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# Semi-Riemannian manifolds with a doubly warped structure

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**Abstract.** We investigate manifolds obtained as a quotient of a doubly warped product. We show that they are always covered by the product of two suitable leaves. This allows us to prove, under regularity hypothesis, that these manifolds are a doubly warped product up to a zero measure subset formed by an union of leaves. We also obtain a necessary and sufficient condition which ensures the decomposition of the whole manifold and use it to give sufficient conditions of geometrical nature. Finally, we study the uniqueness of direct product decomposition in the nonsimply connected case.

## 1. Introduction

Let $(M_i, g_i)$ be two semi-Riemannian manifolds and $\lambda_i : M_1 \times M_2 \to \mathbb{R}^+$ two positive functions, $i = 1, 2$. The doubly twisted product $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ is the manifold $M_1 \times M_2$ furnished with the metric

$$g = \lambda_1^2 \, g_1 + \lambda_2^2 \, g_2 \, .$$

If $\lambda_1 \equiv 1$ or $\lambda_2 \equiv 1$, then it is called a twisted product. On the other hand, when $\lambda_1$ only depends on the second factor and $\lambda_2$ on the first one, it is called a doubly warped product (warped product if $\lambda_1 \equiv 1$ or $\lambda_2 \equiv 1$). The most simple metric, the direct product, corresponds to the case $\lambda_1 \equiv \lambda_2 \equiv 1$.

Using the language of foliations, the classical De Rham–Wu Theorem says that two orthogonally, complementary and geodesic foliations (called a direct product structure) in a complete and simply connected semi-Riemannian manifold give rise to a global decomposition as a direct product of two leaves. This theorem can be generalized in two ways: one way is obtaining more general decompositions than direct products and the second one is removing the simply connectedness

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hypothesis. The most general theorems in the first direction were obtained in [15] and [20], where the authors showed that geometrical properties of the foliations determine the type of decomposition.

In the second direction, P. Wang obtained that a complete semi-Riemannian manifold furnished with a direct product structure is covered by the direct product of two leaves, [25]. Moreover, if the manifold is Riemannian, using the theory of bundle-like metrics, he showed that a necessary and sufficient condition to obtain the global decomposition as a direct product is the existence of two regular leaves which intersect each other at only one point. There are other remarkable works avoiding the simply connectedness but they treat with codimension one foliations, see [10] and references therein.

In this paper we study semi-Riemannian manifolds furnished with a *doubly warped structure*, that is, two complementary, orthogonal and umbilic foliations with closed mean curvature vector fields. If one of the mean curvature vector fields is identically null, then it is called a *warped structure*.

Doubly warped and warped structures appear in different situations. For example, Codazzi tensors with exactly two eigenvalues, one of them constant [5], Killing tensors [9], semi-Riemann submersions with umbilic fibres and some additional hypotheses [8] and certain group actions [1], [16] lead to (doubly) warped structures. On the other hand, the translation of physical content into geometrical language gives rise to warped structures which, by a topological simplification, are supposed global products. In this way are constructed important spaces as Robertson–Walker, Schwarzschild, Kruskal, static spaces…

Manifolds with a doubly warped structure are locally a doubly warped product and, under completeness hypothesis, they are a quotient of a global doubly warped product. This is why, after the preliminaries of section 2, we focus our attention on studying these quotients.

The main tool in this paper is Theorem 3.4, which gives a normal semi-Riemannian covering map with a doubly warped product of two leaves as domain. We use it to obtain a necessary and sufficient condition for a semi-Riemannian manifold with a doubly warped structure to be a global product, which extends the one given in [25] for direct products structures in the Riemannian setting. Other consequence of Theorem 3.4 is that any leaf is covered by a leaf without holonomy of the same foliation.

We study the space of leaves obtaining that, under regularity hypothesis, a manifold with a doubly warped structure is a fiber bundle over the space of leaves. This allows us to compute the fundamental group of the space of leaves and to show that there is an open dense subset which is isometric to a doubly warped product. We also give a result involving the curvature that ensures the global decomposition and apply it to semi-Riemannian submersion with umbilic fibres.

As a consequence of the De Rham–Wu Theorem, it can be ensured the uniqueness of the direct product decomposition of a simply connected manifold under nondegeneracy hypothesis. In the last section, we apply the decomposition results obtained to investigate the uniqueness of the decomposition without the simply connectedness assumption.

## 2. Preliminaries

Given a product manifold $M_1 \times M_2$ and $X \in \mathfrak{X}(M_i)$, we will also denote $X$ to its elevation to $\mathfrak{X}(M_1 \times M_2)$ and $P_i : TM_1 \times TM_2 \to TM_i$ will be the canonical projection. Unless it is explicitly said, all manifolds are supposed to be semi-Riemannian. We write some formulaes about Levi-Civita connection and curvature, which are established in [20].

**Lemma 2.1.** *Let $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ be a doubly twisted product and call $\nabla^i$ the Levi-Civita connection of $(M_i, g_i)$. Given $X, Y \in \mathfrak{X}(M_1)$ and $V, W \in \mathfrak{X}(M_2)$ it holds*

1. $\nabla_X Y = \nabla^1_X Y - g(X,Y)\nabla \ln \lambda_1 + g(X, \nabla \ln \lambda_1)Y + g(Y, \nabla \ln \lambda_1)X.$
2. $\nabla_V W = \nabla^2_V W - g(V,W)\nabla \ln \lambda_2 + g(V, \nabla \ln \lambda_2)W + g(W, \nabla \ln \lambda_2)V.$
3. $\nabla_X V = \nabla_V X = g(\nabla \ln \lambda_1, V)X + g(\nabla \ln \lambda_2, X)V.$

It follows that canonical foliations are umbilic and the mean curvature vector field of the first canonical foliation is $N_1 = P_2(-\nabla \ln \lambda_1)$ whereas that of the second is $N_2 = P_1(-\nabla \ln \lambda_2)$.

**Lemma 2.2.** *Let $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ be a doubly twisted product and take $\Pi_1 = \mathrm{span}(X,Y)$, $\Pi_2 = \mathrm{span}(V,W)$ and $\Pi_3 = \mathrm{span}(X,V)$ nondegenerate planes where $X, Y \in TM_1$ and $V, W \in TM_2$ are unitary and orthogonal vectors. Then the sectional curvature is given by*

1. $K(\Pi_1) = \frac{K^1(\Pi_1)+g(\nabla\lambda_1,\nabla\lambda_1)}{\lambda_1^2} - \frac{1}{\lambda_1}\Big(\varepsilon_X g(h_1(X),X) + \varepsilon_Y g(h_1(Y),Y)\Big).$
2. $K(\Pi_2) = \frac{K^2(\Pi_2)+g(\nabla\lambda_2,\nabla\lambda_2)}{\lambda_2^2} - \frac{1}{\lambda_2}\Big(\varepsilon_V g(h_2(V),V) + \varepsilon_W g(h_2(W),W)\Big).$
3. $K(\Pi_3) = -\frac{\varepsilon_V}{\lambda_1} g(h_1(V),V) - \frac{\varepsilon_X}{\lambda_2} g(h_2(X),X) + \frac{g(\nabla\lambda_1,\nabla\lambda_2)}{\lambda_1\lambda_2},$

*where $K^i$ is the sectional curvature of $(M_i, g_i)$, $h_i$ is the hessian endomorphism of $\lambda_i$ and $\varepsilon_Z$ is the sign of $g(Z,Z)$.*

A vector field is called closed if its metrically equivalent one form is closed.

**Lemma 2.3.** *Let $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ be a doubly twisted product. It is a doubly warped product if and only if $N_i = P_{3-i}(-\nabla \ln \lambda_i)$ is closed, for $i = 1, 2$.*

*Proof.* Suppose $d\omega_1 = 0$, where $\omega_1$ is the equivalent one form to $N_1$. If $X \in \mathfrak{X}(M_1)$ and $V \in \mathfrak{X}(M_2)$, then $XV(\ln \lambda_1) = -X\omega_1(V) = -d\omega_1(X,V) = 0$. Thus there are functions $f_1 \in C^\infty(M_1)$ and $h_1 \in C^\infty(M_2)$ such that $\lambda_1(x,y) = f_1(x)h_1(y)$ for all $(x,y) \in M_1 \times M_2$. Analogously, $\lambda_2(x,y) = f_2(x)h_2(y)$ for certain functions $f_2 \in C^\infty(M_1)$ and $h_2 \in C^\infty(M_2)$. Hence, taking conformal metrics if necessary, $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ can be expressed as a doubly warped product. The *only if* part is trivial. □

We want to generalize the concept of doubly twisted or doubly warped product to manifold which are not necessarily a topological product.

**Definition 2.4.** Two complementary, orthogonal and umbilic foliations $(\mathcal{F}_1, \mathcal{F}_2)$ in a semi-Riemannian manifold is called a *doubly twisted structure*. Moreover, if the mean curvature vectors of the foliations are closed, then it is called a *doubly warped structure*. Finally, we say that it is a *warped structure* if one mean curvature vector is closed and the other one is zero.

Notice that this last case is equivalent to one of the foliations being totally geodesic and the other one spherical, see [20] for the definition.

We call $N_i$ the mean curvature vector field of $\mathcal{F}_i$ and $\omega_i$, which we call *mean curvature form*, to its metrically equivalent one-form. The leaf of $\mathcal{F}_i$ through $x \in M$ is denoted by $F_i(x)$ and $\mathcal{F}_i(x)$ will be the tangent plane of $F_i(x)$ at the point $x$. If there is not confusion or if the point is not relevant, we simply write $F_i$. We always put the induced metric on the leaves.

**Remark 2.5.** If $M$ has a doubly twisted (warped) structure, then we can take around any point an adapted chart to both foliations. Lemma 2.3 and Proposition 3 of [20] show that M is locally isometric to a doubly twisted (warped) product. In the doubly warped structure case, the condition on the mean curvature vectors in Theorem 5.4 of [15] can be easily checked. So, if the leaves of one of the foliations are complete we can apply this theorem to obtain that $M$ is a quotient of a global doubly warped product.

Given a curve $\alpha : [0,1] \to M$ we call $\alpha_t : [0,t] \to M$, $0 \leq t \leq 1$, its restriction.

**Definition 2.6.** Let $M$ be a semi-Riemannian manifold with $\mathcal{F}_1$ and $\mathcal{F}_2$ two orthogonal and complementary foliations. Take $x \in M$, $v \in \mathcal{F}_2(x)$ and $\alpha : [0,1] \to F_1(x)$ a curve with $\alpha(0) = x$. We define the adapted translation of $v$ along $\alpha_t$ as $A_{\alpha_t}(v) = \exp\big(-\int_{\alpha_t} \omega_2\big) W(t)$, where $W$ is the normal parallel translation to $\mathcal{F}_1$ of $v$ along $\alpha$, [17].

In the same way we can define the adapted translation of a vector of $\mathcal{F}_1(x)$ along a curve in $F_2(x)$. Observe that $|A_{\alpha_t}(v)| = |v| \exp\big(-\int_{\alpha_t} \omega_2\big)$.

