1/2} \leq \sqrt{2} \cdot D_2(X, L). \end{aligned}$$

A.2. Proof of Lemma 4.3.1

Given the n-net E_n, let

$$\mathcal{B}'_n = \{B(x, 4 \cdot \alpha_0^n)\}_{x \in E_n}.$$

We note that both $\mathcal{B}'_n$ and $\frac{1}{4}\cdot\mathcal{B}'_n$ cover $\operatorname{supp}(\mu)$ since E_n is an n-net. We take $\mathcal{B}_n$ to be a subfamily of $\mathcal{B}'_n$ such that $\frac{1}{4}\cdot\mathcal{B}_n$ is a maximal, mutually disjoint collection of balls. In this case, we note that $\mathcal{B}_n$ still provides a cover of $\operatorname{supp}(\mu)$ due to the maximality.

The idea is to categorize the elements $B'\in\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]$ according to the first element of $\frac{1}{4}\cdot\mathcal{B}_n$ they intersect, and then use this to take the "appropriate" part of B'. Then, once this is done, for each $j\in\Lambda_n$ the element $P_{n,j}$ is formed by adding these appropriate pieces to the corresponding ball $\frac{1}{4}\cdot B_{n,j}$. We clarify this as follows.

If $\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]=\emptyset$, then we take the partition $\mathcal{P}_n=\{P_{n,j}\}_{j\in\Lambda_m}$, where for fixed $j\in\Lambda_n$

$$P_{n,j}=\operatorname{supp}(\mu)\cap\frac{1}{4}\cdot B_{n,j},\ \text{ for } B_{n,j}\in\mathcal{B}_n.$$

We thus assume that $\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]\neq\emptyset$, and we index the elements of $\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]$ by the set $\Omega_n=\{1,2,\ldots\}$, which is either finite or $\mathbb{N}$, i.e., $\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]=\{B'_m\}_{m\in\Omega_n}$. From this set of balls, we then recursively form the following sets. For $m=1$, let

$$\bar{B}'_1=B'_1\cap\left(\bigcup\frac{1}{4}\cdot\mathcal{B}_n\right)^c,$$

and for $m\geq 2$, let

$$\bar{B}'_m=B'_m\bigcap\left(\bigcup\frac{1}{4}\cdot\mathcal{B}_n\bigcup\bigcup_{i=1}^{m-1}\bar{B}'_i\right)^c.$$

Note that the elements of $\left\{\bar{B}'_m\right\}_{m\in\Omega_n}$ are mutually disjoint, and that $\operatorname{supp}(\mu)$ is covered by the collection of sets

$$\frac{1}{4}\cdot\mathcal{B}_n\bigcup\left\{\bar{B}'_m\right\}_{m\in\Omega}.$$

Let the function $g_n:\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]\to\Lambda_n$ be defined as follows:

$$g_n\left(B'\right)=\min\left\{j\in\Lambda_n:\frac{1}{4}\cdot B_{n,j}\cap B'\neq\emptyset\right\}. \tag{A.7}$$

It follows from the maximality of $\frac{1}{4}\cdot\mathcal{B}_n$ that for every $B'\in\frac{1}{4}\cdot\mathcal{B}'_n$, there exists a $B\in\frac{1}{4}\cdot\mathcal{B}_n$ such that $B\cap B'\neq\emptyset$. Consequently, the minimum of equation (A.7) is obtained at an element of Λ_n, and taking

$$P_{n,j}=\operatorname{supp}(\mu)\bigcap\left(\frac{1}{4}\cdot B_{n,j}\bigcup\bigcup_{g_n(B'_m)=j}\bar{B}'_m\right), \tag{A.8}$$

we note that the sets $P_{n,j}$ are disjoint and cover $\operatorname{supp}(\mu)$. The desired set inclusions follow from the definition of $P_{n,j}$ and observing that $B'\subseteq\frac{3}{4}\cdot B_{n,j}$ for any $B'\in\frac{1}{4}\cdot[\mathcal{B}'_n\setminus\mathcal{B}_n]$ such that $g_n\left(B'\right)=j$.

A.3. Proof of Proposition 8.1

First, for any $X \in \mathbf{S}^1_{k,p}$, along with $Z \in [\mathrm{supp}(\mu)]^{d+2}$ such that $(Z)_0 = (X)_0 = x_0$, as well as $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$ and $1 \leq j \leq k \cdot d$, the following estimate holds

$$\mu\left(B(x_0,r)\right) \geq \mu\left(U_{C_{\mathrm{p}}}(Z,1,\overline{j+1}) \bigcap A_0(x_0,r)\right) \geq \frac{1}{2}\cdot\mu\left(B(x_0,r)\right) > 0. \tag{A.9}$$

