Some revisited results about composition operators on Hardy spaces

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Abstract. On the one hand, we generalize some results known for composition operators on Hardy spaces to the case of Hardy–Orlicz spaces H^{Ψ}: construction of a "slow" Blaschke product giving a non-compact composition operator on H^{Ψ} and yet "nowhere differentiable"; construction of a surjective symbol whose associated composition operator is compact on H^{Ψ} and is, moreover, in all Schatten classes $S_p(H^2)$, $p > 0$. On the other hand, we revisit the classical case of composition operators on H^2, giving first a new, and simpler, characterization of composition operators with closed range, and then showing directly the equivalence of the two characterizations of membership in Schatten classes of Luecking, and Luecking–Zhu.

1. Introduction

The study of composition operators on Hardy spaces is now a classical subject (see [3] and [18] for example). In [8] (see also [7]), we considered a more general setting and studied composition operators on Hardy–Orlicz spaces; we gave there a characterization of their compactness in terms of the Carleson function of their symbol (and in terms of the Nevanlinna counting function in [11]). This work was continued in [10]: we compared compactness on Hardy spaces with compactness on Hardy–Orlicz spaces. For instance, we showed that for every $1 \leq p < \infty$, there is an Orlicz function Ψ such that $H^{p+\varepsilon} \subseteq H^{\Psi} \subseteq H^p$ for every $\varepsilon > 0$, and a composition operator C_φ such that C_φ is compact on H^p and $H^{p+\varepsilon}$, but not compact on H^{Ψ}.

We carry on this study in the present work, taking advantage of our previous results ([8] and [11]). In a first part (Section 3 and Section 4), we shall improve, and extend to the Hardy–Orlicz case, results known for Hardy spaces; we use there results of [8]. In a second part (Section 5 and Section 6), we shall give new lights on some results concerning Hardy spaces, using the results of [11] establishing the

equivalence of the Nevanlinna counting function with the Carleson function of the symbol. More precisely, the content of this paper is as follows.

B. MacCluer[1] and J. Shapiro (Theorem 3.10 of [14]; see also §3.2 of [18]) proved that, when their symbol φ is finitely-valent, the compactness of composition operators C_φ on the Hardy space H^2 can be characterized by the behaviour of the modulus of φ near the frontier of $\mathbb{D}$: compactness is equivalent to the condition $1 - |z| = o\left(1 - |\varphi(z)|\right)$ as $|z| \to 1$. But the latter does not imply compactness in general (Example 3.8 of [14]; see also §10.2 of [18]). In Theorem 5.3 of [11], we gave such a characterization for composition operators, with finitely-valent symbol, on Hardy–Orlicz spaces. In Section 3, we construct a "slow" Blaschke product (generalizing §10.2 of [18] and Proposition 5.5 of [8]) showing that this condition is not sufficient in general. In Section 4, we construct a compact composition operator $C_\varphi \colon H^\Psi \to H^\Psi$ with surjective symbol φ and such that $C_\varphi \colon H^2 \to H^2$ is in all Schatten classes $S_p(H^2)$, $p > 0$. This generalizes and improves a result of B. MacCluer and J. Shapiro (Example 3.12 in [14]; see also §2 in the survey [16]). In Section 5, we give a characterization of composition operators $C_\varphi \colon H^p \to H^p$, $1 \le p < \infty$, with a closed range, simpler than the former ones (see [1] and [20]). Finally, based on the main result of [11], we show directly, in Section 6, the equivalence of Luecking's and Luecking–Zhu's criteria ([12], [13]) for the membership of $C_\varphi \colon H^2 \to H^2$ in Schatten classes.

2. Notation

The open unit disk is denoted by $\mathbb{D} = \{z \in \mathbb{C}\,;\ |z| < 1\}$ and its boundary, the unit circle, by $\mathbb{T} = \{z \in \mathbb{C}\,;\ |z| = 1\}$. The normalized Lebesgue measure $dt/2\pi$ on $\mathbb{T}$ is denoted by m. The normalized area measure $dx\,dy/\pi$ is denoted by A.

The Hardy space H^1 is the space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that $\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|\,d\theta < \infty$. Every $f \in H^1$ has almost everywhere non-tangential limits on $\mathbb{T}$, and we denote by f^* the function on $\mathbb{T}$ given by these boundary values.

An Orlicz function is a convex nondecreasing function $\Psi \colon [0, \infty) \to [0, \infty)$ such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. If μ is a positive measure on some measurable space S, the Orlicz space $L^\Psi(\mu)$ is the set of all (classes of) measurable functions $f \colon S \to \mathbb{C}$ such that $\int_S \Psi(|f|/C)\,d\mu < \infty$ for some $C > 0$; the norm $\|f\|_\Psi$ is defined as the infimum of all positive numbers C for which $\int_S \Psi(|f|/C)\,d\mu \le 1$.

The Hardy–Orlicz space H^Ψ is the linear subspace of $f \in H^1$ such that $f^* \in L^\Psi(m)$ (see [8]).

Every analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ defines a bounded composition operator $C_\varphi \colon f \in H^\Psi \mapsto f \circ \varphi \in H^\Psi$ (see [8]).

For every $\xi \in \mathbb{T}$ and $0 < h < 1$, we set

$$S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \le h\}.$$

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[1] We misspelled her name in our previous papers, and we apologize for this.

The Carleson function ρ_φ of the analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ is defined, for $0 < h < 1$, by

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m\big(\{e^{i\theta} \in \mathbb{T} \, ; \, \varphi^*(e^{i\theta}) \in S(\xi, h)\}\big) \, .$$

Alternatively, $\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m_\varphi[S(\xi, h)]$, where m_φ is the pull-back measure of m by φ. We shall sometimes use the classical Carleson window, namely the set $W(\xi, h) = \{z \in \mathbb{D} \, ; \, |z| \geq 1 - h \text{ and } |\arg(z\bar{\xi})| \leq h\}$ instead of $S(\xi, h)$, because it has an equivalent size.

The Nevanlinna counting function N_φ is defined, for $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, by

$$N_\varphi(w) = \sum_{\varphi(z) = w} \log \frac{1}{|z|} \, ,$$

each term $\log \frac{1}{|z|}$ being repeated according to the multiplicity of z. It may be extended to the whole of $\mathbb{D}$ by $N_\varphi[\varphi(0)] = +\infty$ and $N_\varphi(w) = 0$ for $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$.

