Product kernels adapted to curves in the space

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Abstract

We establish L^p-boundedness for a class of operators that are given by convolution with product kernels adapted to curves in the space. The L^p bounds follow from the decomposition of the adapted kernel into a sum of two kernels with singularities concentrated respectively on a coordinate plane and along the curve.

The proof of the L^p-estimates for the two corresponding operators involves Fourier analysis techniques and some algebraic tools, namely the Bernstein-Sato polynomials.

As an application, we show that these bounds can be exploited in the study of $L^p - L^q$ estimates for analytic families of fractional operators along curves in the space.

1. Introduction

The purpose of this paper is to establish L^p boundedness for a class of product-type convolution operators.

In the last thirty years the theory of singular integrals on product domains has been largely developed. The first case which was considered is that of a convolution operator $Tf = K * f$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, with $K(x, y) = K_1(x)K_2(y)$, $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, K_1 and K_2 being of Calderón-Zygmund type. In this case a simple iteration argument yields the L^p boundedness of T. A more involved situation is that of a convolution operator T, whose kernel K is defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and satisfies all the analogous bounds to those satisfied by $K_1 K_2$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, but cannot be decomposed as product of two kernels $K_1(x)$ and $K_2(y)$. A precise definition of such kernels, which are called "product kernels", was introduced in terms of certain differential inequalities and suitable cancellation conditions.

Several conditions on K, guaranteeing the L^p boundedness of the operator T, have been introduced [4], and many applications of the product theory to the operators arising in certain boundary value problems have been studied [12], [11]. Moreover, the euclidean spaces $\mathbb{R}^{d_j}$, $j = 1, 2$, have been replaced by appropriate nilpotent groups [9], [12], and by smooth manifolds with a geometry determined by a control distance [10].

Recently, one of the authors studied the L^p boundedness for convolution operators with product kernels adapted to curves in the plane [14], that is product-type kernels in $\mathbb{R}^2$ whose singularities are supported along a coordinate axis and a transversal curve. Here we extend these results to higher dimensional spaces. In order not to burden the exposition with notational complexities, we are going to give the full details only for $\mathbb{R}^3 = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, with $d_1 = 1$ and $d_2 = 2$, proving L^p-bounds for convolution operators by kernels singular along a coordinate plane and a transversal curve of polynomial type. In the last section we shall quickly describe how the arguments should be modified in the higher dimensional setting.

We denote an element of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ by the pair (x_1, x), where $x = (x_2, x_3) \in \mathbb{R}^2$. On $\mathbb{R}$ we consider the usual dilations by $\delta > 0$, while on $\mathbb{R}^2$ we consider the anisotropic dilation given by

$$(1.1) \qquad \delta \circ x = \left(\delta^{\frac{1}{2n}} x_2, \delta^{\frac{1}{2m}} x_3 \right), \qquad \text{with} \quad \delta > 0\, , \, m, n \in \mathbb{N}\, , \, m < n\, .$$

We denote by

$$(1.2) \qquad\qquad Q = \frac{1}{2n} + \frac{1}{2m}$$

the homogeneous dimension of $\mathbb{R}^2$ with respect to the dilations (1.1) and by $\rho(x) = x_2^{2n} + x_3^{2m}$ a smooth homogeneous norm on $\mathbb{R}^2$.

In this context the proto-typical example of a product kernel in $\mathbb{R}^3$ (we refer to Section 3 for a precise definition) is given by the distribution

$$(1.3) \qquad H(x_1, x) = C_\mu \, p.v. \frac{1}{x_1} \rho(x)^{-Q+i\mu}\, , \qquad \mu \in \mathbb{R} \setminus \{0\}\, .$$

Throughout the paper we concentrate our attention on the convolution by product-type kernels in $\mathbb{R}^3$ whose singularities are supported on a coordinate plane and on a transversal curve of finite type. A rather simple example of such a kernel, obtained by adapting H to the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(x_1) = (x_1^m, x_1^n)$, is given by

$$K(x_1, x) = C_\mu \, p.v. \left(\frac{1}{x_1} \right) \rho(x - \gamma(x_1))^{-Q+i\mu}\, , \ \mu \in \mathbb{R} \setminus \{0\}\, .$$

Notice that, if $m = n$, the subspace generated by $t \mapsto (t, t^m, t^n)$ is only two-dimensional, a situation which in principle is intermediate between the totally flat case and the one discussed in this paper.

More generally, we introduce the following class of product-type kernels.

Definition 1.1. Assume that K_0 is a product kernel on $\mathbb{R}^3$ and consider the curve $x = \gamma(x_1)$ with $\gamma(x_1) = (x_1^m, x_1^n)$, $x_1 \in \mathbb{R}$, $m, n \in \mathbb{N}$, $2 \leq m < n$. We define a distribution K by

$$(1.4) \qquad \int K(x_1, x) f(x_1, x) \, dx_1 dx := \int K_0(x_1, x) f(x_1, x + \gamma(x_1)) \, dx_1 dx$$

for a Schwartz function f on $\mathbb{R}^3$. K will be called an adapted kernel.

Here with an abuse of notation we write pairings between distributions and test functions as integrals. The kernel K given by the formula (1.4) is a well-defined tempered distribution which is singular on the coordinate plane $x_1 = 0$ and along the curve $x = \gamma(x_1)$, $x_1 \in \mathbb{R}$.

We shall prove the following theorem.

Theorem 1.2. *Let K be the distribution defined by the formula* (1.4). *Then the convolution operator $T : f \mapsto f * K$, initially defined on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, extends to a bounded operator on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.*

To prove Theorem 1.2 we decompose the adapted kernel K of T into the sum of a kernel K_1 with singularities concentrated on the coordinate plane $x_1 = 0$ and of a kernel K_2 singular along the curve $x = \gamma(x_1)$, $x_1 \in \mathbb{R}$. As in [14] we show that the multiplier associated with K_1 satisfies some Marcinkiewicz-type conditions, while K_2 is treated by means of analytic interpolation (our proof is inspired by some arguments used in [16] to prove the L^p boundedness of the Hilbert transform along curves in the plane). In particular, to apply the analytic interpolation method we need to introduce a non-isotropic version of the Riesz potentials

$$\mathcal{I}^z(u_1, u_2) := (\rho(u_1, u_2))^{z-Q} , \qquad z \in \mathbb{C} .$$

and to determine their meromorphic continuation. To extend in a meromorphic way $\mathcal{I}^z$, we study the location of its singularities by using Bernstein-Sato polynomials. Since the reader is not assumed to be familiar with Bernstein-Sato functional identities, we illustrate the definition and the basic properties of this algebraic tool in Section 2.

The results illustrated in this paper find applications in several contexts in harmonic and complex analysis (for more on this, see [12]). In particular, in the last section we show that L^p bounds for convolution by product kernels adapted to curves can be exploited in the study of $L^p - L^q$ estimates for analytic families of fractional operators [2].

Throughout the paper we will use the variable constant convention, and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place.

It is a pleasure to thank Professor Fulvio Ricci for many helpful discussions on the subject of this paper.

2. Bernstein-Sato polynomials and a family of Riesz–type kernels

Consider the polynomial

$$(2.1) \qquad\qquad \rho(u_1, u_2) := u_1^{2n} + u_2^{2m} \,,$$

with $m, n \in \mathbb{N}$, $m \geq 1$, $n > m$. Observe that ρ is homogeneous with respect to the one-parameter family of non isotropic dilations given by (1.1).

We shall often use, in the following, the relations between ρ and the euclidean norm $|\cdot|$ in $\mathbb{R}^2$

$$(2.2) \qquad\qquad A\rho(u)^{\frac{1}{2n}} \leq |u| \leq B\rho(u)^{\frac{1}{2m}} \quad \text{if } \rho(u) > 1$$

and

$$(2.3) \qquad\qquad A'\rho(u)^{\frac{1}{2m}} \leq |u| \leq B'\rho(u)^{\frac{1}{2n}} \quad \text{if } \rho(u) \leq 1$$

for some $A, B, A', B' > 0$.

Define now the distribution

$$(2.4) \qquad\qquad \mathcal{I}^z(u_1, u_2) := (\rho(u_1, u_2))^{z-Q} \,,$$

where $z \in \mathbb{C}$, $\Re e z > 0$.

Observe that both ρ and $\mathcal{I}^z$ depend on m and n. Anyway, for the sake of simplicity, we shall avoid to indicate the dependence on m and n.

We collect in the next proposition some obvious properties of $\mathcal{I}^z$.

Proposition 2.1. *If $\Re e z > 0$, then*

  i) *$\mathcal{I}^z$ is well defined as distribution and locally integrable.*

  ii) *$\mathcal{I}^z$ is a tempered distribution.*

  iii) *$\mathcal{I}^z$ is an analytic family of tempered distributions, that is, given $f \in \mathcal{S}(\mathbb{R}^2)$, the functions $z \mapsto\, <\mathcal{I}^z, f>$ are holomorphic.*

We shall now prove that $\mathcal{I}^z$ admits a meromorphic extension, with poles in a at most countable set, consisting of rational negative points. Our method is based on the theory of Bernstein-Sato polynomials.

It is well-known in algebra that, given a non-zero polynomial $p(u_1, u_2)$ with complex coefficients, there exist a non-zero polynomial $b_p(s) \in \mathbb{C}[s]$ and a differential operator $L(s)$ whose coefficients are polynomials in s, u_1, u_2, such that formally

$$(2.5) \qquad L(s)\big(p(u_1, u_2)\big)^{s+1} = b_p(s)\big(p(u_1, u_2)\big)^s \text{ for all } s \in \mathbb{C}.$$

The set of all polynomials $b_p(s) \in \mathbb{C}[s]$, such that there exists an operator L for which this formal identity is satisfied, forms an ideal, and the unique monic generator of this ideal is called the *Bernstein-Sato polynomial* of p.

In our case, for $\Re e z > 0$ we may write

$$(2.6) \qquad \big(\rho(u_1, u_2)\big)^{z-Q} = \frac{L(z-Q)\,(\rho(u_1, u_2))^{z+1-Q}}{b_\rho(z-Q)}.$$

By repeatedly using the functional equation (2.6) we may extend $\mathcal{I}^z$ to the complex plane in a meromorphic way, with poles whenever $b_\rho(z - Q + k)$ vanishes for a non-negative integer k. Therefore we shall now seek for the zeros of the Bernstein-Sato polynomial $b_\rho(z - Q)$.

According to a theorem of Kashiwara, the roots of the Bernstein-Sato polynomial are negative rational numbers. Moreover, if ρ has the particularly simple form given in (2.1), it is easy to find the roots of $b_\rho(s)$.

Lemma 2.2. *If $b_\rho(s)$ denotes the Bernstein-Sato polynomial associated to $\rho(u_1, u_2) := u_1^{2n} + u_2^{2m}$, $m < n$, $m, n \in \mathbb{N}$, then the roots of $\frac{b_\rho(s)}{s+1}$ are given by*

$$(2.7) \qquad -\frac{p_1}{2n} - \frac{p_2}{2m}, \quad 1 \le p_1 \le 2n - 1, \quad 1 \le p_2 \le 2m - 1,$$

with multiplicity one.

Proof. It is essentially due to Kashiwara [7]. See also [8] and [1, Remark 3.8]. ∎

In the following corollary we collect some observations, which will be useful in the following.

Corollary 2.3. *i) The largest root of $b_\rho(s)$ is $-Q = -\frac{1}{2m} - \frac{1}{2n}$.*
ii) -1 is a root of $b_\rho(s)$ with multiplicity two.
iii) The set of the roots of $b_\rho(s)$ is symmetric with respect to -1.