In other words, there is a sufficient amount of $\mathrm{supp}(\mu)$ in the annulus $A_0(x_0,r)$ satisfying the relaxed two-term inequality defined by the set $U_{C_{\mathrm{p}}}(Z,1,\overline{j+1})$. Furthermore, for such X we have the inequality $\max_{x_0}(X) \leq \mathrm{diam}(X) \leq \mathrm{diam}(\mathrm{supp}(\mu))$, and we note that the radius

$$r = \alpha_0^{k-\lceil \frac{j}{d}\rceil}\cdot \max_{x_0}(X) \tag{A.10}$$

is in the above range for all $1 \leq j \leq k\cdot d$, i.e., $0 < r \leq \mathrm{diam}(\mathrm{supp}(\mu))$. Substituting this choice of radius into equation (A.9) we obtain the estimate

$$\begin{aligned}\mu\left(B(x_0,\alpha_0^{k-\lceil \frac{j}{d}\rceil}\cdot \max_{x_0}(X))\right) \geq&\\ \geq \mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right) \bigcap A_{k-\lceil \frac{j}{d}\rceil}(x_0,\max_{x_0}(X))\right)&\\ \geq \frac{1}{2}\cdot\mu\big(B(x_0,\alpha_0^{k-\lceil \frac{j}{d}\rceil}\cdot \max_{x_0}(X))\big).&\end{aligned} \tag{A.11}$$

Next, we will prove that for $\underline{X} \in \underline{\mathbf{S}^1_{k,p}}$ and $\widetilde{X}_{q-1}$, $1 \leq q \leq k\cdot d$, of the sequence $\widetilde{\Phi}_k(\underline{X})$:

$$\begin{aligned}U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1},1,\overline{q+1}\right) \bigcap A_{k-\lceil \frac{q}{d}\rceil}(x_0,\max_{x_0}(X)) =&\\ = \pi_q\left(T^{-1}_{q-1}\left(x_0,\ldots,y_{q-1}\right)\right).&\end{aligned} \tag{A.12}$$

Taking $Z = \widetilde{X}_{q-1}$ in equation (A.11) and noting equation (A.12) establishes the proposition.

We start by proving equation (A.9). The inequality of the LHS of equation (A.9) is trivial, and to prove the inequality of the RHS of equation (A.9) we note that

$$\begin{aligned}\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right) \bigcap A_0(x_0,r)\right) =&\\ = \mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right) \bigcap B(x_0,r)\right) + \mu\left(A_0(x_0,r)\right)&\\ - \mu\left(\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right) \bigcap B(x_0,r)\right] \bigcup A_0(x_0,r)\right).&\end{aligned} \tag{A.13}$$

By formulating lower bounds for the terms on the RHS of the above equation we can then establish the inequality on the RHS of equation (A.9).

With the above assumptions on X, Z, and r, by Proposition 3.3 we have the inequality

$$\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\cap B(x_0,r)\right)\geq\frac{3}{4}\cdot\mu(B(x_0,r)). \tag{A.14}$$

Furthermore, by the d-regularity and the constant α_0 (equation (2.2)) we have the inequality

$$\mu\left(A_0(x_0,r)\right)\geq\left(1-\alpha_0^d\cdot C_\mu^2\right)\geq\frac{3}{4}\cdot\mu(B(x_0,r)). \tag{A.15}$$

Noting the inclusion

$$\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right]\bigcup A_0(x_0,r)\subseteq B(x_0,r),$$

we obtain

$$\mu\left(\left[U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap B(x_0,r)\right]\bigcup A_0(x_0,r)\right)\leq\mu(B(x_0,r)).$$

Finally, applying this and equations (A.14) and (A.15) to the RHS of equation (A.13) we obtain

$$\mu\left(U_{C_{\mathrm{p}}}\left(Z,1,\overline{j+1}\right)\bigcap A_0(x_0,r)\right)\geq\frac{1}{2}\cdot\mu(B(x_0,r)), \tag{A.16}$$

and thus conclude equation (A.9).

Next, for a fixed $(x_0,\ldots,y_{q-1})=T_{q-1}(\underline{X})$, we establish equation (A.12) via the inclusion

$$\begin{aligned}U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1},1,\overline{q+1}\right)\bigcap A_{k-\lceil\frac{q}{d}\rceil}(x_0,\max\nolimits_{x_0}(X))\subseteq\\ \subseteq\pi_q\left(T_{q-1}^{-1}\left(x_0,\ldots,y_{q-1}\right)\right).\end{aligned} \tag{A.17}$$

The opposite inclusion follows directly from the definitions of the sets $U_{C_{\mathrm{p}}}(\widetilde{X}_{q-1},1,\overline{q+1})$, $1\leq q\leq k\cdot d$, and $\underline{\mathbf{S}^1_{k,p}}$ (see equations (3.9) and (8.10)).

