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Notes on the Group of S^1 Equivariant Homeomorphisms

By

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§ 1. Introduction

Let X be a compact smooth manifold with its first betti number $b_1(X) = 0$. Let P be a principal S^1-bundle over X and $(\mathscr{G}, \mathscr{G}^0)$ be its gauge group and based gauge group, respectively. We can identify $(\mathscr{G}, \mathscr{G}^0) = (\mathrm{Map}(X, S^1), \mathrm{Map}^*(X, S^1))$, where $\mathrm{Map}^*(X, S^1)$ denotes the space consisting of all base point preserving maps $X \to S^1$.

In this paper, we shall consider the topology of the group $\mathrm{Homeo}_{S^1}(P)$ of S^1 equivariant homeomorphisms of P and in particular, we study the case $X = \mathbb{C}P^n$.

It is known ([2], [4]) that there is a fibration

$$(1.1) \qquad \mathscr{G} \to \mathrm{Map}_{S^1}(P, P) \to \mathrm{Map}(X, X)$$

and it is shown in [8] that (1.1) restricts to the fibration

$$(1.2) \qquad \mathscr{G} \to \mathrm{Homeo}_{S^1}(P) \xrightarrow{\pi} \mathrm{Homeo}_P(X)$$

where we take

$$\mathrm{Homeo}_P(X) = \{\varphi \in \mathrm{Homeo}(X) \mid \varphi^* P = P\}.$$

Define the evaluation map $\mu_{x_0} = \mu$: $\mathrm{Homeo}_P(X) \to X$ by $\mu(\varphi) = \varphi(x_0)$, where $x_0 \in X$ is a fixed base point.

Then we have

Proposition 1.1. *There exists a weak homotopy equivalence*

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$$\mathrm{Homeo}_{S^1}(P) \underset{w}{\simeq} \mu^* P$$

Let $\mathrm{Homeo}_P^*(X)$ be the subgroup of $\mathrm{Homeo}_P(X)$ consisting of all base point preserving maps. Then $\mathrm{Homeo}_{S^1, x_0}(P) = \pi^{-1}(\mathrm{Homeo}_P^*(X))$ is the space consisting of maps preserving the fibre over x_0. Let $\mathrm{Homeo}_{S^1}^*(P) \subset \mathrm{Homeo}_{S^1, x_0}(P)$ be the subgroup acting as identity on the fibre over x_0.

Proposition 1.2. *There exist weak homotopy equivalences*

$$B\mathrm{Homeo}_{S^1, x_0}(P) \underset{w}{\simeq} B(\mathrm{Homeo}_P^*(X) \times S^1)$$
$$B\mathrm{Homeo}_{S^1}^*(P) \underset{w}{\simeq} B\mathrm{Homeo}_P^*(X).$$

For any paracompact subgroup $G \subset \mathrm{Homeo}_P(X)$ with homotopy type of CW complexes, since $\mathrm{Map}(X, X)$ is locally contractible and $\mathscr{G}$ acts $\mathrm{Homeo}_{S^1}(P)$ freely on the right, if we restrict (1.2) to G it is a principal bundle. Then we have:

Proposition 1.3. $\mathrm{Homeo}_{S^1}(P)|_G \cong \mu_{x_0}^* P \times \mathscr{G}^0$ *as principal* $\mathscr{G}$ *bundle over* G.

Let P_k be the principal S^1 bundle over $\mathbb{C}P^n$ with $c_1(P_k) = k$. Then, by using [6], we obtain the following theorem.

Theorem 1.4. $\pi_{2i+1}(\mathrm{Homeo}_{S^1}(P_k))$ *has a free part for* $i = 0, 1, \cdots, n$.

A similar result holds for S^3 bundles over $\mathbb{H}P^n$, and we discuss it in the appendix.

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§ 2. Proofs

In this section we shall give the proofs of Proposition 1.1, 1.2 and 1.3.

By [1], there are universal bundles

$$\mathscr{G} \to \mathrm{Map}_{S^1}(P, ES^1) \to \mathrm{Map}_P(X, BS^1)$$
$$\mathscr{G}^0 \to \mathrm{Map}_{S^1}^*(P, ES^1) \to \mathrm{Map}_P^*(X, BS^1).$$

Since $b_1(X) = 0$, $\mathscr{G}^0$ is contractible and $B\mathscr{G}^0 \simeq \mathrm{Map}_P^*(X, BS^1)$ has the homotopy type of CW complexes. Hence $B\mathscr{G}^0$ is contractible and

$$B\mathscr{G} \simeq \mathrm{Map}_P(X, BS^1) \underset{ev}{\simeq} BS^1 \simeq \mathbb{C}P^\infty.$$

