

# Closed ideals of $A^\infty$ and a famous problem of Grothendieck

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*Dedicated to Professor Charles Fefferman on his 60th birthday*

## Abstract

Using Fréchet algebraic technique, we show the existence of a nuclear Fréchet space without basis, thus providing yet another proof (of a different flavor) of a negative answer to a well known problem of Grothendieck from 1955. Using Fefferman’s construction (which is based on complex-variable technique) of a  $C^\infty$ -function on the unit circle with certain properties, we give much simpler, transparent, and “natural” examples of restriction spaces without bases of nuclear Fréchet spaces of  $C^\infty$ -functions; these latter spaces, being classical objects of study, have attracted some attention because of their relevance to the theories of PDE and complex dynamical systems, and harmonic analysis. In particular, the restriction space  $A^\infty(E)$ , being a quotient algebra of the algebra  $A^\infty(\Gamma)$ , is the central one to other examples; the algebras  $A^\infty$  had played a crucial role in solving a well-known problem of Kahane and Katznelson in the negative.

## 1. Introduction and history of the problem

The problem of determining what kinds of ideals and quotients can be found in arbitrary Fréchet algebras arises not only from the aesthetic imperative to understand the internal structure of these algebras but also from certain applications. For example, the proof of the famous Wiener’s Theorem is one of the early celebrated accomplishments of the theory of Banach algebras. The Wiener’s original proof was a good deal more complicated but the use of Banach algebra technique made it very easy.

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In this paper, we will experience same kind of phenomena in respect of *the existence of most easy and natural examples of nuclear Fréchet spaces without bases*. In particular, we are specifically concerned with the determination of those Fréchet algebras which admit a nuclear Fréchet space without a basis as their quotient (see Theorem 2.7). In his famous memoir [15], Grothendieck asked *whether every nuclear Fréchet space has a basis*. The problem has been solved by Mitiagin and Zobin in 1974 [17], and since then there are several various constructions of such spaces which relate to subspaces as well as quotients. (See e.g. [4, 5, 10, 12, 13, 18].) We emphasize that all results given in these references were obtained by assuming *a linear topological space structure*. Though these constructions are ingenious (some of them relatively elementary), all these examples of nuclear Fréchet spaces without bases were constructed on purpose and they are not “natural” spaces of functions, operators or measures appearing in analysis. Recently, Domanski and Vogt [11] showed that the space of real-analytic functions (which is not metrizable) has no basis, and, in the metrizable case, Vogt [23] constructed an example of a nuclear Fréchet space without basis, consisting of  $C^\infty$ -functions.

In our case, it is worth noticeable that the algebraic structure of an algebra plays an important role to produce such examples, reducing more complicated construction. In particular, the non-existence of a cyclic basis in the algebra is due to the totally disconnected spectrum of the algebra; this result is also related to the topological structure of the algebra, that is, to having a special kind of topology on the algebra (see Corollary 2.6). Thus the algebraic structure and the topological structure of the algebra are closely related with each other. Though there are several results of automatic continuity in the literature, the connection between these two structures has been explored in a different manner for the first time in this paper.

We first obtain a few characterizations by investigating Fréchet algebras with a power series generator (defined below). Although the theoretical existence, obtained using the Fréchet algebraic technique, is very easy, the existence of a concrete example seems to be a difficult problem. For this we need to construct a non-zero  $C^\infty$ -function on the unit circle  $\Gamma$  with all negative Fourier coefficients zero and which vanishes on a closed, totally disconnected infinite subset of  $\Gamma$ . The construction of such function is given by C. Fefferman using complex-variable theory. Using this function, we give a few examples of nuclear Fréchet spaces without bases which are restriction spaces of Fréchet spaces of  $C^\infty$ -functions on subsets of either  $\mathbb{R}$  or  $\mathbb{C}$ .

The situation with quotients is somewhat more complicated. In the first place, a Fréchet space which admits a continuous norm can have a quotient which does not. Indeed, in 1936, Eidelheit showed that *any non-Banach*

*Fréchet space has a quotient isomorphic to  $\omega$*  [14]. (Another complication is that in order to obtain results one usually has to assume that the original algebra is separable –the nonseparable case seems to be much more difficult.) On the other hand, Bellenot and Dubinsky [3] showed that *every Fréchet-Montel space not isomorphic to  $\omega$  has a nuclear Köthe quotient, i.e., a quotient with a continuous norm and a basis*. It is then of interest to determine those Fréchet spaces which admit a *nuclear Fréchet space with a continuous norm but without a basis* as their quotient (see e.g. [13]), and, thus, our examples are different from Moscatelli’s examples given in [18] as they are certainly *not quojections*.

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## 2. Existence using Fréchet algebraic technique

All algebras in this paper will be commutative and unital unless otherwise specified. We use many concepts from the standard theory of Fréchet spaces (and algebras). Thus, we recall that a *Fréchet algebra* is a complete metrizable topological algebra  $A$  whose topology may be defined by a sequence  $(p_k)_{k \geq 1}$  (assumed increasing without loss of generality) of submultiplicative seminorms. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which  $A = \varprojlim (A_k; d_k)$  (see [19, §2] for more details). If each  $d_k$  is a surjective operator from  $A_{k+1}$  onto  $A_k$ , then we say that a Fréchet space  $A$  is a *quojection*. If the topology of  $A$  can be defined by a sequence  $(p_k)$  of norms, then we say that  $A$  *admits a continuous norm*. In fact, it is well known and easy to prove that a nuclear Fréchet space admits a continuous norm if and only if it admits a sequence of norms defining its Fréchet topology. A Fréchet algebra  $A$  is said to be a  *$Q$ -algebra* if the set of all its invertible elements is open.  $A$  is *local* if the Gel’fand space  $M(A)$  is singleton.

A sequence  $(x_n)$  in a Fréchet space  $A$  is a *basis* if for each  $y \in A$  there exists a unique expansion of the form  $y = \sum_{n=0}^{\infty} \lambda_n x_n$ ,  $\lambda_n$  complex scalars. An element  $x$  in a Fréchet algebra  $A$  is a *power series generator* (shortly: p.s.g.) for  $A$ , if and only if each  $y \in A$  is of the form  $y = \sum_{n=0}^{\infty} \lambda_n x^n$ ,  $\lambda_n$  complex scalars, such that  $\sum_{n=0}^{\infty} |\lambda_n| p_k(x^n) < \infty$  for all  $k$  ([6]). (A normed algebra with a p.s.g. can analogously be defined.) Such an  $A$  is a commutative, separable and singly generated Fréchet algebra generated by  $x$ ; also, if  $I$  is a closed ideal of  $A$ , then  $A/I$  has a p.s.g.  $x+I$ . We remark that the elements  $y$  in  $A$  may not necessarily have unique expansion of the form  $\sum_{n=0}^{\infty} \lambda_n x^n$ .