Our approach to proving equation (A.17) is to fix $1\leq q\leq k\cdot d$ and take an arbitrary point

$$y'_q\in U_{C_{\mathrm{p}}}\left(\widetilde{X}_{q-1},1,\overline{q+1}\right)\bigcap A_{k-\lceil\frac{q}{d}\rceil}(x_0,\max\nolimits_{x_0}(X)). \tag{A.18}$$

We then iteratively use the inequality of equation (A.9) to construct an element

$$\underline{X'}=(x_0,\ldots,y_{q-1},y'_q,\ldots,y'_{k\cdot d})\in T_{q-1}^{-1}\left(x_0,\ldots,y_{q-1}\right),$$

so that

$$y'_q = \pi_q(\underline{X'}) \in \pi_q\left(T_q\left(\pi_{q-1}^{-1}(\pi_{q-1}(\underline{X}))\right)\right).$$

Fixing $1 \leq q \leq k \cdot d$ and y'_q satisfying equation (A.18), we recursively form the sequence $\{y'_i\}_{i=q+1}^{k \cdot d}$ together with additional elements of an auxiliary sequence $\{\widetilde{X}'_i\}_{i=q}^{k \cdot d}$ as follows. First we initialize the auxiliary sequence by defining

$$\widetilde{X}'_q = \widetilde{X}_{q-1}(y'_q, \overline{q+1}).$$

Next, given $q+1 \leq i \leq k \cdot d$ and assuming that $\{y'_i\}_{i=q}^{i-1}$ and $\{\widetilde{X}'_i\}_{i=q}^{i-1}$ have already been defined, we fix arbitrarily

$$y'_i \in U_{C_{\mathrm{p}}}(\widetilde{X}'_{i-1}, 1, \overline{i+1}) \cap A_{k-\lceil \frac{i}{d} \rceil}(x_0, \max_{x_0}(X)), \tag{A.19}$$

and form

$$\widetilde{X}'_i = \widetilde{X}'_{i-1}(y'_i, \overline{i+1}).$$

This procedure is well defined since equation (A.9) implies that for each $q+1 \leq j \leq k \cdot d$:

$$\mu\left(U_{C_{\mathrm{p}}}\left(\widetilde{X}'_{j-1}, 1, \overline{j+1}\right) \bigcap A_{k-\lceil \frac{j}{d} \rceil}(x_0, \max_{x_0}(X))\right) > 0.$$

Finally, forming

$$\underline{X'} = (x_0, \ldots, y_{q-1}, y'_q, \ldots, y'_{k \cdot d}) = X \times Y'_X \in H^{(k+1) \cdot d+2},$$

we note that Y'_X is a well-scaled element for X and the elements of the sequences

$$\widetilde{\Phi}_k(\underline{X'}) = \left\{\widetilde{X}_q\right\}_{q=0}^{k \cdot d} \quad \text{and} \quad \Phi_k(\underline{X'}) = \{X_q\}_{q=1}^{k \cdot d+1}$$

satisfy the inequality

$$\mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_{j-1}) \leq C_{\mathrm{p}}\big(\mathrm{p}_d\mathrm{sin}_{x_0}(X_j) + \mathrm{p}_d\mathrm{sin}_{x_0}(\widetilde{X}_j)\big).$$

Therefore, $\underline{X'} \in \underline{\mathbf{S}^1_{k,p}}$. Furthermore, $T_{q-1}(\underline{X'}) = (x_0, \ldots, y_{q-1})$, and thus

$$\underline{X'} \in \pi_{q-1}^{-1}(\pi_{q-1}(x_0, \ldots, y_{q-1})).$$

Since $\pi_q(\underline{X'}) = y'_q$, equation (A.17) and consequently equation (A.12) are now established.

A.4. Proof of Proposition 9.3

For $m \geq m(Q)$ we define

$$\widehat{R}^s(m)(Q) = \left\{\overline{X} \in \widehat{R}^s(Q) : \max_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}, \tag{A.20}$$

and this gives the following decomposition of the integral

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} = \\ = \sum_{m \geq m(Q)} \int_{\widehat{R}^s(m)(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}. \tag{A.21}$$

Fixing $m \geq m(Q)$, we partition $\widehat{R}^s(m)(Q)$ according to $\max_{x_0}(X^s)$ in the following way.

If $\overline{X} \in \widehat{R}^s(m)$, then Lemma 9.1.1 and the well scaling of X^s imply

$$\max_{x_0}(X^s) \in \left(\alpha_0^k \cdot \max_{x_0}(X), \alpha_0^{k-3} \cdot \max_{x_0}(X)\right]. \tag{A.22}$$

According to equations (A.20) and (A.22), we use the following scale exponent:

$$\mathrm{sc}(m,k) = m + k - 3 \geq m(Q). \tag{A.23}$$

Since $\left\{P_{\mathrm{sc}(m,k),j}\right\}_{j \in \Lambda_{\mathrm{sc}(m,k)}(Q)}$ covers $Q \cap \mathrm{supp}(\mu)$, we cover $\widehat{R}^s(m)(Q)$ by

$$\widehat{P}_{\mathrm{sc}(m,k),j} = \left\{\overline{X} \in \widehat{R}^s(m) : x_0 \in P_{\mathrm{sc}(m,k),j}\right\}, \text{ for all } j \in \Lambda_{\mathrm{sc}(m,k)(Q)}. \tag{A.24}$$

Hence we obtain the inequality

$$\int_{\widehat{R}^s(Q)} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq \\ \leq \sum_{m \geq m(Q)} \sum_{j \in \Lambda_{\mathrm{sc}(m,k)}(Q)} \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)}. \tag{A.25}$$

Now we fix $m \geq m(Q)$ and $j \in \Lambda_{\mathrm{sc}(m,k)}(Q)$ and concentrate on the individual terms of the RHS of equation (A.25). Obtaining the "proper" bounds here will clearly imply the proposition.

