



Endpoint estimates for first-order Riesz transforms associated to the Ornstein–Uhlenbeck operator

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Abstract. In the setting of Euclidean space with the Gaussian measure γ , we consider all first-order Riesz transforms associated to the infinitesimal generator of the Ornstein–Uhlenbeck semigroup. These operators are known to be bounded on $L^p(\gamma)$, for $1 < p < \infty$. We determine which of them are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $BMO(\gamma)$. Here $H^1(\gamma)$ and $BMO(\gamma)$ are the spaces introduced in this setting by the first two authors. Surprisingly, we find that the results depend on the dimension of the ambient space.

1. Introduction

Denote by γ the normalised Gaussian measure on \mathbb{R}^d having density $x \mapsto \pi^{-d/2} e^{-|x|^2}$ with respect to the Lebesgue measure. The Ornstein–Uhlenbeck operator \mathcal{L} is the closure in $L^2(\gamma)$ of the operator \mathcal{L}_0 , defined on $C_c^\infty(\mathbb{R}^n)$ by

$$\mathcal{L}_0 = -\frac{1}{2} \Delta + x \cdot \nabla,$$

where Δ and ∇ denote the Euclidean Laplacian and gradient, respectively. The operator \mathcal{L} is self-adjoint, and its spectral resolution is

$$\mathcal{L} = \sum_{j=0}^{\infty} j \mathcal{P}_j,$$

where \mathcal{P}_j denotes the orthogonal projection onto the linear span of Hermite polynomials of degree j in d variables. The Ornstein–Uhlenbeck operator generates a diffusion semigroup, which has been the object of many investigations during the last two decades. In particular, efforts have been made to study operators related

to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators (see [25], [12], [19], [10]), Riesz transforms (see [21], [13], [20], [23], [22], [14], [11], [5], [6], [7], [24], [26], [4]) and functional calculus ([8], [9], [17], [15]).

In this paper, we shall concentrate on the first-order Riesz transforms associated with \mathcal{L} . Since the eigenspace associated to the zero eigenvalue is nontrivial, the positive square root of \mathcal{L} is not invertible, and one must define the Riesz transforms carefully. Consider the sequence $M : \mathbb{N} \rightarrow \mathbb{C}$, defined by

$$M(j) = \begin{cases} 0 & \text{if } j = 0 \\ j^{-1/2} & \text{if } j = 1, 2, \dots \end{cases}$$

The operator $M(\mathcal{L})$, spectrally defined on $L^2(\gamma)$, extends to a bounded operator on $L^p(\gamma)$ for every p in $(1, \infty)$ (see, for instance, Theorem 1.2 of [17]). The operator $M(\mathcal{L})$ plays, in this setting, the same role as the operator $(-\Delta)^{-1/2}$ in Euclidean harmonic analysis. We denote by ∂_i the differentiation operator with respect to the variable x_i . The formal adjoint of ∂_i in $L^2(\gamma)$ is the operator $\partial_i^* = 2x_i - \partial_i$. Notice that, at least formally,

$$\sum_{i=1}^d \partial_i^* \partial_i = 2\mathcal{L}.$$

For each i in $\{1, \dots, d\}$, we define the operators R_i , S_i , R_i^* and S_i^* on finite linear combinations of Hermite polynomials by

$$(1.1) \quad \begin{aligned} R_i &= \partial_i M(\mathcal{L}), & S_i &= M(\mathcal{L}) \partial_i, \\ R_i^* &= M(\mathcal{L}) \partial_i^*, & S_i^* &= \partial_i^* M(\mathcal{L}). \end{aligned}$$

It is straightforward to check that R_i^* and S_i^* are the *formal* adjoints in $L^2(\gamma)$ of R_i and S_i , respectively. Recall the action of ∂_i and ∂_i^* on Hermite polynomials:

$$\partial_i H_n = 2n H_{n-1} \quad \text{and} \quad \partial_i^* H_n = H_{n+1},$$

where H_j denotes the Hermite polynomials of degree j in the i^{th} variable. A straightforward argument, which uses these formulae and the expression for the $L^2(\gamma)$ norm of a Hermite polynomial, then shows that R_i , R_i^* , S_i and S_i^* extend to bounded operators on $L^2(\gamma)$, and that R_i^* and S_i^* are the Hilbert space adjoints of R_i and S_i , respectively. Note that, in contrast with the Euclidean case, these transforms are not antisymmetric.

The operators R_i , S_i , R_i^* and S_i^* are all bounded on $L^p(\gamma)$ for each p in $(1, \infty)$ (see [20]). This may be easily proved by using the commutation relations between ∂_i and \mathcal{L} and the spectral multiplier result (see Theorem 1.2 of [17]). It is straightforward to check that none of these operators are bounded on $L^1(\gamma)$ or on $L^\infty(\gamma)$. Weak type 1 estimates for R_i and for S_i^* have been proved in [5] and [1], respectively.