Then the following diagram is commutative up to homotopy:

$$\begin{array}{ccc}
\mathrm{Map}_{S^1}(P, ES^1) & \xrightarrow{-\tilde{f}} & \mathrm{Map}_{S^1}(X\times S^1, ES^1) \\
\downarrow & & \downarrow \\
\mathrm{Map}_P(X, BS^1) & \underset{-f}{\xrightarrow{\cong}} & \mathrm{Map}_0(X, BS^1) \\
{\scriptstyle ev}\downarrow & & \uparrow{\scriptstyle i} \\
BS^1 & = & BS^1
\end{array}$$

and it can be easily shown that

$$i^*\mathrm{Map}_{S^1}(X\times S^1, ES^1) \cong ES^1\times \mathscr{G}^0$$

as $\mathscr{G}$-bundle.

Proof of Proposition 1.3. Fix a base point preserving classifying map f for P, and we identify P and f^*ES^1. Then we have following commutative diagram

$$\begin{array}{ccc}
\mathrm{Homeo}_{S^1}(P)\mid_G & \xrightarrow{\tilde{f}_*} & \mathrm{Map}_{S^1}(P, ES^1) \\
\downarrow & & \downarrow \\
G & \xrightarrow{f_*} & \mathrm{Map}_P(X, BS^1) \\
{\scriptstyle \mu_{x_0}}\downarrow & & \downarrow{\scriptstyle ev} \\
X & \xrightarrow{f} & BS^1
\end{array}$$

and $\tilde{f}_*$ is $\mathscr{G}$-equivariant. Hence

$$\begin{aligned}
\mathrm{Homeo}_{S^1}(P)\mid_G &\cong (\tilde{f}_*)^*\mathrm{Map}_{S^1}(P, ES^1) \\
&\cong (f_*)^*(-f)^*\mathrm{Map}_{S^1}(X\times S^1, ES^1) \\
&\cong (f_*)^*ev^*i^*\mathrm{Map}_{S^1}(X\times S^1, ES^1) \\
&\cong (f_*)^*ev^*ES^1\times \mathscr{G}^0 \\
&\cong \mu_{x_0}^* f^*ES^1\times \mathscr{G}^0 \\
&\cong \mu_{x_0}^*P\times \mathscr{G}^0. \quad \square
\end{aligned}$$

Proof of Proposition 1.1. Fix a base point p_0 in the fibre over x_0. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Homeo}_{S^1}(P) & \xrightarrow{\mu_{p_0}} & P \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_P(X) & \xrightarrow{\mu} & X \end{array}$$

where μ_{p_0} denotes the evaluation map at p_0. Then the above diagram naturally induces a morphism of fibrations

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\simeq} & S^1 \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_{S^1}(P) & \longrightarrow & \mu^*P \\ \downarrow & & \downarrow \\ \mathrm{Homeo}_P(X) & = & \mathrm{Homeo}_P(X). \end{array}$$

By the homotopy exact sequence, we have $\mathrm{Homeo}_{S^1}(P) \underset{w}{\simeq} \mu^*P$. □

Of course when restricted to the subgroup G as in Proposition 1.3, this weak homotopy equivalence is the homotopy equivalence given.

Proof of Proposition 1.2. Identify P_{x_0} with S^1 by

$$S^1 \ni z \to p_0 \cdot z \in P_{x_0},$$

then there is a group isomorphism

$$\Phi : \mathrm{Homeo}_{S^1, x_0}(P) \to \mathrm{Homeo}^*_{S^1}(P) \times S^1$$

given by $\Phi(\tilde{\varphi}) = (\tilde{\varphi} \cdot \mu_{p_0}(\tilde{\varphi})^{-1}, \mu_{p_0}(\tilde{\varphi}))$.

From (1.2), we have a fibration

$$\mathscr{G}^0 \times S^1 \to \mathrm{Homeo}^*_{S^1}(P) \times S^1 \xrightarrow{\pi} \mathrm{Homeo}^*_P(X)$$

where $\pi(\tilde{\varphi}, z) = \pi(\tilde{\varphi})$. Since $\mathscr{G}^0$ is contractible, we easily deduce the results. □

§ 3. S^1 Bundle over $\mathbb{C}P^n$

Let P_k be the principal S^1 bundle over $\mathbb{C}P^n$ with $c_1(P_k) = k$ and consider the subgroup $PU(n+1) \subset \mathrm{Homeo}_{P_k}(\mathbb{C}P^n)$.