We write  $\mathcal{F}$  for the algebra  $\mathbb{C}[[X]]$  of all formal power series in an indeterminate  $X$ , with complex coefficients. The algebra  $\mathcal{F}$  is a Fréchet algebra when endowed with the weak topology defined by the projections  $\pi_m : \mathcal{F} \rightarrow \mathbb{C}$ ,  $m \in \mathbb{Z}^+$ , where  $\pi_m(\sum_{n=0}^{\infty} \lambda_n X^n) = \lambda_m$ . A defining sequence of seminorms for  $\mathcal{F}$  is  $(p'_k)$ , where  $p'_k(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^k |\lambda_n|$  ( $k \in \mathbb{N}$ ). A *Fréchet algebra of power series* is a subalgebra  $A$  of  $\mathcal{F}$  such that  $A$  is a Fréchet algebra containing the indeterminate  $X$  and such that the inclusion map  $A \hookrightarrow \mathcal{F}$  is continuous [19]. In fact, surprising recent results show that the continuity of the inclusion map  $A \hookrightarrow \mathcal{F}$  in this setting is automatic, and hence the time-honored definitions of Banach and Fréchet (and, more generally,  $(F)$ -) algebras of power series contain a redundant clause (see [8, Corollaries 11.3 and 11.4]); this is not possible in the several-variable case by Theorem 12.3 of [8]. Recently, Fréchet algebras of power series –and more generally, the power series ideas in general Fréchet algebras– have acquired significance in understanding the structure of a Fréchet algebra [1, 6, 7, 8, 19, 20].

The initial range of examples of Fréchet algebras of power series having a p.s.g. includes  $\mathcal{F}$ , the Beurling-Fréchet algebras  $\ell^1(\mathbb{Z}^+, W)$ ,  $\text{Hol}(U)$  ( $U$  a domain in  $\mathbb{C}$ ) and  $A^\infty(\Gamma)$ ; some other examples without a p.s.g. are the disc algebra  $A(D)$ ,  $H^\infty(U)$  ( $U$  a bounded domain in  $\mathbb{C}$  containing 0),  $A^k(\Gamma)$  (see [6] for more details), and Fréchet algebras of power series in which polynomials fail to be dense (see [19, Remarks 1 (b) and 2]). Also Fréchet algebras with a cyclic basis generated by  $x$  are realized as examples of Fréchet algebras of power series by identifying the series expansions.

A seminorm  $p$  on a Fréchet (or even metrizable) algebra  $A$  with a cyclic basis generated by a p.s.g.  $x$  is a *power series seminorm* if  $p(\sum_{n=0}^{\infty} \lambda_n x^n) = \sum_{n=0}^{\infty} |\lambda_n| p(x^n)$  for all  $y \in A$ . Now let  $(A, \|\cdot\|)$  be a normed algebra with a p.s.g.  $x$  and let  $A^\sim$  be the completion of  $A$ . Then  $A^\sim$  need not be a Banach algebra with a p.s.g.  $x$ ; but, if there exists a power series norm  $|\cdot|$  on  $A$  equivalent to  $\|\cdot\|$ , then  $A^\sim$  is a Banach algebra with a p.s.g.  $x$  (see [6, Remark 2.4 and Proposition 2.5]). More generally, we have the following lemma whose proof is based on [6, Lemma 2.2].

**Lemma 2.1.** *Let  $A$  be a Fréchet algebra having a p.s.g.  $x$ . Then  $x$  generates a cyclic basis if and only if the topology of  $A$  is defined by a sequence of power series seminorms.*

A weight function on  $\mathbb{Z}^+$  is a function  $\omega : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  such that for all  $m, n \in \mathbb{Z}^+$ ,  $\omega(m+n) \leq \omega(m)\omega(n)$  and  $\omega(n) > 0$ . If  $A$  is a Fréchet algebra with a cyclic basis generated by a p.s.g.  $x$  and we define  $\omega_k(n) = p_k(x^n)$  ( $n \in \mathbb{Z}^+$ ), then  $W = (\omega_k)$  is a separating sequence of functions on  $\mathbb{Z}^+$  satisfying  $\omega_k(m+n) \leq \omega_k(m)\omega_k(n)$ ,  $\omega_k(n) \leq \omega_{k+1}(n)$  and  $\omega_k(n) \geq 0$  for all  $m, n \in \mathbb{Z}^+$  all  $k \in \mathbb{N}$ . Let

$$\ell^1(\mathbb{Z}^+, W) := \left\{ \sum_{n=0}^{\infty} \lambda_n X^n \in \mathcal{F} : \sum_{n=0}^{\infty} |\lambda_n| \omega_k(n) < \infty \text{ for all } k \right\}.$$

Since, by Lemma 2.1, each  $p_k$  is a power series seminorm on  $A$ , the mapping  $\theta : \ell^1(\mathbb{Z}^+, W) \rightarrow A$  defined by  $\theta(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^{\infty} \lambda_n x^n$  is a homeomorphic isomorphism in view of the open mapping theorem. Thus we have the following characterization (see [6, Theorem 2.1] for more details):

**Theorem 2.2.** *Let  $A$  be a Fréchet algebra with a cyclic basis generated by a p.s.g.  $x$ . Then  $A$  is isomorphic to either  $\mathcal{F}$  or the Beurling-Fréchet algebra  $\ell^1(\mathbb{Z}^+, W)$  for an increasing sequence  $W$  of weights on  $\mathbb{Z}^+$ .*

More generally, if  $A$  is a Fréchet algebra with a p.s.g.  $x$ , then  $A \cong \ell^1(\mathbb{Z}^+, W)/\ker\theta$ , and so  $Sp_A(x) = M(A)$  is totally disconnected provided that  $\ker\theta$  is proper (see [6, Theorem 3.7 (1)]). As corollaries, we have the following characterizations. A Banach algebra  $A$  is *uniform* if  $\|x^2\| = \|x\|^2$  for all  $x \in A$ . By a *proper* seminorm we mean a seminorm that is not a norm.

**Corollary 2.3.** *Let  $(A, \|\cdot\|)$  be a Banach algebra of power series such that  $X$  is a p.s.g. for  $A$ . Then  $A$  is not a uniform Banach algebra.*

**Corollary 2.4.** *Let  $A$  be a Fréchet algebra. Then  $A$  is isomorphic to  $\mathcal{F}$  if and only if it is a Fréchet algebra with a p.s.g. and with its Fréchet topology defined by a sequence  $(p_k)$  of proper power series seminorms.*

We remark that the above corollary also holds under the weak assumptions:  $A$  a Fréchet algebra of power series and any sequence  $(p_k)$  of proper seminorms on  $A$  (see [19, Corollary 3.4]). Thus every Fréchet algebra of power series other than  $\mathcal{F}$  admits a continuous norm.

**Corollary 2.5.** *Let  $A$  be a Fréchet algebra with the Gel'fand space  $M(A)$  of continuous characters which is not totally disconnected. If  $A$  has a p.s.g., then  $A$  is a semisimple Fréchet algebra and  $M(A)$  is either a disc in the complex plane or is the whole of  $\mathbb{C}$ .*

**Proof.** Evidently,  $A$  is not a local algebra as  $M(A)$  is not singleton. Further,  $A \cong \ell^1(\mathbb{Z}^+, W)$  since  $M(A)$  is totally disconnected in the other case as remarked earlier after Theorem 2.2. Thus  $M(\ell^1(\mathbb{Z}^+, W))$  is not totally disconnected, and contains at least two distinct points in  $\mathbb{C}$ . Hence  $r > 0$ , where  $r = \sup_k r_k$  and  $r_k = \inf_{n \in \mathbb{Z}^+} \omega_k(n)^{1/n}$ . This shows that  $\ell^1(\mathbb{Z}^+, W)$  is semisimple. Now, by [6, Theorem 3.7 (2)], if  $r = \infty$ , then  $A \cong \text{Hol}(\mathbb{C})$ , the Fréchet algebra of entire functions, and if  $r < \infty$ , then either  $A \cong \text{Hol}(U)$ ,  $U$  an open disc with radius  $r$ , or is a  $Q$ -algebra in which case  $M(A)$  is a closed disc. ■

The above corollary shows that the assumption that  $A$  is semisimple in the statement (2) of Theorem 3.7 of [6] can be replaced by a weaker hypothesis:  $Sp_A(x)$  is not singleton.