Thus, it is natural to ask what further estimates these operators satisfy at the endpoints $p = 1$ and $p = \infty$. In the Euclidean case there are substitute results

at $p = 1$ and $p = \infty$ saying that the operators are bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ and from $L^\infty(\mathbb{R}^d)$ to $\text{BMO}(\mathbb{R}^d)$. In our setting, the spaces $H^1(\gamma)$ and $\text{BMO}(\gamma)$ were defined by Mauceri and Meda in [16], where these authors also developed a theory of singular integral operators in the Gaussian setting (see also [2] for a related theory in a more general setting). As applications, they proved that the imaginary powers of the Ornstein–Uhlenbeck operator are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$, and that the operators R_i are bounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.

The question to be studied below is whether the Riesz transforms defined in (1.1) are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. We find that this boundedness always holds in dimension one. But, surprisingly, in higher dimensions, each Riesz transform has one of these boundedness properties but not the other. Our results for $d \geq 2$ are summarised in the table in Theorem 3.1.

The paper is organised as follows. Section 2 contains background material and a few preliminary results, including the expressions for the kernels of R_i and S_i^* . Our main result, Theorem 3.1, is stated in Section 3. Its proof constitutes Sections 4–7.

2. Preliminaries

We briefly recall the Hardy space $H^1(\gamma)$ from [16]. A Euclidean ball B is called *admissible* if

$$(2.1) \quad r_B \leq \min(1, 1/|c_B|);$$

here and in the sequel r_B and c_B denote the radius and the centre of B , respectively. The collection of all admissible balls will be denoted by \mathcal{B}_1 . When there is equality in (2.1), we say that B is a *maximal ball* in \mathcal{B}_1 .

A (Gaussian) *atom* is either the constant function 1 or a function a in $L^\infty(\gamma)$, supported in an admissible ball B and such that

$$(2.2) \quad \|a\|_\infty \leq \gamma(B)^{-1} \quad \text{and} \quad \int_{\mathbb{R}^n} a \, d\gamma = 0.$$

The space $H^1(\gamma)$ is then the vector space of all functions f in $L^1(\gamma)$ that admit a decomposition of the form $\sum_j \lambda_j a_j$, where the a_j are atoms and the sequence of complex numbers $\{\lambda_j\}$ is summable. The norm of f in $H^1(\gamma)$ is defined as the infimum of $\sum_j |\lambda_j|$ over all representations of f as above. In [16] the space $H^1(\gamma)$ is defined by means of $(1, p)$ -atoms with $1 < p < \infty$, but in [18] it is verified that the space obtained is the same as ours.

Note that $H^1(\gamma)$ is defined much as the atomic space H^1 on spaces of homogeneous type in the sense of R. R. Coifman and G. Weiss [3], but with one difference. Namely, only the exceptional atom and atoms with “small supports”, i.e., with supports contained in admissible balls, appear in the definition of $H^1(\gamma)$. This difference is quite significant and has important consequences.

It is known (Theorem 5.2 of [16]) that the Banach dual of $H^1(\gamma)$ is isomorphic to the space $\text{BMO}(\gamma)$ of all functions of “bounded mean oscillation”, i.e., of all functions in $L^1(\gamma)$ such that

$$(2.3) \quad \|f\|_* = \sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \, d\gamma < \infty,$$

where $f_B = \frac{1}{\gamma(B)} \int_B f \, d\gamma$. A convenient norm on $\text{BMO}(\gamma)$ is the following

$$\|f\|_{\text{BMO}} = \|f\|_1 + \|f\|_*.$$

Remark 2.1. An equivalent norm on $\text{BMO}(\gamma)$ is obtained by replacing in (2.3) balls in \mathcal{B}_1 by cubes of sidelength at most $2 \min(1, 1/|c_B|)$ (see Section 2 of [16]). As a consequence, a function f is in $H^1(\gamma)$ if and only if it admits a decomposition of the form $\sum_j \lambda_j b_j$ where $\{\lambda_j\}$ is summable and the b_j are either the constant function 1 or functions in $L^\infty(\gamma)$ supported in cubes Q of sidelength at most $2 \min(1, 1/|c_B|)$, satisfying

$$\|b\|_\infty \leq \gamma(Q)^{-1} \quad \text{and} \quad \int_{\mathbb{R}^n} b \, d\gamma = 0.$$

An equivalent norm on $H^1(\gamma)$ is then obtained by taking the infimum of $\sum_j |\lambda_j|$ over all representations of f as above.