3. Slow Blaschke products

B. MacCluer and J. Shapiro (Theorem 3.10 of [14]; see also §3.2 in [18]) proved that, when φ is finitely-valent (meaning that, for some $s \geq 1$, the equation $\varphi(z) = w$ has at most s solutions), the composition operator $C_\varphi \colon H^p \to H^p$ is compact, $1 \leq p < \infty$, if and only if φ has an angular derivative at no point of $\mathbb{T}$; that means that

(3.1) $$\lim_{|z| \to 1} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \, .$$

In Theorem 5.3 of [11], we generalized this result to Hardy–Orlicz spaces and proved that if φ is finitely-valent, the composition operator $C_\varphi \colon H^\Psi \to H^\Psi$ is compact if and only if

(3.2) $$\lim_{|z| \to 1} \frac{\Psi^{-1}\left[\dfrac{1}{1 - |\varphi(z)|}\right]}{\Psi^{-1}\left[\dfrac{1}{1 - |z|}\right]} = 0 \, .$$

Without the assumption that φ is finitely-valent, condition (3.2) is no longer sufficient to ensure the compactness of $C_\varphi \colon H^\Psi \to H^\Psi$. Indeed, we are going to construct a Blaschke product satisfying (3.2), but whose associated composition operator is of course not compact on H^Ψ, as this is the case for every inner function. A Blaschke product satisfying (3.1) is constructed in §10.2 of [18]; that construction uses Frostman's Theorem. Our construction, which is more general, is entirely elementary.

Theorem 3.1. *Let $\delta\colon (0,1) \to (0,1/2]$ be any function such that $\lim_{t\to 0} \delta(t) = 0$. Then, there exists a Blaschke product B such that*

$$(3.3) \qquad\qquad 1 - |B(z)| \geq \delta(1 - |z|), \quad \text{for all } z \in \mathbb{D}.$$

Corollary 3.2. *For every Orlicz function Ψ there exists a Blaschke product B which satisfies*

$$\lim_{|z|\to 1} \frac{\Psi^{-1}\Big[\dfrac{1}{1 - |B(z)|}\Big]}{\Psi^{-1}\Big[\dfrac{1}{1 - |z|}\Big]} = 0\,,$$

though the composition operator $C_B\colon H^\Psi \to H^\Psi$ is not compact.

Proof. C_B is not compact since every compact composition operator should satisfy $|\varphi^*| < 1$ *a.e.* (see Lemma 4.8 of [8]). It suffices then to choose $\delta(t) = 1/\Psi\big(\sqrt{\Psi^{-1}(1/t)}\big)$, which satisfies the hypothesis of Theorem 3.1. Moreover,

$$\frac{\Psi^{-1}\big(1/\delta(t)\big)}{\Psi^{-1}(1/t)} = \frac{1}{\sqrt{\Psi^{-1}(1/t)}} \xrightarrow[t\to 0]{} 0\,,$$

and condition (3.3) gives the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.1. We shall essentially construct our Blaschke product B as an infinite product of finite Blaschke products

$$\prod_n B_n\,,$$

where each finite Blaschke product B_n has p_n zeros equidistributed in the circumference of radius r_n. That is, we will have, writing $\theta_k = 2\pi k/p_n$ and $z_k = r_n\, e^{i\theta_k}$, for $k = 1, 2, \ldots, p_n$,

$$(3.4) \qquad B_n(z) = \prod_{k=1}^{p_n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z} = \prod_{k=1}^{p_n} \frac{r_n - e^{-i\theta_k} z}{1 - r_n e^{-i\theta_k} z}\,.$$

We shall need the following estimate for the finite Blaschke product in (3.4).

Lemma 3.3. *Let $p \in \mathbb{N}$, and $0 < r < 1$. Consider the finite Blaschke product*

$$(3.5) \qquad\qquad G(z) = \prod_{k=1}^{p} \frac{r - e^{-i\theta_k} z}{1 - r e^{-i\theta_k} z}\,,$$

where $\theta_k = \frac{2k\pi}{p}$, for $k = 1, 2, \ldots, p$.

   a) *Then, for every $z \in \mathbb{D}$ with $|z| = r$,*

$$(3.6) \qquad\qquad |G(z)| \leq \frac{2r^p}{1 + r^{2p}} = 1 - \frac{(1 - r^p)^2}{1 + r^{2p}}\,.$$

b) *If besides we have $p\,h \leq 1/2$, where $h = 1 - r$, we also have, for every $z \in \mathbb{D}$ with $|z| = r$,*

$$(3.7) \qquad |G(z)| \leq 1 - \frac{(p\,h)^2}{2e}\,.$$

Let us continue the proof of the theorem. Define $\chi\colon (0,1) \to (0,1]$ by

$$(3.8) \qquad \chi(x) = \sup_{t \leq x} \left[\max\{2\delta(t), \sqrt{t}\} \right].$$

Then χ is non-decreasing, $\lim_{x \to 0} \chi(x) = 0$ and $\lim_{x \to 1} \chi(x) = 1$. We can find a decreasing sequence $(h_n)_{n \geq 0}$ of points $h_n \in (0,1)$, such that $\chi(h_n) \leq 2^{-n}$. This sequence converges to 0; in fact, $\sqrt{h_n} \leq \chi(h_n) \leq 2^{-n}$, by (3.8), and hence

$$(3.9) \qquad h_n \leq 2^{-2n}\,.$$

We now define, for every $n \in \mathbb{N}$, a positive integer p_n, by

$$(3.10) \qquad p_n = \min\left\{ p \in \mathbb{N}; \ \frac{p^2 h_n^2}{2e} > 2^{-n} \right\}.$$

We have $p_n > 1$ because $h_n^2/2e < h_n^2 \leq 2^{-4n}$. So, for every n, we have $4(p_n - 1)^2 \geq p_n^2$, and then

$$(3.11) \qquad 4 \cdot 2^{-n} \geq \frac{4(p_n - 1)^2 h_n^2}{2e} \geq \frac{p_n^2 h_n^2}{2e}\,.$$

This yields, for $n \geq 7$, that $(p_n h_n)^2 \leq 8e\,2^{-n} \leq 1/4$. Therefore $p_n h_n \leq 1/2$, and we can use the estimate in part (b) of Lemma 3.3.