Proof. *i*) It follows obviously from (2.7) for $p_1 = 1$ and $p_2 = 1$.

ii) Observe that $-\frac{p_1}{2n} - \frac{p_2}{2m} = -1$ for $p_1 = n$ and $p_2 = m$. Then -1 is a root of multiplicity one for $\frac{b_\rho(s)}{s+1}$, whence the thesis follows.

iii) Suppose that the $-1+\delta := -\frac{\overline{p}_1}{2n} - \frac{\overline{p}_2}{2m}$ is a root of $b_\rho(s)$ for some $\overline{p}_1, \overline{p}_2 \in \mathbb{N}$, $1 \le \overline{p}_1 \le 2n-1$, $1 \le \overline{p}_2 \le 2m-1$ and some $\delta > 0$. Take $\tilde{p}_1 := 2n - \overline{p}_1$ and $\tilde{p}_2 := 2m - \overline{p}_2$. Since $1 \le \tilde{p}_1 \le 2n-1$, $1 \le \tilde{p}_2 \le 2m-1$, then

$$-\frac{\tilde{p}_1}{2n} - \frac{\tilde{p}_2}{2m} = -2 + \frac{\overline{p}_1}{2n} + \frac{\overline{p}_2}{2m} = -1 - \delta$$

is a root of $b_\rho(s)$. ∎

Example 2.4. By means of formula (2.7) it is possible to find the roots of the Bernstein-Sato polynomial associated to $u_1^6 + u_2^4$ and one finds

$$b_\rho(s) = \left(s+1\right)^2 \left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$$
$$\left(s+\frac{5}{12}\right)\left(s+\frac{19}{12}\right)\left(s+\frac{7}{12}\right)\left(s+\frac{17}{12}\right)\left(s+\frac{11}{12}\right)\left(s+\frac{13}{12}\right).$$

Let us now consider the distribution $\mathcal{I}^z$ defined by (2.4). Let $s_1, \ldots, s_h$ be the zeros of $b_\rho(s)$ in $(-Q-1, -Q]$, each counted with its multiplicity and ordered in a decreasing way. Then the meromorphic continuation of $\mathcal{I}^z$ has poles whenever $z = Q + s_j - k$, $k \ge 0$. We remark, in particular, that 0 and $-1+Q$ are always poles for $\mathcal{I}^z$. By gluing all together, we get the following result.

Proposition 2.5. *$\mathcal{I}^z$ may be analytically continued to a meromorphic distribution-valued function of z, also denoted by $\mathcal{I}^z$, with poles in a set A, consisting of rational negative points. More precisely,*

$$A = \{\zeta_{j,k} := Q + s_j - k : k \in \mathbb{N}, j = 1, \ldots, h\},$$

s_j, $j = 1, \ldots, h$, denoting the zeros of the Bernstein-Sato polynomial b_ρ in $(-Q-1, -Q]$, each listed as many times as its multiplicity. Each pole has order one, with the exception of the points $-1 + Q - k$, $k \in \mathbb{N}$, which have order two.

Set $\zeta_j := \zeta_{j,0}$. Observe that $\zeta_1 = 0$ is a pole of order 1 for $\mathcal{I}^z$.

Consider now the function G, given by

$$(2.8) \qquad G(z) := \Gamma\left(z + 1 - Q\right) \cdot \prod_{j=2,\ldots,h} \Gamma(z - \zeta_j).$$

If S denotes the sphere

$$S := \left\{(u_1, u_2) \in \mathbb{R}^2 : \rho(u_1, u_2) = 1\right\}$$

with surface measure $\sigma(S)$, set

$$(2.9) \qquad I^z(u_1, u_2) := \frac{G(0)\,(u_1^{2n} + u_2^{2m})^{z-Q}}{\sigma(S)\,\Gamma(z)G(z)}\,.$$

In the sequel we will denote by $\mathcal{S}(\mathbb{R}^s)$, $s = 2, 3$, the Schwartz space on $\mathbb{R}^s$ endowed with a denumerable family of norms $\|\cdot\|_{(N)}$ given by

$$\|\Phi\|_{(N)} = \sum_{|\alpha| \leq N} \sup_{u \in \mathbb{R}^s} (1 + |u|)^N |\partial_u^\alpha \Phi(u)|.$$

Here we use the conventional notation

$$\partial_u^\alpha = \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_s}}{\partial u_s^{\alpha_s}}\,,$$

with $\alpha = (\alpha_1, \ldots, \alpha_s)$ s-tuple of natural numbers and $|\alpha| = \alpha_1 + \cdots + \alpha_s$.

Proposition 2.6. *The distribution I^z satisfies*

$$I^0 = \delta_0\,.$$

Proof. Take $z \in \mathbb{C}$, $\Re e z > 0$, and $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Set $C_z := \frac{G(0)}{\sigma(S)\Gamma(z)G(z)}$. Then

$$
\begin{aligned}
< I^z, \varphi > &= C_z \int_{\mathbb{R}^2} \rho(u_1, u_2)^{z-Q} \varphi(u_1, u_2)\, du_1\, du_2 \\
&= C_z \Bigg(\int_{\{\rho(u_1, u_2) \leq 1\}} \rho(u_1, u_2)^{z-Q}\, (\varphi(u_1, u_2) - \varphi(0, 0))\, du_1\, du_2 \\
&\quad + \int_{\{\rho(u_1, u_2) \leq 1\}} \rho(u_1, u_2)^{z-Q} \varphi(0, 0)\, du_1\, du_2 \\
&\quad + \int_{\{\rho(u_1, u_2) \geq 1\}} \rho(u_1, u_2)^{z-Q} \varphi(u_1, u_2)\, du_1\, du_2 \Bigg) \\
&= C_z \, (I_1 + I_2 + I_3)\,.
\end{aligned}
$$

By introducing polar coordinates (see [5]) we obtain

$$
\begin{aligned}
I_2 &= \int_S \int_0^1 \varphi(0, 0) \rho(r \circ (v_1, v_2))^{z-Q} r^{Q-1} dr\, d\sigma(v_1\, v_2) \\
&= \int_S \int_0^1 \varphi(0, 0)\, (r \cdot \rho(v_1, v_2))^{z-Q}\, r^{Q-1} dr\, d\sigma(v_1\, v_2) \\
&= \varphi(0, 0) \int_S \int_0^1 r^{z-1} dr\, d\sigma(v_1\, v_2) = \varphi(0, 0) \frac{\sigma(S)}{z}\,,
\end{aligned}
$$

so that

$$(2.10) \qquad C_z I_2 = \varphi(0, 0) \frac{G(0)}{z\Gamma(z)\, G(z)} = \varphi(0, 0) \frac{G(0)}{\Gamma(z+1)\, G(z)}$$

and this expression is well-defined for every z, with $\Re ez > -\min\{-\zeta_2, 1\}$. Now, it it is easy to show that both I_1 and I_3 are absolutely convergent for $\Re ez > -\min\{\frac{1}{2n}, -\zeta_2\}$. Indeed,

$$
\begin{aligned}
|I_1| &\leq \int_{\{\rho(u_1,u_2)\leq 1\}} \rho(u_1, u_2)^{\Re ez-Q} \, |\varphi(u_1, u_2) - \varphi(0,0)| \, du_1 \, du_2 \\
&\leq C||\nabla\varphi||_\infty \int_{\{\rho(u_1,u_2)\leq 1\}} \rho(u_1, u_2)^{\Re ez-Q} \, |(u_1, u_2)| \, du_1 \, du_2 \\
&\leq C||\nabla\varphi||_\infty \int_{\{\rho(u_1,u_2)\leq 1\}} \rho(u_1, u_2)^{\Re ez-Q+\frac{1}{2n}} \, du_1 \, du_2 \\
&= C||\nabla\varphi||_\infty \int_S \int_0^1 \rho\left(r \circ (v_1, v_2)\right)^{\Re ez-Q+\frac{1}{2n}} \, r^{Q-1} \, dr \, d\sigma(v_1, v_2) \\
&= C||\nabla\varphi||_\infty \int_S \int_0^1 r^{\Re ez-Q+\frac{1}{2n}} \, r^{Q-1} \, dr \, d\sigma(v_1, v_2) \\
&= C||\nabla\varphi||_\infty \frac{\sigma(S)}{\frac{1}{2n} + \Re ez},
\end{aligned}
$$

which is well-defined for $\Re ez > -\frac{1}{2n}$. Here, in particular, we used (2.3).

Moreover,

$$
\begin{aligned}
|I_3| &\leq \int_{\{\rho(u_1,u_2)>1\}} \rho(u_1, u_2)^{\Re ez-Q} \, |\varphi(u_1, u_2)| \, du_1 \, du_2 \\
&\leq C||\varphi||_{(N)} \int_{\{\rho(u_1,u_2)>1\}} \rho(u_1, u_2)^{\Re ez-Q-\frac{N}{2n}} \, du_1 \, du_2,
\end{aligned}
$$

since

$$
|\varphi(u_1, u_2)| \leq \frac{||\varphi||_{(N)}}{(1 + |(u_1, u_2)|)^N} \leq \frac{||\varphi||_{(N)}}{|(u_1, u_2)|^N} \leq C \frac{||\varphi||_{(N)}}{(\rho(u_1, u_2))^{\frac{N}{2n}}}
$$

when $\rho(u_1, u_2) > 1$, as a consequence of (2.2). Now by passing to polar coordinates we obtain

$$
\begin{aligned}
|I_3| &\leq C||\varphi||_{(N)} \int_S \int_1^{+\infty} \rho\left(r \circ (v_1, v_2)\right)^{\Re ez-Q-\frac{N}{2n}} \, r^{Q-1} \, dr \, d\sigma(v_1 \, v_2) \\
&= C||\varphi||_{(N)} \frac{\sigma(S)}{\frac{N}{2n} - \Re ez} < +\infty,
\end{aligned}
$$

if N is a positive integer greater than $2n \cdot \Re ez$. Thus, as a consequence of the uniqueness of the analytic continuation, the expression

$$
< I^z, \varphi >= C_z \left(I_1 + I_2 + I_3\right)
$$

defines the action of I^z on a Schwartz function φ in $\mathbb{R}^2$, for $\Re ez > -\min\{\frac{1}{2n}, -\zeta_2, 1\}$, and by using the bounds for I_1 and I_3 and (2.10) one gets the thesis. ∎

Proposition 2.7. *I^z is a homogeneous tempered distribution of degree $-Q + z$.*

We recall that this means that for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ the following equality is satisfied

$$< I^z, \varphi_\delta >= \delta^{z-Q} < I^z, \varphi >,$$

where

$$\varphi_\delta(u_1, u_2) := \delta^{-Q} \varphi \left(\delta^{-1} \circ u \right) = \delta^{-Q} \varphi \left(\delta^{-\frac{1}{2n}} u_1, \delta^{-\frac{1}{2m}} u_2 \right).$$

Thus the Fourier transform of the (tempered and homogeneous) distribution I^z is a well-defined distribution, homogeneous of degree $-Q - (z - Q) = -z$. Moreover, the following holds.

Proposition 2.8. $\widehat{I^z}$ *agrees with a function $C^\infty(\mathbb{R}^2 \backslash \{(0,0)\})$ away from $(0,0)$. Moreover,*

$$(2.11) \qquad |\widehat{I^z}(\xi)| \leq C\rho(\xi)^{-\Re ez}, \qquad \text{for all } \xi \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Proof. It suffices to prove the statement for $0 < \Re ez < Q$; indeed, the other cases can be treated by analytic continuation.