Given a bounded operator T on $L^2(\gamma)$, we denote by K_T the Schwartz kernel of T and by k_T the kernel of T with respect to the Gaussian measure, defined by

$$(2.4) \quad k_T(x, y) = \pi^{d/2} e^{-|y|^2} K_T(x, y),$$

in the sense of distributions in $\mathbb{R}^d \times \mathbb{R}^d$. The reason for introducing k_T is that if K_T is locally integrable in $\mathbb{R}^d \times \mathbb{R}^d$, then T is an integral operator with kernel k_T with respect to γ , i.e.,

$$Tf(x) = \int_{\mathbb{R}^d} k_T(x, y) f(y) \, d\gamma(y) \quad \forall x \in \mathbb{R}^d, \quad \forall f \in C_c(\mathbb{R}^d).$$

One of the results in [16] says that if T is bounded on $L^2(\gamma)$ and k_T is locally integrable off the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$, and satisfies

$$(2.5) \quad \sup_{B \in \mathcal{B}_1} r_B \sup_{y \in B} \int_{(2B)^c} |\nabla_y k_T(x, y)| \, d\gamma(x) < \infty,$$

then T is bounded from $H^1(\gamma)$ to $L^1(\gamma)$, and, consequently, on $L^p(\gamma)$ for all p in $(1, 2)$. Notice that (2.5) is a local Hörmander integral condition. Similarly, if

$$(2.6) \quad \sup_{B \in \mathcal{B}_1} r_B \sup_{x \in B} \int_{(2B)^c} |\nabla_x k_T(x, y)| \, d\gamma(y) < \infty,$$

then T is bounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$, and, consequently, on $L^p(\gamma)$ for all p in $(2, \infty)$.

We shall use these criteria to prove the boundedness results contained in Theorem 3.1, and start by determining the kernels of R_i and of S_i^* . The functions ϕ and ψ , defined by

$$\phi(r, x, y) = \frac{ry - x}{\sqrt{1 - r^2}} \quad \text{and} \quad \psi(r, x, y) = \frac{rx - y}{\sqrt{1 - r^2}}, \quad x, y \in \mathbb{R}^d, \quad r \in (0, 1),$$

will occur frequently. Note that if $d \geq 2$, then ϕ and ψ are vector-valued; their components will be denoted by ϕ_i and ψ_i . The arguments of these two functions will often be suppressed.

As shown in Lemma 2.2 of [7], the Schwartz kernel of the operator $M(\mathcal{L})$, defined in the introduction, is

$$K_{M(\mathcal{L})}(x, y) = \pi^{-(d+1)/2} \int_0^1 \left[\frac{e^{-|\psi|^2}}{(1 - r^2)^{d/2}} - e^{-|y|^2} \right] d\rho(r),$$

where the measure ρ , supported in $[0, 1]$, is defined by

$$(2.7) \quad d\rho(r) = \frac{dr}{r \sqrt{-\log r}}.$$

Hence the Schwartz kernel K_{R_i} of the operator R_i agrees off the diagonal with the function

$$\partial_{x_i} K_{M(\mathcal{L})}(x, y) = -2\pi^{-(d+1)/2} \int_0^1 \frac{r \psi_i e^{-|\psi|^2}}{(1 - r^2)^{(d+1)/2}} d\rho(r).$$

By using (2.4) and the fact that $|\phi|^2 - |\psi|^2 = |x|^2 - |y|^2$, we get that

$$(2.8) \quad k_{R_i}(x, y) = -\frac{2}{\sqrt{\pi}} e^{|x|^2} \int_0^1 \frac{r \psi_i e^{-|\psi|^2}}{(1 - r^2)^{(d+1)/2}} d\rho(r)$$

for all $x \neq y$. To determine $k_{S_i^*}$, we argue similarly, observing that

$$\partial_{x_i}^* e^{-|\psi|^2} = \frac{-2\phi_i}{\sqrt{1 - r^2}} e^{-|\psi|^2},$$

and obtain

$$(2.9) \quad \begin{aligned} k_{S_i^*}(x, y) &= -\frac{2}{\sqrt{\pi}} e^{|y|^2} \int_0^1 \left[\frac{\phi_i e^{-|\psi|^2}}{(1 - r^2)^{(d+1)/2}} + x_i e^{-|y|^2} \right] d\rho(r) \\ &= -\frac{2}{\sqrt{\pi}} e^{|x|^2} \int_0^1 \left[\frac{\phi_i e^{-|\phi|^2}}{(1 - r^2)^{(d+1)/2}} + x_i e^{-|x|^2} \right] d\rho(r) \end{aligned}$$

for all $x \neq y$.

The constants $C < \infty$ and $c > 0$ will depend only on the dimension d , and they may vary from occurrence to occurrence. By $f \sim g$ we mean $c < f/g < C$. Lebesgue measure is denoted λ .

3. Statements of results

The main result of this paper is the following.

Theorem 3.1. *In dimension one, the four operators R_1 , R_1^* , S_1 and S_1^* are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.*

In dimension $d \geq 2$, the boundedness properties of the operators R_i , R_i^ , S_i and S_i^* are given by the following table, where B means “bounded” and U “unbounded”:*

	R_i	S_i	R_i^*	S_i^*
$H^1 \rightarrow L^1$	U	U	B	B
$L^\infty \rightarrow \text{BMO}$	B	B	U	U

Remark 3.2. Note that all the operators R_i and S_i , i in $\{1, \dots, d\}$, have the same boundedness properties, and so do the operators R_i^* and S_i^* . Furthermore, the boundedness properties of R_i^* and S_i^* are “dual” to those of R_i and S_i . This is not an accident, and the proofs of the statements concerning the boundedness properties of R_i^* and S_i will be obtained, by a duality argument, from the properties of R_i and S_i^* .