Now, for $n \geq 7$, let B_n be the finite Blaschke product defined by (3.4), where $r_n = 1 - h_n$. Using b) in Lemma 3.3, the Maximum Modulus Principle and the definition of p_n in (3.10), we have

$$(3.12) \qquad |B_n(z)| \leq 1 - \frac{p_n^2 h_n^2}{2e} < 1 - 2^{-n}, \quad \text{for } |z| \leq r_n.$$

Consider then the Blaschke product D defined by

$$(3.13) \qquad D(z) = \prod_{n=7}^{\infty} B_n(z).$$

This product is convergent since, by (3.11), we have

$$\sum p_n (1 - r_n) = \sum p_n h_n \leq \sum \sqrt{8e\,2^{-n}} < +\infty\,.$$

Finally, take $N \in \mathbb{N}$ big enough to have $r_6^N < 1/2$, and define

$$(3.14) \qquad B(z) = z^N D(z).$$

Thus B is a Blaschke product, and, if $|z| \le r_6$, we have, since $\delta(t) \le 1/2$:

$$(3.15) \qquad |B(z)| \le |z^N| \le r_6^N < 1/2 \le 1 - \delta(1 - |z|).$$

If $1 > |z| > r_6$, there exists $k \ge 7$ such that $r_k \ge |z| > r_{k-1}$. Therefore, thanks to (3.12),

$$(3.16) \qquad |B(z)| \le |D(z)| \le |B_k(z)| \le 1 - 2^{-k}.$$

On the other hand $r_k \ge |z| > r_{k-1}$ implies $h_k \le 1 - |z| < h_{k-1}$, and so

$$(3.17) \qquad \delta(1 - |z|) \le \frac{1}{2}\chi(1 - |z|) \le \frac{1}{2}\chi(h_{k-1}) \le 2^{-k}.$$

Combining (3.16) and (3.17) we get $|B(z)| \le 1 - \delta(1 - |z|)$, when $1 > |z| > r_6$. From this and (3.15), Theorem 3.1 follows. $\qquad\square$

Proof of Lemma 3.3. It is obvious that, for all $a, z \in \mathbb{C}$,

$$\prod_{k=1}^{p}(z - a\mathrm{e}^{i\theta_k}) = z^p - a^p.$$

Using this we have

$$(3.18) \qquad G(z) = \prod_{k=1}^{p} \frac{r - \mathrm{e}^{-i\theta_k}z}{1 - r\mathrm{e}^{-i\theta_k}z} = \prod_{k=1}^{p} \frac{z - r\mathrm{e}^{i\theta_k}}{rz - \mathrm{e}^{i\theta_k}} = \frac{z^p - r^p}{(rz)^p - 1}.$$

Now, if $|z| = r$, we can write $z^p = r^p u$, for some u with $|u| = 1$. Then $|G(z)| = |T(u)|$, where T is the Möbius transformation

$$T(u) = \frac{r^p(u - 1)}{r^{2p}u - 1}.$$

This transformation T maps the unit circle $\partial\mathbb{D}$ onto a circumference C. As T maps the extended real line $\mathbb{R}_\infty$ to itself, and $\partial\mathbb{D}$ is orthogonal to $\mathbb{R}_\infty$ at the intersection points 1 and -1, C is the circumference orthogonal to $\mathbb{R}_\infty$ crossing through the points $T(1) = 0$ and $T(-1) = \alpha$. It is easy to see that $|w| \le |\alpha|$, for every $w \in C$; consequently,

$$|G(z)| \le \sup_{u \in \partial\mathbb{D}} |T(u)| = |T(-1)| = \frac{2r^p}{1 + r^{2p}}.$$

This finishes the proof of the statement (a).

To prove part (b), observe that, $1 + r^{2p} \le 2$, and so, for $|z| = r$,

$$(3.19) \qquad |G(z)| \le 1 - \frac{(1 - r^p)^2}{1 + r^{2p}} \le 1 - \frac{(1 - r^p)^2}{2}.$$

Remember that $r = 1 - h$, so $r \le \mathrm{e}^{-h}$, and $r^p \le \mathrm{e}^{-ph}$. Thus $1 - r^p \ge 1 - \mathrm{e}^{-ph}$. Now, if $x \in [0, 1/2]$, we have, by the Mean Value Theorem,

$$1 - \mathrm{e}^{-x} \ge \frac{x}{\sqrt{\mathrm{e}}}.$$

Since $p\,h \le 1/2$, we can apply this last estimate to (3.19) to get, as promised,

$$|G(z)| \le 1 - \frac{(1 - \mathrm{e}^{-ph})^2}{2} \le 1 - \frac{p^2 h^2}{2\mathrm{e}}\,,$$

ending the proof of Lemma 3.3.   □

Remark. The key point in the proof of Theorem 3.1 is the inequality (3.6) in Lemma 3.3. This inequality may be viewed as a consequence of the strong triangle inequality (applied to $a = z^p$, $b = r^p$ and $c = 0$)

$$(3.20) \qquad d(a, b) \le \frac{d(a, c) + d(c, b)}{1 + d(a, c)\, d(c, b)}$$

for the pseudo-hyperbolic distance $d(u, v) = \frac{|u-v|}{|1-\bar{u}v|}$ on $\mathbb{D}$. Let us recall a proof for the convenience of the reader: by conformal invariance, we may assume that $c = 0$; then

$$1 - [d(a, b)]^2 = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \bar{a}b|^2} \ge \frac{(1 - |a|^2)(1 - |b|^2)}{(1 + |a|\,|b|)^2} = 1 - [d(|a|, -|b|)]^2\,,$$

so that

$$d(a, b) \le d(|a|, -|b|) = \frac{|a| + |b|}{1 + |a|\,|b|}\,,$$

proving (3.20), since $d(a, 0) = |a|$ and $d(0, b) = |b|$.

4. A compact composition operator with a surjective symbol

A well-known result of H. J. Schwartz (Theorem 2.8 of [17]) asserts that the composition operator $C_\varphi \colon H^\infty \to H^\infty$ is compact if and only if $\|\varphi\|_\infty < 1$. In particular, the compactness of $C_\varphi \colon H^\infty \to H^\infty$ prevents the surjectivity of φ. It may therefore be expected that, the bigger Ψ, the more difficult it will be to obtain both the compactness of $C_\varphi \colon H^\Psi \to H^\Psi$ and the surjectivity of φ. The following theorem asserts that this is possible, and the case H^∞ appears really as a singular case (corresponding to an "Orlicz function" which is discontinuous and can take the value infinity).

Theorem 4.1. *For every Orlicz function Ψ, there exists a symbol $\varphi \colon \mathbb{D} \to \mathbb{D}$ which is 4-valent and surjective and such that $C_\varphi \colon H^\Psi \to H^\Psi$ is compact. Moreover, φ can be taken so as $C_\varphi \colon H^2 \to H^2$ is in all Schatten classes $S_p(H^2)$, $p > 0$.*

In the case of H^2 ($\Psi(x) = x^2$), B. MacCluer and J. Shapiro gave an example (Example 3.12 in [14]) based on the Riemann mapping theorem and on the fact that, for a finitely valent symbol φ, we have the equivalence

$$(4.1) \qquad C_\varphi \colon H^2 \to H^2 \text{ compact} \quad \Longleftrightarrow \quad \lim_{|z| \overset{<}{\to} 1} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty.$$

A specific example is as follows. Take

$$(4.2) \qquad R = \big\{ z = x + iy \in \mathbb{C} \, ; \ x > 0 \ \text{and} \ \frac{1}{x} < y < \frac{1}{x} + 4\pi \big\},$$

let $g \colon \mathbb{D} \to R$ be a Riemann map and set $\varphi = \mathrm{e}^{-g}$. Then, φ is 2-valent, $\varphi(\mathbb{D}) = \mathbb{D}^*$ (where $\mathbb{D}^* = \mathbb{D} \backslash \{0\}$), and the validity of (4.1) is tested through the use of the Julia–Carathéodory theorem (see [16] for details). To get a fully surjective mapping φ_1, just compose φ with the square of a Blaschke factor:

$$\varphi_1(z) = B \circ \varphi, \ \ \text{with} \ \ B(z) = \Big(\frac{z - \alpha}{1 - \overline{\alpha} z} \Big)^2, \quad \alpha \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$$

(note that $B(0) = B(2\alpha / 1 + |\alpha|^2)$. Since $C_{\varphi_1} = C_\varphi \circ C_B$, we see that C_{φ_1} is compact as well and we are done.