**Lemma 2.7.** *Let $M = M_1 \times M_2$ be a semi-Riemannian manifold such that the canonical foliations constitute a doubly twisted structure. Take $\alpha : [0,1] \to M_1$ a curve with $\alpha(0) = a$ and $v_b \in T_b M_2$. The adapted translation of $(0_a, v_b)$ along the curve $\gamma(t) = (\alpha(t), b)$ is $A_{\gamma_t}(0_a, v_b) = (0_{\alpha(t)}, v_b)$.*

*Proof.* First we show that formula 3 of Lemma 2.1 is still true in this case. In fact, take $X, Y \in \mathfrak{X}(M_1)$ and $V, W \in \mathfrak{X}(M_2)$. Since $[X, V] = 0$ we have

$$g(\nabla_X V, W) = -g(X, \nabla_V W) = -g(V, W) g(X, N_2),$$

and analogously $g(\nabla_X V, Y) = -g(V, N_1) g(X, Y)$. Therefore, it follows that $\nabla_X V = -\omega_1(V) X - \omega_2(X) V$ for all $X \in \mathfrak{X}(M_1)$ and $V \in \mathfrak{X}(M_2)$.

Take $V(t) = (0_{\alpha(t)}, v_b)$ and $W(t) = \lambda(t) V(t)$, where $\lambda(t) = \exp\big(\int_{\gamma_t} \omega_2\big)$. We only have to check that $W(t)$ is the normal parallel translation of $(0_a, v_b)$ along $\gamma$.

But this is an immediate consequence of

$$\frac{DW}{dt} = \lambda' V + \lambda(-\omega_1(V)\,\gamma' - \omega_2(\gamma')V) = -\lambda\,\omega_1(V)\,\gamma'.$$

□

Given a foliation $\mathcal{F}$ we call $\mathrm{Hol}(F)$ the holonomy group of a leaf $F$ (see [3] for definitions and properties). We say that $F$ has no holonomy if its holonomy group is trivial. The foliation $\mathcal{F}$ has no holonomy if any leaf has no holonomy.

**Lemma 2.8.** *Let $M$ be a semi-Riemannian manifold with $(\mathcal{F}_1, \mathcal{F}_2)$ a doubly twisted structure. Take $x \in M$ and $\alpha : [0,1] \to F_1(x)$ a loop at $x$. If $f \in \mathrm{Hol}(F_1(x))$ is the holonomy map associated to $\alpha$, then $f_{*_x}(v) = A_\alpha(v)$ for all $v \in \mathcal{F}_2(x)$.*

*Proof.* It is sufficient to show it locally. Take an open set of $x$ isometric to a doubly twisted product $U_1 \times_{(\lambda_1,\lambda_2)} U_2$ where $U_i$ is an open set of $F_i(x)$ with $x \in U_i$. If $\alpha(t) \in U_1$, for $0 \le t \le t_0$, then the holonomy map associated to this arc is $f : \{x\} \times U_2 \to \{\alpha(t_0)\} \times U_2$ given by $f(x,y) = (\alpha(t_0), y)$ and clearly $f_{*_x}(0_x, v_x) = (0_{\alpha(t_0)}, v_x) = A_{\alpha_{t_0}}(v_x)$. □

Observe that, in the doubly warped structure case, holonomy maps are homotheties and thus they are determined by their derivative at a point. Therefore, a leaf $F_1$ has no holonomy if and only if for any loop $\alpha$ in $F_1$ it holds $A_\alpha = \mathrm{id}$.

We finish this section relating two doubly twisted structure via a local isometry.

**Lemma 2.9.** *Let $\overline{M}$ and $M$ be two semi-Riemannian manifold with a doubly twisted structure $(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ and $(\mathcal{F}_1, \mathcal{F}_2)$ respectively. Take $f : \overline{M} \to M$ a local isometry which preserves the doubly twisted structure, that is, $f_{*_x}(\overline{\mathcal{F}}_i(x)) = \mathcal{F}_i(f(x))$ for all $x \in \overline{M}$ and $i = 1, 2$. Then $f$ also preserves*

1. *the leaves, $f(\overline{F}_i(x)) \subset F_i(f(x))$;*
2. *the mean curvature vector fields, $f_{*_x}(\overline{N}_i(x)) = N_i(f(x))$;*
3. *the mean curvature forms, $f^*(\omega_i) = \overline{\omega}_i$ ;*
4. *the adapted translation, that is, if $\gamma : [0,1] \to \overline{F}_1(x)$ is a curve with $\gamma(0) = x$ and $v \in \overline{\mathcal{F}}_2(x)$, then $A_{f\circ\gamma_t}(f_{*_x}(v)) = f_{*_{\gamma(t)}}(A_{\gamma_t}(v))$ and analogously for a curve in $\overline{F}_2(x)$.*

*Proof.* (1) It is immediate.

(2) Take $\overline{X}, \overline{Y} \in \overline{\mathcal{F}}_1$ and call $X = f_*(\overline{X})$ and $Y = f_*(\overline{Y})$ in a suitable open set. Since $f$ is a local isometry, $g(X,Y)N_1 = P_2(\nabla_X Y) = f_*(\overline{P}_2(\overline{\nabla}_{\overline{X}}\overline{Y})) = g(X,Y) f_*(\overline{N}_1)$. Hence $N_1 = f_*(\overline{N}_1)$ and analogously for $\overline{N}_2$.

(3) Immediate from point (2).

(4) Take $\overline{W}(t)$ the parallel translation of $v$ normal to $\overline{\mathcal{F}}_1$ along $\gamma$. Since $f$ is a local isometry which preserves the foliations, $f_*(\overline{W}(t))$ is the normal parallel translation of $f_*(v)$ along $f \circ \gamma$. Using point (3), $\int_{\gamma_t} \overline{\omega}_2 = \int_{f\circ\gamma_t} \omega_2$ and therefore $A_{f\circ\gamma_t}(f_{*_x}(v)) = f_{*_{\gamma(t)}}(A_{\gamma_t}(v))$. □

## 3. Quotient of a doubly warped product

From now on, $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ will be a doubly warped product and $\Gamma$ a group of isometries such that:

1. $\Gamma$ acts in a properly discontinuous manner.
2. It preserves the canonical foliations. This implies that if $f \in \Gamma$, then $f = \phi \times \psi : M_1 \times M_2 \to M_1 \times M_2$, where $\phi : M_1 \to M_1$ and $\psi : M_2 \to M_2$ are homotheties with factor $c_1^2$ and $c_2^2$ respectively, such that $\lambda_1 \circ \psi = \frac{1}{c_1}\lambda_1$ and $\lambda_2 \circ \phi = \frac{1}{c_2}\lambda_2$.

The semi-Riemannian manifold $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ has a doubly warped structure, which, as always, we call $(\mathcal{F}_1, \mathcal{F}_2)$. We are going to work with $\mathcal{F}_1$ because all definitions and results are stated analogously for $\mathcal{F}_2$.

If we take the canonical projection $p : M_1 \times M_2 \to M$, which is a semi-Riemannian covering map, applying Lemma 2.9 we have $p(M_1 \times \{b\}) \subset F_1(p(a,b))$ for all $(a,b) \in M_1 \times M_2$. We call $p_1^{(a,b)} : M_1 \times \{b\} \to F_1(p(a,b))$ the restriction of $p$.

**Lemma 3.1.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. We take $p\colon M_1 \times M_2 \to M$ the canonical projection, $(a,b) \in M_1 \times M_2$ and $x = p(a,b)$. Then, the restriction $p_1^{(a,b)} : M_1 \times \{b\} \to F_1(x)$ is a normal semi-Riemannian covering map.*

*Proof.* It is clear that $p_1^{(a,b)} : M_1 \times \{b\} \to F_1(x)$ is a local isometry. Let $\gamma : [0,1] \to F_1(x)$ be a curve with $\gamma(0) = x$. Since $p : M_1 \times M_2 \to M$ is a covering map, there is a lift $\alpha : [0,1] \to M_1 \times M_2$ with $\alpha(0) = (a,b)$. But $p_*(\alpha'(t)) = \gamma'(t) \in \mathcal{F}_1(\gamma(t))$ and $p$ preserves the foliations, so $\alpha(t)$ is a curve in $M_1 \times \{b\}$. Applying Theorem 28, page 201, of [17], we get that $p_1^{(a,b)}$ is a covering map. Now we show that it is normal. Take $a' \in M_1$ such that $p_1^{(a,b)}(a',b) = x$. Then, there exists $f \in \Gamma$ with $f(a',b) = (a,b)$ and since $f$ preserves the canonical foliations, $f(M_1 \times \{b\}) = M_1 \times \{b\}$. So, the restriction of $f$ to $M_1 \times \{b\}$ is a deck transformation of the covering $p_1^{(a,b)}$ which sends $(a',b)$ to $(a,b)$. □

Let $\Gamma_1^{(a,b)}$ be the group of deck transformations of $p_1^{(a,b)} : M_1 \times \{b\} \to F_1(p(a,b))$. When there is not confusion with the chosen point, we simply write $p_1$ and $\Gamma_1$ instead of $p_1^{(a,b)}$ and $\Gamma_1^{(a,b)}$.

If $\phi \in \Gamma_1$, in general, it does not exist $\psi \in \Gamma_2$ with $\phi \times \psi \in \Gamma$ and it does not have to hold that $\lambda_2 \circ \phi = \frac{1}{c_2}\lambda_2$, as it was said at the beginning of this section.

**Lemma 3.2.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. Fix $(a_0,b_0) \in M_1 \times M_2$ and $x_0 = p(a_0,b_0)$ such that the leaf $F_1(x_0)$ has no holonomy. Then*

(1) $\lambda_2 \circ \phi = \lambda_2$ *for all* $\phi \in \Gamma_1^{(a_0,b_0)}$.

(2) $\phi \times \mathrm{id} \in \Gamma$ *for all* $\phi \in \Gamma_1^{(a_0,b_0)}$ *and* $\Gamma_1^{(a_0,b_0)} \times \{\mathrm{id}\}$ *is a normal subgroup of* $\Gamma$.

*Proof.* (1) The mean curvature forms $\overline{\omega}_2$ and $\omega_2$ of the foliations in $M_1 \times M_2$ and $M$ respectively are closed. Thus every point in $M$ has an open neighborhood where $\omega_2 = df_2$ for certain function $f_2$, and analogously every point in $M_1 \times M_2$ has an open neighborhood where $\overline{\omega}_2 = d\overline{f}_2$, being in this case $\overline{f}_2 = -\ln \lambda_2$. Since $p_1 \circ \phi = p_1$, we have $\phi^*(p^*(\omega_2)) = p^*(\omega_2)$ and therefore $f_2 \circ p \circ \phi = f_2 \circ p + k_1$ for certain constant $k_1$. On the other hand, using Lemma 2.9, we have $f_2 \circ p = -\ln \lambda_2 + k_2$ for some constant $k_2$.

Joining the last two equations, we get $\lambda_2 \circ \phi = c\lambda_2$ for certain constant $c$. This formula must be true in the whole $M_1$ with the same constant $c$ and, in fact, $c = \frac{\lambda_2(a_1)}{\lambda_2(a_0)}$ where $a_1 = \phi(a_0)$. Take $\alpha : [0,1] \to M_1 \times \{b_0\}$ with $\alpha(0) = (a_0, b_0)$, $\alpha(1) = (a_1, b_0)$ and $w \in T_{b_0}M_2$ a non lightlike vector. Using Lemma 2.7 we have $A_\alpha(0_{a_0}, w) = (0_{a_1}, w)$ and since $p$ preserves the adapted translation,

$$A_{p\circ\alpha}\big(p_{*(a_0,b_0)}(0_{a_0}, w)\big) = p_{*(a_1,b_0)}(0_{a_1}, w).$$

Using that $F_1(x_0)$ has no holonomy and Lemma 2.8, we obtain

$$p_{*(a_1,b_0)}(0_{a_1}, w) = p_{*(a_0,b_0)}(0_{a_0}, w),$$

taking norms, we get $c = 1$.