We fix an arbitrary d-plane L. If $\overline{X} \in \widehat{P}_{\mathrm{sc}(m,k),j}$, then equations (A.20) and (A.22) imply

$$\mathrm{diam}(X^s) \geq \alpha_0^{m+k+1} = \frac{\alpha_0^4}{8} \cdot \mathrm{diam}(B_{\mathrm{sc}(m,k),j}).$$

Applying this and Proposition 3.2 we get that

$$\text{(A.26)}\quad \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\mathrm{p}_d \sin^2_{x_0}(X^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} \leq$$

$$\leq \frac{2^7 \cdot (d+1)^2 \cdot (d+2)^2}{\alpha_0^{14}} \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{D_2^2\left(X^s, L\right)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \cdot \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right) \cdot \operatorname{diam}(X)^{d(d+1)}}.$$

Hence we focus on the individual terms of

$$\frac{D_2^2\left(X^s, L\right)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k),j}\right)}.$$

We arbitrarily fix $0 \leq t \leq d+1$ and note the cases for the possible values of $(X^s)_t$. Per Lemma 9.1.1 and the construction of the elements X^s we have the following cases.

Case 1: $t \in \{0, 1, d+1\}$. In this case we note that by our construction $(X^s)_0 = x_0$, $(X^s)_1 = x_{i_s}$ for $1 \leq i_s \leq n$, and $(X^s)_{d+1} = x_i$, where $n+1 \leq i \leq d+1$.

Case 2: $2 \leq t \leq d$. In this case, again by the construction we have that $(X^s)_t = z_\ell$, for exactly $d-1$ distinct indices ℓ, $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$.

Assume Case 1. Per Fubini's, the corresponding terms on the RHS of equation (A.26) are

$$\text{(A.27)}\quad \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right) \cdot \operatorname{diam}(X)^{d(d+1)}} =$$

$$= \int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}}.$$

Equation (A.22) and Lemma 9.1.1 imply the set inclusion

$$\text{(A.28)}\qquad T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right) \subseteq \bigcup_{x_0 \in P_{\mathrm{sc}(m,k),j}} \{x_0\} \times \prod_{i=1}^{d+1} B(x_0, \alpha_0^{p_i}),$$

where

$$p_i = \begin{cases} m+k-3, & \text{if } i = i_s, \text{(See Case 1 above)}; \\ m+k, & \text{if } n+1 \leq i \leq d+1; \\ m, & \text{otherwise.} \end{cases}$$

Equation (A.28) then yields the following bound on the RHS of equation (A.27)

$$\text{(A.29)}\quad \int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2(B)} \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}} \leq$$

$$\leq \frac{1}{\alpha_0^{d(d+1)}} \int_{P_{\mathrm{sc}(m,k),j}} \int_{\prod_{i=1}^{d+1} B(x_0,\alpha_0^{p_i})} \frac{\operatorname{dist}^2\left(\left(X^s\right)_t, L\right)}{\operatorname{diam}^2\left(B_{\mathrm{sc}(m,k),j}\right)} \frac{\mathrm{d}\mu(x_{d+1})\cdots\mathrm{d}\mu(x_0)}{[\alpha_0^m]^{d(d+1)}}.$$

Applying the usual calculations (see the proofs of Propositions 6.1 and 6.2) to the RHS of equation (A.29), we obtain the following bound on the RHS of equation (A.27)

$$\text{(A.30)}\qquad \frac{3^d \cdot C_\mu^{d+1} \cdot \alpha_0^{k\cdot d\cdot(d-n+2)}}{\alpha_0^{d(d+1)+3\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right).$$

For Case 2, we iterate the integral to obtain the equality

$$\text{(A.31)}\quad \int_{\widehat{P}_{\mathrm{sc}(m,k),j}} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\right)}{f_k^n\left(\overline{X}\right)} =$$

$$= \int_{T_0\left(\widehat{P}_{\mathrm{sc}(m,k),j}\right)} \left(\int_{\left\{Z_X : X\times Z_X \in \widehat{P}_{\mathrm{sc}(m,k),j}\right\}} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{N_n}(Z_X)}{f_k^n\left(\overline{X}\right)} \right) \cdot \frac{\mathrm{d}\mu^{d+2}(X)}{\operatorname{diam}(X)^{d(d+1)}}.$$

