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Functions of class C^1 subject to a Legendre condition in an enhanced density set

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Abstract. We investigate certain second-order differential properties of functions and forms of class C^1 at the points around which a suitable Legendrian condition is "very densely verified". In particular we provide a generalization of the classical identity $d^2 = 0$ on differential forms and some results about second-order osculating properties of graphs. Particular emphasis is placed on the case when the condition is verified in a locally finite perimeter set. A conjecture about the C^2-rectifiability of the horizontal projection of a Legendrian rectifiable set is discussed.

1. Introduction

In this paper we investigate certain second-order differential properties of functions and forms of class C^1 at the points around which a suitable Legendrian condition is "very densely verified". In order to make more precise the argument behind such a rough expression, we introduce the following definition.

Definition 1.1. Let Ω be a measurable subset of $\mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is said to be a "point of enhanced density of Ω" if

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B(x,r) \backslash \Omega)}{r^{n+1}} = 0.$$

We say that "Ω is an enhanced density set" whenever a.e. $x \in \Omega$ is a point of enhanced density of Ω.

The first application we give of this notion is in Theorem 2.1, which generalizes the classical identity $d^2 = 0$ on smooth differential forms: *If λ and μ are C^1-differential forms in $\mathbb{R}^n$ such that $d\lambda = \mu$ in a measurable set Ω, then $d\mu(x) = 0$ whenever x is a point of enhanced density of Ω.* Combining Theorem 2.1 with the theory developed in [6], we get Theorem 3.1: *If $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$*

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and $\Psi \in C^1(\mathbb{R}^n; M_{k,n})$ satisfy $Df = \Psi$ in a measurable set Ω, then there exists the approximate tangent paraboloid to the graph of $f|\Omega$ at each point of enhanced density of Ω.

In Lemma 4.1 we state that: *Every locally finite perimeter set is an enhanced density set.* This fact (quite unexpected for us) makes enhanced density sets a common topic in many geometric measure theoretic contexts. As a consequence of Lemma 4.1 we obtain the "global versions" of the two results mentioned above, namely Theorem 2.1 and Theorem 3.1, under the additional assumption that Ω has locally finite perimeter. In particular, Corollary 4.2 follows: *If $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$ and $\Psi \in C^1(\mathbb{R}^n; M_{k,n})$ satisfy $Df = \Psi$ in a locally finite perimeter set Ω, then the graph of $f|\Omega$ is a n-dimensional C^2-rectifiable set.*

A conjecture about the C^2-rectifiability of the horizontal projection of a Legendrian rectifiable set is posed in the last section, where a sketch-proof based on Corollary 4.2 is provided too.

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2. Differential forms of class C^1. A generalization of the classical identity $d^2 = 0$

Theorem 2.1. *Let λ and μ be differential forms of class C^1 in $\mathbb{R}^n$, respectively of degree h and $h+1$ (with $h \geq 0$). If x_0 is a point of enhanced density of*

$$K := \{x \in \mathbb{R}^n \mid d\lambda(x) = \mu(x)\}$$

then

(i) *$x_0 \in K$;*

(ii) *μ is closed at x_0, i.e., $d\mu(x_0) = 0$.*

Proof. (i) It follows by observing that K is closed.

(ii) For $h+1 \geq n$ the assertion is trivial, hence we can assume

$$h \leq n-2.$$

Let $\rho \in (0,1)$ and consider $\varphi \in C_c^2(B(0,1))$ such that

$$0 \leq \varphi \leq 1, \quad \varphi|B(0,\rho) \equiv 1$$

and

$$|D_h\varphi| \leq \frac{2}{1-\rho} \quad (h = 1, \dots, n).$$

For $r > 0$ and $x \in \mathbb{R}^n$, define

$$\varphi_r(x) := \varphi\left(\frac{x - x_0}{r}\right).$$

Then

$$D_h\varphi_r(x) = \frac{1}{r}\, D_h\varphi\Big(\frac{x-x_0}{r}\Big)$$

hence

$$|D_h\varphi_r| \le \frac{2}{r(1-\rho)}. \tag{2.1}$$

Given an arbitrary $(n-2-h)$-form θ of class C^2 in $\mathbb{R}^n$, one has

$$d(\varphi_r\,\mu\wedge\theta) = d\varphi_r\wedge\mu\wedge\theta + \varphi_r\,d\mu\wedge\theta + (-1)^{h+1}\varphi_r\,\mu\wedge d\theta$$