**Corollary 2.6.** *Let  $A$  be a non-local Fréchet algebra with a p.s.g.  $x$ . The following are equivalent.*

- (1)  $Sp_A(x)$  is a hemicompact metric space which is not totally disconnected.
- (2) The p.s.g.  $x$  generates a cyclic basis for  $A$ .
- (3) The topology of  $A$  is defined by a sequence of power series seminorms.

**Proof.** (2)  $\Leftrightarrow$  (3). This is an easy consequence of Lemma 2.1.

(1)  $\Rightarrow$  (2). By Corollary 2.5,  $A$  is a semisimple Fréchet algebra isomorphic to the Beurling-Fréchet algebra  $\ell^1(\mathbb{Z}^+, W)$ . Hence the p.s.g.  $x$  generates a cyclic basis.

(2)  $\Rightarrow$  (1). By Theorem 2.2,  $A$  is isomorphic to  $\ell^1(\mathbb{Z}^+, W)$ . The Gel'fand space of  $\ell^1(\mathbb{Z}^+, W)$  is not totally disconnected. ■

**Remark.** We note that the algebra  $C(E)$  of all continuous functions on a totally disconnected compact metric space  $E$  is a uniform Banach algebra with a p.s.g.  $e^{i\theta}|_E$ ,  $0 \leq \theta \leq 2\pi$ , whose topology is not given by a power series norm and the Gel'fand space is  $E$ . In fact,  $C(E)$  is isomorphic to  $A^+(E)$ , a quotient of the algebra  $A^+(\Gamma)$  of absolutely convergent Taylor series. On the other hand, by Corollary 2.3, there does not exist a Banach algebra of power series having a p.s.g., which is also a uniform algebra. But, in the Fréchet case, this can happen, for example in the algebra  $\text{Hol}(U)$ . This is one of the significant differences between Banach algebra of power series and Fréchet algebra of power series at the level of uniform algebras with a p.s.g.

All of the ingredients are now present and we are ready to prove the main result of this paper.

**Theorem 2.7.** *If  $A$  is a non-local, non-Banach Fréchet  $Q$ -algebra with a cyclic basis generated by a p.s.g.  $x$  such that the Fréchet topology of  $A$  is defined by a sequence  $(p_k)_{k \geq 1}$ , where  $p_k(\sum_{n=0}^\infty \lambda_n x^n) = \sup_{n \in \mathbb{Z}^+} |\lambda_n| p_k(x^n)$ , then  $A$  has a quotient which is a nuclear Fréchet space without a basis.*

**Proof.** Let  $A$  satisfy the stated conditions. Then, by Theorem 2.2,  $A$  is isomorphic to  $\ell^1(\mathbb{Z}^+, W)$ . Also, by [21, Theorem 6.1.3],  $A$  is nuclear, and, by Corollary 2.5, the Gel'fand space  $Sp_A(x)$  of continuous characters is isomorphic with the closed unit disc  $D$  as  $A$  is a  $Q$ -algebra. Suppose that  $E$  is a closed, totally disconnected infinite subset of  $\Gamma$  such that

$$I(E) := \{y \in A : \hat{y} \equiv 0 \text{ on } E\}$$

is a proper closed ideal of  $A$ . Consider the restriction algebra  $A/I(E) := A(E)$ . Clearly it is a non-local nuclear Fréchet  $Q$ -algebra with a p.s.g.  $x + I(E)$  and  $E$  as its Gel'fand space. Then, by Corollary 2.6, it has no cyclic basis. Suppose  $A(E)$ , when considered as a nuclear Fréchet space, has some other basis  $(x_n)$ . Then  $(x_n)$  is equicontinuous. So, by the Dynin-Mitiagin Basis Theorem, it is absolute. Hence, by [21, 10.2.2] due to Rolewicz,  $A(E)$  is isomorphic to a nuclear Fréchet sequence (Köthe) space. Thus, by [16, Proposition 28.16],  $A(E) \cong \lambda^\infty(A')$ , where  $A'$  is the Köthe matrix  $(q_k(x_n))_{k,n \in \mathbb{Z}^+}$ . Also each  $z \in A(E)$  admits a unique expansion of the form  $\sum_{n=0}^\infty \mu_n x_n$ , i.e.,  $z = \sum_{n=0}^\infty \lambda_n (x + I(E))^n = \sum_{n=0}^\infty \mu_n x_n$ . Since  $x + I(E)$  does not generate a cyclic basis for  $A(E)$ ,  $z$  can be represented by the system  $(x + I(E))^n$ ,  $n \in \mathbb{Z}^+$ , but not in a unique way, say,  $\sum_{n=0}^\infty \eta_n (x + I(E))^n$  is another expression for  $z$ . So there is at least one  $k \in \mathbb{Z}^+$  such that  $\lambda_k \neq \eta_k$ . For this  $k$ , we have  $\lambda_k (x + I(E))^k = \sum_{n=0}^\infty \mu_n^k x_n$  since  $\lambda_k (x + I(E))^k$  is an element of  $A(E)$ . But then  $\eta_k (x + I(E))^k = \sum_{n=0}^\infty \zeta_n^k x_n$ , where  $\zeta_n^k = \frac{\mu_n^k \eta_k}{\lambda_k}$  for all  $n$ . Thus, we have

$$z = \lim_n \sum_{k=0}^n \lambda_k (x + I(E))^k = \lim_n \lim_m \sum_{k=0}^n \left( \sum_{l=0}^m \mu_l^k x_l \right) = \sum_{m=0}^\infty \mu_m x_m$$

since  $\phi_k(l) = \mu_l^k x_l$  is uniformly Cauchy over  $k$ ; note that, as a consequence of the Grothendieck-Pietsch criterion, the Fréchet topology of  $A(E)$  can be defined by  $(\|\cdot\|_k)$ , where  $\|(\sum_{n=0}^\infty \mu_n x_n)\|_k = \sup_n |\mu_n| q_k(x_n)$ . On the other hand, we have

$$z = \lim_n \sum_{k=0}^n \eta_k (x + I(E))^k = \lim_n \lim_m \sum_{k=0}^n \left( \sum_{l=0}^m \zeta_l^k x_l \right) = \sum_{m=0}^\infty \zeta_m x_m$$

since  $\phi'_k(l) = \zeta_l^k x_l$  is uniformly Cauchy over  $k$ . Hence the two series expansions of  $z$  about the system  $(x_n)$  are different, showing that the system  $(x_n)$  cannot be a basis for  $A(E)$ . Thus we have shown that there does not exist any basis for  $A(E)$ . ■

We give an alternate proof for a special case as follows; one can follow this idea to give proof for a more general case by replacing  $A^\infty(\Gamma)$  and  $A^\infty(E)$  by  $A$  and  $A(E)$ , respectively.