We first construct a partition of unity adapted to dyadic spherical shells. The procedure is standard and we briefly recall it only for the sake of completeness.

Let ψ be a $C_c^\infty(\mathbb{R}^2)$ function, such that

(i) $0 \leq \psi(u_1, u_2) \leq 1$ for every $(u_1, u_2) \in \mathbb{R}^2$;

(ii) $\psi(u_1, u_2) \equiv 0$ if $(u_1, u_2) \notin C_0 := \left\{ (u_1, u_2) \in \mathbb{R}^2 \ : \ \frac{1}{4} \leq \rho(u_1, u_2) \leq 8 \right\}$;

(iii) $\psi(u_1, u_2) \equiv 1$ if $(u_1, u_2) \in C_1 := \left\{ (u_1, u_2) \ : \ \frac{1}{2} \leq \rho(u_1, u_2) \leq 4 \right\}$.

Define now for $(u_1, u_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$(2.12) \qquad \Psi(u_1, u_2) := \sum_{j \in \mathbb{Z}} \psi(2^j \circ (u_1, u_2)).$$

Since there is at most a finite number of nonzero terms in the sum (2.12), Ψ is well-defined and strictly positive on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus we may introduce the functions

$$(2.13) \qquad \eta(u_1, u_2) := \frac{\psi(u_1, u_2)}{\Psi(u_1, u_2)}.$$

It is easy to check that

$$(2.14) \qquad \sum_{j \in \mathbb{Z}} \eta\big(2^j \circ (u_1, u_2)\big) = 1 \quad \text{for every } (u_1, u_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Now using (2.14) we may write

$$
\begin{aligned}
I^z(u_1, u_2) &= C_z \rho(u_1, u_2)^{z-Q} \\
&= C_z \sum_{j \in \mathbb{Z}} \eta(2^j \circ (u_1, u_2)) \rho(2^{-j} \circ 2^j \circ (u_1, u_2))^{z-Q} \\
&= C_z \sum_{j \in \mathbb{Z}} \eta(2^j \circ (u_1, u_2)) 2^{-j(z-Q)} \rho(2^j \circ (u_1, u_2))^{z-Q} \\
&= C_z \sum_{j \in \mathbb{Z}} 2^{-j(z-Q)} f_0(2^j \circ (u_1, u_2)),
\end{aligned}
$$

where we set

$$f_0(u_1, u_2) := \eta(u_1, u_2) \rho(u_1, u_2)^{z-Q}.$$

Since

$$\big(f_0(2^j \circ (\cdot, \cdot))\big)^\frown(\xi_1, \xi_2) = 2^{-jQ} \widehat{f_0}\big(2^{-j} \circ (\xi_1, \xi_2)\big),$$

we obtain formally

$$\sum_{j \in \mathbb{Z}} 2^{-j(z-Q)} \big(f_0(2^j \circ (\cdot, \cdot))\big)^\frown(\xi_1, \xi_2) = \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)).$$

This series is absolutely convergent, since if $(\xi_1, \xi_2) \neq (0,0)$ one has

$$
\begin{aligned}
\left| \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \right| &\leq \sum_{j \in \mathbb{Z}} 2^{-j\Re e z} \left| \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \right| \\
&\leq \left(\sum_{2^{-j}\rho(\xi_1,\xi_2) \leq 1} + \sum_{2^{-j}\rho(\xi_1,\xi_2) > 1} \right) 2^{-j\Re e z} \left| \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \right| \\
&\leq \|\widehat{f_0}\|_{(0)} \sum_{2^{-j}\rho(\xi_1,\xi_2) \leq 1} 2^{-j\Re e z} + \sum_{\rho(2^{-j}\circ(\xi_1,\xi_2)) > 1} 2^{-j\Re e z} \frac{\|\widehat{f_0}\|_{(N)}}{\big(1 + |(2^{-j} \circ (\xi_1, \xi_2))|\big)^N} \\
&\leq C \|\widehat{f_0}\|_{(0)} \rho(\xi)^{-\Re e z} + C \sum_{2^{-j}\rho(\xi_1,\xi_2) > 1} 2^{-j\Re e z} \frac{\|\widehat{f_0}\|_{(N)}}{\big(\rho(2^{-j} \circ (\xi_1, \xi_2))\big)^{\frac{N}{2n}}} \\
&\leq C \|\widehat{f_0}\|_{(0)} \rho(\xi)^{-\Re e z} + \frac{C}{\big(\rho(\xi_1,\xi_2)\big)^{\frac{N}{2n}}} \sum_{2^{-j}\rho(\xi_1,\xi_2) > 1} 2^{-j\Re e z + j\frac{N}{2n}} \\
&\leq C \|\widehat{f_0}\|_{(0)} \rho(\xi)^{-\Re e z} + \frac{C}{\big(\rho(\xi_1,\xi_2)\big)^{\frac{N}{2n}}} \rho(\xi_1,\xi_2)^{-\Re e z + \frac{N}{2n}} \leq C \rho(\xi_1,\xi_2)^{-\Re e z},
\end{aligned}
$$

where we used in particular the fact that

$$|(u_1, u_2)| \geq C\, \rho(u_1, u_2)^{\frac{1}{2n}} \text{ for } |(u_1, u_2)| > 1\,.$$

We set therefore

$$v(\xi_1, \xi_2) := \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \in \mathrm{L}^\infty(\mathbb{R}^2)\,.$$

By the Dominated Convergence Theorem we obtain, given $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} v\varphi = \sum_{j \in \mathbb{Z}} 2^{-jz} \int_{\mathbb{R}^2} \widehat{f_0}(2^{-j} \circ (\cdot, \cdot))\, \varphi\,,$$

that is

$$v(\cdot, \cdot) = \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\cdot, \cdot))$$

in the sense of distributions, whence

$$\widehat{I^z}(\cdot, \cdot) = \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\cdot, \cdot))$$

in the sense of distributions and, moreover,

$$(2.15) \qquad \left| \widehat{I^z}(\xi_1, \xi_2) \right| \leq \rho(\xi_1, \xi_2)^{-\Re ez} \text{ for all } (\xi_1 \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

(observe that this inequality could also be retrieved from the homogeneity). Finally we prove that $\widehat{I^z}$ agrees with a function $C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ away from $(0,0)$. First of all, we observe that f_0 is in the Schwartz space, hence $\widehat{f_0}(2^{-j} \circ (\cdot, \cdot))$ belongs to $\mathcal{C}^\infty(\mathbb{R}^2)$. Moreover, the following estimates hold:

$$\left| \partial_{\xi_1}^k \left(2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \right) \right| \leq C_k\, 2^{-j\left(\Re ez + \frac{k}{2n}\right)} \text{ for all } (\xi_1, \xi_2) \in \mathbb{R}^2$$

$$\left| \partial_{\xi_1}^k \left(2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2)) \right) \right| \leq C_{k,N}\, 2^{-j\left(\Re ez + \frac{k}{2n}\right)} \frac{1}{\left(1 + |(2^{-j} \circ (\xi_1, \xi_2))|\right)^N}$$

$$\leq \frac{C_{k,N}}{\rho(\xi_1, \xi_2)^{\frac{N}{2n}}}\, 2^{j\left(-\Re ez - \frac{k}{2n} + \frac{N}{2n}\right)} \quad \text{if } \rho\left(2^{-j} \circ (\xi_1, \xi_2)\right) > 1\,,$$

for all $k \in \mathbb{N}$. Since analogous bounds hold for $\partial_{\xi_2}^k(2^{-jz} \widehat{f_0}(2^{-j} \circ (\xi_1, \xi_2))$, with $2n$ replaced by $2m$, the series of the partial derivatives of $2^{-jz} \widehat{f_0}(2^{-j} \circ (\cdot, \cdot))$ of any order k converge on the compact subsets of $\mathbb{R}^2 \setminus \{(0,0)\}$. It follows that $\widehat{I^z}(\cdot, \cdot) = \sum_{j \in \mathbb{Z}} 2^{-jz} \widehat{f_0}(2^{-j} \circ (\cdot, \cdot))$ is $\mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$. ∎

3. Some preliminary results

In the following, if $f(x_1, x) \in \mathcal{S}(\mathbb{R}^3)$ we denote by $\mathcal{F}^{-1} f$ the inverse Fourier transform of f and by $\mathcal{F}_2 f$ and $\mathcal{F}_2^{-1} f$ respectively the partial Fourier transform and the inverse of the partial Fourier transform of f with respect to the variable x.

Moreover we denote the dual variables as (ξ_1, ξ) with $\xi = (\xi_2, \xi_3)$.

Characterization of product kernels

As recalled in the Introduction, the precise definition of product kernels involves certain differential inequalities and certain cancellation conditions which are analogous to those satisfied by the kernel $H(x_1, x)$ defined by (1.3). Our study will be based on the following equivalent definition (see [12]).

Definition 3.1. A product kernel K on $\mathbb{R}^3$ is a sum

$$(3.1) \qquad K(x_1, x) = \sum_{J \in \mathbb{Z}^2} 2^{-j_1 - jQ} \, \psi_J(2^{-j_1} x_1, 2^{-j} \circ x), \qquad J = (j_1, j)$$

convergent in the sense of distributions, of smooth functions ψ_J supported on the set where $1/2 \le |x_1| \le 4$ and $1/2 \le \rho(x) \le 4$, satisfying the cancellation conditions

$$(3.2) \qquad\qquad\qquad \int \psi_J(x_1, x) \, dx_1 = 0$$

$$(3.3) \qquad\qquad\qquad \int \psi_J(x_1, x) \, dx = 0$$

identically for every J, and with uniformly bounded C^k norms for every $k \in \mathbb{N}$.

We shall need a characterization of product kernels as dyadic sums of Schwartz functions on $\mathbb{R}^3$ which are compactly supported only in the first variable and that satisfy some moment conditions.

Lemma 3.2. *A product kernel K on R^3 can be written as a sum*

$$K(x_1, x) = \sum_{J \in \mathbb{Z}^2} 2^{-j_1 - jQ} \varphi_J(2^{-j_1} x_1, 2^{-j} \circ x), \qquad J = (j_1, j),$$

convergent in the sense of distributions, of Schwartz functions φ_J such that

(i) *the φ_J have compact x_1-support where $1/2 \le |x_1| \le 4$;*

(ii) *the φ_J form a bounded set of $\mathcal{S}(\mathbb{R}^3)$, that is the Schwartz norms $\|\varphi_J\|_{(N)}$ are uniformly bounded in J for each $N \in \mathbb{N}$;*

(*iii*) *the φ_J satisfy the cancellation conditions*

$$(3.4) \qquad \int x_1^\ell \varphi_J(x_1, x)\, dx_1 = 0$$

for every positive integer $\ell \leq M_1$, for some fixed $M_1 \in \mathbb{N}$, and

$$(3.5) \qquad \int x^\beta \varphi_J(x_1, x)\, dx = 0$$

for every multi-index $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, identically for every $J \in \mathbb{Z}^2$. Here, as usual, $x^\beta = x_2^{\beta_1} x_3^{\beta_2}$.