by the differentiation formula for the wedge product of forms (see, e.g., Section 4.1.6 of [12]). If set for simplicity $B_r := B(x_0, r)$ and observe that

$$\int_{B_r} d(\varphi_r\,\mu\wedge\theta) = \int_{\partial B_r} \varphi_r\,\mu\wedge\theta = 0$$

by the Stokes theorem, we get

$$\begin{aligned}
\int_{B_r} \varphi_r\,d\mu\wedge\theta &= -\int_{B_r} d\varphi_r\wedge\mu\wedge\theta + (-1)^h\int_{B_r}\varphi_r\,\mu\wedge d\theta\\
&= -\int_{B_r\cap K} d\varphi_r\wedge d\lambda\wedge\theta - \int_{B_r\setminus K} d\varphi_r\wedge\mu\wedge\theta\, +\\
&\qquad + (-1)^h\int_{B_r\cap K}\varphi_r\,d\lambda\wedge d\theta + (-1)^h\int_{B_r\setminus K}\varphi_r\,\mu\wedge d\theta\\
&= \int_{B_r} -d\varphi_r\wedge d\lambda\wedge\theta + (-1)^h\varphi_r\,d\lambda\wedge d\theta\, +\\
&\qquad + \int_{B_r\setminus K} d\varphi_r\wedge(d\lambda-\mu)\wedge\theta + (-1)^h\varphi_r\,(\mu-d\lambda)\wedge d\theta.
\end{aligned}$$

But in the last member of this equality the integral over B_r is zero. Indeed, since φ_r and θ are of class C^2, a standard computation based on the differentiation formula for the wedge product of forms (see, e.g., Section 4.1.6 of [12]) shows that

$$-d\varphi_r\wedge d\lambda\wedge\theta + (-1)^h\varphi_r\,d\lambda\wedge d\theta = d\big(d\varphi_r\wedge\lambda\wedge\theta + (-1)^h\varphi_r\,\lambda\wedge d\theta\big)$$

hence

$$\int_{B_r} -d\varphi_r\wedge d\lambda\wedge\theta + (-1)^h\varphi_r\,d\lambda\wedge d\theta = 0$$

by the Stokes theorem.

Thus

$$\int_{B_r}\varphi_r\,d\mu\wedge\theta = \int_{B_r\setminus K} d\varphi_r\wedge(d\lambda-\mu)\wedge\theta + (-1)^h\varphi_r\,(\mu-d\lambda)\wedge d\theta.$$

It follows from (2.1) that there exists a number C, not depending on r and ρ, such that

$$\left|\int_{B_r}\varphi_r\, d\mu\wedge\theta\right|\le C\,\mathcal{L}^n(B_r\backslash K)\left(\frac{1}{r(1-\rho)}+1\right).$$

On the other hand, the triangle inequality implies

$$\left|\int_{B_r}\varphi_r\, d\mu\wedge\theta\right|\ge\left|\int_{B_{\rho r}}\varphi_r\, d\mu\wedge\theta\right|-\left|\int_{B_r\backslash B_{\rho r}}\varphi_r\, d\mu\wedge\theta\right|,$$

hence there are two numbers C_1 and C_2, which do not depend on r and ρ, such that

$$\begin{aligned}\rho^n\left|\fint_{B_{\rho r}} d\mu\wedge\theta\right| &\le \left|\fint_{B_r}\varphi_r\, d\mu\wedge\theta\right|+\frac{1}{\mathcal{L}^n(B_r)}\left|\int_{B_r\backslash B_{\rho r}}\varphi_r\, d\mu\wedge\theta\right|\\ &\le \frac{C_1\mathcal{L}^n(B_r\backslash K)}{r^n}\left(\frac{1}{r(1-\rho)}+1\right)+\frac{C_2(r^n-\rho^n r^n)}{r^n}\\ &= C_1\frac{\mathcal{L}^n(B_r\backslash K)}{r^{n+1}}\left(\frac{1}{1-\rho}+r\right)+C_2(1-\rho^n).\end{aligned}$$

Passing to the limit for $r\downarrow 0$, we obtain

$$\rho^n\big|\langle d\mu(x_0)\wedge\theta(x_0)\,,\,dx_1\wedge\ldots\wedge dx_n\rangle\big|\le C_2(1-\rho^n).$$

Then, letting $\rho\uparrow 1$, we find

$$\langle d\mu(x_0)\wedge\theta(x_0)\,,\,dx_1\wedge\ldots\wedge dx_n\rangle=0.$$

The conclusion follows at once from the arbitrariness of θ. □

Remark 2.1. The classical identity $d^2\omega=0$ for a differential form ω of class C^2 in $\mathbb{R}^n$ follows immediately by applying Theorem 2.1 with $\lambda:=\omega$ and $\mu:=d\omega$.