In addition to R_i , R_i^* , S_i and S_i^* we also consider the operators M_i and M_i^* , defined on Hermite polynomials by

$$\begin{aligned} M_i &= x_i M(\mathcal{L}), \\ M_i^* &= M(\mathcal{L}) x_i. \end{aligned}$$

Corollary 3.3. *The operators M_i and M_i^* extend to bounded operators on $L^p(\gamma)$ for all p in $(1, \infty)$ in any dimension. For $d = 1$, they are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. For $d \geq 2$, they are unbounded both from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.*

Proof. Since

$$M_i = \frac{R_i + S_i^*}{2} \quad \text{and} \quad M_i^* = \frac{R_i^* + S_i}{2},$$

the $L^p(\gamma)$ boundedness of M_i and M_i^* follows from that of R_i , S_i , R_i^* and S_i^* mentioned in the introduction. The remaining parts of the corollary are straightforward consequences of Theorem 3.1. \square

The proof of Theorem 3.1 occupies the remaining part of the paper. We first deal with R_i and S_i^* . In Section 4, it is proved that in dimension one, R_i is bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and S_i^* from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. Section 5 deals with the positive statements concerning these two operators for $d \geq 2$, as indicated in the table. The negative results claimed for R_i and S_i^* are proved in Section 6. Finally, a rather simple duality argument in Section 7 gives all the results for the remaining operators R_i^* and S_i .

4. Boundedness of R_1 and S_1^* in one dimension

Let $d = 1$. We first prove that R_1 is bounded from $H^1(\gamma)$ to $L^1(\gamma)$. Because of (2.5) and (2.8), it suffices to show that there exists a constant C such that for all balls B in \mathcal{B}_1 and all points y in B

$$(4.1) \quad r_B \int_{(2B)^c} d\lambda(x) \left| \int_0^1 \frac{r \partial_y(\psi e^{-\phi^2})}{1-r^2} d\rho(r) \right| \leq C.$$

Next, observe that

$$\partial_y(\psi e^{-\phi^2}) = -\frac{1}{\sqrt{1-r^2}} e^{-\phi^2} - 2\phi \psi \frac{r}{\sqrt{1-r^2}} e^{-\phi^2} = F_1(r, x, y) + F_2(r, x, y).$$

To deal with F_2 , we first note that $\partial\phi/\partial r = -\psi/(1-r^2)$, so that

$$F_2(r, x, y) = -r \sqrt{1-r^2} \partial e^{-\phi^2} / \partial r.$$

Integrating by parts, we get

$$\int_0^1 \frac{(-\log r)^{-1/2}}{1-r^2} F_2(r, x, y) dr = \int_0^1 \frac{\partial}{\partial r} \frac{r(-\log r)^{-1/2}}{\sqrt{1-r^2}} e^{-\phi^2} dr.$$

The integral to the right here is easily seen to be bounded by

$$C \int_0^1 (1-r)^{-2} e^{-\phi^2} dr.$$

Considering also F_1 , we can then estimate the expression in (4.1) by

$$(4.2) \quad C r_B \int_{(2B)^c} \int_0^1 (1-r)^{-2} e^{-\phi^2} dr d\lambda(x).$$

Here we first integrate in r only over $1-r_B/(2(1+|y|)) < r < 1$, which implies

$$|ry - x| = |(r-1)y + y - x| \geq |y - x| - \frac{r_B|y|}{2(1+|y|)} \geq |y - x| - \frac{r_B}{2} \geq \frac{|y - x|}{2}$$

and thus $|\phi| \geq |y - x|/(2\sqrt{1-r})$. This part of (4.2) can thus be estimated by the following expression, where we make the change of variables $t = (1-r)/|x-y|^2$,

$$\begin{aligned} C r_B \int_{(2B)^c} \int_{1-r_B/(2(1+|y|))}^1 \frac{e^{-c|x-y|^2/(1-r)}}{(1-r)^2} dr d\lambda(x) &\leq \\ &\leq C r_B \int_{(2B)^c} \frac{d\lambda(x)}{|x-y|^2} \int_0^\infty t^{-2} e^{-c/t} dt \leq C. \end{aligned}$$

In the remaining part of (4.2), we switch the order of integration and find

$$\begin{aligned} r_B \int_{(2B)^c} d\lambda(x) \int_0^{1-r_B/(2(1+|y|))} \frac{e^{-\phi(r,x,y)^2}}{(1-r)^2} dr &\leq \\ &\leq r_B \int_0^{1-r_B/(2(1+|y|))} \frac{dr}{(1-r)^2} \int_{\mathbb{R}^d} e^{-\phi(r,x,y)^2} d\lambda(x) \\ &\leq C r_B \int_0^{1-r_B/(2(1+|y|))} \frac{dr}{(1-r)^{3/2}}, \end{aligned}$$

which is easily seen to be bounded by a constant independent of y in B and of B in \mathcal{B}_1 . The proof of the boundedness of R_1 from $H^1(\gamma)$ to $L^1(\gamma)$ is complete.