(2) Take $\phi \in \Gamma_1$. Since $\lambda_2 \circ \phi = \lambda_2$, it follows that $\phi \times \mathrm{id}$ is an isometry of $M_1 \times_{(\lambda_1,\lambda_2)} M_2$. Now, to show that $p \circ (\phi \times \mathrm{id}) = p$ it is enough to check $(p \circ \phi \times \mathrm{id})_{*(a_0,b_0)} = p_{*(a_0,b_0)}$. Using $p_1 \circ \phi = p_1$, we see that

$$(p \circ (\phi \times \mathrm{id}))_{*(a_0,b_0)}(v, 0_{b_0}) = p_{*(a_0,b_0)}(v, 0_{b_0})$$

for all $v \in T_{a_0}M_1$. Given $w \in T_{b_0}M_2$, we take $\alpha : [0,1] \to M_1 \times \{b_0\}$ a curve from $(a_0, b_0)$ to $(a_1, b_0)$, where $\phi(a_0) = a_1$. Then $A_\alpha(0_{a_0}, w) = (0_{a_1}, w)$ and

$$\begin{aligned}(p \circ (\phi \times \mathrm{id}))_{*(a_0,b_0)}(0_{a_0}, w) &= p_{*(a_1,b_0)}(0_{a_1}, w) = p_{*(a_1,b_0)}\big(A_\alpha(0_{a_0}, w)\big) \\ &= A_{p\circ\alpha}\big(p_{*(a_0,b_0)}(0_{a_0}, w)\big) = p_{*(a_0,b_0)}(0_{a_0}, w),\end{aligned}$$

where the last equality holds because $F_1(x_0)$ has no holonomy. Therefore $\phi \times \mathrm{id} \in \Gamma$ and $\Gamma_1 \times \{\mathrm{id}\}$ is a subgroup of $\Gamma$.

Now to prove that it is a normal subgroup, we take $\phi \in \Gamma_1$ and $f \in \Gamma$ and show that $f^{-1} \circ (\phi \times \mathrm{id}) \circ f \in \Gamma_1$. Since $f$ preserves the foliations, $f^{-1} \circ (\phi \times \mathrm{id}) \circ f$ takes $M_1 \times \{b_0\}$ into $M_1 \times \{b_0\}$ and therefore we can consider $h = f^{-1} \circ (\phi \times \mathrm{id}) \circ f\big|_{M_1\times\{b_0\}} \in \Gamma_1$. But $h \times \mathrm{id}$ coincides with $f^{-1} \circ (\phi \times \mathrm{id}) \circ f$ at $a_0$, thus $f^{-1} \circ (\phi \times \mathrm{id}) \circ f = h \times \mathrm{id} \in \Gamma_1 \times \{\mathrm{id}\}$. □

**Remark 3.3.** Observe that we have used that $F_1(x_0)$ has no holonomy only to ensure $A_{p\circ\alpha} = \mathrm{id}$, thus we have a little bit more general result. Suppose that $\gamma : [0,1] \to F_1(x)$ is a loop at $x \in M$ such that its associated holonomy map is trivial. Take $\alpha : [0,1] \to M_1 \times \{b\}$ a lift through $p_1 : M_1 \times \{b\} \to F_1(x)$ with basepoint $(a, b)$ and suppose $\alpha(1) = (a', b)$. If $\phi \in \Gamma_1$ with $\phi(a) = a'$, then it can be proven, identically as in the above lemma, that for this deck transformation it holds $\lambda_2 \circ \phi = \lambda_2$ and $\phi \times \mathrm{id} \in \Gamma$. This will be used in Theorem 3.10.

Now, we can give the following theorem, which is the main tool of this paper (compare with Theorem 2 of [25] and Theorem 7 of [23]).

**Theorem 3.4.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. Fix $(a_0, b_0) \in M_1 \times M_2$ and $x_0 = p(a_0, b_0)$. The leaf $F_1(x_0)$ has no holonomy if and only if there exists a semi-Riemannian normal covering map $\Phi : F_1(x_0) \times_{(\lambda_1,\rho_2)} M_2 \to M$, where $\rho_2 : F_1(x_0) \to \mathbb{R}^+$. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\;p\;} & M \\ {\scriptstyle p_1^{(a_0,b_0)} \times \mathrm{id}} \big\downarrow & \nearrow_{\Phi} & \\ F_1(x_0) \times M_2 & & \end{array}$$

*In particular, $\Phi(x, b_0) = x$ for all $x \in F_1(x_0)$.*

*Proof.* Suppose that $F_1(x_0)$ has no holonomy. Since $\Gamma_1 \times \{\mathrm{id}\}$ is a normal subgroup of $\Gamma$, there exists a normal covering map

$$\Phi : \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right) / \left(\Gamma_1 \times \{\mathrm{id}\}\right) \to M.$$

But $\left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right) / \left(\Gamma_1 \times \{\mathrm{id}\}\right)$ is isometric to $F_1(x_0) \times_{(\lambda_1,\rho_2)} M_2(x_0)$ for certain function $\rho_2$ with $\rho_2 \circ p_1 = \lambda_2$ and by construction $\Phi \circ (p_1 \times \mathrm{id}) = p$.

Conversely, we suppose the existence of such semi-Riemannian covering. Take $\alpha : [0, 1] \to F_1(x_0)$ a loop at $x_0$, $w \in T_{x_0} F_2(x_0)$ and $v \in T_{b_0} M_2$ with $p_{*(a_0,b_0)}(0, v) = w$. Then

$$\Phi_{*(x_0,b_0)}(0, v) = \Phi_{*(x_0,b_0)}\Big((p_1 \times \mathrm{id})_{*(a_0,b_0)}(0, v)\Big) = p_{*(a_0,b_0)}(0, v) = w.$$

Now, using Lemmas 2.7 and 2.9, and that the holonomy in $F_1(x_0) \times_{(\lambda_1,\rho_2)} M_2$ is trivial,

$$A_\alpha(w) = \Phi_{*(x_0,b_0)}\left(A_{(\alpha,b_0)}(0, v)\right) = \Phi_{*(x_0,b_0)}(0, v) = w$$

and therefore $F_1(x_0)$ has no holonomy. □

We say that $x_0 \in M$ has no holonomy if $F_1(x_0)$ and $F_2(x_0)$ have no holonomy.

**Corollary 3.5.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. Fix $(a_0, b_0) \in M_1 \times M_2$ and $x_0 = p(a_0, b_0)$. The point $x_0$ has no holonomy if and only if there is a semi-Riemannian normal covering map*

$$\Phi : F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0) \to M,$$

*where $\rho_1 : F_2(x_0) \to \mathbb{R}^+$ and $\rho_2 : F_1(x_0) \to \mathbb{R}^+$. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\;p\;} & M \\ {\scriptstyle p_1^{(a_0,b_0)} \times p_2^{(a_0,b_0)}} \big\downarrow & \nearrow_{\Phi} & \\ F_1(x_0) \times F_2(x_0) & & \end{array}$$

*In particular, $\Phi(x, x_0) = x$ and $\Phi(x_0, y) = y$ for all $x \in F_1(x_0)$ and $y \in F_2(x_0)$.*

**Remark 3.6.** It is known ([6]) that the set of leaves without holonomy is dense on $M$. Thus, we can always take a point $x_0 \in M$ without holonomy and apply Corollary 3.5.

**Theorem 3.7.** *Let* $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ *be a quotient of a doubly warped product. If* $x_0 \in M$ *has no holonomy then*

$$\mathrm{card}(F_1(x_0) \cap F_2(x_0)) = \mathrm{card}(\Phi^{-1}(x_0)),$$

*where* $\Phi : F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0) \to M$ *is the semi-Riemannian covering map of Corollary* 3.5.

*Proof.* Recall that, by construction, $\Phi(x, y) = p(a, b)$ where $a \in M_1$ with $p(a, b_0) = x$ and $b \in M_2$ with $p(a_0, b) = y$. Now we define $\Lambda : \Phi^{-1}(x_0) \to F_1(x_0) \cap F_2(x_0)$ by $\Lambda(x, y) = x$. First we show that $\Lambda$ is well defined. If $(x, y) \in \Phi^{-1}(x_0)$ then $p(a, b) = x_0$, where $p(a, b_0) = x$ and $p(a_0, b) = y$. Hence $p(\{a\} \times M_2) \subset F_2(p(a, b)) = F_2(x_0)$ and $p(M_1 \times \{b_0\}) \subset F_1(p(a_0, b_0)) = F_1(x_0)$, thus

$$x = p(a, b_0) = p(M_1 \times \{b_0\} \cap \{a\} \times M_2) \in F_1(x_0) \cap F_2(x_0).$$

Now we check that $\Lambda$ is onto. Take $x \in F_1(x_0) \cap F_2(x_0)$. Since $p_1 : M_1 \times \{b_0\} \to F_1(x_0)$ is a covering map there exists $a \in M_1$ such that $p(a, b_0) = x$. But $p_2^{(a,b_0)} : \{a\} \times M_2 \to F_2(x) = F_2(x_0)$ is a covering map too, therefore there is $(a, b) \in \{a\} \times M_2$ such that $p(a, b) = x_0$. If we call $y = p(a_0, b)$, then $\Phi(x, y) = x_0$ and $\Lambda(x, y) = x$.

Finally, we show that $\Lambda$ is injective. Take $(x, y), (x, y') \in \Phi^{-1}(x_0)$ and $a \in M_1$, $b, b' \in M_2$ such that $p(a, b_0) = x$, $p(a_0, b) = y$ and $p(a_0, b') = y'$. Consider the covering $p_2^{(a,b_0)} : \{a\} \times M_2 \to F_2(x_0)$. Since $p_2^{(a,b_0)}(a, b) = p_2^{(a,b_0)}(a, b')$ and this covering is normal, there exists a deck transformation $\psi \in \Gamma_2^{(a,b_0)}$ such that $\psi(a, b) = (a, b')$. But $F_2(x_0)$ has no holonomy, so Lemma 3.2 assures that $\mathrm{id} \times \psi \in \Gamma$. Now, $(\mathrm{id} \times \psi)(a_0, b) = (a_0, b')$ and thus $y = y'$. □

Now we give a necessary and sufficient condition for a doubly warped structure to be a global doubly warped product, which extends the one given in [25] for direct products and Riemannian manifolds.

**Corollary 3.8.** *Let* $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ *be a quotient of a doubly warped product and* $x_0 \in M$. *Then* $M$ *is isometric to the doubly warped product* $F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0)$ *if and only if* $x_0$ *has no holonomy and* $F_1(x_0) \cap F_2(x_0) = \{x_0\}$.

Condition $F_1(x_0) \cap F_2(x_0) = \{x_0\}$ alone is not sufficient to split $M$ as a product $F_1(x_0) \times F_2(x_0)$, as intuition perhaps suggests. The Möbius trip illustrates this point.

**Theorem 3.9.** *Let* $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ *be a quotient of a doubly warped product. If* $x_0$ *has no holonomy then*

$$\mathrm{card}(F_1(x) \cap F_2(x)) \leq \mathrm{card}(F_1(x_0) \cap F_2(x_0))$$

*for all* $x \in M$.