Remark 2.2. The closure of μ at x_0 is false, in general, if one simply assumes that x_0 is a density point (instead of an enhanced density point). For example, consider

$$\mu(x):=-x_2\,dx_1+x_1\,dx_2,\quad x=(x_1,x_2)\in\mathbb{R}^2.$$

By Theorem 1 in [1] we can find $\lambda\in C^1(\mathbb{R}^2)$ supported in $B(0,1)$ and such that

$$\mathcal{L}^2\big(\{x\in B(0,1)\,|\,d\lambda(x)=\mu(x)\}\big)\ge 1.$$

But $\{x\in B(0,1)\,|\,d\lambda(x)=\mu(x)\}$ is just the set K defined in the statement of Theorem 2.1, hence $\mathcal{L}^2(K)\ge 1$. So the density points of K form a set of positive measure and at each of them (as long as at each point in $\mathbb{R}^2$) the form μ is not closed. We observe, incidentally, that K does not contain enhanced density points.

3. Second order osculating properties of EC^1 graphs

3.1. A Whitney-type extension problem

We begin this subsection with a definition.

Definition 3.1. Let Ω be any subset of $\mathbb{R}^n$. Then $EC^1(\Omega, \mathbb{R}^k)$ denotes the class of maps $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$ for which there exists $\Psi \in C^1(\mathbb{R}^n; M_{k,n})$ such that $\Psi|\Omega = (Df)|\Omega$.

A Whitney-type extension problem concerning $f \in EC^1(\Omega, \mathbb{R}^k)$ arises naturally:

> **Problem.** *Does $f|\Omega$ have an extension of class C^2, namely a map $h \in C^2(U, \mathbb{R}^k)$ with U open subset of $\mathbb{R}^n$, $\Omega \subset U$ and $h|\Omega = f|\Omega$?*

Remark 3.1. In the special case when Ω is open, the answer to the question set in Problem above is trivially "yes"(with $U = \Omega$ and $h = f|\Omega$). In general the answer is "no". A counterexample can be obtained for $n = k = 1$ by considering the function introduced in the Appendix of [4] and defined as

$$f(x) := \int_0^x \operatorname{dist}(t, \Omega)^{1/2} dt \quad (x \in \mathbb{R})$$

where Ω is a suitable Cantor-like closed subset of $[0, 1]$ of positive measure. Then f belongs to $EC^1(\Omega, \mathbb{R})$, because it satisfies the assumptions in Definition 3.1 with $\Psi := 0$. However it has been proved in [4] that the graph of $f|\Omega$ has zero-measure intersection with every C^2 graph, namely it is not a 1-dimensional C^2-rectifiable. In particular $f|\Omega$ has no extension of class C^2.

3.2. Parabolic blow-up of a $EC^1(\Omega, \mathbb{R}^k)$ graph over a point of enhanced density

Let Ω be a subset of $\mathbb{R}^n$, $f \in EC^1(\Omega, \mathbb{R}^k)$ and Ψ be as in Definition 3.1. Then consider $x_0 \in \mathbb{R}^n$ and the second degree polynomial Γ^{x_0} defined as

$$\Gamma^{x_0}(\xi) := \frac{1}{2} \sum_{i,j=1}^{n} \xi_i \xi_j D_j \Psi_{*i}(x_0) \quad (\xi \in \mathbb{R}^n)$$

where $\Psi_{*i} := (\Psi_{1i}, \dots, \Psi_{ki})$.

Remark 3.2. If Ω is an open subset of $\mathbb{R}^n$ then $f|\Omega$ is of class C^2 and $\Psi|\Omega = D(f|\Omega)$, compare with Remark 3.1. Hence, for $x_0 \in \Omega$, the map Γ^{x_0} is just the second degree term in the second order Taylor polynomial of $f|\Omega$ at x_0.