Now we prove that if $d = 1$, then S_1^* is bounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. Recall that the functions ϕ and ψ are now scalar valued. By (2.6) and (2.9) it suffices to show that there exists a constant C , independent of x in B and of B in \mathcal{B}_1 , such that

$$(4.3) \quad r_B \int_{(2B)^c} d\lambda(y) \left| \int_0^1 \frac{1}{1-r^2} \partial_x (\phi e^{-\psi^2} + (1-r^2) x e^{-y^2}) d\rho(r) \right| \leq C.$$

This is analogous to (4.1), with the following three differences. The variables x and y , and thus also ϕ and ψ , are swapped; there is an extra factor $1/r$; there is an extra term $(1-r^2)xe^{-y^2}$.

For that part of the expression in (4.3) obtained by integrating only over $1/2 < r < 1$, we can copy the argument used to verify (4.1) above. The extra factor $1/r$ is harmless in this interval, and so is the term $(1-r^2)xe^{-y^2}$. In the integration by parts with respect to r , one will now get an integrated term at $r = 1/2$, equal to $C e^{-\psi(1/2,x,y)^2}$ and easy to handle.

Consider then the integral over $0 < r < 1/2$. We have

$$(4.4) \quad \begin{aligned} \partial_x (\phi e^{-\psi^2} + (1-r^2) x e^{-y^2}) &= \\ &= -(1-r^2) \left[\frac{e^{-\psi^2}}{(1-r^2)^{3/2}} - e^{-y^2} \right] - \frac{2\phi\psi r}{(1-r^2)^{1/2}} e^{-\psi^2}. \end{aligned}$$

Notice first that $ry - x = r(y - rx) + (r^2 - 1)x$, so that

$$(4.5) \quad |\phi| \leq |\psi| + |x|.$$

This allows us to estimate the last term in (4.4) by $C r(1+|x|)e^{-\psi^2/2}$ for $0 < r < 1/2$. The corresponding part of the expression in (4.3) can then be seen not to be larger than $C r_B(1+|x|) \leq C$, since $x \in B$. For the first term in the right-hand side of (4.4), we write

$$\frac{e^{-\psi^2}}{(1-r^2)^{3/2}} - e^{-y^2} = \frac{e^{-\psi^2} - e^{-y^2}}{(1-r^2)^{3/2}} + \left[\frac{1}{(1-r^2)^{3/2}} - 1 \right] e^{-y^2}.$$

The last term here is easily seen to be harmless and will be neglected. For the first term to the right, we observe that $e^{-y^2} = e^{-\psi(0,x,y)^2}$, so that

$$e^{-\psi(r,x,y)^2} - e^{-y^2} = \int_0^r \frac{\partial}{\partial s} e^{-\psi(s,x,y)^2} ds = \int_0^r \frac{2\phi(s,x,y)\psi(s,x,y)}{1-s^2} e^{-\psi(s,x,y)^2} ds.$$

Since $s < r < 1/2$ here, (4.5) implies that the last integral is at most

$$C \int_0^r (1+|x|) e^{-\psi(s,x,y)^2/2} ds.$$

Integrating this expression with respect to y , we get at most $C r(1+|x|)$. This will give a contribution to the expression in (4.3) which is at most $C r_B(1+|x|) \int_0^{1/2} (-\log r)^{-1/2} dr$, and this is uniformly bounded with respect to x in B and B in \mathcal{B}_1 .

This concludes the proof that S_1^* is bounded from $L^\infty(\gamma)$ to $BMO(\gamma)$.

5. Boundedness of R_i and S_i^* in higher dimensions

The boundedness of R_i from $L^\infty(\gamma)$ to $BMO(\gamma)$ for any d was proved in Theorem 7.2 (ii) of [16].

To prove that S_i^* is bounded from $H^1(\gamma)$ to $L^1(\gamma)$ in all dimensions, we follow the lines of the proof of Theorem 7.2 in [16]. By (2.5) and (2.9), it is enough to verify that there exists a constant C , independent of y in B and of B in \mathcal{B}_1 such that

$$(5.1) \quad r_B \int_{(2B)^c} \left| \int_0^1 \nabla_y \left(\frac{\phi_i e^{-|\phi|^2}}{(1-r^2)^{(d+1)/2}} + x_i e^{-|x|^2} \right) d\rho(r) \right| d\lambda(x) \leq C.$$

It is straightforward to verify that all the components of the gradient with respect to y that appear here are bounded in absolute value by

$$C \frac{r(1+|\phi|^2)}{(1-r^2)^{(d+2)/2}} e^{-|\phi|^2} \leq C \frac{r}{(1-r^2)^{(d+2)/2}} e^{-|\phi|^2/2}.$$