First, we note that by using the formula

$$\langle \nu, f \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\nu_n$$

for  $\nu = \{\nu_n\} \in C_0$ , the Banach space of all complex sequences converging to zero at infinity, we identify  $A^+(\Gamma)$  with the dual of  $C_0$ . As in the proof above, if  $A^\infty(E)$  has some other basis  $(F_n)$ , then it is isomorphic to a nuclear Fréchet sequence (Köthe) space. Hence, following the above argument, we can identify  $A^\infty(E)$  with the dual of some suitable Fréchet sequence space  $C$ , which is a subspace of  $C_0$ . Further, using the above formula, it is easy to see that if  $F = f|_E = f + I(E)$  is an element of  $A^\infty(E)$ , then  $\langle \nu, f + g \rangle = \langle \nu, f \rangle$  for  $g \in I(E)$  and for a fixed element  $\nu \in C \subset C_0$ . By varying  $\nu$ , we see that  $\langle \nu, g \rangle = 0$  for all  $\nu \in C$ , and hence  $g = 0$ , implying that  $I(E) = \{0\}$ , a contradiction of the fact that  $I(E)$  is a proper closed ideal of  $A^\infty(\Gamma)$  (see §3).

We again emphasize that we have considered only separable algebras, and, for the separability condition,  $x$  being a p.s.g. for  $A$  is sufficient (but not necessary as evidenced by the examples before Lemma 2.1). Also the specific Fréchet topology on  $A$  is required to ensure that  $A$  is nuclear. We remark that the uniqueness of the Fréchet topology for Fréchet algebras of power series has been established in Corollary 4.2 of [19], and the uniqueness of the Fréchet topology for  $(F)$ -algebras of power series has been established in Corollary 11.7 of [8] by another approach. The following counterexamples show that the assumptions on  $A$  cannot be dropped.

**Example 1.** We note that  $\mathcal{F}$  is a local algebra, satisfying all other stated conditions. The non-zero ideals of  $\mathcal{F}$  are just the principal ideals  $\mathcal{F}X^k$  ( $k \geq 0$ ); each of these is closed in  $\mathcal{F}$  and therefore the quotient algebras are finite-dimensional algebras. Hence, it is of interest to also note that  $\mathcal{F}$  is the only Fréchet algebra of finite type among Fréchet algebras of power series [19, p. 132].

**Example 2.** If  $A$  was a non-local Banach algebra with a cyclic basis generated by a p.s.g.  $x$ , then, by Theorem 2.2,  $A$  is isomorphic to the semisimple Beurling-Banach algebra  $\ell^1(\mathbb{Z}^+, \omega)$ . Hence,  $A$  cannot be nuclear being an infinite-dimensional normed algebra (Dvoretzky-Rogers Theorem), and therefore the quotient algebras of  $A$  with a p.s.g.  $x + I$  cannot be nuclear for the same reason.



**Example 3.** Suppose that  $A$  is not a  $\mathbb{Q}$ -algebra, satisfying all other stated conditions. Then, by Corollary 2.5,  $A$  is isomorphic to  $\ell^1(\mathbb{Z}^+, W)$  which is semi-simple, and therefore, by [6, Theorem 3.7], it is isomorphic to either  $\text{Hol}(U)$  or  $\text{Hol}(\mathbb{C})$ . If  $E$  is a closed, totally disconnected infinite subset of  $U$  (or  $\mathbb{C}$ ), then elementary complex-variable theory shows that  $I(E) = \{0\}$  for  $\text{Hol}(U)$  (or  $\text{Hol}(\mathbb{C})$ ).

### 3. Fefferman’s construction of a $C^\infty$ -function

We now show that  $I(E)$  of Example 4.1 is, indeed, proper for  $E$  given below. We remark that the construction given here is different from a construction of outer functions in  $A^\infty(\Gamma)$  for Carleson sets (see [22, Theorem 3.3]).

We consider the following subsets of  $\Gamma$ .

Let  $\epsilon_N = 64^{-N}$  for  $N \geq 1$ . Let  $E = \{ \exp(i \sum_{n=1}^\infty \sigma_n \epsilon_n) : \text{each } \sigma_n = \pm 1 \}$  and  $E_N = \{ \exp(i \sum_{n=1}^N \sigma_n \epsilon_n) : \text{each } \sigma_n = \pm 1 \}$  for  $N \geq 1$ . For  $1 \leq M \leq N$  and  $\zeta = \exp(i \sum_{n=1}^M \sigma_n \epsilon_n) \in E_M$ , let

$$E_N(\zeta) = \left\{ \exp \left( i \sum_{n=1}^N \hat{\sigma}_n \epsilon_n \right) : \text{each } \hat{\sigma}_n = \pm 1, \text{ and } \hat{\sigma}_n = \sigma_n \text{ if } n \leq M \right\}.$$

It is easy to see that  $E$  is a closed, totally disconnected subset of  $\Gamma$ . Note that  $\zeta \in E_M, \zeta' \in E_N(\zeta)$  imply  $|\zeta - \zeta'| \leq \epsilon_M$  and  $\zeta \in E$  implies  $\text{dist}(\zeta, E_N) \leq \epsilon_N$ . Also  $E_N$  consists of  $2^N$  points on  $\Gamma$  and  $\zeta \in E_N$  implies  $\text{dist}(\zeta, E) \leq \epsilon_N$ . For any  $z \in \mathbb{C}$ , and for  $M \leq N$ , we have  $|\text{dist}(z, E_M) - \text{dist}(z, E_N)| \leq \epsilon_M$ .

Next, we define a basic analytic function and give some important properties of that function.

For  $0 < \epsilon < 1$  and  $z \in U$  an open unit disc, define

$$\tilde{G}_\epsilon(z) = \int_{-\pi}^\pi \frac{|1 + \epsilon - e^{i\theta}|^{-1/2} d\theta}{1 - e^{-i\theta} z}.$$

**Lemma 3.1.** *For each  $0 < \epsilon < 1$ , the function  $\tilde{G}_\epsilon(z)$ , defined initially for  $z \in U$ , extends to a function  $G_\epsilon(z)$ , defined for  $z \in D$  a closed unit disc, and having the following properties:*

- (1)  $G_\epsilon(z)$  is analytic in  $U$  and  $C^\infty$  on  $D$ .
- (2)  $\left| \left( \frac{d}{dz} \right)^k G_\epsilon(z) \right| \leq c(k) [ |1 - z| + \epsilon ]^{-1/2 - k}$  for  $z \in D$ ,  $k \geq 0$  and  $0 < \epsilon < 1$ ; where  $c(k)$  is a constant depending only on  $k$ , but not on  $z$  or  $\epsilon$ .
- (3)  $\text{Re} G_\epsilon(z) \geq c [ |1 - z| + \epsilon ]^{-1/2}$  for  $z \in D$  and  $0 < \epsilon < 1$ ; where  $c$  is a positive constant, independent of  $z$  and  $\epsilon$ .