Proof. Let K be a product kernel on $\mathbb{R}^3$. By Definition 3.1 we can write K as a sum

$$K(x_1, x) = \sum_{(j_1, i) \in \mathbb{Z}^2} 2^{-j_1 - iQ} \psi_{(j_1, i)}(2^{-j_1} x_1, 2^{-i} \circ x),$$

convergent in the sense of distributions, of smooth functions $\psi_{(j_1, i)}$ supported on the set where $1/2 \leq |x_1| \leq 4$ and $1/2 \leq \rho(x) \leq 4$, satisfying the cancellation conditions (3.2) and (3.3) identically for every $(j_1, i) \in \mathbb{Z}^2$, and with uniformly bounded C^k norms for every $k \in \mathbb{N}$.

Let

$$\widehat{K}(\xi_1, \xi) = \sum_{(j_1, i) \in \mathbb{Z}^2} \widehat{\psi}_{(j_1, i)}(2^{j_1} \xi_1, 2^i \circ \xi)$$

be the corresponding product multiplier.

Consider a smooth function ζ on the real line, supported on the interval $[1, 4]$ and such that $\sum_{k \in \mathbb{Z}} \zeta(2^k t) = 1$ for every $t > 0$. For $J = (j_1, j) \in \mathbb{Z}^2$, define

$$\mu_J(\xi_1, \xi) := \sum_{i \in \mathbb{Z}} \widehat{\psi}_{(j_1, i)}(\xi_1, 2^{i-j} \circ \xi) \zeta(\rho(\xi)).$$

It can be easily proved that the μ_J form a bounded set of $\mathcal{S}(\mathbb{R}^3)$. In addition, a direct computation shows that

$$\widehat{K}(\xi_1, \xi) = \sum_{J \in \mathbb{Z}^2} \mu_J(2^{j_1} \xi_1, 2^j \circ \xi)$$

in the sense of distributions. Setting

$$\varphi_J(x_1, x) := (\mathcal{F}^{-1} \mu_J)(x_1, x) = \mathcal{F}_2^{-1}\Big(\sum_{i \in \mathbb{Z}} (\mathcal{F}_2 \psi_{(j_1, j)})(x_1, 2^{i-j} \circ \cdot) \zeta(\rho(\cdot)) \Big)(x),$$

it is possible to write the product kernel K as the sum

$$(3.6) \qquad K(x_1, x) = \sum_{J \in \mathbb{Z}^2} 2^{-j_1 - jQ} \varphi_J(2^{-j_1} x_1, 2^{-j} \circ x),$$

convergent in the sense of distributions, of functions φ_J that form a bounded set of $\mathcal{S}(\mathbb{R}^3)$ and have compact x_1-support where $1/2 \leq |x_1| \leq 4$. Finally, the fact that $\mu_J(0, \xi) = 0$ and $(\partial_\xi^\beta \mu_J)(\xi_1, 0) = 0$ for every multi-index $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, identically for every $J \in \mathbb{Z}^2$, yields (3.4) for $m = 0$ and (3.5).

In fact, we can choose φ_J so that a finite number of moments in the variable x_1 vanish. This follows from a slight modification of the arguments in Lemma 2.2.3 in [12]. More explicitly, denote by φ each function φ_J in the decomposition (3.6). Then each function φ may be written as a series

$$(3.7) \qquad \varphi(x_1, x) = \sum_{k \in \mathbb{Z}} 2^{-k} A_k(2^{-k} x_1, x),$$

convergent in the sense of distributions, of functions A_k which form a bounded set of $\mathcal{S}(\mathbb{R}^3)$ with norms that decay exponentially in k as $k \to \pm\infty$, have compact x_1-support on the set $\{x_1 \in \mathbb{R} : 1/2 \leq |x_1| \leq 4\}$, and satisfy (3.4) with $\ell = 1$.

To prove this fact, consider a function $\eta \in \mathcal{C}_0^\infty(\mathbb{R})$, supported on the set $[-4, -1] \cup [1, 4]$, such that $\sum_{k \in \mathbb{Z}} \eta(2^k t) = 1$ for every $t \neq 0$ and $\int t\, \eta(t)\, dt \neq 0$. Set

$$\chi_k(x_1) := \eta(2^{-k} x_1),$$

$$\tilde{\chi}_k(x_1) := \frac{\chi_k(x_1)}{\int x_1 \chi_k(x_1)\, dx_1},$$

$$a_k(x) = \int x_1\, \chi_k(x_1)\, \varphi(x_1, x)\, dx_1,$$

$$S_k(x) = \sum_{j \geq k} a_j(x).$$

Then write φ as

$$\varphi(x_1, x) = \sum_{k \in \mathbb{Z}} \big(\varphi(x_1, x)\chi_k(x_1) - a_k(x)\tilde{\chi}_k(x_1)\big) + \sum_{k \in \mathbb{Z}} \big(S_k(x) - S_{k+1}(x)\big)\tilde{\chi}_k(x_1)$$

$$= \sum_{k \in \mathbb{Z}} \big(\varphi(x_1, x)\chi_k(x_1) - a_k(x)\tilde{\chi}_k(x_1)\big) + \sum_{k \in \mathbb{Z}} S_k(x)\big(\tilde{\chi}_k(x_1) - \tilde{\chi}_{k-1}(x_1)\big)$$

$$= \sum_{k \in \mathbb{Z}} A_k(x_1, x),$$

where the series converges in the sense of distributions, the functions A_k satisfy the moment conditions $\int x_1 A_k(x_1, x)\, dx_1 = 0$ for all $k \in \mathbb{Z}$, and the Schwartz norms decay exponentially in k as $k \to \pm\infty$. Now by rescaling x_1 we obtain (3.7). Iterating this argument yields (3.4) for all $\ell \leq M_1$, for some fixed $M_1 \in \mathbb{N}$. ∎

A result analogous to Lemma 3.2 can be stated by interchanging the role of x_1 and x.

Estimates on oscillatory integrals

In the following, we prove some estimates on certain oscillatory integrals related to our problem. Let $||| \cdot |||$ denote any homogeneous norm with respect to the family of non-isotropic dilations

$$(3.8) \qquad \delta \bullet (\xi_1, \xi) = (\delta \xi_1, \delta^m \xi_2, \delta^n \xi_3), \qquad \delta > 0,$$

e.g. we may choose

$$(3.9) \qquad |||(\xi_1, \xi)||| = \max\{|\xi_1|, |\xi_2|^{\frac{1}{m}}, |\xi_3|^{\frac{1}{n}}\}.$$

We observe in passing that

$$(3.10) \qquad \delta \bullet (\xi_1, \xi) = (\delta \xi_1, \delta^{2mn} \circ \xi).$$

Consider the integral

$$(3.11) \qquad I(\xi_1, \xi, \eta) := \int f(x_1, \eta) e^{-i(\xi_1, \xi) \cdot (x_1, \gamma(x_1))} \, dx_1,$$

where $\gamma(x_1) = (x_1^m, x_1^n)$ and f is such that

(h1) f belongs to $\mathcal{S}(\mathbb{R}^3)$ and is x_1-compactly supported on the interval $1/2 \le |x_1| \le 4$;

(h2) $f(x_1, 0) = 0$ for all x_1 such that $1/2 \le |x_1| \le 4$.

The constants C_N occurring in the following inequalities depend on the Schwartz norms of f.

Lemma 3.3. *Under the hypotheses* (h1) *and* (h2) *the following estimate holds for the integral* $I(\xi_1, \xi, \eta)$ *defined by* (3.11)

$$(3.12) \qquad |I(\xi_1, \xi, \eta)| \le C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for every integer $N \ge 0$.

Proof. Since f is a Schwartz function, by using (h2), (2.2) and (2.3) we deduce that for every $(x_1, \eta) \in \{x_1 \in \mathbb{R} : 1/2 \le |x_1| \le 4\} \times \mathbb{R}^2$

$$(3.13) \qquad |\partial_{x_1}^k f(x_1, \eta)| \le C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for all $N \in \mathbb{N}$ and $k \in \mathbb{N}$. For $k = 0$ this inequality yields then (3.12). ∎

Lemma 3.4. *Let* I *be the oscillatory integral defined by* (3.11). *Assume that* (h1) *and* (h2) *are satisfied. Then*

$$(3.14) \quad |I(\xi_1, \xi, \eta)| \leq C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N |||(\xi_1, \xi)|||^{\frac{1}{n}}} \qquad if \quad |||(\xi_1, \xi)||| > 1$$

for every integer $N \geq 0$.

Under the additional assumption

$$(3.15) \qquad\qquad \int f(x_1, x) \, dx_1 = 0 \,, \ x \in \mathbb{R}^2 \,,$$

the following estimate holds

$$(3.16) \quad |I(\xi_1, \xi, \eta)| \leq C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N} |||(\xi_1, \xi)||| \qquad if \quad |||(\xi_1, \xi)||| \leq 1$$

for every integer $N \geq 0$.

Proof. Let $|||(\xi_1, \xi)|||$ be the homogeneous norm defined in (3.9). We divide the phase space ξ_1, ξ into two regions, depending on whether $|||(\xi_1, \xi)||| \leq 1$ or $|||(\xi_1, \xi)||| > 1$.

Assume that $|||(\xi_1, \xi)||| > 1$ and write $(\xi_1, \xi) = \lambda\omega$, where $\lambda = |||(\xi_1, \xi)|||$ and $\omega = (\xi_1, \xi)/|||(\xi_1, \xi)|||$.

With these notation, the oscillatory integral $I(\xi_1, \xi, \eta)$ becomes

$$(3.17) \qquad I(\lambda\omega, \eta) = \int_{\frac{1}{2} \leq |x_1| \leq 4} f(x_1, \eta) \ e^{-i\lambda\omega \cdot (x_1, \gamma(x_1))} \, dx_1 \,.$$

Set

$$(3.18) \qquad \Phi(x_1) = \omega \cdot (x_1, \gamma(x_1)) = \frac{1}{|||(\xi_1, \xi)|||}(\xi_1 x_1 + \xi_2 x_1^m + \xi_3 x_1^n),$$

for $1/2 \leq |x_1| \leq 4$.

We observe that, since the curve $x_1 \mapsto (x_1, x_1^m, x_1^n)$ for $1/2 \leq |x_1| \leq 4$ is of finite type n, for every $x_1 \in \mathbb{R}$ with $1/2 \leq |x_1| \leq 4$ there exists a positive integer $\overline{n} \leq n$ such that $\Phi^{(\overline{n})}(x_1) \neq 0$ [15].

A standard application of the Van der Corput's lemma to the oscillatory integral (3.17), the compactness of the x_1-support , and

$$(3.19) \qquad\qquad |\Phi^{(\overline{n})}(x_1)| \geq C$$

for some $C > 0$, for all x_1 such that $\frac{1}{4} \leq |x_1| \leq 4$, yield

$$(3.20) \qquad |I(\lambda\omega, \eta)| \leq C\lambda^{-\frac{1}{n}} \|\partial_{x_1} f(x_1, \eta)\|_{L^1(\mathbb{R}_{x_1})} \,.$$

Since the function $f(x_1, \eta)$ is smooth and compactly supported in x_1 where $\frac{1}{2} \leq |x_1| \leq 4$, by using the estimate (3.13) in the case $k = 1$ we see that

$$\|\partial_{x_1} f(x_1, \eta)\|_{L^1(\mathbb{R}_{x_1})} \leq C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for every $N \in \mathbb{N}$. As a consequence,

$$|I(\lambda\omega, \eta)| \leq C_N \lambda^{-\frac{1}{n}} \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for every $N \in \mathbb{N}$. Since $\lambda = |||(\xi_1, \xi)|||$, we get (3.14).