Here, we can no longer rely on the Julia–Carathéodory theorem. But we shall use the following necessary and sufficient condition, in terms of the maximal Carleson function ρ_φ, which is valid for any symbol, finitely-valent or not (see Theorem 4.18 of [8] – or Théorème 4.2 of [7], where a different, but equivalent, formulation is given):

$$(4.3) \qquad C_\varphi \colon H^\Psi \to H^\Psi \ \text{compact} \quad \Longleftrightarrow \quad \lim_{h \searrow 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}\big(1/\rho_\varphi(h)\big)} = 0 \,.$$

For the sequel, we shall set

$$(4.4) \qquad\qquad \Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}\big(1/\rho_\varphi(h)\big)} \,.$$

Our strategy will be to elaborate on the previous example to produce a (nearly) surjective φ such that $\rho_\varphi(h)$ is very small (depending on Ψ) for small h. The tool will be the notion of harmonic measure for certain open sets of the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called *hyperbolic* (see Definition 19.9.3 in [2]); for example, every conformal image of $\mathbb{D}$ is hyperbolic (see Proposition 19.9.2 (d) and Theorem 19.9.7 of [2]). If G is a hyperbolic domain and $a \in G$, the *harmonic measure* of G at a is the probability measure $\omega_G(a, \, . \,)$ supported by ∂G (here, and throughout the rest of this section, boundaries and closures will be taken in $\hat{\mathbb{C}}$) such that

$$u(a) = \int_{\partial G} u(z) \, d\omega_G(a, z)$$

for each bounded and continuous function u on $\overline{G}$, which is harmonic in G (see Definition 21.1.3 in [2]). The harmonic measure at a of a Borel set $A \subseteq \partial G$ will be denoted by $\omega_G(a, A)$. Clearly,

$$\omega_{\mathbb{D}}(0, \, . \,) = m,$$

the Haar measure (*i.e.* normalized Lebesgue measure) of $\partial \mathbb{D}$.

R. Nevanlinna (see Proposition 21.1.6 in [2]) showed that harmonic measures share a *conformal invariance property*. Namely, assume that G is a simply connected domain, in which the Dirichlet problem can be solved (a *Dirichlet domain*), and $\tau\colon \overline{\mathbb{D}} \to \overline{G}$ is a continuous function which maps conformally $\mathbb{D}$ onto G; then τ maps $\partial\mathbb{D}$ onto ∂G, and, if $\tau(0) = a$:

$$(4.5) \qquad\qquad \omega_G(a, A) = m\big(\tau^{-1}(A)\big)$$

for every Borel set $A \subseteq \partial G$. This explains why harmonic measures enter the matter when we consider composition operators C_φ: such an operator induces a map $H^\Psi \to L^\Psi(m_\varphi)$, where $m_\varphi = \varphi^*(m)$ appears as an image measure of m, as it happens for the harmonic measure of G at a in (4.5).

Lemma 4.2 (Hole principle). *Let G_0 and G_1 be two hyperbolic open sets and $H \subseteq \partial G_0$ a Borel set such that*

$$G_0 \subseteq G_1 \quad and \quad \partial G_0 \subseteq \partial G_1 \cup H.$$

Then, for every $a \in G_0$, we have the following inequality:

$$(4.6) \qquad\qquad \omega_{G_1}(a, \partial G_1 \setminus \partial G_0) \leq \omega_{G_0}(a, H).$$

Proof.[2] By using Corollary 21.1.14 of [2], with $\Delta = \partial G_0 \cap \partial G_1$, one has $\omega_{G_0}(a, \Delta) \leq \omega_{G_1}(a, \Delta)$. But $\partial G_1 \setminus \Delta = \partial G_1 \setminus \partial G_0$, and hence, since harmonic measures are probability measures,

$$\omega_{G_1}(a, \partial G_1 \setminus \partial G_0) = \omega_{G_1}(a, \partial G_1 \setminus \Delta) = 1 - \omega_{G_1}(a, \Delta) \leq 1 - \omega_{G_0}(a, \Delta);$$

we get the result since $\partial G_0 = H \cup \Delta$, which implies $1 \leq \omega_{G_0}(a, H) + \omega_{G_0}(a, \Delta)$. □

We also shall need the following result (see Proposition 21.1.17 in [2]).

Proposition 4.3 (Continuity principle). *If G is a hyperbolic open set and $a \in G$, then the harmonic measure $\omega_G(a, \,.\,)$ is atomless.*

Proof of Theorem 4.1. It will be enough to construct a 2-valent mapping $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(\mathbb{D}) = \mathbb{D}^*$ and $C_\varphi\colon H^\Psi \to H^\Psi$ is compact. We can then modify φ by the same trick as the one used by B. MacCluer and J. Shapiro. Note that every point in $\mathbb{D}^*$ is the image by e^{-z} of two distinct points of R, except those which are the image of points of the hyperbola $y = (1/x) + 2\pi$, which have only one pre-image.

For a positive integer n, set

$$(4.7) \qquad\qquad b_n = \frac{1}{4n\pi}\,,$$

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[2] A more intuitive way to see why this result holds is to use the fact, due to S. Kakutani, and completed by J. Doob (see page 454 of [19] and Appendix F, page 477 of [6]) that the harmonic measure of A at a is the probability that the Brownian motion starting at a exits from G through the Borel set $A \subseteq \partial G$.
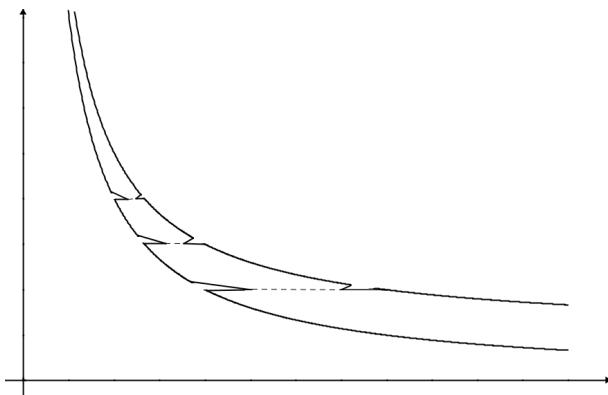
and let $\varepsilon_n > 0$ such that

$$(4.8) \qquad \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \le \frac{1}{n} \, .$$

We now modify the domain R, including "barriers" in it (not in the sense of potential theory, nor of Perron!) in the following way.