By an easy computation, $\mu^*_{x_0}$ is mod $n+1$ reduction,

$$\begin{array}{ccccc} 0 & \longleftarrow & H^2(PU(n+1);\mathbb{Z}) & \xleftarrow{\mu_{x_0}^*} & H^2(\mathbb{C}P^n;\mathbb{Z}) \\ & & \| & & \| \\ & & \mathbb{Z}/n+1 & & \mathbb{Z}. \end{array}$$

The bundle P_1 is given by the following Hopf fibration (for simplicity we orient $\mathbb{C}P^n$ like this)

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$

Since $P_k = P_1^{\otimes k}$, we have

$$P_k = S^{2n+1} \times_{\rho_k} S^1$$

where

$$\rho_k : S^1 \to S^1$$

is defined by $\rho_k(z) = z^k$.

Since there is a commutative diagram

$$\begin{array}{ccc} S^1 & = & S^1 \\ \downarrow & & \downarrow \\ U(n+1) & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow \\ PU(n+1) & \xrightarrow{\mu_{x_0}} & \mathbb{C}P^n, \end{array}$$

we have

$$\mu_{x_0}^* P_k = U(n+1) \times_{\rho_k} S^1 = SU(n+1) \times_k S^1$$

where

$$k = k\times : \mathbb{Z}/n+1 \to \mathbb{Z}/n+1 \subset S^1.$$

Note that if $k \equiv k' (\text{mod } n+1)$, then $U(n+1) \times_{\rho_k} S^1 \cong U(n+1) \times_{\rho_{k'}} S^1$ as groups.

By Proposition 1.3, we have

Proposition 3.1. $\text{Homeo}_{S^1}(P_k)\,|_{PU(n+1)} \cong (U(n+1) \times_{\rho_k} S^1) \times \mathscr{G}^0$ as *$\mathscr{G}$ bundle over $PU(n+1)$.*

In fact these are isomorphic as groups. We construct the isomorphism explicitly. Consider the following action

$$\rho\colon (U(n+1)\times S^1)\times(S^{2n+1}\times S^1)\to(S^{2n+1}\times S^1)$$

given by $\rho((g, z), (x, z')) = (g\cdot x, z'z)$. This induces the action

$$\rho\colon (U(n+1)\times_{\rho_k} S^1)\times P_k\to P_k.$$

This gives desired isomorphism

$$F\colon (U(n+1)\times_{\rho_k} S^1)\times\mathscr{G}^0\to \mathrm{Homeo}_{S^1}(P_k)\,|_{PU(n+1)}$$

by

$$F(g, u)(p) = \rho(g, p)\cdot u(x)$$

where $p\in P_k$ and $\pi(p) = x\in\mathbb{C}P^n$.

Theorem 3.2. $\mathrm{Homeo}_{S^1}(P_k)\,|_{PU(n+1)}\cong(U(n+1)\times_{\rho_k} S^1)\times\mathscr{G}^0$ *as group. In particular, there exists a homotopy equivalence*

$$B(\mathrm{Homeo}_{S^1}(P_k)\,|_{PU(n+1)})\simeq B(U(n+1)\times_{\rho_k} S^1). \quad\square$$

Note that for $n = 1$, $PU(2) = SO(3)$ and $\mathbb{C}P^1 = S^2$. By [5], the inclusion $SO(3)\hookrightarrow\mathrm{Homeo}^+(S^2)$ is a homotopy equivalence.

Moreover there are group isomorphisms

$$U(2)\times_{\rho_{2k}} S^1 = SO(3)\times S^1$$
$$U(2)\times_{\rho_{2k+1}} S^1 = Spin^c(3)$$

and we have

Proposition 3.3. *There exist weak homotopy equivalences*

$$B\mathrm{Homeo}_{S^1}(P_k)\underset{w}{\simeq}\begin{cases} B(SO(3)\times S^1) & k\colon \textit{even}\\ BSpin^c(3) & k\colon \textit{odd}.\end{cases}\quad\square$$

The cases for $k = 0, 1$ are studied in [8].

Finally we consider the homotopy groups.

Proposition 3.4. *The following map is injective for all i.*

$$\pi_i(U(n+1)\times_{\rho_k}S^1)\otimes\mathbb{Q}\to\pi_i(\mathrm{Homeo}_{S^1}(P_k))\otimes\mathbb{Q}.$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ U(n+1)\times_{\rho_k}S^1 & \xrightarrow{i} & \mathrm{Homeo}_{S^1}(P_k) & \xrightarrow{j} & \mathrm{Map}(P_k, P_k) \\ \downarrow & & \downarrow & & \downarrow \\ PU(n+1) & \xrightarrow{i} & \mathrm{Homeo}_{P_k}(\mathbb{C}P^n) & \xrightarrow{j} & \mathrm{Map}(\mathbb{C}P^n, \mathbb{C}P^n). \end{array}$$