Assume now that $|||(\xi_1, \xi)||| \leq 1$. The cancellation condition (3.15), the Mean value theorem and the estimate (3.13) can be used to prove that

$$|I(\xi_1, \xi, \eta)| \leq \int_{\frac{1}{2} \leq |x_1| \leq 4} |f(x_1, \eta)| \left| e^{-i(\xi_1, \xi) \cdot (x_1, \gamma(x_1))} - 1 \right| dx_1$$

$$\leq C_N \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N} |(\xi_1, \xi)|, \quad \text{for every } N \in \mathbb{N}.$$

Since by hypothesis $|||(\xi_1, \xi)||| \leq 1$, we have that $|(\xi_1, \xi)| \leq 3|||(\xi_1, \xi)|||$. This inequality, together with the previous estimate, yields (3.16). ∎

The estimate (3.14) in Lemma 3.4 can be improved in the region of the space (ξ_1, ξ) where the first derivative of the phase (3.18) never vanishes for $1/2 \leq |x_1| \leq 4$, as the following lemma shows.

Lemma 3.5. *Let I be the oscillatory integral defined by (3.11). Under the hypotheses* (h1) *and* (h2) *there exists a constant $\tilde{C} > 1$ such that for every integer $N \geq 0$*

(3.21) $|I(\xi_1, \xi, \eta)| \leq C_N \dfrac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N |||(\xi_1, \xi)|||^N}$ *when* $|||(\xi_1, \xi)||| > 1$

and

$$|\xi_1| > \tilde{C}(|\xi_2| + |\xi_3|), \quad or \quad |\xi_2| > \tilde{C}(|\xi_1| + |\xi_3|), \quad or \quad |\xi_3| > \tilde{C}(|\xi_1| + |\xi_2|).$$

Proof. We use for the integral I the notation introduced in formula (3.17).

In order to improve the estimate (3.14), we have to determine the subsets of the phase space (ξ_1, ξ) where

(3.22) $$\Phi'(x_1) = \xi_1 + m\xi_2 x_1^{m-1} + n\xi_3 x_1^{n-1} = 0$$

for some $x_1 \in \mathbb{R}$ such that $\frac{1}{2} \leq |x_1| \leq 4$. Some elementary estimates show that we can find a constant $\tilde{C} > 1$ sufficiently large, such that for any fixed point $(\xi_1, \xi) \in \mathbb{R}^3$, satisfying $|||(\xi_1, \xi)||| > 1$ and

$$|\xi_1| > \tilde{C}(|\xi_2| + |\xi_3|), \quad \text{or} \quad |\xi_2| > \tilde{C}(|\xi_1| + |\xi_3|), \quad \text{or} \quad |\xi_3| > \tilde{C}(|\xi_1| + |\xi_2|),$$

there exists a constant $C_\omega > 0$ such that

$$(3.23) \qquad\qquad |\Phi'(x_1)| \geq C_\omega$$

for every $x_1 \in \mathbb{R}$ with $1/2 \leq |x_1| \leq 4$.

Let D denote the differential operator

$$Df(x_1, \eta) = (-i\lambda\Phi'(x_1))^{-1} \frac{\partial f}{\partial x_1}(x_1, \eta)$$

and let ${}^tD$ denote its transpose

$${}^tDf(x_1, \eta) = \frac{\partial}{\partial x_1}\left(\frac{f}{i\lambda\Phi'(x_1)}\right).$$

Since $D^N(e^{-i\lambda\Phi(x_1)}) = e^{-i\lambda\Phi(x_1)}$ for every $N \in \mathbb{N}$, integration by parts shows that

$$I(\lambda\omega, \eta) = \int_{\frac{1}{2} \leq |x_1| \leq 4} f(x_1, \eta) D^N\left(e^{-i\lambda\Phi(x_1)}\right) dx_1$$

$$= \int_{\frac{1}{2} \leq |x_1| \leq 4} ({}^tD)^N f(x_1, \eta) \, e^{-i\lambda\Phi(x_1)} dx_1.$$

Since $f(x_1, \eta)$ is a smooth function with compact support in the x_1 variable in the region where $1/2 \leq |x_1| \leq 4$, f satisfies the estimate (3.13), and $\Phi(x_1)$ is a smooth function satisfying the inequality (3.23), we can verify that

$$\left|({}^tD)^N f(x_1, \eta)\right| \leq C_{N,\omega} \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N} \lambda^{-N}$$

for every $N \in \mathbb{N}$.

Therefore we conclude that

$$\left|I(\lambda\omega, \eta)\right| \leq C_{N,\omega} \, \lambda^{-N} \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for every $N \in \mathbb{N}$. By a compactness argument we can show that the previous estimate is independent of ω, so that

$$\left|I(\lambda\omega, \eta)\right| \leq C_N \lambda^{-N} \frac{\rho(\eta)^{\frac{1}{2n}}}{(1 + \rho(\eta))^N}$$

for every $N \in \mathbb{N}$. Since $\lambda = |||(\xi_1, \xi)|||$, we obtain the inequality (3.21). ∎

Remark 3.6. In the sequel, we shall systematically apply the estimates (3.12), (3.14), (3.16), and (3.21) to the oscillatory integral (3.11) with the integrand $f(x_1, \eta)$ of the form $x_1^\alpha (\mathcal{F}_2(x^\beta \varphi_J))(x_1, \eta)$, where the functions φ_J are given by Lemma 3.2. In particular, the Schwartz norms $\| \cdot \|_{(N)}$ of the functions φ_J are uniformly bounded in $J \in \mathbb{Z}^2$ for every $N \in \mathbb{N}$.

We observe that the functions $x_1^\alpha (\mathcal{F}_2(x^\beta \varphi_J))(x_1, \eta)$ fulfill the hypotheses (h1) and (h2), as a consequence of the cancellation condition (3.5).

Moreover we have

$$\int x_1^\ell (\mathcal{F}_2(x^\beta \varphi_J))(x_1, \eta) dx_1 = 0$$

as a consequence of the cancellation property (3.4) for all $\ell \leq M_1$ for some fixed $M_1 \in \mathbb{N}$, so that (3.15) is satisfied.

4. L^2-boundedness

Let K be the kernel defined by (1.4) and T the operator given by $T : f \mapsto f * K$. In this section we prove that T is bounded on $L^2(\mathbb{R}^3)$.

Let $J = (j_1, j)$. We proved in Lemma 3.2 that the product kernel K_0 can be written as a sum

$$K_0(x_1, x) = \sum_{J \in \mathbb{Z}^2} 2^{-j_1 - jQ} \varphi_J(2^{-j_1} x_1, 2^{-j} \circ x)$$

convergent in the sense of distributions, of Schwartz functions $\{\varphi_J\}_{J \in \mathbb{Z}^2}$ on $\mathbb{R}^3$, satisfying the properties (i), (ii) and (iii) of Lemma 3.2.

Proposition 4.1. *The series*

$$(4.1) \qquad K(x_1, x) = \sum_{J \in \mathbb{Z}^2} 2^{-j_1 - jQ} \varphi_J \left(2^{-j_1} x_1, 2^{-j} \circ (x - \gamma(x_1)) \right)$$

converges in the sense of distributions and the corresponding convolution operator T is bounded on $L^2(\mathbb{R}^3)$.

Proof. Let $m_J(\xi_1, \xi)$ be the Fourier transform of the J-th summand of the series (4.1). With a change of variables, we may write

$$
\begin{aligned}
m_J(\xi_1, \xi) &= \int_{1/2 \leq |x_1| \leq 4} \int \varphi_J(x_1, x) \, e^{-i(\xi_1, \xi) \cdot (2^{j_1} x_1, 2^j \circ x + \gamma(2^{j_1} x_1))} \, dx_1 dx \\
&= \int_{1/2 \leq |x_1| \leq 4} (\mathcal{F}_2 \varphi_J)(x_1, 2^j \circ \xi) \, e^{-i(2^{j_1} \xi_1, 2^{2mn \, j_1} \circ \xi) \cdot (x_1, \gamma(x_1))} \, dx_1.
\end{aligned}
$$
(4.2)

Let $|||(\xi_1, \xi)|||$ be any norm homogeneous with respect to the family of non-isotropic dilations (3.8), *e.g.* we may choose the norm defined by (3.9). We decompose the series

$$(4.3) \qquad \sum_{J \in \mathbb{Z}^2} m_J(\xi_1, \xi)$$

as

$$(4.4) \qquad \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} m_J(\xi_1, \xi) + \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ j \in \mathbb{Z}}} m_J(\xi_1, \xi) \,.$$

It suffices to prove that each of the two series in (4.4) converges boundedly to a bounded function. To do this, we apply Lemma 3.4 to the oscillatory integral

$$(4.5) \qquad I_J(\xi_1, \xi, \eta) := \int_{\frac{1}{2} \leq |x_1| \leq 4} (\mathcal{F}_2(\varphi_J))(x_1, \eta) e^{-i(\xi_1, \xi) \cdot (x_1, \gamma(x_1))} \, dx_1 \,.$$

The functions $(\mathcal{F}_2(\varphi_J))(x_1, \eta)$ have Schwartz norms $\|\varphi_J\|_{(N)}$ uniformly bounded in $J \in \mathbb{Z}^2$ for every $N \in \mathbb{N}$. As observed in Remark 3.6, they satisfy (h1), (h2) and (3.15), so that both estimates (3.14) and (3.16) hold, with costants C_N independent of N.

More precisely, consider the first series in (4.4). Since

$$(4.6) \qquad m_J(\xi_1, \xi) = I_J(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi),$$

by applying (3.16) we obtain

$$\sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} |m_J(\xi_1, \xi)| \leq C_N \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} \frac{\rho(2^j \circ \xi)^{\frac{1}{2n}}}{(1 + \rho(2^j \circ \xi))^N} |||(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi)|||$$

$$= \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} \frac{(2^j \rho(\xi))^{\frac{1}{2n}}}{(1 + 2^j \rho(\xi))^N} 2^{j_1} |||(\xi_1, \xi)|||$$

$$\leq C_N \sum_{j \in \mathbb{Z}} \frac{(2^j \rho(\xi))^{\frac{1}{2n}}}{(1 + 2^j \rho(\xi))^N},$$

for every $N \in \mathbb{N}$. Since the series

$$\sum_{j \in \mathbb{Z}} \frac{(2^j b)^{\frac{1}{2n}}}{(1 + 2^j b)^N}$$

is uniformly bounded in b, it follows that the series in the previous formula converges boundedly to a bounded function.

By using the identity (4.6) and the inequality (3.14) we prove that also the second sum on the right-hand side of (4.4) converges boundedly to a bounded function.

This proves that the series in (4.3) converges boundedly (and hence in the sense of distributions) to a bounded function $m(\xi_1, \xi)$. As a consequence the series (4.1) converges in the sense of distributions to the distribution $K = \mathcal{F}^{-1}(m)$.