Let, for every $n \ge 1$, M_n be the intersection point of the horizontal line $y = 4\pi n$ and of the hyperbola $y = (1/x) + 2\pi$, that is $M_n = \frac{1}{4\pi n - 2\pi} + 4\pi ni$.

Define inductively closed sets P_n^+ and P_n^-, which are like small points of swords (two segments and a piece of hyperbola), in the following way:

- The lower part of P_n^+ and P_n^- are horizontal segments of altitude $4n\pi$.
- Those two horizontal segments are separated by a small open horizontal segment H_n whose middle is M_n.
- The upper part of P_n^+ is a slant segment whose upper extremity c_n^+ lies on the hyperbola $y = 1/x$.
- The upper part of P_n^- is a slant segment whose upper extremity c_n^- lies on the hyperbola $y = (1/x) + 4\pi$.
- The curvilinear part of P_n^+ is supported by the hyperbola $y = 1/x$.
- The curvilinear part of P_n^- is supported by the hyperbola $y = (1/x) + 4\pi$.
- One has $4(n+1)\pi - \Im c_n^{\pm} > 2\pi$.

The size of the small horizontal holes will be determined inductively in the following way. Fix once and for all $a \in R$ such that $\Im a < 4\pi$. Suppose that $H_1, H_2, \ldots, H_{n-1}$ have already been determined. Set

$$(4.9) \qquad \Omega_n = \Big\{ z \in R \setminus \bigcup_{j<n} (P_j^+ \cup P_j^-) \, ; \ \Im z < 4n\pi \Big\}.$$

We can adjust H_n so small that

$$(4.10) \qquad \omega_{\Omega_n}(a, H_n) \le \varepsilon_n.$$

Indeed, Ω_n is bounded above by the horizontal segment $[b_n + 4in\pi, b_{n-1} + 4in\pi]$, where the point M_n lies. If $H_n = [M_n - \delta, M_n + \delta]$, we see that H_n decreases to the singleton $\{M_n\}$ as δ decreases to zero. Therefore, by Proposition 4.3, we can adjust δ so as to realize (4.10).

We now define our modified open set Ω by the formula

$$(4.11) \qquad \Omega = R \setminus \bigcup_{n \geq 1} (P_n^+ \cup P_n^-) = \bigcup_{n \geq 1} \Omega_n.$$

It is useful to observe that

$$(4.12) \qquad \inf_{w \in \partial \Omega_n} \mathfrak{Re}\, w = b_n \,.$$

This is obvious by the way we defined the upper part of $\partial \Omega_n$.

Now, we can easily finish the proof. Fix $h \leq b_1/2$ and let n be the integer such that

$$(4.13) \qquad b_{n+1} < 2h \leq b_n \,.$$

Let $g \colon \mathbb{D} \to \Omega$ be a conformal mapping such that $g(0) = a$. Since $\partial_\infty \Omega$ is connected, Caratheodory's Theorem (see [15]) ensures that g can be continuously extended from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. More explicitly, using the Möbius transformation $T(z) = 1/z$, we see that there exists an automorphism of the extended complex plane such that $\overline{\Omega}$ is sent onto a compact subset of $\mathbb{C}$; so, we can apply to Ω many results stated for bounded domains. For instance, the boundary of Ω is a continuous path in the extended plane; so, by Theorem 14.5.5 of [2], g can be extended to a continuous function (for the extended plane topology) $g \colon \overline{\mathbb{D}} \to \overline{G}$. In particular, g has boundary values g^*.

We define $\varphi = \mathrm{e}^{-g}$.

As in the proof of B. MacCluer and J. Shapiro ([14]), we have that φ is 2-valent (see the remark made at the beginning of this proof), and we still have $\varphi(\mathbb{D}) = \mathbb{D}^*$, since, in the process for constructing Ω from R, for every point of $\mathbb{D}^*$, at least one of the preimages by e^{-z} in R has not been removed. Observe that, in particular, we did not remove any point in the hyperbola $y = (1/x) + 2\pi$, thanks to the choice of M_n.

Moreover, Ω is a Dirichlet domain (because each component of $\partial \Omega$ has more than one point: see the comment after Definition 19.7.1 in [2]), so we can use the conformal invariance. Then by (4.5), (4.10), (4.12) and by the hole principle, we see that, if $A = \{\mathfrak{Re}\, g^*(\mathrm{e}^{it}) < 2h\}$,

$$(4.14) \qquad \begin{aligned} \rho_\varphi(h) &\leq m_\varphi(\{|z| > 1 - h\}) = m(\{\mathrm{e}^{-\mathfrak{Re}\, g^*(\mathrm{e}^{it})} > 1 - h\}) \\ &= m(\{\mathfrak{Re}\, g^*(\mathrm{e}^{it}) < \log(1/1 - h)\}) \\ &\leq m(\{\mathfrak{Re}\, g^*(\mathrm{e}^{it}) < 2h\}) = \omega_{\mathbb{D}}(0, A) \\ &= \omega_{g(\mathbb{D})}\big(g(0), g(A)\big) = \omega_\Omega(a, \{\mathfrak{Re}\, w < 2h\}) \\ &\leq \omega_\Omega(a, \{\mathfrak{Re}\, w \leq b_n\}) \\ &\leq \omega_\Omega(a, \partial \Omega \setminus \partial \Omega_n) \leq \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n. \end{aligned}$$

It remains to observe that

$$\Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} \leq \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n} \leq Ch\,,$$

in view of (4.8) and of the choice of n, C being a numerical constant. We should point out the fact that we applied the hole principle to the domains $G_0 = \Omega_n$ and $G_1 = \Omega$ and that this was licit because the assumptions of the hole principle (in particular the inclusion $\partial\Omega_n \subseteq \partial\Omega \cup H_n$) are satisfied. We have therefore proved that

$$\lim_{h \to 0} \Delta(h) = 0\,,$$

and this ends, as we already explained, the first part of the proof of Theorem 4.1.

To prove the last part, let us remark that in (4.8) we may take ε_n arbitrarily small. If one takes $\varepsilon_n \leq \mathrm{e}^{-n}$, one has, for some constant $c > 0$, $\rho_\varphi(h) \leq \mathrm{e}^{-c/h}$, by using (4.13) and (4.14). In particular, $\rho_\varphi(h) \leq C\,h^\alpha$ for every $\alpha > 1$. By Luecking's criterion, that implies that $C_\varphi \in S_p(H^2)$ for every $p > 0$ (see Corollary 3.2 in [9]). □

Remark. Let us note that our result is stronger than MacCluer–Shapiro's, since our C_φ is in all Schatten classes $S_p(H^2)$, $p > 0$. Though our construction follows MacCluer–Shapiro's, it is the introduction of the "barriers" P_n^+ and P_n^- which allows us to get this improvement.