To uniformly bound for the inner integral, it is sufficient to uniformly bound

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,q),j}\right)} \right)^2 \frac{\mathrm{d}\mu(z_\ell)}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}\left(x_0,\ldots,z_{\ell-1}\right)\right)\right)}.$$

Proposition 9.1, equation (A.20), and the d-regularity of μ imply

$$\text{(A.32)}\qquad \mu\left(\pi_\ell\left(T_{\ell-1}^{-1}\left(x_0,\ldots,z_{\ell-1}\right)\right)\right) \geq \frac{1}{2\cdot C_\mu^2} \cdot \left(\frac{\alpha_0^4}{4}\right)^d \cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right).$$

Furthermore, $\pi_\ell\left(T_{\ell-1}^{-1}\left(x_0,\ldots,z_{\ell-1}\right)\right) \subseteq B_{\mathrm{sc}(m,k),j}$, and thus

$$\text{(A.33)}\quad \int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)} \frac{\operatorname{dist}^2(z_\ell, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu(z_{\ell_i})}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right)} \leq$$

$$\leq \frac{2\cdot 4^d \cdot C_\mu^2}{\alpha_0^{4\cdot d}} \cdot \beta_2^2(B_{\mathrm{sc}(m,k),j}, L).$$

Therefore the inner integral on the RHS of equation (A.31) is bounded by

$$\int_{\{Z_X : X\times Z_X \in \widehat{P}_{\mathrm{sc}(m,k),j}\}} \frac{\operatorname{dist}^2(z_{\ell_i}, L)}{\operatorname{diam}^2(B_{\mathrm{sc}(m,k),j})} \frac{\mathrm{d}\mu^{N_n}(Z_X)}{f_k^n\left(\overline{X}\right)} \leq \\ \leq \frac{2\cdot 4^d \cdot C_\mu^2}{\alpha_0^{4\cdot d}} \cdot \beta_2^2(B_{\mathrm{sc}(m,k),j}). \tag{A.34}$$

Applying equation (A.28) gives the following upper bound for the RHS of equation (A.31):

$$\frac{2\cdot 4^d \cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+6\cdot d}} \cdot \alpha_0^{k\cdot d\cdot (d-n+2)} \cdot \beta_2^2(B_{\mathrm{sc}(m,k),j}, L)\cdot \mu\left(B_{\mathrm{sc}(m,k),j}\right). \tag{A.35}$$

Applying equations (A.30) and (A.35) to the RHS of equation (A.26) and then taking the infimum over all d-planes L establishes the proposition.

A.5. Proof of Proposition 9.4

Fixing $2 \leq k' \leq k-1$, $1 \leq q \leq k'\cdot d + 1$, and $m \geq m(Q)$, we define

$$\underline{R_{k'}^s}(m)(Q) = \left\{\overline{X}\times Y_{X^s} \in \underline{R_{k'}^s(Q)} : \max\nolimits_{x_0}(X) \in (\alpha_0^{m+1}, \alpha_0^m]\right\}. \tag{A.36}$$

This gives the following decomposition of the integral on the LHS of equation (9.14)

$$\int_{\underline{R_{k'}^s}(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X_q^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)} = \\ = \sum_{m\geq m(Q)} \int_{\underline{R_{k'}^s}(m)(Q)} \frac{\mathrm{p}_d\sin^2_{x_0}(X_q^s)}{\operatorname{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)}. \tag{A.37}$$

Fixing $m \geq m(Q)$, we partition $\underline{R_{k'}^s}(m)(Q)$ according to $\max_{x_0}\left(X_q^s\right)$ in the following way. By Lemma 9.1.1, for $\overline{X} \in \underline{R_{k'}^s}$ we have that

$$\alpha_0^{m+k-k'+2} \leq \max\nolimits_{x_0}(X^s) < \alpha_0^{m+k-k'-2}. \tag{A.38}$$

If $k' = k-1$, then the upper bound on the RHS of equation (A.38) is too large since we always have that $\max_{x_0}(X^s) \leq \alpha_0^m < \alpha_0^{m-1}$. So, we amend equation (A.38) in the following way. Let

$$e(m,k') = \begin{cases} m+k-k'-2, & \text{if } 2\leq k'\leq k-2;\\ m, & \text{if } k' = k-1.\end{cases} \tag{A.39}$$

We note that $e(m,k') \geq m$ and

$$\alpha_0^{e(m,k')+4} \leq \max_{x_0}(X^s) \leq \alpha_0^{e(m,k')}. \tag{A.40}$$

Now, combining Lemma 8.1.1 and equation (A.40) we have the following estimate

$$\max_{x_0}\left(X_q^s\right) \in \begin{cases} \left(\alpha_0^{k'+e(m,k')-\lceil \frac{q}{d}\rceil+5}, \alpha_0^{k'+e(m,k')-\lceil \frac{q}{d}\rceil}\right], & \text{if } 1 \leq q \leq k' \cdot d; \\ \left(\alpha_0^{e(m,k')+1}, \alpha_0^{e(m,k')}\right], & \text{if } q = k' \cdot d + 1. \end{cases} \tag{A.41}$$