Finally, by Plancherel's theorem, the boundedness of m implies that the corresponding operator T is bounded on $L^2(\mathbb{R}^3)$. ∎

5. L^p-boundedness

In this section we prove the L^p-boundedness of the operator T. For this we split the sum (4.1) into two parts

$$K(x_1, x) = \sum_{2mnj_1 \leq j} 2^{-j_1 - jQ} \varphi_J \left(2^{-j_1} x_1, 2^{-j} \circ (x - \gamma(x_1)) \right)$$

(5.1)
$$+ \sum_{2mnj_1 > j} 2^{-j_1 - jQ} \varphi_J \left(2^{-j_1} x_1, 2^{-j} \circ (x - \gamma(x_1)) \right)$$

$$=: K_1(x_1, x) + K_2(x_1, x).$$

Correspondingly we break the operator T into the sum

$$Tf = f * K_1 + f * K_2 =: T_1 f + T_2 f$$

and we prove that T_1 and T_2 are bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

Proposition 5.1. *The operator T_1 is bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.*

Proof. Let $m_J(\xi_1, \xi)$ be the multiplier given in (4.2). We show that the series

$$\widehat{K_1}(\xi_1, \xi) = \sum_{2mnj_1 \leq j} m_J(\xi_1, \xi)$$

defines a Marcinkiewicz multiplier on $\mathbb{R} \times \mathbb{R}^2$ adapted to the dilations (1.1) on $\mathbb{R}^2$. As a consequence, we will obtain that T_1 is bounded on $L^p(\mathbb{R}^3)$. This is part of the folklore, for a formal proof see [13].

It suffices to show that $\widehat{K_1}(\xi_1, \xi)$ is a bounded function on $\mathbb{R}^3$ such that for each $s_1 \in \{0, 1\}$ and for each multi-index $s = (s_2, s_3) \in \mathbb{N}^2$ with $|s| \leq 2$ there is a positive constant $C_{s_1, s}$ for which

(5.2)
$$\left| \partial_{\xi_1}^{s_1} \partial_{\xi}^s \widehat{K_1}(\xi_1, \xi) \right| \leq C_{s_1, s} |\xi_1|^{-s_1} \rho(\xi)^{-\frac{s_2}{2n} - \frac{s_3}{2m}}$$

for every $(\xi_1, \xi) \in \mathbb{R} \times \mathbb{R}^2$ with $\xi_1 \neq 0$ and $\xi \neq 0$.

We already proved in Proposition 4.1 that $\widehat{K}_1(\xi_1, \xi)$ is a bounded function on $\mathbb{R}^3$.

We give the proof of the differential inequalities (5.2) for $s_1 = 1$ and $s = (0,0)$ and for $s_1 = 0$ and $s = (1,0)$, the other cases being essentially the same, with the extra disadvantages of more complicated notation and computations.

Set

$$(5.3) \quad I_J^{\alpha,\beta}(\xi_1, \xi, \eta) := \int_{\frac{1}{2} \le |x_1| \le 4} x_1^\alpha (\mathcal{F}_2(x^\beta \varphi_J))(x_1, \eta) e^{-i(\xi_1, \xi) \cdot (x_1, \gamma(x_1))} \, dx_1 \,,$$

where $\alpha \in \mathbb{N}$, $\alpha \le 2$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ with $|\beta| \le 2$.

We first consider the case $s_1 = 1$ and $s = (0,0)$. Since

$$\partial_{\xi_1} m_J(\xi_1, \xi) = -i 2^{j_1} I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi) \,,$$

we write

$$\sum_{2mn\,j_1 \le j} |\partial_{\xi_1} m_J(\xi_1, \xi)| \le$$

$$\le |\xi_1|^{-1} \Bigg(\sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \le 1 \\ j \in \mathbb{Z}}} 2^{j_1} |\xi_1| \, |I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)|$$

$$+ \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ 2mn\,j_1 \le j \\ 2^{j_1} |\xi_1| \le \tilde{C}(2^{m\,j_1} |\xi_2| + 2^{n\,j_1} |\xi_3|)}} 2^{j_1} |\xi_1| \, |I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)|$$

$$+ \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ 2mn\,j_1 \le j \\ 2^{j_1} |\xi_1| > \tilde{C}(2^{m\,j_1} |\xi_2| + 2^{n\,j_1} |\xi_3|)}} 2^{j_1} |\xi_1| \, |I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)| \Bigg)$$

$$=: \Sigma_1 + \Sigma_2 + \Sigma_3 \,.$$

Since $2^{j_1} |\xi_1| \le 2^{j_1} |||(\xi_1, \xi)|||$, the convergence of Σ_1 follows from the estimate (3.12) applied to the integral $I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)$.

The inequalities $2^{j_1} |\xi_1| \le \tilde{C}(2^{m\,j_1} |\xi_2| + 2^{n\,j_1} |\xi_3|)$ and $2mn\,j_1 \le j$ imply that $2^{j_1} |\xi_1| \le \tilde{C}(2^{\frac{j}{2n}} |\xi_2| + 2^{\frac{j}{2m}} |\xi_3|) \le \tilde{C}((2^j \rho(\xi))^{\frac{1}{2n}} + (2^j \rho(\xi))^{\frac{1}{2m}})$. This fact, together with the estimate (3.14) for the integral $I_J^{1,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)$ shows that also Σ_2 converge to a bounded function. Finally, the sum Σ_3 converges because of (3.21). Therefore (5.2) holds for $s_1 = 1$ and $s = (0,0)$.

Now, assume that $s_1 = 0$ and $s = (1,0)$, then we have that

$$\partial_{\xi_2} m_J(\xi_1, \xi) = -i 2^{j/2n} I_J^{0,(1,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)$$
$$- i 2^{m\,j_1} I_J^{m,(0,0)}(2^{j_1} \xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi).$$

We write

$$\sum_{2mn\,j_1 \leq j} |\partial_{\xi_2} m_J(\xi_1, \xi)| \leq$$

$$\leq \rho(\xi)^{-\frac{1}{2n}} \Bigg(\sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} (2^j \rho(\xi))^{\frac{1}{2n}} |I_J^{0,(1,0)}(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)|$$

$$+ \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ j \in \mathbb{Z}}} (2^j \rho(\xi))^{\frac{1}{2n}} |I_J^{0,(1,0)}(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)|$$

$$+ \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ j \in \mathbb{Z}}} (2^j \rho(\xi))^{\frac{1}{2n}} |I_J^{m,(0,0)}(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)|$$

$$(5.4) \qquad + \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ j \in \mathbb{Z}}} (2^j \rho(\xi))^{\frac{1}{2n}} |I_J^{m,(0,0)}(2^{j_1}\xi_1, 2^{2mn\,j_1} \circ \xi, 2^j \circ \xi)| \Bigg).$$

By using the estimate (3.16) we can easily prove that the first and the third series on the right-hand side of (5.4) converge to a bounded function. Also, the second and the fourth series on the right-hand side of (5.4) converge to a bounded function as we can see by applying the estimate (3.14). Hence (5.2) holds for $s_1 = 0$ and $s = (1, 0)$. ∎

We now prove the L^p-boundedness of the operator T_2 by means of the analytic interpolation method. We start constructing an analytic family of linear operators $T_{2,z}$.

For $z \in \mathbb{C}$ we consider the kernel and its analytic continuation, defined in (2.9),

$$I^z(u) = \frac{G(0)\rho(u)^{z-Q}}{\sigma(S)\Gamma(z)G(z)}$$

on $\mathbb{R}^2$, where $u = (u_2, u_3)$, $\sigma(S)$ denotes the surface measure of the sphere

$$S := \left\{ u \in \mathbb{R}^2 : \rho(u) = 1 \right\},$$

and $G(z)$ has been defined in (2.8).

Example 5.2. In the light of Example 2.4, when $m = 2$ and $n = 3$ we have

$$(5.5) \qquad \begin{aligned} G(z) &:= \left(\Gamma\left(z + \frac{7}{12}\right)\right)^2 \cdot \Gamma\left(z + \frac{1}{6}\right)\Gamma\left(z + \frac{1}{4}\right)\Gamma\left(z + \frac{1}{3}\right)\Gamma\left(z + \frac{5}{12}\right) \\ &\quad \Gamma\left(z + \frac{1}{2}\right)\Gamma\left(z + \frac{2}{3}\right)\Gamma\left(z + \frac{3}{4}\right)\Gamma\left(z + \frac{5}{6}\right)\Gamma\left(z + \frac{11}{12}\right). \end{aligned}$$

We shall denote by B_r, $r > 0$, the non-isotropic ball

$$(5.6) \qquad B_r := \left\{ (u_1, u_2) \in \mathbb{R}^2 \ : \ u_1^{2n} + u_2^{2m} \le r \right\}.$$

Let θ be a smooth compactly supported function on $\mathbb{R}^2$ whose support is contained in the ball $B_{\frac{1}{4}}$ and which is identically one in a neighbourhood of the origin. Let B^z be the distribution defined by

$$(5.7) \qquad \langle B^z, h \rangle := \langle I^z, \theta h \rangle, \qquad h \in \mathcal{S}(\mathbb{R}^2).$$

Observe that $B^0 = \delta_0$, since $I^0 = \delta_0$ and θ is identically one in a neighbourhood of the origin.

By considering the convolution between $2^{-j_1 - jQ} \varphi_J (2^{-j_1} x_1, 2^{-j} \circ (x - \gamma(x_1)))$ and $2^{-(m+n)j_1} (\delta_0 \otimes B^z)(x_1, 2^{-2mn j_1} \circ x)$ we get the kernel

$$K_{2,z}(x_1, x) = \sum_{2mn\, j_1 > j} 2^{-(m+n+1)j_1 - jQ} \times$$

$$\times \int \varphi_J(2^{-j_1} x_1, 2^{-j} \circ (u - \gamma(x_1))) B^z (2^{-2mn j_1} \circ (x - u)) \, du \,.$$

If we set

$$(5.8) \quad \lambda_J(x_1, x) := 2^{(2mn\, j_1 - j)Q} \int \varphi_J(x_1, 2^{2mn\, j_1 - j} \circ (x - u - \gamma(x_1))) B^z(u) \, du \,,$$

then the kernel $K_{2,z}$ may be written, by a change of variable, as

$$(5.9) \qquad K_{2,z}(x_1, x) := \sum_{2mn\, j_1 > j} 2^{-(m+n+1)j_1} \lambda_J(2^{-j_1} x_1, 2^{-2mn j_1} \circ x) \,.$$

We consider the analytic family of operators (of admissible growth)

$$(5.10) \qquad T_{2,z} f := f * K_{2,z}, \qquad f \in \mathcal{S}(\mathbb{R}^3) \,,$$

and we observe that $T_{2,0} f = f * K_{2,0} = T_2 f$.

In order to prove the L^p-boundedness of T_2 we need some preliminary results. The first result is an L^1- Lipschitz condition for the distribution B^z defined by (5.7).

Lemma 5.3. *For all* $0 < \Re e\, z < Q$

$$(5.11) \qquad \int \left| B^z(u + h) - B^z(u) \right| du \le C_z \rho(h)^{\Re e\, z}$$

for all $h \in \mathbb{R}$.