5. Composition operators with closed range

In [1], J. Cima, J. Thomson and W. Wogen gave a characterization of composition operators $C_\varphi \colon H^p \to H^p$ with closed range. This characterization involves the Radon–Nikodym derivative of the restriction to $\partial\mathbb{D}$ of m_φ. They found it not satisfactory, and asked a characterization with the range of φ itself. N. Zorboska ([20]) gave such a characterization, but her statement is somewhat complicated. We shall give here more explicit characterizations, either in terms of the Nevanlinna counting function N_φ, or in terms of the Carleson measure m_φ.

Theorem 5.1. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant analytic self map. Then the composition operator $C_\varphi\colon H^p \to H^p$, $1 \leq p < \infty$, has closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(5.1) \qquad \frac{1}{A\big(S(\xi,h)\big)} \int_{S(\xi,h)} N_\varphi(z)\,dA(z) \geq c\,h\,, \quad \forall \xi \in \partial\mathbb{D}\,.$$

We shall use the following result (Theorem 4.2 of [11]).

Theorem 5.2. *There are universal constants $C, c > 1$ such that*

$$m_\varphi\big(S(\xi,h)\big) \leq C\,\frac{1}{A\big(S(\xi,ch)\big)} \int_{S(\xi,ch)} N_\varphi(z)\,dA(z)$$

for every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, every $\xi \in \partial\mathbb{D}$, and $0 < h < (1 - |\varphi(0)|)/8$.

Then Theorem 5.1 will follow immediately from this theorem and the next theorem, applied to $\mu = m_\varphi$.

Theorem 5.3. *Let μ be a finite positive measure on $\overline{\mathbb{D}}$. Assume that the canonical map $J\colon H^p \to L^p(\mu)$ is continuous, $1 \le p < \infty$. Then J is one-to-one and has a closed range if and only if there is a constant $c > 0$ such that, for $0 < h < 1$,*

$$(5.2) \qquad \mu\big[W(\xi, h)\big] \ge c\,h\,, \quad \forall \xi \in \partial\mathbb{D}\,.$$

Proof. 1) Assume that J has a closed range. By making a rotation on the variable z, we only have to find a constant $c > 0$ such that

$$(5.3) \qquad \mu(S_h) \ge c\,h\,,$$

for $h > 0$ small enough, where $S_h = S(1, h)$.

Since J is one-to-one with closed range, the open mapping theorem implies that there is a constant $C > 0$ such that

$$(5.4) \qquad \|f\|_{L^p(\mu)}^p \ge C^p\,\|f\|_p^p\,, \quad \forall f \in H^p.$$

We are going to test (5.4) on

$$(5.5) \qquad f_N(z) = \left(\frac{1+z}{2}\right)^N.$$

It is classical that there is a constant $c_p > 0$ such that

$$(5.6) \qquad \|f_N\|_p^p = \int_{-\pi}^{\pi} \left|\cos\frac{t}{2}\right|^{pN} dt \ge \frac{c_p}{\sqrt{N}}\,.$$

Now, since $|z + 1|^2 + |z - 1|^2 = 2(|z|^2 + 1) \le 4$ for every $z \in \overline{\mathbb{D}}$, one has

$$|f_N(z)| \le \left(1 - \frac{|z - 1|^2}{4}\right)^{N/2} \le \mathrm{e}^{-\frac{N}{8}|z-1|^2}\,.$$

Hence, using $|f_N(z)| \le 1$ when $|z - 1| \le h$, one has

$$\|f_N\|_{L^p(\mu)}^p \le \mu(S_h) + \int_{|z-1|>h} \mathrm{e}^{-p\frac{N}{8}|z-1|^2}\,d\mu$$

$$= \mu(S_h) + \int_0^{\mathrm{e}^{-pNh^2/8}} \mu\big(\{\mathrm{e}^{-p\frac{N}{8}|z-1|^2} > u\}\big)\,du\,,$$

that is, making the change of variable $u = \mathrm{e}^{-p\frac{N}{8}x^2}$,

$$\|f_N\|_{L^p(\mu)}^p \le \mu(S_h) + \int_h^\infty \mu(\{|z - 1| \le x\})\,\frac{pN}{4}\,x\,\mathrm{e}^{-p\frac{N}{8}x^2}\,dx\,.$$

Now, the continuity of J means, by Carleson's Theorem (see Theorem 9.3 in [4]), that there is a constant $K > 0$ such that

$$(5.7) \qquad \sup_{|\xi|=1} \mu\big(S(\xi, x)\big) \leq K\, x, \quad 0 \leq x < 1\,.$$

We get hence

$$\|f_N\|_{L^p(\mu)}^p \leq \mu(S_h) + \int_h^\infty K\, x\, \frac{pN}{4}\, x\, \mathrm{e}^{-p\frac{N}{8} x^2}\, dx$$

$$= \mu(S_h) + \frac{K\sqrt{8}}{\sqrt{p}}\, \frac{1}{\sqrt{N}} \int_{h\sqrt{\frac{pN}{8}}}^\infty y^2\, \mathrm{e}^{-y^2}\, dy\,.$$

We take now for N the smaller integer $> 1/h^2$, multiplied by some constant integer a_p, large enough to have

$$\frac{K\sqrt{8}}{\sqrt{p}} \int_{\sqrt{\frac{p\, a_p}{8}}}^\infty y^2\, \mathrm{e}^{-y^2}\, dy \leq \frac{c_p\, C^p}{2}\,.$$

We get then, from (5.4) and (5.6),

$$\mu(S_h) \geq \frac{C^p\, c_p}{2}\, \frac{1}{\sqrt{N}}\,,$$

which gives (5.3).

2) Conversely, assume that (5.2) holds. Since the disk algebra $A(\mathbb{D})$ is dense in H^p, it suffices to show that there exists a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \geq C\, \|f\|_p$ for every $f \in A(\mathbb{D})$.