*Proof.* Take $(a_0,b_0)\in M_1\times M_2$ such that $p(a_0,b_0)=x_0$. First suppose that $x\in F_1(x_0)$ and $F_1(x)\cap F_2(x)=\{x_i\colon i\in I\}$. If we take $a\in M_1$ such that $p(a,b_0)=x$, then we know that $p_2^{(a,b_0)}\colon\{a\}\times M_2\to F_2(x)$ is a covering map, so we can take $b_i\in M_2$ with $p_2^{(a,b_0)}(a,b_i)=x_i$. If we call $y_i=p_2^{(a_0,b_0)}(a_0,b_i)$, then both $x_i,y_i\in p(M_1\times\{b_i\})=F_1(p(a,b_i))=F_1(x_0)$, and moreover, since $y_i\in F_2(x_0)$ we have $y_i\in F_1(x_0)\cap F_2(x_0)$.

Now, we show that the map $\Lambda:F_1(x)\cap F_2(x)\to F_1(x_0)\cap F_2(x_0)$ given by $\Lambda(x_i)=y_i$ is injective. If $y_i=p_2^{(a_0,b_0)}(a_0,b_i)=p_2^{(a_0,b_0)}(a_0,b_j)=y_j$ for $i\neq j$ then there is $\psi\in\Gamma_2^{(a_0,b_0)}$ such that $\psi(a_0,b_i)=(a_0,b_j)$. Since $F_2(x_0)$ has no holonomy (Lemma 3.2), $\mathrm{id}\times\psi\in\Gamma$ and it sends $(a,b_i)$ to $(a,b_j)$. Therefore $x_i=x_j$. This shows that $\mathrm{card}(F_1(x)\cap F_2(x))\leq\mathrm{card}(F_1(x_0)\cap F_2(x_0))$ when $x\in F_1(x_0)$.

Take now an arbitrary point $x\in M$ and $(a,b)\in M_1\times M_2$ with $p(a,b)=x$. We have that $F_2(x)$ intersects $F_1(x_0)$ at some point $z=p(a,b_0)$. In the same way as above, using that $F_1(x_0)$ has no holonomy, we can show that $\mathrm{card}(F_1(x)\cap F_2(x))\leq\mathrm{card}(F_1(z)\cap F_2(z))$, but we have already proven that $\mathrm{card}(F_1(z)\cap F_2(z))\leq\mathrm{card}(F_1(x_0)\cap F_2(x_0))$. □

Take $x_0=p(a_0,b_0)\in M$ such that $F_1(x_0)$ has no holonomy and let

$$\Phi:F_1(x_0)\times_{(\lambda_1,\rho_2)}M_2\to M$$

be the semi-Riemannian covering map constructed in Theorem 3.4, which has $\Omega=\Gamma/\big(\Gamma_1^{(a_0,b_0)}\times\{\mathrm{id}\}\big)$ as deck transformation group. Take $x\in F_2(x_0)$ and a point $b\in M_2$ with $\Phi(x_0,b)=x$. Applying Lemma 3.1,

$$\Phi_1^{(x_0,b)}:F_1(x_0)\times\{b\}\to F_1(x),$$

the restriction of $\Phi$, is a normal semi-Riemannian covering map. Call $\Omega_1^{(x_0,b)}$ its deck transformations group.

**Theorem 3.10.** *In the above situation, the following sequence is exact*

$$0\longrightarrow\pi_1(F_1(x_0),x_0)\xrightarrow{\Phi_{1\#}^{(x_0,b)}}\pi_1(F_1(x),x)\xrightarrow{H}\mathrm{Hol}\,(F_1(x))\longrightarrow 0,$$

*where* $H:\pi_1(F_1(x),x)\longrightarrow\mathrm{Hol}\,(F_1(x))$ *is the usual holonomy homomorphism. In particular we have* $\Omega_1^{(x_0,b)}=\mathrm{Hol}\,(F_1(x))$.

*Proof.* It is clear that $\Phi_{1\#}$ is injective and $H$ is onto, so we only prove that $\mathrm{Ker}\,H=\mathrm{Im}\,\Phi_{1\#}$.

Take $[\gamma]\in\pi_1(F_1(x),x)$ such that $H([\gamma])=1$, i.e., $f_\gamma=\mathrm{id}$, where $f_\gamma$ is the associated holonomy map. Take $\alpha$ a lift of $\gamma$ in $F_1(x_0)\times\{b\}$ with basepoint $(x_0,b)$ and $\phi\in\Omega_1^{(x_0,b)}$ such that $\phi(x_0,b)=\alpha(1)$. Since $f_\gamma=\mathrm{id}$ it follows that $\rho_2\circ\phi=\rho_2$ and $\phi\times\mathrm{id}\in\Omega$, see Remark 3.3.

Therefore, taking into account that $\Phi(x,b_0)=x$ for all $x\in F_1(x_0)$, we get $x_0=\Phi_1(x_0,b_0)=\Phi_1(\phi(x_0),b_0)=\phi(x_0)$. Hence $\alpha$ is a loop at $x_0$ which holds $\Phi_{1\#}([\alpha])=[\gamma]$. This shows that $\mathrm{Ker}\,H\subset\mathrm{Im}\,\Phi_{1\#}$. The other inclusion is trivial because the holonomy of the first canonical foliation in the product $F_1(x_0)\times M_2$ is trivial and $\Phi$ preserves the foliations. □

Summarizing, we obtain

**Corollary 3.11.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product and take $x_0 \in M$ such that $F_1(x_0)$ has no holonomy.*

(1) *For any leaf $F_1$ there exists a normal semi-Riemannian covering map $\Phi : F_1(x_0) \to F_1$ with deck transformation group* $\mathrm{Hol}(F_1)$.

(2) *All leaves without holonomy are homothetic.*

*Proof.* For the first point, notice that given any leaf $F_1$ it always exists $x \in F_2(x_0)$ such that $F_1 = F_1(x)$. For the second statement, just notice that $F_1(x_0)$ and $F_1(x_0) \times \{b\}$ are homothetic. □

**Corollary 3.12.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. If there is a noncompact leaf, then any compact leaf has nontrivial holonomy.*

**Corollary 3.13.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product and $x_0 = p(a_0, b_0) \in M$. If $F_1(x_0)$ has no holonomy, then $\pi_1(F_1(x_0), x_0)$ is a normal subgroup of $\pi_1(M, x_0)$.*

*Proof.* From Theorem 3.4 it is immediate that $\pi_1(F_1(x_0), x_0)$ is a subgroup of $\pi_1(M, x_0)$. Take $[\alpha] \in \pi_1(F_1(x_0), x_0)$ and $[\gamma] \in \pi_1(M, x_0)$. We show that $[\gamma\cdot\alpha\cdot\gamma^{-1}]$ is homotopic to a loop in $F_1(x_0)$. Take the covering map $\Phi : F_1(x_0) \times M_2 \to M$ and $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$ a lift of $\gamma$ starting at $(x_0, b_0)$. Since $\Phi(\tilde{\gamma}(1)) = x_0$, using the above corollary we have $\Phi : F_1(x_0) \times \{\tilde{\gamma}_2(1)\} \to F_1(x_0)$ is an isometry, thus we can lift the loop $\alpha$ to a loop $\tilde{\alpha}$ starting at $\tilde{\gamma}(1)$. Therefore, the lift of $\gamma \cdot \alpha \cdot \gamma^{-1}$ to $F_1(x_0) \times M_2$ starting at $(x_0, b_0)$ is $\tilde{\gamma} \cdot \widetilde{\alpha} \cdot \tilde{\gamma}^{-1}$. But it is clear that this last loop is homotopic to a loop in $F_1(x_0) \times \{b_0\}$ and so $\gamma \cdot \alpha \cdot \gamma^{-1}$ is homotopic to a loop in $F_1(x_0)$. □

**Example 3.14.** Lemma 3.2, and thus the results that depend on it, does not hold if we consider more general products than doubly warped product, as the following example shows.

First, we are going to construct a function $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^+$ step by step. Take $h\colon \mathbb{R} \to \mathbb{R}$, such that $h = \mathrm{id}$ in a neighborhood of 0 and $h'(x) > 0$ for all $x \in \mathbb{R}$, and $\lambda : (-\varepsilon, \varepsilon) \times \mathbb{R} \to \mathbb{R}^+$ any $C^\infty$ function for $\varepsilon < \frac{1}{2}$. We extend $\lambda$ to the trip $(1-\varepsilon, 1+\varepsilon)\times\mathbb{R}$ defining $\lambda(x, y) = \lambda(x-1, h(y))h'(y)$ for every $(x, y) \in (-\varepsilon, \varepsilon)\times\mathbb{R}$. Now extend it again to $[\varepsilon, 1-\varepsilon]\times\mathbb{R}$ in any way such that $\lambda : (-\varepsilon, 1+\varepsilon) \times \mathbb{R} \to \mathbb{R}^+$ is $C^\infty$. Thus, we have a function with $\lambda(x, y) = \lambda(x - 1, h(y))h'(y)$ for all $(x, y) \in (1 - \varepsilon, 1 + \varepsilon) \times \mathbb{R}$, or equivalently, $\lambda(x, y) = \lambda(x + 1, f(y))f'(y)$ for all $(x, y) \in (-\varepsilon, \varepsilon) \times \mathbb{R}$, where $f$ is the inverse of $h$.

Now, we define $\lambda$ in $[1 + \varepsilon, \infty)$ recursively by $\lambda(x, y) = \lambda(x - 1, h(y))h'(y)$, and in $(-\infty, -\varepsilon]$ by $\lambda(x, y) = \lambda(x + 1, f(y))f'(y)$. It is easy to show that $\lambda : \mathbb{R}^2 \to \mathbb{R}^+$ is $C^\infty$.

Take $\mathbb{R}^2$ endowed with the twisted metric $dx^2 + \lambda(x, y)^2dy^2$ and $\Gamma$ the group generated by the isometry $\phi(x, y) = (x + 1, f(y))$, which preserves the canonical foliations and acts in a properly discontinuous manner. Take $p : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma = M$ the projection. The leaf of the first foliation through $p(0, 0)$ is diffeomorphic to $\mathbb{S}^1$

and has no holonomy. But Theorem 3.4 does not hold because if $\Phi : \mathbb{S}^1 \times \mathbb{R} \to M$ were a covering map, then $\mathbb{S}^1$ would be a covering of all leaves of the first foliation (Corollary 3.11). But this is impossible because for a suitable choice of $h$, there are leaves diffeomorphic to $\mathbb{R}$.

We finish this section with a cohomological obstruction to the existence of a quotient of a doubly warped product with compact leaves. If $M_1$ and $M_2$ are $n$-dimensional, compact and oriented manifolds, Künneth formula implies that the $n$-th Betti number of the product $M_1 \times M_2$ is greater or equal than 2. The following theorem shows that the same is true for any oriented quotient of a doubly warped product with $n$-dimensional compact leaves.

**Theorem 3.15.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be an oriented quotient of a doubly warped product such that the leaves of both foliations on $M$ are $n$-dimensional and compact submanifolds of $M$. Then, the $n$-th Betti number of $M$ satisfies $b_n \geq 2$.*

*Proof.* Take a point $(a_0, b_0) \in M_1 \times M_2$ such that $x_0 = p(a_0, b_0)$ has no holonomy and $\Phi \colon F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0) \longrightarrow M$ the covering map given in Corollary 3.5. Since $M$ is oriented, $M_1$ and $M_2$ are orientable and $\Gamma$ preserves the orientation. But $\Gamma_i^{(a_0,b_0)}$ is a normal subgroup of $\Gamma$ and therefore it preserves the orientation of $M_i$. Thus $F_i(x_0) = M_i/\Gamma_i^{(a_0,b_0)}$ is orientable.