**Proof.** For  $z \in U$ , we have

$$\tilde{G}_\epsilon(z) = \frac{1}{i} \oint_{|\zeta|=1} \frac{|1 + \epsilon - \zeta|^{-1/2} d\zeta}{\zeta - z}.$$

It is easy to define single-valued branches of the analytic functions

$$(1 + \epsilon - \zeta)^{-1/4}$$

in  $\{|\zeta| < 1 + \epsilon\}$  and  $(1 + \epsilon - \zeta^{-1})^{-1/4}$  in  $\{|\zeta| > (1 + \epsilon)^{-1}\}$  so that we have

$$|1 + \epsilon - \zeta|^{-1/2} = (1 + \epsilon - \zeta)^{-1/4} (1 + \epsilon - \zeta^{-1})^{-1/4}$$

for  $\zeta \in \Gamma$ . Hence, for  $z \in U$ , we have

$$\begin{aligned} \tilde{G}_\epsilon(z) &= \frac{1}{i} \oint_{|\zeta|=1} \frac{(1 + \epsilon - \zeta)^{-1/4} (1 + \epsilon - \zeta^{-1})^{-1/4} d\zeta}{\zeta - z} \\ &= \frac{1}{i} \oint_{|\zeta|=r} \frac{(1 + \epsilon - \zeta)^{-1/4} (1 + \epsilon - \zeta^{-1})^{-1/4} d\zeta}{\zeta - z} \end{aligned}$$

for  $1 < r < 1 + \epsilon$ . The last integral clearly defines a function of  $z$  analytic in the disc  $\{|z| < r\}$ . Since  $r$  may be taken arbitrarily close to  $1 + \epsilon$ , we conclude that  $\tilde{G}_\epsilon(z)$ , defined initially for  $z \in U$ , extends to an analytic function on  $\{|z| < 1 + \epsilon\}$ . This proves (1).

To prove (2), we consider integrals of the form

$$F(\phi) = \int_{-\pi}^{\pi} K(\phi - \theta) f(\theta) d\theta, \quad \phi \in \mathbb{R},$$

under various assumptions on  $K$  and  $f$ . Regarding  $f$ , we assume that

$$\left| \left( \frac{d}{d\theta} \right)^k f(\theta) \right| \leq c(k) [|\theta| + \epsilon]^{-1/2-k}$$

for  $k \geq 0$ , where  $c(k)$  is a constant depending only on  $k$ . Then we use the following two propositions.

**Proposition 3.2.** *Let  $F$  be defined as above, where  $f$  satisfies the assumption given above. Let  $\phi \in \mathbb{R}$ . Under any of the following hypotheses, we have*

$$\left| \left( \frac{d}{d\phi} \right)^k F(\phi) \right| \leq c'(k) [|\phi| + \epsilon]^{-1/2-k}$$

for  $k \geq 0$ , where  $c'(k)$  depends only on  $k$  and on the constants  $c_0$  and  $c_0(k)$  below.

- (H1)  $|\phi| \leq \epsilon$ ,  $K$  is even, and  $\int_{-\pi}^{\pi} |K(\theta)| d\theta \leq c_0$ .
- (H2)  $|\phi| \leq \epsilon$ ,  $K$  is odd,  $\text{supp } K \subset [-2\epsilon, 2\epsilon]$ , and  $|K(\theta)| \leq \frac{c_0}{|\theta|}$  for all  $\theta$ .
- (H3)  $|\phi| \leq \epsilon$ ,  $K$  is odd,  $\text{supp } K \subset \{\theta : |\theta| \geq \epsilon\}$ , and  $|K^{(k)}(\theta)| \leq \frac{c_0(k)}{|\theta|^{1+k}}$  for all  $\theta$  and  $k$ .
- (H4)  $|\phi| \geq \epsilon$ ,  $\text{supp } f \subset \{\theta : |\theta| \leq \frac{|\phi|}{2}\}$ , and  $|K^{(k)}(\theta)| \leq \frac{c_0(k)}{|\theta|^{1+k}}$  for all  $\theta \in \mathbb{R}$  and  $k \geq 0$ .
- (H5)  $|\phi| \geq \epsilon$ ,  $\text{supp } f \subset \{\theta : \frac{1}{4}|\phi| \leq |\theta| \leq 4|\phi|\}$ ,  $K$  is even and  $\int_{-\pi}^{\pi} |K(\theta)| d\theta \leq c_0$ .
- (H6)  $|\phi| \geq \epsilon$ ,  $\text{supp } f \subset \{\theta : \frac{1}{4}|\phi| \leq |\theta| \leq 4|\phi|\}$ ,  $K$  is odd,  $\text{supp } K \subset \{\theta : |\theta| \leq \frac{1}{8}|\phi|\}$ , and  $|K(\theta)| \leq \frac{c_0}{|\theta|}$  for all  $\theta$ .
- (H7)  $|\phi| \geq \epsilon$ ,  $\text{supp } f \subset \{\theta : \frac{1}{4}|\phi| \leq |\theta| \leq 4|\phi|\}$ ,  $K$  is odd,  $\text{supp } K \subset \{\theta : |\theta| \geq \frac{1}{16}|\phi|\}$ , and  $|K(\theta)| \leq \frac{c_0}{|\theta|}$  for all  $\theta$ .
- (H8)  $|\phi| \geq \epsilon$ ,  $\text{supp } f \subset \{\theta : |\theta| \geq 3|\phi|\}$ , and  $|K(\theta)| \leq \frac{c_0}{|\theta|}$  for all  $\theta$ .

**Sketch of Proof of Proposition 3.2.** Under each of the above hypotheses (H1),..., (H8), one of the following formulas yields the result at once.

- F(1)  $F^{(k)}(\phi) = \int_{-\pi}^{\pi} K(\theta) f^{(k)}(\phi - \theta) d\theta$ ,
- F(2)  $F^{(k)}(\phi) = \int_{-\pi}^{\pi} K(\theta) f^{(k)}(\phi - \theta) d\theta = \frac{1}{2} \int_{-\pi}^{\pi} K(\theta) [f^{(k)}(\phi - \theta) - f^{(k)}(\phi + \theta)] d\theta$   
provided  $K$  is odd,
- F(3)  $F^{(k)}(\phi) = \int_{-\pi}^{\pi} K^{(k)}(\theta) f(\phi - \theta) d\theta$ , and
- F(4)  $F^{(k)}(\phi) = \int_{-\pi}^{\pi} K^{(k)}(\phi - \theta) f(\theta) d\theta$ .

Formulas F(1), ..., F(4) are trivial consequences of the definition of  $F$  above. When we apply F(2) to treat (H2) and (H6), we note that

$$|f^{(k)}(\phi - \theta) - f^{(k)}(\phi + \theta)| \leq 2|\theta| \max\{|f^{(k+1)}(\xi)| : \xi \in (\phi - \theta, \phi + \theta)\}.$$

We apply F(1) to treat (H1), (H5), (H7) and (H8). We apply F(3) to treat (H3), and apply F(4) to treat (H4). For example, under (H6), we

argue as follows.

$$\begin{aligned}
 |F^{(k)}(\phi)| &= \left| \frac{1}{2} \int_{-\pi}^{\pi} K(\theta) [f^{(k)}(\phi - \theta) - f^{(k)}(\phi + \theta)] d\theta \right| \\
 &\leq \int_{-\pi}^{\pi} |K(\theta)| |\theta| \max \{ |f^{(k+1)}(\xi)| : \xi \in (\phi - \theta, \phi + \theta) \} d\theta \\
 &\leq \int_{\{|\theta| \leq \frac{1}{8}|\phi|\}} |K(\theta)| |\theta| c(k) \left[ \frac{7}{8}|\phi| \right]^{-1/2-(k+1)} d\theta \\
 &\leq c_0 |\phi| c(k) \left[ \frac{7}{8}|\phi| \right]^{-1/2-(k+1)} \leq c'(k) [|\phi| + \epsilon]^{-1/2-k}.
 \end{aligned}$$

The remaining cases are no harder than the above, and are left to the reader. ■

Now let  $K(\theta)$  be a function on  $[-\pi, \pi]$ , satisfying the following conditions:

- (i)  $|K^{(k)}(\theta)| \leq \frac{c_0(k)}{|\theta|^{1+k}}$  for all  $\theta \in [-\pi, \pi]$ ,  $k \geq 0$ , with  $c_0(k)$  depending only on  $k$ , and
- (ii)  $\int_{-\pi}^{\pi} |K(\theta) + K(-\theta)| d\theta \leq c_0$ .