Let $f \in A(\mathbb{D})$ such that $\|f\|_p = 1$. Choose an integer N such that

$$\frac{1}{N} \sum_{n=1}^N |f(\mathrm{e}^{2\pi i n/N})|^p \geq \frac{1}{2} \int_{\partial\mathbb{D}} |f(\xi)|^p\, dm(\xi) = \frac{1}{2}\,,$$

and such that, due to the uniform continuity of f,

$$z, z' \in \overline{\mathbb{D}} \quad \text{and} \quad |z - z'| \leq \frac{2\pi}{N} \quad \Longrightarrow \quad |f(z) - f(z')| \leq \frac{1}{2^{(p+1)/p}}\,.$$

Then, setting $W_n = W(\mathrm{e}^{2\pi i n/N}, \pi/N)$, $1 \leq n \leq N$, one has

$$\|f\|_{L^p(\mu)}^p = \int_{\mathbb{D}} |f|^p\, d\mu \geq \sum_{n=1}^N \int_{W_n} |f|^p\, d\mu\,.$$

If we choose $z_n \in W_n$ such that $|f(z_n)| = \min_{z \in W_n} |f(z)|$, we get, using (5.2),

$$\|f\|_{L^p(\mu)}^p \geq \sum_{n=1}^N |f(z_n)|^p\, \mu(W_n) \geq \frac{c\pi}{N} \sum_{n=1}^N |f(z_n)|^p\,.$$

Since $A^p \leq 2^{p-1}[(A - B)^p + B^p]$, by Hölder's inequality, one has

$$|f(z_n)|^p \geq \frac{1}{2^{p-1}}\, |f(\mathrm{e}^{2\pi i n/N})|^p - |f(z_n) - f(\mathrm{e}^{2\pi i n/N})|^p$$

and hence

$$\|f\|_{L^p(\mu)}^p \geq \frac{c\pi}{N} \sum_{n=1}^{N} \left[\frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - |f(z_n) - f(e^{2\pi in/N})|^p \right].$$

Now, since $z_n \in W_n$, one has

$$|z_n - e^{2\pi in/N}| \leq \left| z_n - \frac{z_n}{|z_n|} \right| + \left| \frac{z_n}{|z_n|} - e^{2\pi in/N} \right| \leq \frac{\pi}{N} + \frac{\pi}{N} = \frac{2\pi}{N} \,;$$

therefore $|f(z_n) - f(e^{2\pi in/N})| \leq 1/2^{p+1}$ and we get

$$\|f\|_{L^p(\mu)}^p \geq c\pi \left[\frac{1}{N} \sum_{n=1}^{N} \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - \frac{1}{2^{p+1}} \right]$$

$$\geq c\pi \left(\frac{1}{2^{p-1}} \frac{1}{2} - \frac{1}{2^{p+1}} \right) = \frac{c\pi}{2^{p+1}} \,.$$

That ends the proof of Theorem 5.3. □

Remark. Cima–Thomson–Wogen's criterion (Theorem 2 of [1]) says that if φ is a non-constant analytic self-map of $\mathbb{D}$, the composition operator C_φ has closed range if and only if the Radon–Nikodym derivative dm_φ/dm is essentially bounded away from 0. To make the link with this criterion, we shall see that condition (5.2) implies that the restriction of μ to the boundary $\mathbb{T} = \partial \mathbb{D}$ of the disk dominates the Lebesgue measure m. In fact, let I be an arc of $\mathbb{T}$. If $m(I) = h$, we can write

$$I = \bigcap_{n \geq 1} \bigcup_{j=1}^{n} W(\xi_{n,j}, h/2n) \,,$$

with disjoint windows $W(\xi_{n,1}, h/2n), \ldots, W(\xi_{n,n}, h/2n)$; hence,

$$\mu(I) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu[W(\xi_{n,j}, h/2n)] \geq \lim_{n \to \infty} c \sum_{j=1}^{n} \frac{h}{2n} = \frac{c}{2} h \,.$$

6. Composition operators in Schatten classes

In [12], D. Luecking characterized composition operators $C_\varphi \colon H^2 \to H^2$ which are in the Schatten classes, by using, essentially, the m_φ-measure of Carleson windows. Five years later, D. Luecking and K. Zhu ([13]) characterized them by using the Nevanlinna counting function of φ. We shall see in this section how the results of [11] (equivalence of the Carleson function of φ with its Nevanlinna counting function: we shall be more explicit within the proof of Theorem 6.1) makes these two characterizations directly equivalent.

It will be convenient here to work with *modified* Carleson windows, namely

$$W_{n,j} = \left\{ z \in \overline{\mathbb{D}}; \; 1 - 2^{-n} \leq |z| \leq 1 \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

$(j = 0, 1, \ldots, 2^n - 1, \, n = 1, 2, \ldots)$. We shall say that $W_{n,j}$ is the Carleson window centered at $e^{2\pi ij/2^n}$ with size 2^{-n}.

Theorem 6.1. *For $p > 0$ the two following conditions are equivalent:*

a) $\dfrac{N_\varphi(z)}{\log(1/|z|)} \in L^{p/2}(\lambda)$, *where $d\lambda(z) = (1 - |z|)^{-2}\, dA(z)$ and A is the normalized area measure on $\mathbb{D}$;*

b) $\displaystyle\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \left[2^n\, m_\varphi(W_{n,j})\right]^{p/2} < \infty$.

Condition b) in the last theorem yields that $\lim_{n\to\infty} \max_j 2^n m_\varphi(W_{n,j}) = 0$, and it is not difficult to see that this implies that $m_\varphi(\partial\mathbb{D}) = 0$, or equivalently, that $|\varphi^*| < 1$ almost everywhere on $\partial\mathbb{D}$. In this situation we know (see Proposition 3.3 of [9]) that b) in Theorem 6.1 is equivalent to Luecking's condition in [12]. In fact the characterization of belonging to a Schatten class in [12] includes the requirement $m_\varphi(\partial\mathbb{D}) = 0$.

Proof. We may, and do, assume that $\varphi(0) = 0$.

1) Assume first that condition b) is satisfied. Let

$$R_{n,j} = \left\{ z \in \mathbb{D}\, ;\ 1 - \frac{1}{2^n} \le |z| < 1 - \frac{1}{2^{n+1}} \ \text{ and }\ \frac{(2j-1)\pi}{2^n} \le \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

be the (disjoint) Luecking windows ($0 \le j \le 2^n - 1$, $n \ge 0$). One has $R_{n,j} \subseteq W_{n,j}$.

By Theorem 3.1 of [11], there are a constant $C > 0$ and an integer K such that

$$N_\varphi(z) \le C\, m_\varphi(\widetilde{W}_{n,j}), \quad \text{for every } z \in R_{n,j},$$

where $\widetilde{W}_{n,j}$ is the window centered at $e^{2\pi i j/2^n}$, as $W_{n,j}$, but with size 2^{K-n}. The windows $W_{n-K,j}$, $j = 0, 1, \ldots, 2^{n-K} - 1$, have the same size as the windows $\widetilde{W}_{n,j}$, but may have a different center; nevertheless, each $\widetilde{W}_{n,j}$ can be covered with two windows $W_{n-K,l}$: for $n > K$,

$$\widetilde{W}_{n,j} \subseteq W_{n-K,l} \cup W_{n-K,l+1},$$

for some $l = 1, 2, \ldots, 2^{n-K}$ (where $l + 1$ is understood as 0 if $l = 2^{n-K} - 1$), we get (we shall use $\lesssim$ to mean $\le$ up to a constant)

$$
\begin{aligned}
\int_{\mathbb{D}} \frac{\left(N_\varphi(z)\right)^{p/2}}{(1 - |z|)^{\frac{p}{2}+2}}\, dA(z) &\le \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} \left(N_\varphi(z)\right)^{p/2} dA(z) \\
&\lesssim \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} \left(m_\varphi(\widetilde{W}_{n,j})\right)^{p/2} dA(z) \\
&\lesssim \sum_{n,j} (2^n)^{p/2} \left(m_\varphi(\widetilde{W}_{n,j})\right)^{p/2} \\
&\lesssim \sum_{\nu,l} (2^\nu)^{p/2} \left(m_\varphi(W_{\nu,l})\right)^{p/2} < \infty,
\end{aligned}
$$

and a) holds.