Hence we define the scale exponent as follows

$$\mathrm{sc}(m,k',q) = \mathrm{sc}(m,k,k',q) = \begin{cases} k' + e(m,k') - \left\lceil \frac{q}{d} \right\rceil, & \text{if } 1 \leq q \leq k' \cdot d; \\ e(m,k'), & \text{if } q = k' \cdot d + 1. \end{cases} \tag{A.42}$$

We note that the scale exponent is independent of s, and furthermore, we have the inequality

$$\mathrm{sc}(m,k',q) \;\geq\; e(m,k'), \text{ for all } 1 \leq q \leq k' \cdot d + 1. \tag{A.43}$$

Next, $\left\{P_{\mathrm{sc}(m,k',q),j}\right\}_{j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)}$ covers $Q \cap \mathrm{supp}(\mu)$, and so we cover $\underline{R_{k'}^s}(m)(Q)$ by

$$\underline{P_{\mathrm{sc}(m,k',q),j}} = \left\{\overline{X} \times Y_{X^s} \in \underline{R_{k'}^s}(m) : x_0 \in P_{\mathrm{sc}(m,k',q),j}\right\}, \tag{A.44}$$

for $j \in \Lambda_{\mathrm{sc}(m,k',q)(Q)}$. Letting $m \geq m(Q)$ and j vary we obtain the inequality

$$\begin{aligned} &\int_{\underline{R_{k'}^s}} \frac{\mathrm{p}_d \sin_{x_0}^2(X_q^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f_k^n\left(\overline{X}\right) \cdot f_{k'}^d\left(\underline{X^s}\right)} \leq \\ &\leq \sum_{m \geq m(Q)} \sum_{j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)} \int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{\mathrm{p}_d \sin_{x_0}^2(X_q^s)}{\mathrm{diam}(X)^{d(d+1)}} \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X} \times Y_{X^s})}{f_k^n\left(\overline{X}\right) \cdot f_{k'}^d\left(\underline{X^s}\right)}. \end{aligned} \tag{A.45}$$

Fixing $m \geq m(Q)$ and $j \in \Lambda_{\mathrm{sc}(m,k',q)}(Q)$, we now concentrate on the terms of the RHS of equation (A.45). We note that the "proper" control on these terms implies the proposition.

Let L be an arbitrary d-plane. If $\overline{X} \times Y_{X^s} \in \underline{P_{\mathrm{sc}(m,k',q),j}}$, then by equations (A.41) and (A.42)

$$\mathrm{diam}\left({X^s}_q\right) \geq \alpha_0^{\mathrm{sc}(m,k',q)+5} = \frac{\alpha_0^5}{8} \cdot \mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right).$$

Applying this and Proposition 3.2 we obtain the inequality

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \mathrm{p}_d \sin^2_{x_0}\left(X^s_q\right) \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{\mathrm{diam}(X)^{d(d+1)}\cdot f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)} \leq \tag{A.46}$$

$$\leq \frac{2^7\cdot(d+1)^2\cdot(d+2)^2}{\alpha_0^{16}} \int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \frac{D_2^2\left(X^s_q, L\right)}{\mathrm{diam}^2\left(B_{\mathrm{sc}(m,k',q),j}\right)} \cdot \frac{\mathrm{d}\mu^{M_n+k'\cdot d}(\overline{X}\times Y_{X^s})}{\mathrm{diam}(X)^{d(d+1)}\cdot f^n_k\left(\overline{X}\right)\cdot f^d_{k'}\left(\underline{X^s}\right)}.$$

To bound the RHS of equation (A.46), we focus on the individual terms of

$$\frac{D_2^2\left(X^s_q, L\right)}{\mathrm{diam}^2\left(B_{\mathrm{sc}(m,k',q)}\right)}.$$

Fixing $0\leq t\leq d+1$, per equations (8.6)-(8.7) and Lemma 9.1.1, we have the following cases:

Case 1: $\left(X^s_q\right)_t = x_0$. In this case q has no restriction, that is, $1\leq q\leq k'\cdot d$.

Case 2: $\left(X^s_q\right)_t = x_{i_s}$. In this case $q = k'\cdot d+1$ because x_{i_s} is the handle of X^s, and only the last element of the sequence has this handle. Hence $\mathrm{sc}(m,k',q) = k'+e(m,k')$ by equation (A.42).

Case 3: $\left(X^s_q\right)_t = x_i$, for $n+1\leq i\leq d+1$. In this case $1\leq q\leq d$ since only the first d elements of the sequence contain the tines X^s.

Case 4: $\left(X^s_q\right)_t = z_\ell$, where $1\leq \ell\leq 2^{n-2}+\left\lceil\frac{s}{2}\right\rceil-1$. We again have $1\leq q\leq d$ as in case 3.