**Proposition 3.3.** *Let  $F$  and  $f$  be as in Proposition 3.2. Let  $K$  satisfy (i) and (ii) above. Then*

$$\left| \left( \frac{d}{d\phi} \right)^k F(\phi) \right| \leq c'(k) [|\phi| + \epsilon]^{-1/2-k}$$

for  $k \geq 0$  and  $\phi \in [-\pi, \pi]$ , where  $c'(k)$  depends only on  $k$  and on the constants  $c_0(k')$ ,  $c(k')$  ( $k' \geq 0$ ), and on the constant  $c_0$  appearing in (H1), (H6) and (H7). In particular,  $c'(k)$  does not depend on  $\epsilon$  or  $\phi$ .

**Proof.** First suppose  $|\phi| \leq \epsilon$ . Then, by a partition of unity, we may write  $K = K_{\text{even}} + K_{\text{odd,in}} + K_{\text{odd,out}}$ , where  $K_{\text{even}}$ ,  $K_{\text{odd,in}}$  and  $K_{\text{odd,out}}$  satisfy (H1), (H2), (H3) respectively. Hence, Proposition 3.3 in this case follows easily from Proposition 3.2. Similarly, suppose  $|\phi| \geq \epsilon$ . Then, by using a partition of unity, we can write  $f = f_{\text{in}} + f_{\text{med}} + f_{\text{out}}$ , with  $f_{\text{in}}, f_{\text{med}}, f_{\text{out}}$  satisfying the same assumption imposed on  $f$  (with different constants  $c(k)$ ), and with  $\text{supp} f_{\text{in}} \subset \{ \theta : |\theta| \leq \frac{|\phi|}{2} \}$ ,  $\text{supp} f_{\text{med}} \subset \{ \theta : \frac{1}{4}|\phi| \leq |\theta| \leq 4|\phi| \}$ , and  $\text{supp} f_{\text{out}} \subset \{ \theta : |\theta| \geq 3|\phi| \}$ . Also, by a partition of unity, we can express  $K = K_{\text{even}} + K_{\text{odd,in}} + K_{\text{odd,out}}$ , where  $K_{\text{even}}, K_{\text{odd,in}}, K_{\text{odd,out}}$  satisfy the same hypotheses as  $K$  (with different constants), and  $K_{\text{even}}$  is even;  $K_{\text{odd,in}}$  is odd

and is supported in  $\{\theta : |\theta| \leq \frac{1}{8} |\phi|\}$ ; and  $K_{\text{odd,out}}$  is odd and is supported in  $\{\theta : |\theta| \geq \frac{1}{16} |\phi|\}$ . We then write

$$\begin{aligned} F(\phi) &= \int_{-\pi}^{\pi} K(\phi - \theta) f(\theta) d\theta \\ &= \int_{-\pi}^{\pi} K(\phi - \theta) f_{\text{in}}(\theta) d\theta + \int_{-\pi}^{\pi} K_{\text{even}}(\phi - \theta) f_{\text{med}}(\theta) d\theta \\ &\quad + \int_{-\pi}^{\pi} K_{\text{odd,in}}(\phi - \theta) f_{\text{med}}(\theta) d\theta + \int_{-\pi}^{\pi} K_{\text{odd,out}}(\phi - \theta) f_{\text{med}}(\theta) d\theta \\ &\quad + \int_{-\pi}^{\pi} K(\phi - \theta) f_{\text{out}}(\theta) d\theta. \end{aligned}$$

The terms on the right satisfy the conclusions of Proposition 3.2, thanks to (H4), (H5), (H6), (H7), (H8) respectively. Therefore, Proposition 3.3 holds also for  $|\phi| \geq \epsilon$ . This completes the proof of Proposition 3.3.  $\blacksquare$

Finally, we return to the setting of Lemma 3.1, and establish (2). First suppose  $z \in U$ , say  $z = \rho e^{i\phi}$ . Then

$$G_\epsilon(z) = \tilde{G}_\epsilon(z) = \int_{-\pi}^{\pi} K(\phi - \theta) f(\theta) d\theta,$$

where  $K(\theta) = \frac{1}{1 - \rho e^{i\theta}}$  for all  $\theta$ , and  $f(\theta) = |1 + \epsilon - e^{i\theta}|^{-1/2}$ . One checks that  $K$  and  $f$  satisfy the assumptions of Proposition 3.3, whenever  $\rho \in (0, 1)$ . Moreover, the constants  $c_0, c_0(k)$ , etc appearing in the hypotheses of Proposition 3.3 may be taken to be independent of  $\rho \in (0, 1)$ . Hence, Proposition 3.3 shows that

$$\left| \left( \frac{d}{d\phi} \right)^k \tilde{G}_\epsilon(\rho e^{i\phi}) \right| \leq c'(k) \quad \text{for } \epsilon, \rho \in (0, 1), \phi \in [-\pi, \pi], k \geq 0.$$

Here,  $c'(k)$  depends only on  $k$ . Since  $\tilde{G}_\epsilon(z)$  is analytic in  $U$ , it follows that  $\left| \left( \frac{d}{dz} \right)^k \tilde{G}_\epsilon(z) \right| \leq c''(k)$  for  $k \geq 0, \frac{1}{2} \leq |z| < 1$ , where  $c''(k)$  depends only on  $k$ . In fact, this inequality holds for  $z \in U$  due to the maximum modulus principle. It now follows at once from (1) that it holds also for  $z \in D$ . This proves (2).

To prove (3), first suppose  $z = \rho e^{i\phi}$  with  $\rho < 1$ . Then

$$\begin{aligned} \text{Re } G_\epsilon(z) &= \text{Re } \tilde{G}_\epsilon(z) = \int_{-\pi}^{\pi} |1 + \epsilon - e^{i\theta}|^{-1/2} \text{Re} \left\{ \frac{1}{1 - \rho e^{i(\phi - \theta)}} \right\} d\theta \\ &= \int_{-\pi}^{\pi} |1 + \epsilon - e^{i\theta}|^{-1/2} \left\{ \frac{1}{2} + \frac{1}{2} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\phi - \theta)} \right\} d\theta. \end{aligned}$$

We have  $1 + \rho^2 - 2\rho \cos(\phi - \theta) \leq c[(1 - \rho)^2 + |\phi - \theta|^2]$  for a universal constant  $c$ . Also,  $|1 + \epsilon - e^{i\theta}| \leq \epsilon + |\theta|$ . Therefore,

$$\operatorname{Re} G_\epsilon(\rho e^{i\phi}) \geq c \int_{-\pi}^{\pi} [\epsilon + |\theta|]^{-1/2} \frac{(1 - \rho)}{(1 - \rho)^2 + |\phi - \theta|^2} d\theta$$

for a universal constant  $c$ .