Let $G_{f|\Omega}$ and $G_{\Gamma^{x_0}}$ denote the graphs of $f|\Omega$ and Γ^{x_0}, respectively. Also consider the family of nonhomogeneous dilatations

$$T^{x_0}_\rho : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k \quad (\rho > 0)$$

defined by

$$T_\rho^{x_0}(x,y) := \left(\frac{x-x_0}{\rho}, \frac{y-f(x_0)-Df(x_0)(x-x_0)}{\rho^2} \right).$$

The following theorem holds.

Theorem 3.1. *Let Ω be measurable and x_0 be a point of enhanced density of Ω. Then Γ^{x_0} is independent from the choice of Ψ and one has*

$$\mathcal{H}^n \llcorner T_\rho^{x_0}(G_{f|\Omega}) \to \mathcal{H}^n \llcorner G_{\Gamma^{x_0}} \quad (as\ \rho \downarrow 0)$$

in the weak sense of measures.*

Proof. The assertion about the independence of Γ^{x_0} from the choice of Ψ follows at once by Lemma 3.1 below. In order to prove the second claim, consider the 1-forms of class C^1 in $\mathbb{R}^n$

$$\omega_i := \sum_{j=1}^n \Psi_{ij} dx_j \quad (i=1,\dots,k)$$

and the corresponding sets

$$K_i := \{x \in \mathbb{R}^n \,|\, df_i(x) = \omega_i(x)\}.$$

Observe that

$$\mathcal{L}^n(\Omega \backslash K_i) = 0 \quad (i=1,\dots,k)$$

hence x_0 has to be a point of enhanced density of each K_i. Theorem 2.1 implies that the forms ω_i are closed at x_0, namely the Schwartz-like equality

$$\frac{\partial \Psi_{ij}}{\partial x_m}(x_0) = \frac{\partial \Psi_{im}}{\partial x_j}(x_0)$$

holds for all $i=1,\dots,k$ and $j,m=1,\dots,n$. The conclusion follows easily from Corollary 4.2 of [6]. □

Remark 3.3. The argument (based on Theorem 2.1) used in the proof of Theorem 3.1 shows that Corollary 4.2 of [6] holds even without assuming the Schwartz-like condition.

The following lemma is completely elementary. However, since we have no reference to cite for it, we will provide a short proof.

Lemma 3.1. *Let a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, a subset Ω of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ be given such that:*

(i) *φ is differentiable (hence continuous!) at x;*

(ii) *x is a point of density of $\Omega \cap \varphi^{-1}(0)$.*

Then $\nabla\varphi(x) = 0$.

Proof. Assume $\nabla\varphi(x) \neq 0$ and show that a contradiction follows. For all $j = 1, 2, \ldots$, we can find $x_j \in \Omega \cap B(x, 1/j)\backslash\{x\}$ such that $\varphi(x_j) = 0$ and

$$u_j := \frac{x_j - x}{\|x_j - x\|} \to \frac{\nabla\varphi(x)}{\|\nabla\varphi(x)\|} \quad (\text{as } j \to \infty).$$

On the other hand,

$$0 = \varphi(x_j) = \varphi(x + \|x_j - x\| u_j) = \|x_j - x\| \, \nabla\varphi(x) \cdot u_j + o(\|x_j - x\|),$$

namely

$$\nabla\varphi(x) \cdot u_j \to 0 \quad (\text{as } j \to \infty).$$

It follows that $\|\nabla\varphi(x)\| = 0$, which contradicts our assumption. □

3.3. The case when Ω is an enhanced density set: C^2-rectifiability of $EC^1(\Omega, \mathbb{R}^k)$ graphs

Recall that a subset of a Euclidean space is said to be a n-dimensional C^2-rectifiable set if $\mathcal{H}^n$-almost all of it may be covered by countably many n-dimensional submanifolds of class C^2. The notion of C^k-rectifiable set has been introduced in [4] and provides a natural setting for the description of singularities of convex functions and convex surfaces [2], [3]. More generally, it can be used to study the singularities of surfaces with generalized curvatures [3]. Rectifiability of class C^2 is strictly related to the context of Legendrian rectifiable subsets of $\mathbb{R}^N \times \mathbb{S}^{N-1}$ (see [13], [14], [7], [8]). The level sets of a $W^{k,p}_{\text{loc}}$ mapping between manifolds are rectifiable sets of class C^k [5]. Applications to geometric variational problems can be found in [9].

We know from Remark 3.1 that the graph of $f|\Omega$, with $f \in EC^1(\Omega, \mathbb{R}^k)$, does not need to be C^2-rectifiable. The following result shows that C^2-rectifiability occurs whenever Ω is an enhanced density set.