2) Conversely, assume that a) is satisfied. We shall use the following inequality, whose proof will be postponed (for $p \geq 2$, (6.1) follows directly from Theorem 4.2 of [11], see Theorem 5.2 above, and Hölder's inequality):

$$(6.1) \qquad [m_\varphi(W_{n,j})]^{p/2} \lesssim \frac{1}{A(\widetilde{W}_{n,j})} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} \, dA(z) \,,$$

where $\widetilde{W}_{n,j}$ is a window with the same center as $W_{n,j}$ but with a bigger proportional size; say of size 2^{-n+L}. We get

$$\sum_{n,j} [2^n \, m_\varphi(W_{n,j})]^{p/2} \lesssim \sum_{n,j} 2^{np/2} \, 2^{2n} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} \, dA(z)$$

$$= \int_{\mathbb{D}} \left(\sum_n 2^{n(2+\frac{p}{2})} \Big[\sum_j \mathbb{1}_{\widetilde{W}_{n,j}}(z) \Big] \right) [N_\varphi(z)]^{p/2} \, dA(z) \,.$$

Let $k = 0, 1, \dots$ such that $1 - 2^{-k+1} < |z| \leq 1 - 2^{-k}$. One has $z \in \widetilde{W}_{n,j}$ only if $n \leq k + L$, and then, for each such n, z is at most in 2^L windows $\widetilde{W}_{n,j}$. It follows that

$$\sum_n 2^{n(2+\frac{p}{2})} \sum_j \mathbb{1}_{\widetilde{W}_{n,j}}(z) \leq 2^{(k+L+1)(2+\frac{p}{2})} \times 2^L \,.$$

But $|z| \geq 1 - 2^{-k+1}$ implies $2^{(k+L+1)(2+\frac{p}{2})} \leq C_p / (1 - |z|)^{2+\frac{p}{2}}$; hence,

$$\sum_{n,j} [2^n \, m_\varphi(W_{n,j})]^{p/2} \lesssim \int_{\mathbb{D}} \frac{[N_\varphi(z)]^{p/2}}{(1-|z|)^{\frac{p}{2}+2}} \, dA(z) < \infty \,,$$

and b) holds.

It remains to show (6.1).

By Theorem 4.1 of [11], we can find a window W with the same center as $W_{n,j}$, but with greater size ch ($h = 2^{-n}$ is the size of the window $W_{n,j}$), such that

$$m_\varphi(W_{n,j}) \lesssim \sup_{w \in W} N_\varphi(w).$$

There is hence some $w_0 \in W$ such that:

$$m_\varphi(W_{n,j}) \lesssim N_\varphi(w_0).$$

Take $R = |w_0| + ch$ (one has $R \geq 1$ since $w_0 \in W$ and W has size ch) and set $\varphi_0(z) = \varphi(z)/R$. One has $N_{\varphi_0}(z) = N_\varphi(Rz)$ for $|z| < 1/R$ and $N_{\varphi_0}(z) = 0$ if $|z| \geq 1/R$.

Let now u be the upper subharmonic regularization of N_{φ_0} (see Lemma 1 and its proof on page 1140 in [13]): u is a subharmonic function on $\mathbb{D} \setminus \{0\}$ such that $u \geq N_{\varphi_0}$ and $u = N_{\varphi_0}$ almost everywhere, with respect to dA.

A result of C. Fefferman and E. M. Stein (Lemma 2 of [5]), generously attributed by them to Hardy and Littlewood, asserts that for any $q > 0$, there exists a constant $C = C(q)$ such that

$$(6.2) \qquad [u(a)]^q \leq \frac{C}{A\big(D(a,r)\big)} \int_{D(a,r)} [u(z)]^q \, dA(z)$$

for every nonnegative subharmonic function u on a domain G and every disk $D(a,r) \subseteq G$ (see also Lemma 3 in [13]).

If Δ is the disk centered at w_0/R and of radius $1 - |w_0|/R$ (which is contained in $\mathbb{D} \setminus \{0\}$ since $R > |w_0|$), one has, by (6.2),

$$
\begin{aligned}
[N_\varphi(w_0)]^{p/2} = [N_{\varphi_0}(w_0/R)]^{p/2} &\leq [u(w_0/R)]^{p/2} \\
&\leq \frac{C}{A(\Delta)} \int_\Delta [u(z)]^{p/2} \, dA(z) \\
&= \frac{C}{A(\Delta)} \int_\Delta [N_{\varphi_0}(z)]^{p/2} \, dA(z) \\
&= \frac{C}{A(\Delta)} \int_{\Delta \cap D(0,1/R)} [N_\varphi(Rz)]^{p/2} \, dA(z) \\
&= \frac{C}{A(\tilde\Delta)} \int_{\tilde\Delta \cap \mathbb{D}} [N_\varphi(w)]^{p/2} \, dA(w) \,,
\end{aligned}
$$

where

$$\tilde\Delta = D(w_0, R - |w_0|) = D(w_0, ch).$$

Since the center w_0 of $\tilde\Delta$ is in $\mathbb{D}$, $\tilde\Delta \cap \mathbb{D}$ contains more than a quarter of $\tilde\Delta$ (at least for $ch \leq 1$), and hence

$$A(\tilde\Delta \cap \mathbb{D}) \geq A(\tilde\Delta)/4 = c^2 h^2/4\pi.$$

Now, let $\tilde W_{n,j}$ be the window with the same center as $W_{n,j}$ and of size $2ch$. Since $2ch \geq ch + (1 - |w_0|)$, $\tilde W_{n,j}$ contains $\tilde\Delta \cap \mathbb{D}$ and $A(\tilde W_{n,j}) \approx h^2 \approx A(\tilde\Delta)$ ($\approx$ meaning that the ratio is between two absolute constants). We therefore get

$$[N_\varphi(w_0)]^{p/2} \lesssim \frac{1}{A(\tilde W_{n,j})} \int_{\tilde W_{n,j}} [N_\varphi(w)]^{p/2} \, dA(w) \,,$$

proving (6.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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