Let $[\varpi_1], [\varpi_2] \in H^n(M)$ be the Poincaré dual of $F_1(x_0)$ and $F_2(x_0)$ respectively. The submanifolds $S_i = \Phi^{-1}(F_i(x_0))$ are closed in $F_1(x_0) \times F_2(x_0)$ and therefore they are compact. With the appropriate orientation, they have Poincaré duals $[\sigma_i] = \Phi^*([\varpi_i])$, [2].

Call $\pi_i \colon F_1(x_0) \times F_2(x_0) \to F_i(x_0)$ the canonical projection, $\Phi_i \colon S_i \to F_i(x_0)$ the restriction of $\Phi$ to $S_i$, and $i_j \colon S_j \to F_1(x_0) \times F_2(x_0)$ the canonical inclusion. Consider the following commutative diagram:

$$
\begin{array}{ccc}
S_j & \xrightarrow{\ \Phi_j\ } & F_j(x_0) \\
{\scriptstyle i_j}\downarrow & \nearrow{\scriptstyle \pi_j} & \\
F_1(x_0) \times F_2(x_0) & &
\end{array}
$$

If $\Theta_1$ is a volume form of $F_1(x_0)$, then $\Phi_1^*(\Theta_1) = i_1^*(\pi_1^*(\Theta_1))$ is a volume form in $S_1$. Therefore

$$0 \neq \int_{S_1} i_1^* \, \pi_1^*(\Theta_1) = \int_{F_1 \times F_2} \pi_1^*(\Theta_1) \wedge \sigma_1,$$

thus $[\sigma_1]$ is not null. In the same way we can show that $[\sigma_2]$ is not null.

Now if $\sigma_1 - c\sigma_2 = d\tau$ for some $0 \neq c \in \mathbb{R}$ and $\tau \in \Lambda^{n-1}(M)$, then

$$\int_{F_1 \times F_2} \pi_1^*(\Theta_1) \wedge \sigma_1 = c \int_{F_1 \times F_2} \pi_1^*(\Theta_1) \wedge \sigma_2 = c \int_{S_2} i_2^* \, \pi_1^*(\Theta_1) = 0,$$

which is a contradiction. Therefore $[\sigma_1]$ and $[\sigma_2]$ are linearly independent, so the same is true for $[\varpi_1]$ and $[\varpi_2]$. □

Observe that if the dimension of the foliations are $n$ and $m$ with $n \neq m$, then we can only conclude that the $n$-th and $m$-th Betti numbers of $M$ satisfy $b_n, b_m \geq 1$. In the category of four dimensional Lorentzian manifolds we have the following result.

**Corollary 3.16.** *In the conditions of the above theorem, if $M$ is a four dimensional Lorentzian manifold, then its first and second Betti numbers satisfies $b_1, b_2 \geq 2$.*

*Proof.* It is clear that $M$ is compact (Corollary 3.5). The existence of a Lorentz metric implies that the Euler characteristic is null, thus $2b_1 = 2 + b_2$. □

## 4. Space of leaves

Given a foliation $\mathcal{F}$ on a manifold $M$, a point $x$ is called regular if it exists an adapted chart $(U, \varphi)$ to $\mathcal{F}$, with $x \in U$, such that each leaf of the foliation intersects $U$ in an unique slice. The open set $U$ is also called a regular neighborhood of $x$. If all points are regular (i.e., $\mathcal{F}$ is a regular foliation), then the space of leaves $\mathfrak{L}$ of $\mathcal{F}$ is a manifold except for the Hausdorffness, and the canonical projection $\eta : M \to \mathfrak{L}$ is an open map, [14], [19].

Given $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ a quotient of a doubly warped product, we call $\mathfrak{L}_i$ the space of leaves of the induced foliations $\mathcal{F}_i$ on $M$. Take $x_0 \in M$ without holonomy and the normal covering map $\Phi : F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0) \to M$, whose group of deck transformation is $\Psi = \Gamma/(\Gamma_1 \times \Gamma_2)$. The set $\Sigma_{x_0}$ formed by those maps $\psi \in \mathrm{Diff}(F_2(x_0))$ such that there exists $\phi \in \mathrm{Diff}(F_1(x_0))$ with $\phi \times \psi \in \Psi$ is a group of homotheties of $F_2(x_0)$.

**Lemma 4.1.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. Suppose that the foliation $\mathcal{F}_1$ has no holonomy and take $x_0 \in M$ such that $F_2(x_0)$ has no holonomy. Then the action of $\Sigma_{x_0}$ on $F_2(x_0)$ is free.*

*Proof.* Take $\psi \in \Sigma_{x_0}$ and suppose that it has a fixed point $x \in F_2(x_0)$. If $\phi \in \mathrm{Diff}(F_1(x_0))$ with $\phi \times \psi \in \Psi$, then $\Phi(z, x) = \Phi(\phi(z), \psi(x)) = \Phi(\phi(z), x)$ for all $z \in F_1(x_0)$, but since $\mathcal{F}_1$ has no holonomy, applying Corollary 3.11, $\Phi : F_1(x_0) \times \{x\} \to F_1(x)$ is an isometry. Therefore $\phi = \mathrm{id}$ and hence $\psi = \mathrm{id}$. □

**Theorem 4.2.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product such that $\mathcal{F}_1$ is a regular foliation. If $F_2(x_0)$ has no holonomy then*

(1) *The group $\Sigma_{x_0}$ acts in a properly discontinuous manner (in the topological sense) on $F_2(x_0)$.*

(2) *The restriction $\eta_{x_0} = \eta|_{F_2(x_0)} : F_2(x_0) \to \mathfrak{L}_1$ is a normal covering map with $\Sigma_{x_0}$ as deck transformation group.*

*Proof.* (1) Suppose that $\Sigma_{x_0}$ does not act in a properly discontinuous manner. Then, there exists $x \in F_2(x_0)$ such that for all neighborhood $U$ of $x$ in $F_2(x_0)$ there is $\psi \in \Sigma_{x_0}$, $\psi \neq \mathrm{id}$, with $U \cap \psi(U) \neq \emptyset$.

Take $V \subset M$ a regular neighborhood of $x$ adapted to $\mathcal{F}_1$. Since $\Phi(x_0, x) = x$, we can lift $V$ through the covering $\Phi : F_1(x_0) \times F_2(x_0) \to M$ and suppose that there are $U_i \subset F_i(x_0)$ open sets with $x_0 \in U_1$, $x \in U_2$ and $\Phi : U_1 \times U_2 \to V$ an isometry. Using that $\Sigma_{x_0}$ does not act in a properly discontinuous manner, there is $\psi \in \Sigma_{x_0}$, $\psi \neq \mathrm{id}$, with $y = \psi(z)$ for certain $y, z \in U_2$. Moreover, $z \neq y$ since $\psi$ does not have fixed points (Lemma 4.1).

If we take $\phi$ with $\phi \times \psi \in \Psi$, then $z = \Phi(x_0, z) = \Phi(\phi(x_0), y)$ and thus $F_1(z) = F_1(y)$. Now, $\Phi(U_1 \times \{y\})$ and $\Phi(U_1 \times \{z\})$ are two different slices of $\mathcal{F}_1$ in $V$ which belong to the same leaf $F_1(z)$. Contradiction.

(2) It is quite easy to show that $F_2(x_0)/\Sigma_{x_0} = \mathfrak{L}_1$, where the identification is given by $[x] \longleftrightarrow F_1(x)$. □

**Corollary 4.3.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product such that $\mathcal{F}_1$ is a regular foliation. If the space of leaves $\mathfrak{L}_1$ is simply connected, then $M$ is isometric to a global doubly warped product $F_1 \times_{(\rho_1,\rho_2)} F_2$.*

We give some conditions for $\mathfrak{L}_1$ to be a true manifold. Recall that given $(a, b) \in F_1(x_0) \times F_2(x_0)$ with $\Phi(a, b) = x$ we denote $\Psi_1^{(a,b)}$ the deck transformation group of the restriction $\Phi_1^{(a,b)} \colon F_1(x_0) \times \{b\} \to F_1(x)$ of the covering map $\Phi \colon F_1(x_0) \times F_2(x_0) \to M$.

**Theorem 4.4.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product. If $M_2$ is a complete Riemann manifold and $\mathcal{F}_1$ a regular foliation, then the space of leaves $\mathfrak{L}_1$ is a Riemannian manifold.*

*Proof.* Take $x_0 \in M$ without holonomy. We show that $\Sigma_{x_0}$ is a group of isometries. Take $\psi \in \Sigma_{x_0}$ and $\phi : F_1(x_0) \to F_1(x_0)$ such that $f = \phi \times \psi \in \Psi$. As we already said, there exists a constant $c$ such that $\psi^*(g_2) = c^2 g_2$ and $\rho_2 = c\rho_2 \circ \phi$.

Suppose $c \neq 1$. Taking the inverse of $\psi$ if it were necessary, we can suppose $c < 1$. Then $\psi : F_2(x_0) \to F_2(x_0)$ is a contractive map and it is assured the existence of a fixed point $b \in F_2(x_0)$. Therefore $f(F_1(x_0) \times \{b\}) = F_1(x_0) \times \{b\}$ and $f|_{F_1(x_0)\times\{b\}} \in \Psi_1^{(a,b)}$, where $a \in F_1(x_0)$ is some point. Using Lemma 3.2, $c = 1$ and we get a contradiction.

Using the above theorem, $\Sigma_{x_0}$ acts in a properly and discontinuously manner on $F_2(x_0)$ in the topological sense, but since $F_2(x_0)$ is Riemannian and $\Sigma_{x_0}$ a group of isometries, it actually acts in a proper and discontinuous manner in the differentiable sense, i.e., points in different orbits have open neighborhood with disjoint orbits. Thus, $\mathfrak{L}_1$ is a Riemannian manifold. □

**Remark 4.5.** Given a nondegenerate foliation $\mathcal{F}$, it is called semi-Riemannian (or metric) when, locally, the leaves coincide with the fibers of a semi-Riemannian submersion, [21], [24]. If the orthogonal distribution is integrable, then $\mathcal{F}$ is a semi-Riemannian foliation if and only if $\mathcal{F}^\perp$ is totally geodesic, [13]. In the case of a doubly warped product $F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0)$, the first canonical foliation is semi-Riemannian for the conformal metric $\left(\frac{\rho_1}{\rho_2}\right)^2 g_1 + g_2$.

In the hypotheses of the above theorem, $\rho_2$ is invariant under $\Psi$ and thus there exists a function $\sigma_2 : M \to \mathbb{R}^+$ such that $\sigma_2 \circ \Phi = \rho_2$. In this case, it is easy to show that $\mathcal{F}_1$ is semi-Riemannian for the conformal metric $\frac{1}{\sigma_2^2} g$, where $g$ is the induced metric on $M$. Observe that in the Riemannian case, under regularity hypothesis, it is known that the space of leave of a Riemannian foliation is a true manifold and, moreover, the manifold is a fiber bundle over it [21], but there is not an analogous in the semi-Riemannian case.