In [6], Sasao proved that

$$(ji)_*:\pi_i(PU(n+1))\otimes\mathbb{Q}\to\pi_i(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\otimes\mathbb{Q}$$

is an isomorphism for all i, and

$$\pi_1(PU(n+1))\cong\pi_1(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\cong\mathbb{Z}/n+1$$
$$\pi_2(\mathrm{Map}(\mathbb{C}P^n,\mathbb{C}P^n))\cong\mathbb{Z}/2.$$

Then the result follows from the commutatuvity of the diagram. □

Proof of Theorem 1.4. Theorem 1.4 immediately follows from this proposition. □

Appendix

In this appendix, we study S^3 bundles over $\mathbb{H}P^n$.
Let P be a principal S^3 bundle over $\mathbb{H}P^n$. Then we have

Theorem A. $\pi_{4i+3}(\mathrm{Homeo}_{S^3}(P))$ *has a free part for* $i=1, 2, \cdots, n$.

The proof is similar to that of Proposition 3.4. Here we use the result of [7]. Consider the natural action of $Sp(n+1)$ on $\mathbb{H}P^n$, which induces the map

$$Sp(n+1)/\mathbb{Z}_2\xrightarrow{i}\mathrm{Homeo}_P(\mathbb{H}P^n)\xrightarrow{j}\mathrm{Map}(\mathbb{H}P^n,\mathbb{H}P^n).$$

Then we have

Theorem [7]. *The induced homomorphism*

$$(ji)_*\otimes 1: \pi_i(Sp(n+1)/\mathbb{Z}_2)\otimes\mathbb{Q}\to \pi_i(\mathrm{Map}(\mathbb{H}P^n, \mathbb{H}P^n))\otimes\mathbb{Q}$$

is an isomorphism for all $i > 4$.

Moreover for $1 \leq i \leq 6$,

$$\pi_i\mathrm{Map}(\mathbb{H}P^n, \mathbb{H}P^n))\otimes\mathbb{Q} = 0.$$

Proof of Theorem A. For a space Y, denote the space Y localized at 0 by $Y_{(0)}$. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathscr{G} & = & \mathscr{G} & = & \mathscr{G} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Homeo}_{S^3}(P)\,|_{S_p} & \longrightarrow & \mathrm{Homeo}_{S^3}(P) & \longrightarrow & \mathrm{Map}_{S^3}(P, P) \\
\downarrow & & \downarrow & & \downarrow \\
Sp(n+1)/\mathbb{Z}_2 & \longrightarrow & \mathrm{Homeo}_P(\mathbb{H}P^n) & \longrightarrow & \mathrm{Map}(\mathbb{H}P^n, \mathbb{H}P^n)
\end{array}$$

which induces

$$\begin{array}{ccc}
\mathscr{G}_{(0)} & = & \mathscr{G}_{(0)} \\
\downarrow & & \downarrow \\
(\mathrm{Homeo}_{S^3}(P)\,|_{Sp})_{(0)} & \longrightarrow & \mathrm{Map}_{S^3}(P, P)_{(0)} \\
\downarrow & & \downarrow \\
(Sp(n+1)/\mathbb{Z}_2)_{(0)} & \longrightarrow & \mathrm{Map}(\mathbb{H}P^n, \mathbb{H}P^n)_{(0)}
\end{array}$$

where $(\mathrm{Homeo}_{S^3}(P)\,|_{Sp})_{(0)} = \mathrm{Map}_{S^3}(P, P)_{(0)}\,|_{(Sp(n+1)/\mathbb{Z}_2)_{(0)}}$ and $\mathscr{G}_{(0)} = \Omega(B\mathscr{G}_{(0)})$.

Recall that ([1], [3])

$$\begin{aligned}
B\mathscr{G}_{(0)} &\simeq \mathrm{Map}_P(\mathbb{H}P^n, BS^3)_{(0)} \\
&\simeq \mathrm{Map}_P(\mathbb{H}P^n, BS^3_{(0)}) \\
&\simeq \mathrm{Map}_P(\mathbb{H}P^n, K(\mathbb{Q}, 4)) \\
&\simeq K(\mathbb{Q}, 4)
\end{aligned}$$

where the last line is due to the R. Thom's famous result. Hence

$$\mathscr{G}_{(0)} \simeq K(\mathbb{Q}, 3).$$

Therefore the following map is an isomorphism for all $i \neq 3$

$$\pi_i((\mathrm{Homeo}_{S^3}(P)\,|_{Sp})_{(0)}) \to \pi_i(\mathrm{Map}_{S^3}(P, P)_{(0)}).$$

Then

$$\pi_i(\mathrm{Homeo}_{S^3}(P)\,|_{Sp}) \otimes \mathbb{Q} \to \pi_i(\mathrm{Homeo}_{S^3}(P)) \otimes \mathbb{Q}$$

is injective for $i \neq 3$ and this complete the proof. □

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