Theorem 3.2. *Let Ω be an enhanced density subset of $\mathbb{R}^n$ and $f \in EC^1(\Omega, \mathbb{R}^k)$. Then the graph of $f|\Omega$ is a n-dimensional C^2-rectifiable set.*

Proof. Let M denote the graph of $f|\Omega$ and x_0 be a point of enhanced density of Ω. Then it will be enough to show that the conditions (i), (ii) and (iii) in the statement (c) of Theorem 3.5 in [4] are verified at $(x_0, f(x_0))$. Since (i) and (iii) are trivial, we just have to prove the existence of the approximate tangent paraboloid to M at such a point. To this aim let Ψ be as in Definition 3.1, hence $Df(x_0) = \Psi(x_0)$. Without affecting the generality of the argument below, we can assume that

$$x_0 = 0, \quad f(x_0) = f(0) = 0.$$

Let Π denote both the tangent space to the graph of f at $(x_0, f(x_0)) = (0, 0)$ and the orthogonal projection from $\mathbb{R}^n \times \mathbb{R}^k$ onto the tangent space itself. Similarly, the orthogonal complement of Π in $\mathbb{R}^n \times \mathbb{R}^k$ and the corresponding orthogonal projection operator will both be denoted by $\Pi^\perp$. Consider the map $\varphi \in C^1(\mathbb{R}^n, \Pi)$ defined as

$$\varphi(x) := \Pi(x, f(x)) \quad (x \in \mathbb{R}^n)$$

and let $r > 0$ be such that

$$\varphi | B(0,r) : B(0,r) \to U := \varphi\big(B(0,r)\big)$$

is invertible, with inverse $\mu \in C^1(U, B(0,r))$. Then define

$$\widetilde{\Omega} := \varphi\big(\Omega \cap B(0,r)\big)$$
$$\tilde{f}(u) := \Pi^\perp\big(\mu(u), f(\mu(u))\big) \quad (u \in U)$$

and

$$\widetilde{\Psi}(u) := \big[\Pi^\perp \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\big)\big] \circ \big[\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\big)\big]^{-1} \quad (u \in U).$$

Observe that, for $u \in \widetilde{\Omega}$, one has

$$\begin{aligned} \mathrm{Id}_\Pi &= D^\Pi\big((\varphi|B(0,r)) \circ \mu\big)(u) = D\varphi(\mu(u))\, D^\Pi \mu(u) \\ &= \big[\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, Df(\mu(u))\big)\big] D^\Pi \mu(u) \\ &= \big[\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\big)\big] D^\Pi \mu(u). \end{aligned}$$

In particular, it follows that $\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(0)\big)$ is invertible. Hence, by making r smaller if need be, we obtain that $\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(x)\big)$ is invertible for all $x \in B(0,r)$ and also that

$$\big[\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\big)\big]^{-1} = D^\Pi \mu(u)$$

whenever $u \in \widetilde{\Omega}$. We get

$$\begin{aligned} D^\Pi \tilde{f}(u) &= \big[\Pi^\perp \circ \big(\mathrm{Id}_{\mathbb{R}^n}, Df(\mu(u))\big)\big] D^\Pi \mu(u) \\ &= \big[\Pi^\perp \circ \big(\mathrm{Id}_{\mathbb{R}^n}, Df(\mu(u))\big)\big] \circ \big[\Pi \circ \big(\mathrm{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\big)\big]^{-1} \\ &= \widetilde{\Psi}(u) \end{aligned}$$

for all $u \in \widetilde{\Omega}$.

Observe that, in our notation, the family of nonhomogeneous dilatations considered in [4] takes the following form ($\rho > 0$):

$$\widetilde{T}_\rho : \Pi \times \Pi^\perp \to \Pi \times \Pi^\perp, \quad \widetilde{T}_\rho(u,v) := \left(\frac{u}{\rho}, \frac{v}{\rho^2}\right).$$

Since

$$\tilde{f}(0) = 0, \quad D^\Pi \tilde{f}(0) = 0,$$

one has

$$\widetilde{T}_\rho(u,v) = \left(\frac{u}{\rho}, \frac{v - \tilde{f}(0) - D^\Pi \tilde{f}(0) u}{\rho^2}\right).$$