**Corollary 4.6.** *Let $M = \left(M_1 \times_{(\lambda_1,1)} M_2\right)/\Gamma$ be a quotient of a warped product, where $M_2$ is a complete Riemannian manifold. If $\mathcal{F}_1$ is a regular foliation, then the projection $\eta : M \to \mathfrak{L}_1$ is a semi-Riemannian submersion.*

*Proof.* We already know that $\mathfrak{L}_1$ is a Riemannian manifold and $\eta_{x_0} : F_2(x_0) \to \mathfrak{L}_1$ a local isometry, where $x_0$ has no holonomy. Given $x \in F_1(x_0)$, the following diagram is commutative:

$$\begin{array}{ccc} \{x\} \times F_2(x_0) & \xrightarrow{\ \Phi\ } & F_2(x) \\ {\scriptstyle \eta_{x_0} \circ pr_2} \big\downarrow & \swarrow {\scriptstyle \eta_x} & \\ \mathfrak{L}_1 & & \end{array}$$

Since $\lambda_2 = 1$, the map $\Phi \colon \{x\} \times F_2(x_0) \to F_2(x)$ is a local isometry for all $x \in F_1(x_0)$. Thus, $\eta_x \colon F_2(x) \to \mathfrak{L}_1$ is a local isometry for all $x \in M$ and therefore, $\eta \colon M \to \mathfrak{L}_1$ is a semi-Riemannian submersion. □

Observe that in the corollary, the fibres are the leaves of a warped structure, thus they are automatically umbilic.

**Theorem 4.7.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product such that $\mathcal{F}_1$ is a regular foliation. Then*

(1) *The projection $\eta : M \to \mathfrak{L}_1$ is a fiber bundle. Moreover, we have $\pi_1(\mathfrak{L}_1, F_1) = \pi_1(M, x)/\pi_1(F_1, x)$ where $x \in F_1 \in \mathfrak{L}_1$.*

(2) *There exists an open dense subset $W \subset M$ globally isometric to a doubly warped product.*

*Proof.* (1) Take $F_1 \in \mathfrak{L}_1$ and $x_0 \in F_1$ a point without holonomy. Since $\eta_{x_0} \colon F_2(x_0) \to \mathfrak{L}_1$ is a covering map, there are open sets $U \subset F_2(x_0)$ and $V \subset \mathfrak{L}_1$ with $x_0 \in U$ and $F_1 \in V$ such that $\eta_{x_0} : U \to V$ is a diffeomorphism.

Now we show that $\Phi(F_1(x_0) \times U) = \eta^{-1}(V)$. If $(a, b) \in F_1(x_0) \times U$, then $\eta(\Phi(a, b)) = \eta(\Phi(x_0, b)) = \eta_{x_0}(b) \in V$. Given $x \in \eta^{-1}(V)$, if we call $b = \eta_{x_0}^{-1}(\eta(x)) \in U$, then $\eta(x) = \eta_{x_0}(b)$ and $\Phi \colon F_1(x_0) \times \{b\} \to F_1(x)$ is an isometry because $F_1$ has no holonomy (Corollary 3.11). Thus, there exists $a \in F_1(x_0)$ with $\Phi(a, b) = x$.

The map $\Phi : F_1(x_0) \times U \to \eta^{-1}(V)$ is injective (and therefore a diffeomorphism). In fact, if $(a, b), (a', b') \in F_1(x_0) \times U$ with $\Phi(a, b) = \Phi(a', b')$ then

$$\eta_{x_0}(b) = \eta_{x_0}(\Phi(x_0, b)) = \eta_{x_0}(\Phi(a, b)) = \eta_{x_0}(\Phi(a', b')) = \eta_{x_0}(\Phi(x_0, b')) = \eta_{x_0}(b').$$

But since $b, b' \in U$, we get that $b = b'$. Now, using that $\Phi : F_1(x_0) \times \{b\} \to F_1(b)$ is an isometry, we deduce that $a = a'$.

The map $h_V$ that makes commutative the following diagram

$$\begin{array}{ccc} F_1(x_0) \times U & & \\ \Phi \downarrow & \searrow^{\mathrm{id} \times \eta_{x_0}} & \\ \eta^{-1}(V) & \xrightarrow{h_V} & F_1(x_0) \times V \end{array}$$

shows that $M$ is locally trivial.

Finally, using Theorem 4.41 of [12], $\eta_{\#} \colon \pi_1(M, F_1, x_0) \to \pi_1(\mathfrak{L}_1, F_1)$ is an isomorphism. But $\pi_1(F_1, x_0)$ is a normal subgroup of $\pi_1(M, x_0)$ (Corollary 3.13), hence $\pi_1(M, F_1, x_0) = \pi_1(M, x_0)/\pi_1(F_1, x_0)$.

(2) Since $\eta_{x_0} : F_2(x_0) \to \mathfrak{L}_1$ is a covering map, we can take an open dense set $\Theta \subset \mathfrak{L}_1$ and an open set $U \subset F_2(x_0)$ such that $\eta_{x_0} : U \to \Theta$ is a diffeomorphism. Given $F_1 \in \Theta$ we have $\Phi(x_0, \eta_{x_0}^{-1}(F_1)) = \eta_{x_0}^{-1}(F_1) \in F_1$ and thus the restriction $\Phi : F_1(x_0) \times \{\eta_{x_0}^{-1}(F_1)\} \to F_1$ is an isometry. Now, since $\Theta$ is dense, $W = \eta^{-1}(\Theta)$ is dense in $M$, and taking $V = \Theta$ and $W = \eta^{-1}(\Theta)$ in the above proof we get the result. □

Recall that this open set $W$ is obtained removing a suitable set of leaves of $\mathcal{F}_1$ from $M$. This is false for more general product, as twisted products (see example 3.14).

**Remark 4.8.** In [22] a notion of local warped product on a manifold is given as follows. Take a fiber bundle $\Pi : M \to B$ with fibre $F$, where $M$, $B$ and $F$ are semi-Riemannian manifolds. Suppose that there is a function $\lambda : B \to \mathbb{R}^+$ such that we can take a covering $\{U_i : i \in I\}$ of trivializing open sets of $B$ with $(\Pi^{-1}(U_i), g) \to (U_i \times F, g_B + \lambda^2 g_F)$ an isometry for all $i \in I$. Then it is said that $M$ is a local warped product.

It follows that the orthogonal distribution to the fibre is integrable and using that $M$ is locally isometric to a warped product, it is easy to show that these two foliations constitute a warped structure in the sense of definition 2.4. But not all warped structures arise in this way, since the foliation induced by the fibres of a fibre bundle has no holonomy (in fact, it is a regular foliation).

## 5. Global decomposition

Given a product manifold $M_1 \times M_2$, a plane $\Pi = \mathrm{span}(X, V)$, where $X \in TM_1$ and $V \in TM_2$, is called a mixed plane. In this section, we show how the sign of sectional curvature of this kind of planes determines the global decomposition of a doubly warped structure.

**Lemma 5.1.** *Let $M$ be a complete semi-Riemannian manifold of index $\nu$. Take $\lambda : M \to \mathbb{R}^+$ a smooth function and $h_\lambda$ its hessian endomorphism. If $\nu < \dim M$ and $g(h_\lambda(X), X) \leq 0$ for all spacelike vector $X$ (or $0 < \nu$ and $g(h_\lambda(X), X) \leq 0$ for all timelike vector $X$), then $\lambda$ is constant.*

*Proof.* Suppose $\nu < \dim M$. Take $x \in M$ and $V \ni x$ a normal convex neighborhood. Call $S(x)$ the set formed by the points $y \in V$ such that there exists a nonconstant spacelike geodesic inside $V$ joining $x$ with $y$. It is obvious that $S(x)$ is an open set for all $x \in M$ and it does not contain $x$. Let $\gamma : \mathbb{R} \to M$ be a spacelike geodesic with $\gamma(0) = x$. If we call $y(t) = \lambda(\gamma(t))$ then $y''(t) \leq 0$ and $y(t) > 0$ for all $t \in \mathbb{R}$, which implies that $\lambda$ is constant in $S(x)$. Take $x_1 \in S(x)$. In the same way, $\lambda$ is constant in $S(x_1)$, which is an open neighborhood of $x$. Since $x$ is arbitrary, $\lambda$ is constant. The case $0 < \nu$ is similar taking timelike geodesics. □

**Proposition 5.2.** *Let $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ be a doubly warped product with $M_1$ and $M_2$ complete semi-Riemannian manifold of index $\nu_i < \dim M_i$. If $K(\Pi) \geq 0$ for all spacelike mixed plane $\Pi$, then $\lambda_1$ and $\lambda_2$ are constant.*

*Proof.* First note that if $f \in C^\infty(M_1)$, then $g(h_f(X), X) = g_1(h^1_f(X), X)$ for all $X \in \mathfrak{X}(M_1)$, where $h_f$ is the hessian respect to the doubly warped metric $g$ and $h^1_f$ respect to $g_1$.

Suppose there exists a point $p \in M_1$ and a spacelike vector $X_p \in T_pM_1$ such that $0 \leq g(h_2(X), X)$. Given an arbitrary spacelike vector $V_q \in T_qM_2$, we have $0 \leq K(X, V) + \frac{1}{\lambda_2} g(h_2(X), X) = -\frac{1}{\lambda_1} g(h_1(V), V)$ and applying the above lemma, $\lambda_1$ is constant. Therefore, $0 \leq -\frac{1}{\lambda_2} g(h_2(X), X)$ for all spacelike vector $X$ and applying the above lemma again, $\lambda_2$ is constant too.

Suppose now the contrary case: for all spacelike vector $X \in TM_1$ we have $g(h_2(X), X) < 0$. Then, the above lemma gives us that $\lambda_2$ is constant. Thus $0 \leq -\frac{1}{\lambda_1} g(h_1(V), V)$ for all spacelike vector $V$ and the above lemma ensures that $\lambda_1$ is constant too. □

**Theorem 5.3.** *Let $M = \left(M_1 \times_{(\lambda_1,\lambda_2)} M_2\right)/\Gamma$ be a quotient of a doubly warped product, being $M_1$ a complete Riemannian manifold and $M_2$ a semi-Riemannian manifold with $0 < \nu_2$. Suppose that $\mathcal{F}_2$ has no holonomy, $K(\Pi) < 0$ for all mixed nondegenerate plane $\Pi$ and $\lambda_2$ has some critical point. Then $M$ is globally a doubly warped product.*

*Proof.* Suppose that there is a nonlightlike vector $V \in TM_2$ with $\varepsilon_V g(h_1(V), V) \leq 0$. Given an arbitrary non zero vector $X \in TM_1$, span$\{X, V\}$ is a nondegenerate plane, thus

$$-\frac{1}{\lambda_2} g(h_2(X), X) - \frac{\varepsilon_V}{\lambda_1} g(h_1(V), V) = K(X, V) < 0$$

and therefore $0 < g(h_2(X), X)$ for all $X \in TM_1$, $X \neq 0$.

In the opposite case, $0 < \varepsilon_V g(h_1(V), V)$ for all non lightlike vector $V \in TM_2$. Applying Lemma 5.1, we get that $\lambda_1$ is constant, and therefore $g(h_2(X), X) = -\lambda_2 K(X, V) > 0$ for all $X \in TM_1$, $X \neq 0$. In any case $h_2$ is positive definite and so $\lambda_2$ has exactly one critical point.

Take $x_0 \in M$ without holonomy and the associated covering map

$$\Phi : F_1(x_0) \times_{(\rho_1,\rho_2)} F_2(x_0) \to M\,.$$

Let $x_1 \in F_1(x_0)$ be the only critical point of $\rho_2$. If $\phi \times \psi$ is a deck transformation of this covering, then $\rho_2 \circ \phi = c\rho_2$ for some constant $c$, and it follows that $\phi(x_1) \in F_1(x_0)$ is a critical point of $\rho_2$ too. Thus $\phi(x_1) = x_1$, but since $\mathcal{F}_2$ has no holonomy, applying Lemma 4.1, we get $\phi \times \psi = \mathrm{id}$. Thus $\Phi$ is an isometry. □

Observe that in the conditions of the above theorem we can prove that $M = M_1 \times_{(\rho_1,\lambda_2)} (M_2/\Gamma_2)$. In fact, let $(a_0, b_0) \in M_1 \times M_2$ such that $p(a_0, b_0) = x_0$. Since the points of the fibre $p_1^{-1}(x_1)$ are critical points of $\lambda_2$, being $p_1 : M_1 \times \{b_0\} \to F_1(x_0)$ the covering map given in Lemma 3.1, and $\lambda_2$ has only one critical point, it follows that $p_1$ is an isometry.