Restricting the region of integration to  $\{\theta : |\theta - \phi| \leq 1 - \rho\}$ , we conclude that

$$\operatorname{Re} G_\epsilon(\rho e^{i\phi}) \geq c \int_{\{|\theta - \phi| \leq 1 - \rho\}} [\epsilon + |\theta|]^{-1/2} d\theta (1 - \rho)^{-1}.$$

If  $|\phi| \geq 2(1 - \rho)$ , then the above inequality yields

$$\operatorname{Re} G_\epsilon(\rho e^{i\phi}) \geq c [\epsilon + |\phi|]^{-1/2} \geq c' [\epsilon + |\rho e^{i\phi} - 1|]^{-1/2},$$

and if  $|\phi| < 2(1 - \rho)$ , then it yields

$$\operatorname{Re} G_\epsilon(\rho e^{i\phi}) \geq c [\epsilon + (1 - \rho)]^{-1/2} \geq c' [\epsilon + |\rho e^{i\phi} - 1|]^{-1/2}.$$

Thus, in either case, we obtain

$$\operatorname{Re} G_\epsilon(z) \geq c' [\epsilon + |1 - z|]^{-1/2} \text{ for } z \in U.$$

Since  $G_\epsilon(z)$  and  $[\epsilon + |1 - z|]^{-1/2}$  are continuous on  $D$ , we conclude that this inequality holds also for  $z \in D$ . This proves (3), completing the proof of Lemma 3.1. ■

Next, we take average over finitely many rotations. For  $N \geq 1$  and  $z \in D$ , define  $H_N(z) = 2^{-N} \sum_{\zeta \in E_N} G_{\epsilon_N}(\bar{\zeta}z)$ . Since  $E_N$  consists of  $2^N$  points on  $\Gamma$ , Lemma 3.1 immediately shows the following properties of  $H_N$ :

- (1)  $H_N(z)$  is analytic in  $U$  and  $C^\infty$  on  $D$ .
- (2)  $|(\frac{d}{dz})^k H_N(z)| \leq c(k) [\operatorname{dist}(z, E_N) + \epsilon_N]^{-1/2-k}$  for  $z \in D$  and  $k \geq 0$ ; where  $c(k)$  is a constant depending only on  $k$ , but not on  $z$  or  $N$ .
- (3)  $\operatorname{Re} H_N(z) \geq c 2^{-N} \sum_{\zeta \in E_N} [|\zeta - z| + \epsilon_N]^{-1/2}$  for  $z \in \Gamma$ , where  $c$  is a positive constant, independent of  $z$  and  $N$ .

From the above estimate for  $\operatorname{Re} H_N(z)$ , we obtain the following lemma.

**Lemma 3.4.**  $\operatorname{Re} H_N(z) \geq c' [\operatorname{dist}(z, E_N) + \epsilon_N]^{-1/4}$  for  $z \in \Gamma$ , where  $c'$  is a positive constant, independent of  $z$  and  $N$ .

**Proof.** We look separately at the following cases.

Case (1): Suppose  $\text{dist}(z, E_N) \geq \epsilon_1$ . Then we have

$$[\text{dist}(z, E_N) + \epsilon_N]^{-1/4} \leq \epsilon_1^{-1/4}.$$

Hence, the conclusion of the lemma follows in this case, if we can show that  $\text{Re } H_N(z) \geq c' \epsilon_1^{-1/4}$  for  $z \in \Gamma$ . For  $z \in \Gamma$  and  $\zeta \in E_N$ , we have  $|\zeta - z| \leq 2$ . Hence,  $[|\zeta - z| + \epsilon_N]^{-1/2} \geq 2^{-1/2}$ , and therefore, by the property (3) of  $H_N$ ,  $\text{Re } H_N(z) \geq c 2^{-1/2}$  for  $z \in \Gamma$ . Taking  $c'$  small enough that  $c 2^{-1/2} > c' \epsilon_1^{-1/4}$ , we find that the lemma holds in this case.

Case (2): Suppose, for some  $M$  ( $1 \leq M < N$ ),  $\epsilon_{M+1} \leq \text{dist}(z, E_N) < \epsilon_M$ . Let  $\hat{\zeta} \in E_M$  be the closest point of  $E_M$  to  $z$ . Since

$$|\text{dist}(z, E_N) - \text{dist}(z, E_M)| \leq \epsilon_M,$$

we have  $\text{dist}(z, E_M) \leq 2\epsilon_M$ , and thus  $|\hat{\zeta} - z| \leq 2\epsilon_M$ . We have  $|\zeta - \hat{\zeta}| \leq \epsilon_M$  for each  $\zeta \in E_N(\hat{\zeta})$ , and therefore  $|\zeta - z| \leq 3\epsilon_M$  for each  $\zeta \in E_N(\hat{\zeta})$ . Consequently,  $[|\zeta - z| + \epsilon_N]^{-1/2} \geq [|\zeta - z| + \epsilon_M]^{-1/2} \geq [4\epsilon_M]^{-1/2}$  for any  $\zeta \in E_N(\hat{\zeta})$ . Since there are  $2^{N-M}$  points in  $E_N(\hat{\zeta})$ , by the property (3) of  $H_N$ , we have  $\text{Re } H_N(z) \geq c 2^{-N} 2^{N-M} [4\epsilon_M]^{-1/2} = (\frac{c}{2}) 2^{-M} \epsilon_M^{-1/2}$ . Recalling that  $\epsilon_n = 64^{-n}$ , we see that

$$\text{Re } H_N(z) > \left(\frac{c}{8}\right) [\text{dist}(z, E_N) + \epsilon_N]^{-1/4}$$

since  $\epsilon_{M+1} \leq \text{dist}(z, E_N)$  in this case. If we take  $c' < \frac{c}{8}$ , then the lemma holds in this case.

Case (3): Suppose  $\text{dist}(z, E_N) < \epsilon_N$ . Then for some  $\hat{\zeta} \in E_N$ , we have  $|\hat{\zeta} - z| < \epsilon_N$ . Hence  $[|\hat{\zeta} - z| + \epsilon_N]^{-1/2} > [2\epsilon_N]^{-1/2}$ . Consequently, by the property (3) of  $H_N$  and definition of  $\epsilon_n$ , we have

$$\text{Re } H_N(z) \geq c 2^{-N} [2\epsilon_N]^{-1/2} = \left(\frac{c}{\sqrt{2}}\right) 2^{-N} \epsilon_N^{-1/2} \geq \left(\frac{c}{\sqrt{2}}\right) [\text{dist}(z, E_N) + \epsilon_N]^{-1/4}.$$

If we take  $c' < \frac{c}{\sqrt{2}}$ , then the lemma also holds in this case. The proof of the lemma is complete. ■