Thus the left-hand side of (5.1) is no larger than

$$(5.2) \quad C r_B \int_{(2B)^c} d\lambda(x) \int_0^1 \frac{r e^{-|\phi|^2/2}}{(1-r^2)^{(d+2)/2}} d\rho(r).$$

We split the inner integral into integrals over $(0, r_0)$ and $(r_0, 1)$, with $r_0 = 1 - r_B^2/4 > 1/2$. In each of the two resulting double integrals, we switch the order of integration. That part of (5.2) corresponding to $(0, r_0)$ will be less than

$$\begin{aligned} C r_B \int_0^{r_0} \frac{(-\log r)^{-1/2}}{(1-r^2)^{(d+2)/2}} dr \int_{\mathbb{R}^d} e^{-|\phi|^2/2} d\lambda(x) &\leq C r_B \int_0^{r_0} (-\log r)^{-1/2} (1-r^2)^{-1} dr \\ &\leq C r_B (1-r_0)^{-1/2} \leq C. \end{aligned}$$

To deal with the integral over $(r_0, 1)$, observe that $y \in B$ implies $|y| \leq c_B + r_B \leq 2r_B^{-1}$, in view of the definition of \mathcal{B}_1 . For $x \notin 2B$ and $r_0 < r < 1$ we then have

$$|ry - x| = |y - x - (1-r)y| > r_B - |y|r_B^2/4 \geq r_B/2.$$

The remaining part of (5.2) is thus at most

$$(5.3) \quad C r_B \int_{r_0}^1 \frac{r}{(1-r^2)^{(d+2)/2}} d\rho(r) \int_{|ry-x|>r_B/2} e^{-|\phi|^2/2} d\lambda(x).$$

The inner integral in (5.3) is dominated by

$$(1-r^2)^{d/2} \int_{|z|>r_B/(2\sqrt{1-r^2})} e^{-|z|^2} dz < C (1-r^2)^{d/2} \left(r_B/\sqrt{1-r^2} \right)^{-2},$$

and the whole expression (5.3) by

$$C r_B^{-1} \int_{r_0}^1 (1-r)^{-1/2} dr \leq C r_B^{-1} (1-r_0)^{1/2} \leq C.$$

This concludes the proof of the boundedness of S_i^* from $H^1(\gamma)$ to $L^1(\gamma)$.

6. Unboundedness results

In this section, we shall prove that if $d \geq 2$, then R_1 is unbounded from $H^1(\gamma)$ to $L^1(\gamma)$ and S_1^* is unbounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. The proofs for R_2, \dots, R_d and S_2^*, \dots, S_d^* are similar and are omitted.

Consider first R_1 . Let Q be the cube with centre $(\xi, 0, \dots, 0)$ and sidelength $2\xi^{-1}$, for some large $\xi > 0$. Denote by Q_+ and Q_- those halves of Q where $y_2 > 0$ and $y_2 < 0$, respectively. Set $b = \gamma(Q)^{-1}(\mathbf{1}_{Q_+} - \mathbf{1}_{Q_-})$. By Remark 2.1, $\|b\|_{H^1(\gamma)}$ is bounded independently of ξ . Writing $\tilde{y} = (y_1, -y_2, y_3, \dots, y_d)$, one can easily see from (2.8) that

$$\begin{aligned} e^{-|x|^2} R_1 a(x) &= \\ &= \frac{2}{\sqrt{\pi} \gamma(Q)} \int_{Q_+} \int_0^1 \frac{r(y_1 - rx_1)}{(1-r^2)^{(d+2)/2}} (e^{-|\phi(r,x,y)|^2} - e^{-|\phi(r,x,\tilde{y})|^2}) d\rho(r) d\gamma(y) \\ &= \frac{2}{\sqrt{\pi} \gamma(Q)} \int_{Q_+} \int_0^1 \frac{r(y_1 - rx_1)}{(1-r^2)^{(d+2)/2}} e^{-|\phi(r,x,y)|^2} (1 - e^{-\tau(r,x_2,y_2)}) d\rho(r) d\gamma(y); \end{aligned}$$

here and in the rest of the proof we write $\tau(r, x_2, y_2)$ in place of $4rx_2y_2/(1-r^2)$. Similarly, we write $\eta(\xi, x_1)$ instead of $\sqrt{(\xi - x_1)/\xi}$. We shall evaluate $R_1 a$ at points x in the set

$$A_Q = \left\{ x \in \mathbb{R}^d : x_1 \in \left(\xi - 1, \xi - \frac{4}{\xi} \right), x_2 \in \left[\frac{\eta(\xi, x_1)}{2}, \eta(\xi, x_1) \right], \right. \\ \left. |x_k| \leq \eta(\xi, x_1), k = 3, \dots, d \right\}.$$