Case 5: $\left(X^s_q\right)_t = y_i$, where $1\leq i\leq k'\cdot d$. In this case, for each $1\leq q\leq k'\cdot d+1$, we have the following restriction on the quantity $\left\lceil\frac{i}{d}\right\rceil$, just as in equation (8.31):

$$\max\left\{1, \left\lceil\frac{q}{d}\right\rceil-1\right\} \leq \left\lceil\frac{i}{d}\right\rceil \leq \left\lceil\frac{q}{d}\right\rceil. \tag{A.47}$$

For the first three cases, per Fubini's the corresponding terms of equation (A.46) reduce to

$$\int_{T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)} \frac{\mathrm{dist}^2\left(\left(X^s_q\right)_t, L\right)}{\mathrm{diam}^2\left(B_{\mathrm{sc}(m,k',q),j}\right)} \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}}. \tag{A.48}$$

For the set $\underline{P_{\mathrm{sc}(m,k',q),j}}$, equation (A.38) and Lemma 9.1.1 imply the set inclusion

$$T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right) \subseteq \bigcup_{x_0 \in P_{\mathrm{sc}(m,k',q),j}} \{x_0\} \times \prod_{i=1}^{d+2} B(x_0, \alpha_0^{p_i}), \tag{A.49}$$

where

$$p_i = \begin{cases} m+k, & \text{if } n+1 \leq i \leq d+1; \\ e(m,k'), & \text{if } i = i_s; \\ m, & \text{otherwise.} \end{cases} \tag{A.50}$$

Via the usual computations and noting the values of $\mathrm{sc}(m,k',q)$ and $e(m,k')$, the RHS of equation (A.48) has the bound

$$\frac{3^d \cdot C_\mu^{d+1}}{\alpha_0^{d(d+1)+3\cdot d}} \cdot \alpha_0^{k\cdot d\cdot(d-n+1)} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right) \cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \tag{A.51}$$

Assume Case 4. Fix $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$ and iterate the integral to obtain

$$\begin{aligned} &\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \left(\frac{\mathrm{dist}\left(z_\ell, L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{M_n+k'\cdot d}\left(\overline{X} \times Y_{X^s}\right)}{\mathrm{diam}(X)^{d(d+1)} \cdot f_k^n\left(\overline{X}\right) \cdot f_{k'}^d\left(\underline{X^s}\right)} \\ &= \int_{T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)} \left(\int_{\left\{Z_X : X\times Z_X \in R_{k'}^s\right\}} \left(\frac{\mathrm{dist}\left(z_\ell, L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{N_n}\left(Z_X\right)}{f_k^n(\overline{X})}\right) \\ &\qquad \cdot \frac{\mathrm{d}\mu^{d+2}(X)}{\mathrm{diam}(X)^{d(d+1)}}. \end{aligned} \tag{A.52}$$

Again we want to control the inner integral, and this clearly reduces to controlling the quantity

$$\int_{\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)} \left(\frac{\mathrm{dist}\left(z_\ell, L\right)}{\mathrm{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu(z_\ell)}{\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right)}. \tag{A.53}$$

To do this, we use the definitions of $\mathrm{sc}(m,k',q)$ and $e(m,k')$ obtaining

$$m+k-3 \leq \mathrm{sc}(m,k',q) \leq m+k-2.$$

Hence, Proposition 9.1 and the d-regularity of μ imply the following for all $1 \leq \ell \leq 2^{n-2} + \left\lceil \frac{s}{2} \right\rceil - 1$:

$$\mu\left(\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\ldots,z_{\ell-1})\right)\right) \geq \frac{\alpha_0^{4\cdot d}}{2\cdot 4^d \cdot C_\mu^2} \cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \tag{A.54}$$

Moreover, for fixed $(x_0,\dots,z_{\ell-1}) \in T_{\ell-1}\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$, we have that

$$\pi_\ell\left(T_{\ell-1}^{-1}(x_0,\dots,z_{\ell-1})\right) \subseteq B\left(x_0, \alpha_0^k \cdot \max\nolimits_{x_0}(X)\right) \subseteq \frac{3}{4}\cdot B_{\mathrm{sc}(m,k',q),j}. \tag{A.55}$$

Applying equations (A.54) and (A.55) to equation (A.53), we see that equation (A.53), and hence the inner integral of equation (A.52) is bounded (uniformly in $X \in T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$) by

$$\frac{2\cdot 4^d \cdot C_\mu^2}{\alpha_0^{4\cdot d}} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right). \tag{A.56}$$

Applying equations (A.56), (A.49), and the usual computations to the RHS of equation (A.52) gives

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \left(\frac{\operatorname{dist}(z_\ell, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\times Y_{X^s}\right)}{\operatorname{diam}(X)^{d(d+1)}\cdot f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)} \leq \\ \leq \frac{2\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+4\cdot d}} \cdot \alpha_0^{k\cdot d\cdot(d-n+1)} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right)\cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \tag{A.57}$$

At last we consider Case 5. Here we must be a little bit careful in how we use notation, in the sense that we must remember the pertinent "variables". In this case we make the following harmless abuse of notation for the truncation T_0, taking

$$T_0(\underline{X^s}) = T_0(\overline{X}\times Y_{x^s}) = \overline{X},$$

instead of $T_0(\underline{X^s}) = X^s$ as we originally defined the notion in Section 8.2.1.