**Example 5.4.** The Kruskal space has warping function with exactly one critical point. Thus, the last part of the above proof shows that any quotient without holonomy is a global warped product.

Now we apply the above results to semi-Riemannian submersions. We denote $\mathcal{H}$ and $\mathcal{V}$ the horizontal and vertical spaces and $E^v$ (resp. $E^h$) will be the vertical (resp. horizontal) projections of a vector $E$.

**Lemma 5.5.** *Let $\pi : M \to B$ be a semi-Riemannian submersion with umbilic fibres and $T$ and $A$ the O'Neill tensors of $\pi$. Then for arbitrary $E, F \in \mathfrak{X}(M)$ and $X \in \mathcal{H}$, it holds*

(1) $T(E,F) = g(E^v, F^v)N - g(N,F)E^v$,

(2) $(\nabla_X T)(E,F) = g(F, A(X,E^*))N - g(N,F)A(X,E^*) + g(E^v,F^v)\nabla_X N - g(\nabla_X N, F)E^v$,

*where $N$ is the mean curvature vector field of the fibres and $E^* = E^v - E^h$.*

*Proof.* The first point is immediate. For the second, we have $(\nabla_X T)(E,F) = \nabla_X T(E,F) - T(\nabla_X E, F) - T(E, \nabla_X F)$. We compute each term

$$\begin{aligned} \nabla_X T(E,F) &= \nabla_X (g(E^v,F^v)N - g(N,F)E^v) \\ &= \big(g(\nabla_X E^v, F^v) + g(E^v, \nabla_X F^v)\big)N + g(E^v,F^v)\nabla_X N \\ &\quad - \big(g(\nabla_X N, F) + g(N, \nabla_X F)\big)E^v - g(N,F)\nabla_X E^v, \\ T(\nabla_X E, F) &= g((\nabla_X E)^v, F^v)N - g(N,F)(\nabla_X E)^v, \\ T(E, \nabla_X F) &= g(E^v, (\nabla_X F)^v)N - g(N, \nabla_X F)E^v. \end{aligned}$$

Rearranging terms and using that $\nabla_X E^v - (\nabla_X E)^v = A(X, E^*)$, we obtain

$$\begin{aligned} (\nabla_X T)(E,F) = \big(&g(A(X,E^*), F^v) + g(E^v, A(X,F^*))\big)N - g(N,F)A(X,E^*) \\ &+ g(E^v,F^v)\nabla_X N - g(\nabla_X N, F)E^v. \end{aligned}$$

But

$$\begin{aligned} &g(A(X,E^*),F^v) + g(E^v, A(X,F^*)) = -g(A(X,E^h),F^v) - g(E^v, A(X,F^h)) \\ &= -g(A(X,E^h),F^v) + g(A(X,E^v),F^h) = g(A(X,-E^h),F) + g(A(X,E^v),F) \\ &= g(A(X,E^*),F). \end{aligned}$$

And we obtain the result. □

We need to introduce the lightlike curvature of a degenerate plane in a Lorentzian manifold $(M, g)$, [11]. Fix a timelike and unitary vector field $\xi$ and take a degenerate plane $\Pi = \mathrm{span}(u, v)$, where $u$ is the unique lightlike vector in $\Pi$ with $g(u, \xi) = 1$. We define the lightlike sectional curvature of $\Pi$ as

$$\mathcal{K}_\xi(\Pi) = \frac{g(R(v, u, u), v)}{g(v, v)}.$$

This sectional curvature depends on the choice of the unitary timelike vector field $\xi$, but its sign does not change if we choose another vector field. Thus, it makes sense to say positive lightlike sectional curvature or negative lightlike sectional curvature.

**Lemma 5.6.** *Let $(M, g)$ and $(B, h)$ be a Lorentzian and a Riemannian manifold respectively and $\pi : M \to B$ a semi-Riemannian submersion with umbilic fibres. If $\xi \in \mathcal{V}$ is an unitary timelike vector field and $\Pi = \mathrm{span}(u, X)$ is a degenerate plane with $u \in \mathcal{V}$, $X \in \mathcal{H}$, $g(u, u) = 0$ and $g(u, \xi) = 1$, then*

$$\mathcal{K}_\xi(\Pi) = \frac{g(A(X, u), A(X, u))}{g(X, X)}.$$

*Proof.* Using the formulaes of [18], we have

$$g(X, X)\mathcal{K}_\xi(\Pi) = g((\nabla_X T)(u, u), X) - g(T(u, X), T(u, X)) + g(A(X, u), A(X, u)).$$

Since $u$ is lightlike, the first two terms are null by the above lemma. □

Given a warped product $M_1 \times_{(1,\lambda_2)} M_2$, the projection $\pi : M_1 \times M_2 \to M_1$ is a semi-Riemannian submersion with umbilic fibres. The following theorem assures the converse fact.

**Theorem 5.7.** *Let $M$ be a complete Lorentzian manifold, $B$ a Riemannian manifold and $\pi : M \to B$ a semi-Riemannian submersion with umbilic fibres of dimension greater than one and mean curvature vector $N$. If $K(\Pi) < 0$ for all mixed spacelike plane of $M$, and $N$ is closed with some zero, then $M$ is globally a warped product.*

*Proof.* By continuity, it follows that $M$ has nonpositive lightlike curvature for all mixed degenerated plane and thus, applying the above lemma, $A(X, u) = 0$ for all $X \in \mathcal{H}$ and all lightlike $u \in \mathcal{V}$. Therefore $A \equiv 0$ and $\mathcal{H}$ is integrable and necessarily totally geodesic (see [18]), which gives rise to a warped structure, since $N$ is closed. But being $M$ complete $M = (M_1 \times_{(1,\lambda_2)} M_2)/\Gamma$ (see Remark 2.5). Now, using the formulaes of Lemma 2.2 we can easily check that the curvature of a mixed plane $\Pi = \mathrm{span}(X, V)$ is independent of the vertical vector $V$ and thus $K(\Pi) < 0$ for all mixed nondegenerate plane. Finally, since $N$ has some zero, $\lambda_2$ has some critical point and applying Theorem 5.3 we get the result. □

## 6. Uniqueness of product decompositions

In [7], the uniqueness of direct product decompositions of a non necessarily simply connected Riemannian manifold is studied, where the uniqueness is understood in the following sense: a decomposition is unique if the corresponding foliations are uniquely determined. The authors use a short generating set of the fundamental group in the sense of Gromov, which is based in the Riemannian distance. So, the techniques employed can not be used directly in the semi-Riemannian case. In this section we apply the results of this paper to study the uniqueness problem in the semi-Riemannian setting.

**Proposition 6.1.** *Let $M = F_1 \times \cdots \times F_k$ be a semi-Riemannian direct product and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ the canonical foliations. Take $S$ an umbilic/geodesic submanifold of $M$ and suppose that there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{F}_i(x) \cap T_xS$ is a nondegenerate subspace with constant dimension for all $x \in S$. Then the distributions $\mathcal{T}_1$ and $\mathcal{T}_2$ on $S$ determined by $\mathcal{T}_1(x) = \mathcal{F}_i(x) \cap T_xS$ and $\mathcal{T}_2(x) = \mathcal{T}_1^{\perp}(x) \cap T_xS$ are integrable. Moreover, $\mathcal{T}_1$ is a regular and umbilic/geodesic foliation and $\mathcal{T}_2$ is a geodesic one.*

*Proof.* It is clear that $\mathcal{T}_1$ is integrable. We show that $\mathcal{T}_2$ is integrable and geodesic in $S$.

Consider the tensor $J$ given by $J(v_1, \ldots, v_i, \ldots, v_k) = (-v_1, \ldots, v_i, \ldots, -v_k)$, where $(v_1, \ldots, v_k) \in TF_1 \times \cdots \times TF_k$, and take $X, V, W \in \mathfrak{X}(S)$ with $X_x \in \mathcal{T}_1(x)$ and $V_x, W_x \in \mathcal{T}_2(x)$ for all $x \in S$. Since $\nabla J = 0$, we have $0 = (\nabla_V J)(X) = \nabla_V X - J(\nabla_V X)$, which means that $\nabla_{V_x} X \in \mathcal{F}_i(x)$ since $\nabla_V X$ is invariant under $J$. Using that $S$ is umbilical and $X, V$ are orthogonal, we have $\nabla_{V_x} X = \nabla^S_{V_x} X \in T_xS$. Therefore $\nabla_{V_x} X \in \mathcal{T}_1(x)$ for all $x \in S$.

Now, we have $g(\nabla^S_V W, X) = g(\nabla_V W, X) = -g(W, \nabla_V X) = 0$. Thus, $\nabla^S_V W \in \mathcal{T}_2$ for all $V, W \in \mathcal{T}_2$ which means that $\mathcal{T}_2$ is integrable and geodesic in $S$.

To see that $\mathcal{T}_1$ is umbilic, take $X, Y \in \mathfrak{X}(S)$ with $g(X, Y) = 0$ and $X_x, Y_x \in \mathcal{T}_1(x)$ for all $x \in S$. It is easy to show that $\nabla_{X_x} Y \in \mathcal{F}_i(x)$ and since $S$ is umbilic, we have

$$\nabla_{X_x} Y = \nabla^S_{X_x} Y \in \mathcal{F}_i(x) \cap T_xS = \mathcal{T}_1(x)$$

for all $x \in S$. Therefore the second fundamental form of the leaves of $\mathcal{T}_1$ inside $M$ satisfies $\mathbb{I}(X, Y) = 0$ for every couple of orthogonal vectors $X, Y \in \mathcal{T}_1$, which is equivalent to be umbilic submanifolds of $M$. The same argument with the second fundamental form of $\mathcal{T}_1$ as a foliation of $S$ shows that $\mathcal{T}_1$ is an umbilic foliation of $S$. Observe that if $S$ is geodesic it is clear that $\mathcal{T}_1$ is also geodesic.

Finally, we show that $\mathcal{T}_1$ is a regular foliation. Take the map $P : F_1 \times \cdots \times F_k \to F_1 \times \cdots \times F_{i-1} \times F_{i+1} \times \cdots \times F_k$ given by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ and $i : T_2(p) \to F_1 \times \cdots \times F_k$ the canonical inclusion where $p \in S$ is a fixed point. The map $P \circ i$ is locally injective, since $\mathrm{Ker}(P \circ i)_{*x} = \mathcal{F}_i(x) \cap \mathcal{T}_2(x) = 0$ for all $x \in T_2(p)$. Therefore, we can take a neighborhood $U \subset S$ of $p$ adapted to both foliations $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $(P \circ i)|_V$ is injective, being $V$ the slice of $T_2(p)$ in $U$ through $p$. Since $P$ is constant through the leaves of $\mathcal{T}_1$, it follows that $U$ is a regular neighborhood of $p$. $\square$

**Remark 6.2.** Observe that if $S$ is geodesic then $\dim T_xS \cap \mathcal{F}_i(x)$ is constant for all $x \in S$.

We say that a semi-Riemannian manifold is decomposable if it can be expressed globally as a direct product. In the contrary case it is indecomposable.