Observe that if $x \in A_Q$, $y \in Q_+$ and $r \in [0, 1]$, then $y_1 - rx_1 > 0$ and the integrands in the double integrals above are positive. With ξ large and $x \in A_Q$, the integration in r will be restricted to the interval

$$(6.1) \quad I(x_1) = \left\{ r \in (0, 1) : \left| r - \frac{x_1}{\xi} \right| < \frac{\eta(\xi, x_1)}{2\xi} \right\} \subset (1/2, 1).$$

For $x \in A_Q$ a lower bound for $e^{-|x|^2} R_1 a(x)$ is

$$\frac{c}{\gamma(Q)} \int_{Q_+} \int_{I(x_1)} \frac{y_1 - rx_1}{(1 - r^2)^{(d+3)/2}} e^{-|\phi|^2} (1 - e^{-\tau(r, x_2, y_2)}) dr d\gamma(y).$$

Therefore,

$$(6.2) \quad \begin{aligned} \|R_1 a\|_1 &\geq \int_{A_Q} R_1 a(x) e^{-|x|^2} d\lambda(x) \\ &\geq \frac{c}{\gamma(Q)} \int_{A_Q} \int_{Q_+} \int_{I(x_1)} \frac{y_1 - rx_1}{(1 - r^2)^{(d+3)/2}} e^{-|\phi|^2} (1 - e^{-\tau(r, x_2, y_2)}) dr d\gamma(y) d\lambda(x). \end{aligned}$$

To prove that the operator R_1 is unbounded from $H^1(\gamma)$ to $L^1(\gamma)$, we only need to show that the last expression here tends to infinity with ξ . We claim that if ξ is sufficiently large, one has for x in A_Q , y in Q_+ and r in $I(x_1)$

$$(6.3) \quad 1 - r^2 \sim \eta(\xi, x_1)^2,$$

$$(6.4) \quad y_1 - rx_1 \geq c (\xi - x_1),$$

$$(6.5) \quad e^{-|\phi(r, x, y)|^2} \geq c,$$

$$(6.6) \quad 1 - e^{-\tau(r, x_2, y_2)} \sim y_2/\eta(\xi, x_1).$$

Deferring momentarily their proofs, we show how these inequalities yield the desired conclusion. Indeed, applying them and observing that

$$\frac{1}{\gamma(Q)} \int_{Q_+} y_2 d\gamma(y) \geq \frac{c}{\xi},$$

we see from (6.2) that

$$\begin{aligned} \|R_1 a\|_1 &\geq \frac{c}{\gamma(Q)} \int_{A_Q} \int_{Q_+} \eta(\xi, x_1)^{-(d+2)} \xi y_2 |I(x_1)| d\gamma(y) d\lambda(x) \\ &\geq c \int_{A_Q} \eta(\xi, x_1)^{-(d+1)} \xi^{-1} d\lambda(x) = c \int_{\xi-1}^{\xi-4/\xi} (\xi - x_1)^{-1} dx_1 \geq c \log \xi, \end{aligned}$$

which tends to infinity with ξ as desired.

It remains to prove the inequalities (6.3)–(6.6). Observe first that for $x \in A_Q$ one has $4/\xi < \xi - x_1$. Considering also the geometric mean of these two quantities, we have

$$(6.7) \quad 4/\xi < 2\eta(\xi, x_1) < \xi - x_1.$$

To verify (6.3), write for $r \in I(x_1)$

$$1 - r^2 \sim 1 - r = \eta(\xi, x_1)^2 - \left(r - \frac{x_1}{\xi}\right).$$

Observe that $|r - x_1/\xi| < \eta(\xi, x_1)^2/4$, because of the definition of $I(x_1)$ and (6.7), and (6.3) follows.

For (6.4), notice that similarly

$$\begin{aligned} y_1 - rx_1 &> \xi - \frac{1}{\xi} - \left(\frac{x_1}{\xi} + \frac{\xi - x_1}{4\xi}\right)x_1 \\ &> \xi - \frac{\xi - x_1}{4} - \frac{x_1^2}{\xi} - \frac{\xi - x_1}{4} > \frac{\xi^2 - x_1^2}{\xi} - \frac{\xi - x_1}{2} > \frac{\xi - x_1}{2}. \end{aligned}$$

Aiming at (6.5), we write

$$\begin{aligned} |x_1 - ry_1| &\leq |r(y_1 - \xi)| + \left|r - \frac{x_1}{\xi}\right|\xi \\ &< \frac{1}{\xi} + \frac{1}{2\xi}\eta(\xi, x_1)\xi < \eta(\xi, x_1) \leq C\sqrt{1 - r^2}, \end{aligned}$$

the last two inequalities in view of (6.7) and (6.3), respectively. Since $x \in A_Q$ and $y \in Q_+$, we similarly get for $k = 2, \dots, d$

$$|x_k - ry_k| \leq |x_k| + |y_k| \leq \eta(\xi, x_1) + \xi^{-1} \leq C\sqrt{1 - r^2}.$$

Thus $|x - ry|^2/(1 - r^2) \leq C$, which implies (6.5).