Then, via the usual computations, for $1 \leq i \leq k'\cdot d$ and $\overline{X} \in T_0\left(\underline{P_{\mathrm{sc}(m,k',q),j}}\right)$ we have

$$\int_{\left\{Y_{X^s}:\overline{X}\times Y_{X^s}\in \underline{P_{\mathrm{sc}(m,k',q),j}}\right\}} \left(\frac{\operatorname{dist}(y_i, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{k'\cdot d}\left(Y_{X^s}\right)}{f_{k'}^d\left(\underline{X^s}\right)} \leq \\ \leq \frac{2\cdot 4^d\cdot C_\mu^2}{\alpha_0^{6\cdot d}}\cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right). \tag{A.58}$$

Hence, iterating the integral over $\underline{P_{\mathrm{sc}(m,k',q),j}}$ gives the inequality

$$\int_{\underline{P_{\mathrm{sc}(m,k',q),j}}} \left(\frac{\operatorname{dist}(y_i, L)}{\operatorname{diam}\left(B_{\mathrm{sc}(m,k',q),j}\right)}\right)^2 \frac{\mathrm{d}\mu^{M_n}\left(\overline{X}\times Y_{X^s}\right)}{\operatorname{diam}(X)^{d(d+1)}\cdot f_k^n\left(\overline{X}\right)\cdot f_{k'}^d\left(\underline{X^s}\right)} \\ \leq \frac{2\cdot 4^d\cdot C_\mu^{d+3}}{\alpha_0^{d(d+1)+6\cdot d}} \cdot \alpha_0^{k\cdot d\cdot(d-n+1)} \cdot \beta_2^2\left(B_{\mathrm{sc}(m,k',q),j}, L\right)\cdot \mu\left(B_{\mathrm{sc}(m,k',q),j}\right). \tag{A.59}$$

At long last, applying equations (A.51), (A.57), and (A.59) to the RHS of equation (A.46) and taking the infimum over all d-planes L proves the desired proposition.

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References

[1] David, G.: *Wavelets and singular integrals on curves and surfaces.* Lecture Notes in Mathematics **1465**. Springer-Verlag, Berlin, 1991.

[2] David, G. and Semmes, S.: Singular integrals and rectifiable sets in $\mathbb{R}^n$: au-delà des graphes Lipschitziens. *Astérisque* **193** (1991), 152 pp.

[3] David, G. and Semmes, S.: *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs **38**. American Mathematical Society, Providence, RI, 1993.

[4] Greengard, L. and Rokhlin, V.: A fast algorithm for particle simulations. *J. Comput. Phys.* **73** (1987), no. 2, 325–348.

[5] Gritzman, P. and Lassak, M.: Estimates for the minimal width of polytopes inscribed in convex bodies. *Discrete Comput. Geom.* **4** (1989), no. 1, 627–635.

[6] Jones, P. W.: Characterizing one-dimensional uniform rectifiability of measures by their Menger curvature. Unpublished result, recorded by [14, Chapter 3, Section 3].

[7] Jones, P. W.: Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1990), no. 1, 1–15.

[8] Lerman, G. and Whitehouse, J. T.: High-dimensional Menger-type curvatures. Part II: d-separation and a menagerie of curvatures. *Constr. Approx.* **30** (2009), no. 3, 325–360.

[9] Lerman, G. and Whitehouse, J. T.: Least squares approximations of measures via geometric condition numbers. Submitted to *Mathematika*. Current version is available at arxiv.org: `arXiv:1008.2041`.

[10] Lerman, G. and Whitehouse, J. T.: On d-dimensional d-semimetrics and simplex-type inequalities for high-dimensional sine functions. *J. Approx. Theory* **156** (2009), no. 1, 52–81.

[11] Mattila, P., Melnikov, M. and Verdera, J.: The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2)* **144** (1996), no. 1, 127–136.

[12] Mattila, P. and Preiss, D.: Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals. *J. London Math. Soc. (2)* **52** (1995), no. 3, 482–496.

[13] Menger, K.: Untersuchungen über allgemeine Metrik. *Math. Ann.* **103** (1930), no. 1, 466–501.

[14] Pajot, H.: *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral*. Lecture Notes in Mathematics **1799**. Springer-Verlag, Berlin, 2002.

[15] Schul, R.: Subsets of rectifiable curves in Hilbert space —the analyst's TSP. *J. Anal. Math.* **103** (2007), 331–375.

[16] Tolsa, X.: Uniform rectifiability, Calderón-Zygmund operators with odd kernel and quasiorthogonality. *Proc. Lond. Math. Soc. (3)* **98** (2009), no. 2, 393–426.

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