Proof. First we split the integral in (5.11) in two parts

$$\int \left| B^z(u+h) - B^z(u) \right| du = \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left| B^z(u+h) - B^z(u) \right| du$$

$$+ \int_{\rho(h) > \frac{\rho(u)}{2}} \left| B^z(u+h) - B^z(u) \right| du =: I + II \,.$$

Now

$$I \leq \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left| C_z \theta(u+h) \left(\rho^{z-Q}(u+h) - \rho^{z-Q}(u) \right) \right| du$$

$$+ \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left| C_z \left(\theta(u+h) - \theta(u) \right) \rho^{z-Q}(u) \right| du =: I_a + I_b \,.$$

To estimate I_a, we use the Mean Value Theorem [5, p. 11], obtaining

$$I_a \leq C \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left(\rho(u) \right)^{\Re ez - Q - 1} \rho(h) \, du$$

$$= C \rho(h) \int_{2\rho(h)}^{+\infty} r^{\Re ez - Q - 1} r^{Q-1} dr \int_S \rho(v)^{\Re ez - Q - 1} d\sigma(v)$$

$$\leq C \rho(h) \rho(h)^{\Re ez - 1} = C \rho(h)^{\Re ez} \,.$$

To estimate I_b, we observe that

$$I_b \leq C \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left| \theta(u+h) - \theta(u) \right| \left(\rho(u) \right)^{\Re ez - Q} du$$

$$\leq C \rho(h)^{\Re ez - Q} \int_{\rho(h) \leq \frac{\rho(u)}{2}} \left| \theta(u+h) - \theta(u) \right| du \leq 2C \rho(h)^{\Re ez - Q} \int \left| \theta(u) \right| du$$

$$\leq C \rho(h)^{\Re ez - Q} \,.$$

Finally, if k denotes a positive constant such that $\rho(x+y) \leq k \left(\rho(x) + \rho(y) \right)$ we observe that by [5, p. 14]

$$II = \int_{\rho(h) > \frac{\rho(u)}{2}} \left| B^z(u+h) - B^z(u) \right| du \leq 2C \int_{\rho(h) > \frac{\rho(u)}{3k}} \left(\rho(u) \right)^{\Re ez - Q} du$$

$$= 2C \int_0^{3k\rho(h)} r^{\Re ez - Q} r^{Q-1} dr \int_S \rho(v)^{\Re ez - Q} \, d\sigma(v) \leq C \rho(h)^{\Re ez} \,.$$

This inequality, combined with the bounds for I, yields (5.11). ∎

Then we need to recall the definition of non-isotropic Besov spaces [15].

Definition 5.4. In $\mathbb{R}^3$ we consider the family of one-parameter non-isotropic dilations defined in (3.8). Let $\widetilde{\rho}(x_1, x)$ be any homogeneous norm with respect to these dilations.

We denote by $B_{1,\infty}^\alpha$ the non-isotropic Besov space of functions $f \in L^1(\mathbb{R}^3)$ which satisfy an L^1-Lipschitz condition of order α, $0 < \alpha < 1$, i.e. there exists a positive constant C such that

$$\int |f(x_1 + h_1, x + h) - f(x_1, x)|\, dx_1 dx \leq C\widetilde{\rho}(h_1, h)^\alpha$$

for every $(h_1, h) \in \mathbb{R} \times \mathbb{R}^2$, where $h = (h_2, h_3)$.

If $f \in B_{1,\infty}^\alpha$ we set

$$\|f\|_{B_{1,\infty}^\alpha} := \|f\|_1 + \sup_{(h_1, h) \neq (0,0)} \widetilde{\rho}(h_1, h)^{-\alpha} \int |f(x_1 + h_1, x + h) - f(x_1, x)|\, dx_1 dx.$$

Our proof hinges on the following result.

Theorem 5.5. *Let $\{\psi_l\}_{l \in \mathbb{Z}}$ be a family of functions such that for some positive constants C, α, ε the following hypotheses hold uniformly in l:*

(i) $\{\psi_l\}_{l \in \mathbb{Z}} \subset L^1(\mathbb{R}^3)$;

(ii) $\int |\psi_l(x_1, x)|(1 + \widetilde{\rho}(x_1, x))^\varepsilon\, dx_1 dx \leq C$;

(iii) $\int \psi_l(x_1, x)\, dx_1 dx = 0$;

(iv) $\|\psi_l\|_{B_{1,\infty}^\alpha} \leq C$.

Then the series $\sum_{l \in \mathbb{Z}} 2^{-(m+n+1)l}\psi_l(2^{-l} \bullet \cdot)$ converges in the sense of distributions to a Calderón-Zygmund kernel.

Finally we can state and prove the L^p-bounds for T_2.

Proposition 5.6. *The operator T_2 is bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.*

Proof. The proof is by complex interpolation. We first prove that $T_{2,z}$ is bounded on $L^2(\mathbb{R}^3)$ for $-\frac{1}{2mn^2} < \Re e\, z < 0$.

En easy computation shows that

$$\left(2^{-(m+n+1)j_1}\lambda_J(2^{-j_1}\cdot, (2^{-2mn\, j_1} \circ \cdot))\right)\hat{}\,(\xi_1, \xi) = \widehat{\lambda_J}\left(2^{j_1}\xi_1, 2^{2mnj_1} \circ \xi\right)$$
$$= \widehat{B^z}\left(2^{2mnj_1} \circ \xi\right) m_J(\xi_1, \xi),$$

where m_J is the Fourier transform of $2^{-j_1 - jQ}\varphi_J\left(2^{-j_1}x_1, 2^{-j} \circ (x - \gamma(x_1))\right)$ and it is given by (4.2), λ_J is defined by (5.8), and $\widehat{B^z}$ is the Fourier transform of $2^{-(m+n)j_1}\left(\delta_0 \otimes B^z\left(\xi_1, 2^{-2mnj_1} \circ x\right)\right)$.

It follows from Proposition 2.8 that

(5.12)
$$\left| \widehat{B^z}(\xi) \right| \leq C(1 + \rho(\xi))^{-\Re ez} \quad .$$

Our aim is now to prove that the series

$$\sum_{2mnj_1 > j} \widehat{B^z}\left(2^{2mnj_1} \circ \xi\right) m_J(\xi_1, \xi),$$

corresponding to the Fourier transform of (5.9), converges boundedly to a bounded function. The inequality (5.12) yields

$$\left| \sum_{2mnj_1 > j} \widehat{B^z}\left(2^{2mnj_1} \circ \xi\right) m_J(\xi_1, \xi) \right| \leq$$

$$\leq C_z \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ 2mn \, j_1 > j}} \left(1 + \rho(2^{2mnj_1} \circ \xi)\right)^{|\Re ez|} \left| m_J(\xi_1, \xi) \right|$$

$$+ \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ 2mn \, j_1 > j}} \left(1 + \rho(2^{2mnj_1} \circ \xi)\right)^{|\Re ez|} \left| m_J(\xi_1, \xi) \right| =: \mathrm{J}_1 + \mathrm{J}_2.$$

Since
$$2^{2mnj_1} \rho(\xi) = 2^{2mnj_1}(\xi_2^{2n} + \xi_3^{2m}) \leq 2 \cdot \left(2^{j_1} |||(\xi_1, \xi)|||\right)^{2mn},$$

we have

(5.13)
$$\mathrm{J}_1 \leq C \, 3^{|\Re ez|} \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| \leq 1 \\ 2mn \, j_1 > j}} \left| m_J(\xi_1, \xi) \right| \leq C_z,$$

In the light of what has been proved in Proposition 4.1, to estimate J_2 observe that

$$\mathrm{J}_2 \leq C_z \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ 2mn \, j_1 > j}} \left(2^{j_1} |||(\xi_1, \xi)|||\right)^{2mn|\Re ez|} \left| m_J(\xi_1, \xi) \right|.$$

Now, by using (4.6) and estimate (3.14) for I_J, we obtain

$$\mathrm{J}_2 \leq C_z \sum_{\substack{2^{j_1} |||(\xi_1, \xi)||| > 1 \\ 2mn \, j_1 > j}} \frac{\left(2^{j_1} |||(\xi_1, \xi)|||\right)^{2mn|\Re ez|}}{\left(2^{j_1} |||(\xi_1, \xi)|||\right)^{1/n}} \frac{\rho(2^j \circ \xi)^{1/2n}}{\left(1 + \rho(2^j \circ \xi)\right)^N},$$

so that the operator $T_{2,z}$ is bounded on $L^2(\mathbb{R}^3)$ if $-\frac{1}{2mn^2} < \Re ez < 0$.

We will now show that the operator $T_{2,z}$ is bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ for $0 < \Re ez < Q$.

By setting $j - 2mn\,j_1 = k$ and using (3.10), (5.9) may be written as

$$K_{2,z}(x_1, x) = \sum_{j_1 \in \mathbb{Z}} 2^{-(m+n+1)j_1} \sum_{k=-\infty}^{0} \lambda_{(j_1, k+2mn\,j_1)}(2^{-j_1} \bullet (x_1, x))$$

where

$$\lambda_{(j_1, k+2mn\,j_1)}(x_1, x) =$$

$$(5.14) \qquad = 2^{-Qk} \int \varphi_{(j_1, k+2mn\,j_1)}(x_1, 2^{-k} \circ (x - u - \gamma(x_1))) B^z(u)\, du.$$

We shall now prove that the functions

$$\psi_{j_1}(x_1, x) := \sum_{k=-\infty}^{0} \lambda_{(j_1, k+2mn\,j_1)}(x_1, x)$$

satisfy the hypotheses of Theorem 5.5 uniformly in j_1. Since the estimates on the functions $\psi_{j_1}(x_1, x)$ that we will prove later are independent of j_1, it suffices to prove that

$$\psi_0(x_1, x) = \sum_{k=-\infty}^{0} \lambda_{(0,k)}(x_1, x)$$

satisfies the hypotheses of Theorem 5.5.

We begin proving that ψ_0 belongs to $L^1(\mathbb{R}^3)$. As a consequence of (3.5) with $\beta = 0$ and a change of variable we obtain

$$\psi_0(x_1, x) = \sum_{k=-\infty}^{0} \int \varphi_{(0,k)}(x_1, v) \big(B^z(x - 2^k \circ v - \gamma(x_1)) - B^z(x - \gamma(x_1)) \big)\, dv,$$

whence

$$\int |\psi_0(x_1, x)|\, dx_1 dx \leq$$

$$\leq \sum_{k=-\infty}^{0} \int \Big(\int |\varphi_{(0,k)}(x_1, v)| \cdot \big| B^z(x - 2^k \circ v - \gamma(x_1)) - B^z(x - \gamma(x_1)) \big|\, dv \Big) dx_1 dx$$

$$= \sum_{k=-\infty}^{0} \int |\varphi_{(0,k)}(x_1, v)| \Big(\int \big| B^z(x - 2^k \circ v) - B^z(x) \big|\, dx \Big) dx_1 dv$$

$$\leq C_z \sum_{k=-\infty}^{0} \int |\varphi_{(0,k)}(x_1, v)| \rho(2^k \circ v)^{\Re ez}\, dx_1 dv$$

$$= C_z \sum_{k=-\infty}^{0} 2^{k\Re ez} \int |\varphi_{(0,k)}(x_1, v)| \rho(v)^{\Re ez}\, dx_1 dv \leq C_z,$$

by Lemma 5.3 and since $\{\varphi_{(0,k)}\}$ is a bounded set in $\mathcal{S}(\mathbb{R}^3)$.

Now we shall prove that ψ_0 satisfies the hypothesis $ii)$ of Theorem 5.5 with $\varepsilon = 1$ and

$$(5.15) \qquad \widetilde{\rho}(x_1, x) = |x_1| + |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} .$$

We split the integral as

$$\int |\psi_0(x_1, x)|(1 + \widetilde{\rho}(x_1, x)) dx_1 dx$$

$$= \left(\int_{\substack{\frac{1}{2} \le |x_1| \le 4 \\ |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} \le M}} + \int_{\substack{\frac{1}{2} \le |x_1| \le 4 \\ |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M}} \right) |\psi_0(x_1, x)|(1 + \widetilde{\rho}(x_1, x)) \, dx_1 dx$$

$$= \Im_1 + \Im_2 \, ,$$

where the costant M will be chosen later.