Finally, we prove (6.6). Because of (6.3) and the facts that $x \in A_Q$ and $y \in Q_+$, one has

$$\tau(r, x_2, y_2) \sim \frac{\xi}{\xi - x_1}\eta(\xi, x_1)y_2 = \eta(\xi, x_1)y_2 < \frac{1}{\sqrt{\xi(\xi - x_1)}} < 1,$$

and (6.6) follows.

This concludes the proof that R_1 is unbounded from $H^1(\gamma)$ to $L^1(\gamma)$.

Next, we prove that in higher dimension, S_1^* is unbounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$. We shall modify slightly the counterexample constructed above. The kernel is

$$k_{S_1^*}(x, y) = \frac{2}{\sqrt{\pi}}e^{|x|^2} \int_0^1 \left[\frac{x_1 - ry_1}{(1 - r^2)^{(d+2)/2}} e^{-|\phi|^2} + x_1 e^{-|x|^2} \right] d\rho(r).$$

With $\xi > 0$ large, we let the cube Q , its subset Q_+ and the region A_Q be as above. But now f will be the characteristic function of A_Q , and S_1^*f will be evaluated in Q . For $x \in Q_+$ we write $\tilde{x} = (x_1, -x_2, x_3, \dots, x_d)$. We shall verify that

$$(6.8) \quad \frac{1}{\gamma(Q)} \int_{Q_+} (S_1^*f(x) - S_1^*f(\tilde{x})) d\gamma(x)$$

tends to $+\infty$ as ξ tends to $+\infty$. This would clearly mean that $\|S_1^* f\|_{\text{BMO}}$ also tends to infinity, although $\|f\|_\infty = 1$ for all ξ . Notice that for x in Q_+

$$\begin{aligned} S_1^* f(x) - S_1^* f(\tilde{x}) &= \\ &= \frac{2}{\sqrt{\pi}} \int_{A_Q} \int_0^1 \frac{x_1 - ry_1}{(1-r^2)^{(d+3)/2}} e^{-|\psi|^2} (1 - e^{-\tau(r, x_2, y_2)}) d\rho(r) d\lambda(y); \end{aligned}$$

recall that we write $\tau(r, x_2, y_2)$ in place of $4rx_2y_2/(1-r^2)$. Here the integrand is positive. We now integrate with respect to $d\gamma(x)$ over $x \in Q_+$, and restrict the integral in r to the interval $I(y_1) \subset (1/2, 1)$, defined like $I(x_1)$ (see (6.1)). For such r , one can replace the measure $d\rho(r)$ by $(1-r^2)^{-1/2} dr$. This leads to a lower estimate of the quantity (6.8) in terms of a triple integral rather similar to that in (6.2). The only differences are that x and y have switched roles and that we now have also an insignificant factor $1/r$. Thus the integral will tend to ∞ with ξ , like that in (6.2).

This concludes the proof that S_1^* is unbounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.

7. Duality arguments

All the statements contained in Theorem 3.1 that have not been proved yet will follow from those proved above by a simple duality argument, based on the following elementary lemma. We point out that by a result of [18], $L^2(\gamma)$ is a subspace of $H^1(\gamma)$, which is dense because it contains all the atoms.

Lemma 7.1. *Suppose that T is a bounded operator on $L^2(\gamma)$. Then T extends to a bounded operator from $H^1(\gamma)$ to $L^1(\gamma)$ if and only if its (Hilbert space) adjoint operator T^* is bounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.*

Proof. Suppose that T extends to a bounded operator from $H^1(\gamma)$ to $L^1(\gamma)$. Then for g in $L^\infty(\gamma)$

$$(7.1) \quad \|T^*g\|_{\text{BMO}} = \sup \{ |\langle f, T^*g \rangle| : f \in L^2(\gamma), \|f\|_{H^1} \leq 1 \},$$

and

$$|\langle f, T^*g \rangle| = |\langle Tf, g \rangle| \leq \|Tf\|_1 \|g\|_\infty \leq \|T\|_{H^1, L^1} \|f\|_{H^1} \|g\|_\infty.$$

Thus T^* is bounded from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$.

The converse implication is proved similarly. We omit the details. \square

Now, the boundedness in one dimension of R_1^* from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$ follows by Lemma 7.1 from that of R_1 from $H^1(\gamma)$ to $L^1(\gamma)$, proved in Section 4. Similarly, the unboundedness in dimension $d \geq 2$ of R_i^* from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$ follows from the unboundedness of R_i from $H^1(\gamma)$ to $L^1(\gamma)$ proved in Section 6, and the boundedness of R_i^* from $H^1(\gamma)$ to $L^1(\gamma)$ for any d follows from that of R_i from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$ proved in Section 5.

A similar argument gives the boundedness of S_i from $H^1(\gamma)$ to $L^1(\gamma)$ when $d = 1$, that of S_i from $L^\infty(\gamma)$ to $\text{BMO}(\gamma)$ for any d , and the unboundedness of S_i from $H^1(\gamma)$ to $L^1(\gamma)$ when $d \geq 2$.

The theorem is completely proved.

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