**Lemma 6.3.** *Let $M = F_1 \times \cdots \times F_k$ be a complete semi-Riemannian direct product and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ the canonical foliations. Suppose $\mathcal{S}$ is a nondegenerate foliation of dimension greater than one and invariant by parallel translation such that $\mathcal{F}_i(p) \cap \mathcal{S}(p) = \{0\}$ for all $i \in \{1, \ldots, k\}$ and some $p \in M$. Then the leaves of $\mathcal{S}$ are flat and decomposable.*

*Proof.* Being all foliations invariant by parallel translation, the property supposed at $p$ is in fact true at any other point of $M$. Take $x = (x_1, \ldots, x_k) \in F_1 \times \cdots \times F_k$ and suppose there is a loop $\alpha_i : [0,1] \to F_i$ at $x_i$ and $v \in \mathcal{S}(x)$ such that $P_\gamma(v) \neq v$, where $\gamma(t) = (x_1, \ldots, \alpha_i(t), \ldots, x_k)$. If we decompose $v = \sum_{j=1}^k v_j \in \bigoplus_{j=1}^k \mathcal{F}_j(x)$, then $P_\gamma(v) = P_\gamma(v_i) + \sum_{j\neq i}^k v_j$ and so $0 \neq v - P_\gamma(v) = v_i - P_\gamma(v_i) \in \mathcal{S}(x) \cap \mathcal{F}_i(x)$, which is a contradiction. Therefore, $P_\gamma(v) = v$ for all $v \in \mathcal{S}(x)$ and all loops $\gamma$ of the form $\gamma(t) = (x_1, \ldots, \alpha_i(t), \ldots, x_k)$. Since $M = F_1 \times \cdots \times F_k$ has the direct product metric, $P_\gamma(v) = v$ for all $v \in \mathcal{S}(x)$ and an arbitrary loop $\gamma$ at $x$. In particular, the parallel translation along any loop of a leaf $S$ is trivial. But this implies that it splits as a product of factors of the form $\mathbb{R}$ or $\mathbb{S}^1$. □

Given a curve $\gamma : [0,1] \to M$ we define $v_\gamma : [0,1] \to T_{\gamma(0)}M$ by

$$v_\gamma(t) = P^{-1}_{\gamma,\gamma(0),\gamma(t)}(\gamma'(t)),$$

where $P$ is the parallel translation. We will denote $\Omega_p^M(t_1, \ldots, t_m)$ the set of broken geodesics in $M$ which start at $p$ and with breaks at $t_i$, where $0 < t_1 < \cdots < t_m < 1$. If $\gamma \in \Omega_p^M(t_1, \ldots, t_m)$ then $v_\gamma$ is a piecewise constant function,

$$v_\gamma(t) = \begin{cases} v_0 \text{ if } 0 \leq t \leq t_1 \\ \vdots \\ v_m \text{ if } t_m \leq t \leq 1 \end{cases}$$

which we will denote by $(v_0, \ldots, v_m)$. On the other hand, if $M$ is complete, given $(v_0, \ldots, v_m) \in (T_pM)^{m+1}$ we can construct a broken geodesic $\gamma \in \Omega_p^M(t_1, \ldots, t_m)$ with $v_\gamma \equiv (v_0, \ldots, v_m)$.

Now, suppose that a semi-Riemannian manifold $M$ splits as a direct product in two different manners, $M = F_1 \times \cdots \times F_k = S_1 \times \cdots \times S_{k'}$. We call $\mathcal{F}_1, \ldots, \mathcal{F}_k$ and $\mathcal{S}_1, \ldots, \mathcal{S}_{k'}$ the canonical foliations of each decomposition and $\pi_i : M \to F_i$, $\sigma_i : M \to S_i$ will be the canonical projections.

Observe that given a point $p \in M$, the leaf of $\mathcal{F}_i$ through $p$ is $F_i(p) = \{\pi_1(p)\} \times \cdots \times F_i \times \cdots \times \{\pi_k(p)\}$. We will denote by $\Pi_i^p$ the projection $\Pi_i^p : M \to F_i(p)$ given by $\Pi_i^p(x) = (\pi_1(p), \ldots, \pi_i(x), \ldots, \pi_k(p))$. Analogously, $\Sigma_i^p : M \to S_i(p)$ is given by $\Sigma_i^p(x) = (\sigma_1(p), \ldots, \sigma_i(x), \ldots, \sigma_{k'}(p))$.

**Theorem 6.4.** *Let $M = F_0 \times \cdots \times F_k$ be a complete semi-Riemannian direct product with $F_0$ a maximal semi-euclidean factor and each $F_i$ indecomposable for $i > 0$. If $M = S_0 \times \cdots \times S_{k'}$ is another decomposition with $S_0$ a maximal semi-euclidean factor and each $S_j$ indecomposable for $j > 0$ such that $\mathcal{F}_i(p) \cap \mathcal{S}_j(p)$ is zero or a nondegenerate space for some $p \in M$ and all $i, j$, then $k = k'$ and, after rearranging, $\mathcal{F}_i = \mathcal{S}_i$ for all $i \in \{0, \ldots, k\}$.*

*Proof.* Fix $x \in M$ and suppose that $dim S_1(x) > 1$ and $\mathcal{S}_1(x) \neq \mathcal{F}_i(x)$ for all $i \in \{0, \ldots, k\}$. Using the above lemma we have that $\mathcal{S}_1(x) \cap \mathcal{F}_i(x) \neq 0$ for some $i \in \{0, \ldots, k\}$. Moreover, since $\mathcal{S}_1(x) \neq \mathcal{F}_i(x)$ it holds $\mathcal{S}_1(x) \cap \mathcal{F}_i(x) \neq \mathcal{S}_1(x)$ or $\mathcal{S}_1(x) \cap \mathcal{F}_i(x) \neq \mathcal{F}_i(x)$. We suppose the first one (the second case is similar).

Proposition 6.1 ensures that $\mathcal{T}_1 = \mathcal{F}_i \cap \mathcal{S}_1$ is a regular foliation and, since $S_1(x)$ is a geodesic submanifold, $\mathcal{T}_1$ and $\mathcal{T}_2 = \mathcal{T}_1^{\perp} \cap \mathcal{S}_1$ are two geodesic and nondegenerate foliations in $S_1(x)$. We can choose $p \in S_1(x)$ such that the leaf $T_2(p)$ of $\mathcal{T}_2$ has no holonomy. We want to show that $T_1(p) \cap T_2(p) = \{p\}$ and apply Corollary 3.8. For this, fix an orthonormal basis in $T_pM$ and take a definite positive metric such that this basis is orthonormal too. Denote by $|\cdot|$ its associated norm. Given $\gamma \in \Omega_p^M(t_1, \ldots, t_m)$ with $v_\gamma \equiv (v_0, \ldots, v_m)$ we call $|\gamma| = \sum_{j=0}^m |v_j|$.

Suppose there is $q \in T_1(p) \cap T_2(p)$ with $p \neq q$. Then it exists a curve in $\Omega_p^{T_2(p)}(t_1, \ldots, t_m)$ joining $p$ and $q$ for certain $0 < t_1 < \cdots < t_m < 1$ and so we can define

$$r = \inf\{|\gamma| : \gamma \in \Omega_p^{T_2(p)}(t_1, \ldots, t_m) \text{ and } \gamma(1) = q\}.$$

We have that

- $r > 0$. In fact, if $r = 0$ then it exists $\gamma \in \Omega_p^{T_2(p)}(t_1, \ldots, t_m)$ with $\gamma(1) = q$ which lays in a neighborhood of $p$ adapted to both foliations $\mathcal{T}_1$ and $\mathcal{T}_2$ and regular for $\mathcal{T}_1$. But since $T_1(p) = T_1(q)$ and $\gamma$ is a curve is $T_2(p)$, the only possibility is $p = q$, which is a contradiction.
- $r$ is a minimum. Take a sequence $\gamma_n \in \Omega_p^{T_2(p)}(t_1, \ldots, t_m)$ with $v_{\gamma_n} \equiv (v_0^n, \ldots, v_m^n)$, $\gamma_n(1) = q$ and $|\gamma_n| \to r$. Then we can extract a convergent subsequence of $(v_0^n, \ldots, v_m^n)$ to, say, $(v_0, \ldots, v_m)$. Take $\gamma_0 \in \Omega_p^{T_2(p)}(t_1, \ldots, t_m)$ with $v_{\gamma_0} \equiv (v_0, \ldots, v_m)$. Using the differentiable dependence of the solution respect to the initial conditions and the parameters of an ordinary differential equation (see Appendix I of [14]), it is easy to show that $\gamma_0(1) = \lim_{n\to\infty} \gamma_n(1) = q$. Since $|\gamma_0| = r$, the infimum is reached.

Now, take the map $\eta = \Sigma_1^p \circ \Pi_i^p : M \to S_1(p)$, which holds

- $\eta(T_2(p)) \subset T_2(p)$, since $\eta$ takes geodesics into geodesics and $\eta_{*p}(\mathcal{T}_2(p)) = \mathcal{T}_2(p)$.
- $\eta(p) = p$ and $\eta(q) = q$.
- $|\eta_{*p}(v)| \leq |v|$ and the equality holds if and only if $v \in \mathcal{T}_1(p)$.

Consider the broken geodesic $\alpha = \eta \circ \gamma_0 \in \Omega_p^{T_2(p)}(t_1, \ldots, t_m)$. Then, using that $\eta$ commutes with the parallel translation along any curve, we have $v_\alpha(t) = \eta_{*p}(v_{\gamma_0}(t)) \equiv (\eta_{*p}(v_0), \ldots, \eta_{*p}(v_m))$, and so $|\alpha| < |\gamma_0|$. Since $\alpha(1) = q$ we get a contradiction.

Therefore $T_1(p) \cap T_2(p) = \{p\}$ and $S_1(p)$ can be decomposed as $T_1(p) \times T_2(p)$, which is a contradiction because $S_1$ is indecomposable. The contradiction comes from supposing that $\mathcal{S}_1(x) \neq \mathcal{F}_i(x)$ for all $i \in \{0, \ldots, k\}$, thus it has to hold that $\mathcal{S}_1(x) = \mathcal{F}_1(x)$ for example. But this means $\mathcal{S}_1 = \mathcal{F}_1$.

Applying repeatedly the above reasoning we can eliminate the factors with dimension greater than one, except $S_0$, in the decomposition $S_0 \times \cdots \times S_{k'}$, reducing the problem to prove the uniqueness of the decomposition of a semi-Riemannian direct product $S_0 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, where $S_0$ is semi-euclidean. But, in this product, we can trivially change the metric to obtain a Riemannian direct product where we can apply [7]. □

Observe that the nondegeneracy hypothesis is redundant in the Riemannian case. On the contrary, in the semi-Riemannian case it is necessary as the following example shows.

**Example 6.5.** Take $L$ a complete and simply connected Lorentzian manifold with a parallel lightlike vector field $U$, but such that $L$ can not be decomposed as a direct product, (for example a plane fronted wave, [4]). Take $M = L \times \mathbb{R}$ with the product metric and $X = U + \partial_t$. Then $X$ is a spacelike and parallel vector field and since $M$ is complete and simply connected, $M$ splits as a direct product with the integral curves of $X$ as a factor. Thus $M$ admits two different decomposition as direct product, although $L$ is indecomposable.

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