Moreover, 0 is a point of enhanced density of $\widetilde{\Omega}$ (relatively to Π) and the graph of $\widetilde{f}|\widetilde{\Omega}$ coincides with the graph of $f|B(0,r)\cap\Omega$. Then the argument used in the proof of Theorem 3.1 (based on Corollary 4.2 of [6]) shows that

$$\mathcal{H}^n \llcorner \widetilde{T}_\rho(M) \to \mathcal{H}^n|G_{\widetilde{\Gamma}} \quad (\text{as } \rho\downarrow 0)$$

where $G_{\widetilde{\Gamma}}$ is the graph of the second degree polynomial

$$\widetilde{\Gamma}(\xi) := \frac{1}{2}\sum_{i,j=1}^{n} \xi_i\xi_j D_j\widetilde{\Psi}_{*i}(0) \quad (\xi\in\Pi).$$

This definitely proves that condition (ii) in the statement (c) of Theorem 3.5 in [4] holds. □

4. The case of Caccioppoli sets, corollaries

The following somehow unexpected (nevertheless very simple to prove!) result holds.

Lemma 4.1. *Let Ω be a locally finite perimeter subset of $\mathbb{R}^n$. Then*

$$\lim_{r\downarrow 0}\frac{\mathcal{L}^n(B(x,r)\backslash\Omega)}{r^{n+\frac{n}{n-1}}}=0$$

at a.e. $x\in\Omega$. In particular Ω is an enhanced density set.

Proof. By applying Theorem 1 in § 6.1 of [11] (with $f=\varphi_\Omega$), one has

$$0=\lim_{r\downarrow 0}\frac{1}{r}\left(\frac{1}{r^n}\int_{B(x,r)}|\varphi_\Omega-1|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}=\lim_{r\downarrow 0}\left(\frac{1}{r^{n+\frac{n}{n-1}}}\int_{B(x,r)}\varphi_{\mathbb{R}^n\backslash\Omega}\right)^{\frac{n-1}{n}}$$

for a.e. $x\in\Omega$. Hence the conclusion follows. □

From Lemma 4.1 and Theorem 2.1 we get at once the following result.

Corollary 4.1. *Let λ and μ be differential forms of class C^1 in $\mathbb{R}^n$, respectively of degree h and $h+1$ (with $h\geq 0$). Assume that $d\lambda=\mu$ almost everywhere in a locally finite perimeter subset Ω of $\mathbb{R}^n$. Then μ is closed at almost every point in Ω.*

The combination of Lemma 4.1 and Theorem 3.2 gives the second corollary.

Corollary 4.2. *Let Ω be a locally finite perimeter subset of $\mathbb{R}^n$ and $f\in EC^1(\Omega,\mathbb{R}^k)$. Then the graph of $f|\Omega$ is a n-dimensional C^2-rectifiable set.*

Remark 4.1. Let Ω be a measurable subset of $\mathbb{R}^n$ and let $f\in C^1(\mathbb{R}^n,\mathbb{R}^k)$. Denote by $[\![G_{f|\Omega}]\!]$ the multiplicity one rectifiable current naturally associated to the graph $G_{f|\Omega}$ of $f|\Omega$, namely the one carried by $G_{f|\Omega}$ and oriented by η such that

$$\eta(x,f(x)) = \frac{\wedge^n(\mathrm{Id}_{\mathbb{R}^n}, Df(x))(e_1\wedge\ldots\wedge e_n)}{\|\wedge^n(\mathrm{Id}_{\mathbb{R}^n}, Df(x))(e_1\wedge\ldots\wedge e_n)\|} \quad \text{(at a.e. } x\in\Omega)$$

where $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{R}^n$. In particular, for the purely-horizontal stratum $\eta_{(0)}$ of η, the following equality holds:

$$\|\wedge^n(\mathrm{Id}_{\mathbb{R}^n}, Df(x))(e_1\wedge\ldots\wedge e_n)\|\,\eta_{(0)}(x,f(x)) = e_1\wedge\ldots\wedge e_n \quad \text{(at a.e. } x\in\Omega).$$

Observe that, if $\pi:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ denotes the orthogonal projection, then $\pi_\#[\![G_{f|\Omega}]\!]$ is just the current $[\![\Omega]\!]$ corresponding to the canonically oriented integration of forms over Ω. Indeed, for every smooth differential n-form ω with compact support in $\mathbb{R}^n$, one has (compare with Section 26 of [16])

$$\begin{aligned}\pi_\#[\![G_{f|\Omega}]\!](\omega) &= \int_{G_{f|\Omega}} \langle\omega\circ\pi,\eta_{(0)}\rangle\, d\mathcal{H}^n\\ &= \int_\Omega \langle\omega(x),\eta_{(0)}(x,f(x))\rangle\,\|\wedge^n(\mathrm{Id}_{\mathbb{R}^n}, Df(x))(e_1\wedge\ldots\wedge e_n)\|\, d\mathcal{H}^n(x)\\ &= \int_\Omega \langle\omega, e_1\wedge\ldots\wedge e_n\rangle\, d\mathcal{H}^n = [\![\Omega]\!](\omega).\end{aligned}$$