Now, let  $F_N(z) = \exp(-H_N(z))$  for  $N \geq 1$ . Then  $F_N(z)$  is analytic in  $U$  and  $C^\infty$  on  $D$ . Since  $\text{dist}(z, E_N) \leq \epsilon_N$  for  $z \in E$ , by Lemma 3.4,  $|F_N(z)| \leq \exp(-c'' (\epsilon_N)^{-1/4})$  for all  $z \in E$ , where  $c''$  is a positive constant, independent of  $z$  and  $N$ . Further, we will estimate  $F_N$  and its derivatives on  $\Gamma$ . For  $k \geq 0$  and  $z \in \Gamma$ , we note that  $(\frac{d}{dz})^k F_N(z)$  is a sum of terms of the form

$$\pm \prod_{j=1}^l \left[ \left(\frac{d}{dz}\right)^{k_j} H_N(z) \right] \exp(-H_N(z)),$$

with  $1 \leq k_j \leq k$  for each  $j$ , and with  $k_1 + k_2 + \dots + k_l = k$ . Thus, by the property (2) of  $H_N$  and Lemma 3.4, the absolute value of each such term is at most  $\prod_{j=1}^l [(c(k_j) X^{-1/2-k_j}) \exp(-c' X^{-1/4})]$ , where  $X = [\text{dist}(z, E_N) + \epsilon_N]$ . Consequently, each such term has absolute value at most  $c_*$ , where  $c_*$  is a constant determined by  $c(k_1), \dots, c(k_l)$  and  $c'$ . Since  $(\frac{d}{dz})^k F_N(z)$  is a sum of each such term, it follows that  $|(\frac{d}{dz})^k F_N(z)| \leq c_*(k)$  for  $k \geq 0$  and  $z \in \Gamma$  (and hence also for  $z \in D$ ), where  $c_*(k)$  depends only on  $k$ , but not on  $z$  or  $N$ . Finally, for  $k = 0$ , by the property (2) of  $H_N$ ,

$$|H_N(0)| \leq c(0) [1 + \epsilon_N]^{-1/2} \leq c(0)$$

since  $E_N$  is a subset of  $\Gamma$ . Exponentiating it, we find that  $|F_N(0)| \geq \hat{c}$ , where  $\hat{c}$  is a positive constant, independent of  $N$ .

Passing to a suitable convergent subsequence of  $(F_N(z))$  and taking the limit, we obtain a function  $F(z)$  on  $D$  with the properties: (1)  $F(z)$  is analytic on  $U$  and  $C^\infty$  on  $D$ ; (2)  $F(0) \neq 0$ ; and (3)  $F(z) = 0$  for all  $z \in E$ .

### 4. Examples

**4.1.** Suppose that  $E$  is a closed, totally disconnected infinite subset of  $\Gamma$  (in particular,  $E$  is a Carleson set) such that

$$I(E) := \{f \in A^\infty(\Gamma) : f \equiv 0 \text{ on } E\}$$

is a proper closed ideal of  $A^\infty(\Gamma)$ . Then the restriction algebra  $A^\infty(E)$  is an easy and natural example of a nuclear Fréchet space without a basis. The Fréchet topology of  $A^\infty(E)$  is defined by a sequence  $(\|\cdot\|_k)$  of quotient norms, where

$$\|g\|_k := \inf \{ \|G\|_{C^k} : G \in A^\infty(\Gamma), G|_E = g \}, \quad g \in A^\infty(E).$$

Hence  $A^\infty(E)$  is a semisimple, non-Banach Fréchet algebra which admits a continuous norm. Thus it cannot be a quojection. It is also a Q-algebra as its Gel'fand space  $E$  is compact, and therefore all maximal ideals in  $A^\infty(E)$  are closed. In fact, they are principal, being the images of principal maximal ideals in  $A^\infty(\Gamma)$  [20, 4.2 (a)].

We note that the algebra  $A^\infty(\Gamma)$  is the technical main-spring of [20], in which the author has given sufficient conditions for the existence of *local analytic structure* in the spectrum of a Fréchet algebra, characterizing *locally Riemann algebras*. Not only this, but, following the arguments of [20, 4.2 (a)], it is easy to see that the algebra  $A^\infty(E)$  is another example with no analytic structure in its spectrum  $E$ , and therefore it is not a locally



Riemann algebra [20, 4.4], showing that the quotient algebras of locally Riemann algebras need not be locally Riemann algebras. Thus, it is of interest to obtain a criterion when a quotient algebra of a locally Riemann algebra is itself a locally Riemann algebra.

Using a function in  $A^\infty(E)$  and the usual Fréchet algebraic technique, Theorem 2 of [2] gives a negative solution to a well known problem of Kahane and Katznelson (whether for every sequence of positive numbers  $(\gamma_n)_{n \in \mathbb{Z}^+}$  such that  $\gamma_n = o(\exp(\epsilon n))$ ,  $n \rightarrow \infty$ , for all  $\epsilon > 0$ , there exist a  $ZA^+$  set  $E$  and a function  $f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$ ,  $e^{i\theta} \in \Gamma$ , such that  $\sum_{n \in \mathbb{Z}} |c_n| \gamma_{|n|} < \infty$ , but  $f|_E$  cannot be interpolated by a function in  $A^+(\Gamma)$ ) for Carleson sets; similarly, in the Banach case, the algebras  $A^+(\Gamma)$  and  $A^+(E)$  (see Remark after Corollary 2.6) have been used in Theorems 1 and 3 of [2], giving a negative solution to this problem for  $ZA^+$  sets. Thus this shows that the algebras  $A^\infty$  have played an important role in solving two old and famous problems of different areas.

**4.2.** More generally, take  $A = F(\widetilde{W})$  as in §6 of [7]. ( $A^\infty(\Gamma)$  is a particular example of  $F(\widetilde{W})$ .) Then  $A$  satisfies all the conditions stated in Theorem 2.7. Hence it has a nuclear quotient space without a basis which is not a quojection.

**4.3.** Consider the space  $s$  of rapidly decreasing sequences in  $\mathcal{F}$ . Then  $s$  is a Fréchet subalgebra of  $\mathcal{F}$  with a p.s.g.  $X$ ; the Fréchet topology is defined by a sequence  $(p_k)$ , where  $p_k(\sum_{n=0}^\infty \lambda_n X^n) = \sum_{n=0}^\infty |\lambda_n| n^k$ . Evidently,  $s \cong A^\infty(\Gamma)$  and therefore  $D$  is its Gel'fand space. Now, by 4.1,  $s$  has a nuclear quotient space without a basis which is not a quojection.

We can now find a particular closed, totally disconnected subset  $E'$  of  $\mathbb{R}$  and a proper closed ideal  $I(E')$  providing a restriction space without a basis in the following cases. We here note that Fréchet spaces of  $C^\infty$ -functions on subsets of either  $\mathbb{R}$  or  $\mathbb{C}$ , being classical objects of study, have attracted some attention in recent time mainly because of their relevance to the theories of partial differential equations and complex dynamical systems.

**4.4.** Consider the closed subalgebra of  $C^\infty(\mathbb{R})$  having functions of  $2\pi$ -period on  $\mathbb{R}$  and with negative Fourier coefficients equal to 0. It is isomorphic to  $A^\infty(\Gamma)$ . Thus it is a Fréchet algebra with a p.s.g. Let  $E$  be as in 4.1. Then consider  $E' \subset \mathbb{R}$  such that  $E' = p^{-1}(E)$ , where  $p$  is a natural projection of  $\mathbb{R}$  onto  $\Gamma$ . Since  $I(E)$  is a proper closed ideal in  $A^\infty(\Gamma)$ , so is  $I(E')$ .

**4.5.** Consider the closed subspace of  $C^\infty(\mathbb{R})$  having functions of  $2\pi$ -period on  $\mathbb{R}$ . It is a closed subalgebra, isomorphic with  $s$ . Hence it is a Fréchet algebra with a p.s.g. Clearly its Gel'fand space is isomorphic with  $\Gamma$ .

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