Hence and by recalling Sections 26.21 of [16] or Proposition 3, Section 2.3, Chapter 2 in [15], we easily obtain that Ω is a locally finite perimeter set if and only if $\partial[\![G_{f|\Omega}]\!]$ has locally finite mass.

By virtue of the previous remark, Corollary 4.2 can be restated in the following form (which is interesting in view of the next section).

Corollary 4.3. *Let Ω be a measurable subset of $\mathbb{R}^n$, $f\in EC^1(\Omega,\mathbb{R}^k)$ and assume that $\partial[\![G_{f|\Omega}]\!]$ has locally finite mass. Then $G_{f|\Omega}$ is a n-dimensional C^2-rectifiable set.*

Remark 4.2. The example considered in Remark 3.1 shows that without the finiteness conditions assumed in Corollaries 4.2 and 4.3, the C^2-rectifiability of $G_{f|\Omega}$ fails to be true (in general).

5. Legendrian rectifiable subsets of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$. C^2-rectifiability of the horizontal projection (open problem)

5.1. Legendrian rectifiable subsets of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$

Set

$$X := \mathbb{R}^{n+1}, \quad Y := \mathbb{R}^{n+1}.$$

Throughout this section X and Y will also denote, respectively, the projection maps

$$X\times Y\to X, \quad (x;y)\mapsto x$$

and

$$X \times Y \to Y, \quad (x; y) \mapsto y.$$

Let x_j and y_j be the canonical coordinates of X and Y, respectively. If $e_1, \ldots, e_{n+1}$ is the canonical basis of X, then define

$$X_* := \text{span}\{e_1, \ldots, e_n\}.$$

Indicate by J the trivial map identifying X and Y, namely

$$J : X \to Y, \quad J(x_1, \ldots, x_{n+1}) := (x_1, \ldots, x_{n+1}).$$

Consider the usual contact form on $X \times Y$

$$\alpha = \sum_{j=1}^{n+1} y_j \, dx_j .$$

Then define the notion of Legendrian rectifiable subset of $X \times Y$, compare with [13].

Definition 5.1. A n-dimensional rectifiable subset G of $X \times (Y \backslash \{0\})$ is called "Legendrian" if

$$\alpha(x; y) | T_{(x;y)} G = 0 \tag{5.1}$$

for $\mathcal{H}^n \llcorner G$ almost every $(x; y)$. Here $T_{(x;y)}G$ denotes the approximate tangent plane to G at $(x; y)$ in $X \times Y$.

Remark 5.1. Let R be the set of points $(x; y)$ in a Legendrian rectifiable set G at which $T_{(x;y)}G$ exists and $X|T_{(x;y)}G$ has rank n. Then for almost every $(x; y) \in R$ there exists $T_x(XG)$ (i.e., the approximate tangent plane to XG at x) and one has

$$T_x(XG) = X(T_{(x;y)}G).$$

As a consequence, the condition (5.1) yields

$$y \perp T_x(XG)$$

at $\mathcal{H}^n \llcorner R$ almost every $(x; y)$. The area formula (see, e.g., Section 3.2 of [12] and Chapter 2 of [16]) yields $\mathcal{H}^n(X(G \backslash R)) = 0$. Hence $X(R)$ and $X(G)$ coincide, possibly except for a zero set.

We have the following characterization.

Proposition 5.1. *Let Σ be a n-dimensional rectifiable subset of X. Then Σ is the image through X of a Legendrian rectifiable subset of $X \times Y$ if and only if the following condition is verified:*
There exist countably many measurable subsets Ω_j of X_*, functions $f_j \in EC^1(\Omega_j, \mathbb{R})$ and linear isometries L_j in X such that

$$\Sigma = \bigcup_j L_j \Gamma_j, \quad \Gamma_j := \{(x_*, f_j(x_*)) \in X \mid x_* \in \Omega_j\} \tag{5.2}$$

possibly except for a zero set.

Proof. Assume that $\Sigma = X(G)$ where G is a Legendrian rectifiable set in $X \times Y$. Consider

$$R := \left\{(x;y) \in G \,\middle|\, T_{(x;y)}G \text{ exists and } X|T_{(x;y)}G \text{ has rank } n\right\}$$

and observe that

$$X(R) = \Sigma$$

possibly except for a zero set. By Remark 5.1, we may find countably many functions $f_j \in C^1(X_*)$, measurable subsets Ω_j of X_*, maps $\Phi_j \in C^1(X_*, X\backslash\{0\})$ and linear isometries L_j in X such that:

- One has

  $$R = \bigcup_j R_j, \quad R_j := \left\{\left(L_j(x_*, f_j(x_*)); JL_j\Phi_j(x_*)\right) \,\middle|\, x_* \in \Omega_j\right\}$$

  possibly except for a zero set, hence (5.2) holds. Observe that

  $$L_j\left(T_{(x_*,f_j(x_*))}\Gamma_j\right) = T_{(x_*,f_j(x_*))}X(R_j) = T_{(x_*,f_j(x_*))}\Sigma$$

  at a.e. $x_* \in \Omega_j$ and for all j;

- The vector $\Phi_j(x_*)$ is orthogonal to $T_{(x_*,f_j(x_*))}\Gamma_j$ at a.e. $x_* \in \Omega_j$ and for all j.

It follows that, for every j, there exists a measurable function $c_j : \Omega_j \to \mathbb{R}$ satisfying

$$\Phi_j = c_j\left(\nabla f_j - e_{n+1}\right)$$

a.e. in Ω_j. Since Φ_j does not vanish, c_j does not vanish too. In particular, $\Phi_j \cdot e_{n+1}$ is nonzero a.e. in Ω_j. Hence, without loss of generality, we can even assume that $\Phi_j \cdot e_{n+1}$ does not vanish in all of X_*. Then the map

$$\Psi_j := -\frac{\Phi_j}{\Phi_j \cdot e_{n+1}} + e_{n+1}$$

belongs to $C^1(X_*)$ and one has $\nabla f_j = \Psi_j$ a.e. in Ω_j. It follows that $f_j \in EC^1(\Omega_j, \mathbb{R})$.

Vice versa, assume there exist countably many measurable subsets Ω_j of X_*, functions $f_j \in EC^1(\Omega_j, \mathbb{R})$ and linear isometries L_j in X satisfying (5.2). Then, for every j, there exists $\Psi_j \in C^1(X_*)$ such that $\nabla f_j = \Psi_j$ a.e. in Ω_j. If one defines

$$\Phi_j := \Psi_j - e_{n+1} \in C^1(X_*, X\backslash\{0\})$$

the rectifiable set

$$R := \bigcup_j \left\{\left(L_j(x_*, f_j(x_*)); JL_j\Phi_j(x_*)\right) \,\middle|\, x_* \in \Omega_j\right\} \subset X \times Y$$

is Legendrian and one has $X(R) = \Sigma$. □

5.2. A conjecture

The facts stated above raise interesting questions related to the density properties of rectifiable sets carrying locally integral currents (in the sense of §4.1.24 of [12]). As for the case of Legendrian rectifiable sets, in particular, we believe that the following conjecture holds (with the notation of the previous subsection).

Conjecture 5.1. *Let G be a Legendrian n-dimensional rectifiable subset of $X \times Y$ such that $X(G)$ carries a locally integral current. Then $X(G)$ is a n-dimensional C^2-rectifiable set.*

The reason for our "belief" can be summarized by the following rough argument which is, however, far from being a complete proof. First of all, by Proposition 5.1, there must exist countably many measurable subsets Ω_j of X_*, functions $f_j \in EC^1(\Omega_j, \mathbb{R})$ and linear isometries L_j in X such that

$$X(G) = \bigcup_j L_j \Gamma_j, \quad \Gamma_j := \{(x_*, f_j(x_*)) \in X \mid x_* \in \Omega_j\}$$

possibly except for a zero set. Now, as a consequence of the additional assumption about $X(G)$, we expect that the Ω_j can be chosen to be enhanced density sets. We suppose that Lemma 4.1 and the decomposition theorem (§27.6 of [16]) could be useful tools for proving such an assertion. It will be the subject of future work!

The conjecture above, in a slightly less general form, was first posed and discussed in [14]. Strictly related to this subject are also the papers [7], [8], [9], [10].

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