Now $\Im_1$ is bounded by some positive costant C, since $\psi_0(x_1, x)$ belongs to $L^1(\mathbb{R}^3)$ and $\widetilde{\rho}(x_1, x)$ is bounded on the integration set. To study $\Im_2$, observe that

$$\Im_2 \le \sum_{k=-\infty}^{0} 2^{-kQ} \int_{\substack{\frac{1}{2} \le |x_1| \le 4 \\ |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M}} \int_{\rho(u) \le \frac{1}{4}} \left| \varphi_{(0,k)}(x_1, 2^{-k} \circ (x - u - \gamma(x_1))) \right| \times$$

$$\times \left| B^z(u) \right| du \left(5 + |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} \right) dx_1 dx$$

$$\le C \sum_{k=-\infty}^{0} 2^{-kQ} \int_{\substack{\frac{1}{2} \le |x_1| \le 4 \\ |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M}} \int_{\rho(u) \le \frac{1}{4}} \frac{||\varphi_{(0,k)}||_{(N)} |B^z(u)|}{\left(1 + \left| 2^{-k} \circ (x - u - \gamma(x_1)) \right| \right)^N} du \times$$

$$\times \left(5 + |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} \right) dx_1 dx$$

$$\le C \sum_{k=-\infty}^{0} 2^{-kQ} \int_{\substack{\frac{1}{2} \le |x_1| \le 4 \\ |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M}} \left(5 + |x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} \right) \times$$

$$\times \int_{\rho(u) \le \frac{1}{4}} \frac{||\varphi_{(0,k)}||_{(N)} |B^z(u)|}{\left(1 + \left| 2^{-k} \circ x \right| \right)^N} du \, dx_1 dx \, ,$$

where the last inequality follows from the fact that for some constant $C > 0$ we have

$$1 + \left| 2^{-k} \circ (x - u - \gamma(x_1)) \right| \ge C \left(1 + \left| 2^{-k} \circ x \right| \right)$$

if $|x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M$, if M is sufficiently large, $\rho(u) \le \frac{1}{4}$, $\frac{1}{2} \le |x_1| \le 4$.

Next (2.2) yields

$$(5.16) \qquad \left| 2^{-k} \circ x \right| \ge A \left(\rho(2^{-k} \circ x) \right)^{\frac{1}{2n}}$$

for some $A > 0$, if $k \le 0$ and $|x_2|^{\frac{1}{m}} + |x_3|^{\frac{1}{n}} > M$.

Now (5.16) and the local integrability of B^z give

$$\Im_2 \leq C \sum_{k=-\infty}^{0} 2^{-kQ} \int_{|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}>M} (5+|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}) \frac{1}{\left(1+\rho\left(2^{-k}\circ x\right)\right)^{\frac{N}{2n}}} dx$$

$$\leq C \sum_{k=-\infty}^{0} 2^{-kQ} \int_{|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}>M} (5+|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}) \frac{1}{\left(1+2^{-k}\rho(x)\right)^{\frac{N}{2n}}} dx$$

$$\leq C \sum_{k=-\infty}^{0} 2^{-kQ} 2^{\frac{kN}{2n}} \int_{|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}>M} (5+|x_2|^{\frac{1}{m}}+|x_3|^{\frac{1}{n}}) \frac{1}{\rho(x)^{\frac{N}{2n}}} dx \leq C\,,$$

yielding $ii)$.

Property $iii)$ is an immediate consequence of (3.5).

Finally we shall prove that ψ_0 satisfies property $iv)$ in Theorem 5.5.

We start proving that

$$(5.17) \qquad \int \left|\psi_0(x_1, x+h) - \psi_0(x_1, x)\right| dx_1\, dx \leq C_z' \rho(h)^{\frac{\Re ez}{4mn}}\,, \quad h \in \mathbb{R}^2\,.$$

Indeed we have

$$\int \left|\psi_0(x_1, x+h) - \psi_0(x_1, x)\right| dx_1\, dx \leq$$

$$\leq \sum_{k=-\infty}^{0} \int \left(\int \left|\varphi_{(0,k)}(x_1, v)\left(B^z(x+h-2^k\circ v-\gamma(x_1))-B^z(x+h-\gamma(x_1))\right)\right.\right.$$

$$\left.\left. - \varphi_{(0,k)}(x_1, v)\left(B^z(x-2^k\circ v-\gamma(x_1)) - B^z(x-\gamma(x_1))\right)\right| dxdx_1\right) dv$$

$$=: \sum_{k=-\infty}^{0} \int J_k(v) dv\,.$$

Now observe that

$$J_k(v) \leq 2 \int \left|\varphi_{(0,k)}(x_1, v)\left(B^z(x-2^k\circ v-\gamma(x_1)) - B^z(x-\gamma(x_1))\right)\right| dxdx_1$$

$$= 2 \int \left|\varphi_{(0,k)}(x_1, v)\right| \left(\int \left|B^z(y-2^k\circ v) - B^z(y)\right| dy\right) dx_1$$

$$(5.18)$$

$$\leq C_{z,N} \frac{\rho(2^k\circ v)^{\Re ez}}{(1+|v|)^N} = C_{z,N} 2^{k\Re ez} \frac{\rho(v)^{\Re ez}}{(1+|v|)^N}\,, \quad N \in \mathbb{N}\,,$$

where we used both the Lemma 5.3 and the fact that the functions $\varphi_{(0,k)}$ are a bounded set in $\mathcal{S}(\mathbb{R}^3)$.

On the other hand, as a consequence of Lemma 5.3

$$J_k(v) \leq 2 \int \Big| \varphi_{(0,k)}(x_1, v) B^z(y + h - \gamma(x_1)) - \varphi_{(0,k)}(x_1, v) B^z(y - \gamma(x_1)) \Big| dy dx_1$$

$$(5.19) \qquad \leq C_{z,N} \frac{\rho(h)^{\Re e z}}{(1 + |v|)^N}, \quad N \in \mathbb{N}, h \in \mathbb{R}^2.$$

By taking a suitable mean between estimates (5.18) and (5.19) we obtain

$$(5.20) \quad J_k(v) = J_k(v)^{\frac{4mn-1}{4mn}} \cdot J_k(v)^{\frac{1}{4mn}} \leq C_{z,N} 2^{\frac{4mn-1}{4mn} k \Re e z} \rho(h)^{\frac{\Re e z}{4mn}} \frac{\rho(v)^{\frac{4mn-1}{4mn} \Re e z}}{(1 + |v|)^N},$$

for $N \in \mathbb{N}, h \in \mathbb{R}$, so that

$$\int \Big| \psi_0(x_1, x + h) - \psi_0(x_1, x) \Big| dx_1 \, dx \leq \sum_{k=-\infty}^{0} \int J_k(v) dv$$

$$\leq C_{z,N} \sum_{k=-\infty}^{0} 2^{\frac{4mn-1}{4mn} k \Re e z} \rho(h)^{\frac{\Re e z}{4mn}} \int \frac{\rho(v)^{\frac{4mn-1}{4mn} \Re e z}}{(1 + |v|)^N} dv \leq C_{z,N} \rho(h)^{\frac{\Re e z}{4mn}},$$

proving (5.17).

In an analogous way we can prove the inequality

$$(5.21) \qquad \int \Big| \psi_0(x_1 + h_1, x) - \psi_0(x_1, x) \Big| dx_1 \, dx \leq C'_z |h_1|^{\frac{\Re e z}{2}}.$$

Finally we have

$$\int \Big| \psi_0(x_1 + h_1, x + h) - \psi_0(x_1, x) \Big| dx_1 \, dx \leq$$

$$\leq \int \Big| \psi_0(x_1 + h_1, x + h) - \psi_0(x_1 + h_1, x) \Big| dx_1 \, dx$$

$$+ \int \Big| \psi_0(x_1 + h_1, x) - \psi_0(x_1, x) \Big| dx_1 \, dx$$

$$\leq C_z \left(|h_1|^{\frac{\Re e z}{2}} + \rho(h)^{\frac{\Re e z}{4mn}} \right)$$

Since

$$|h_1|^{\frac{\Re e z}{2}} + \rho(h)^{\frac{\Re e z}{4mn}} \leq C \left(|h_1| + \rho(h)^{\frac{1}{2mn}} \right)^{\frac{\Re e z}{2}}$$

and, by a standard inequality,

$$\left(h_2^{2n} + h_3^{2m} \right)^{\frac{1}{2mn}} \leq |h_2|^{\frac{1}{m}} + |h_3|^{\frac{1}{n}},$$

we finally get

$$|h_1|^{\frac{\Re e z}{2}} + \rho(h)^{\frac{\Re e z}{4mn}} \leq C \tilde{\rho}(h_1, h)^{\frac{\Re e z}{2}},$$

proving (iv) with $\alpha = \Re e z / 2$. As a consequence of Theorem 5.5 the operator $T_{2,z}$ is bounded on $L^p(\mathbb{R}^3)$, $1 < p < \infty$, for $0 < \Re e z < Q$.

Finally choose $p_0 \in (1, 2)$ and fix $q_0 \in (1, p_0)$ such that

$$\frac{1}{p_0} = \frac{b}{2(b-a)} - \frac{a}{q_0(b-a)} \,.$$

Then the operator $T_{2,0} = T_2$ is bounded on $L^{p_0}(\mathbb{R}^3)$. By the arbitrariness of p_0 and by duality we conclude that $T_{2,0} = T_2$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. ∎

6. Final remarks

Remark 1

We observe that our results also hold in the more general situation in which the curve $\gamma(x_1) = (x_1^m, x_1^n)$ is perturbed to $\hat{\gamma}(x_1) = (x_1^m + \lambda_2(x_1), x_1^n + \lambda_3(x_1))$ with λ_2 and λ_3 smooth and satisfying $\lambda_2(x_1) = o(x_1^m)$ and $\lambda_3(x_1) = o(x_1^n)$. In fact, given a product kernel K_0 on $\mathbb{R}^3$, we define the distribution K by

$$\int K(x_1, x) f(x_1, x)\, dx_1 dx := \int K_0(x_1, x) f(x_1, x + \hat{\gamma}(x_1))\, dx_1 dx$$

for all Schwartz functions f on $\mathbb{R}^3$. Then we break the integral on the right-hand side as

$$\left(\int_{\mathbb{R}} \int_{-1}^{1} + \int_{\mathbb{R}} \int_{|x_1|>1} \right) K_0(x_1, x) f(x_1, x + \hat{\gamma}(x_1))\, dx_1 dx \,.$$

The first term, by means of a Taylor expansion, may be reduced to the polynomial case, while the latter one is of Calderon-Zygmund type.

Remark 2

Our results still hold for convolution operators on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ with kernels adapted to curves of the form $\gamma(x_1) = (x_1^{m_1}, x_1^{m_2}, \ldots, x_1^{m_{d-1}})$, $1 < m_1 < m_2 < \cdots < m_{d-1}$, $m_j \in \mathbb{N}$, $j = 1, \ldots, d-1$, with values in $\mathbb{R}^{d-1}$. Since notation in the higher dimensional case is more cumbersome, we gave full details of the proof only for the space $\mathbb{R}^3$.

On $\mathbb{R}^{d-1}$ we introduce the dilations given by

(6.1) $\qquad \delta \circ x = (\delta^{1/2m_{d-1}} x_2, \delta^{1/2m_{d-2}} x_3, \ldots, \delta^{1/2m_1}) \,,$

where $\delta > 0$ and $x = (x_2, \ldots, x_{d-1})$. Moreover we equip $\mathbb{R}^{d-1}$ with the smooth homogeneous norm

$$\rho(x) = x_2^{2m_{d-1}} + x_3^{2m_{d-2}} + \cdots + x_d^{2m_1}$$

on $\mathbb{R}^{d-1}$. From this point on, the proof follows the same pattern as in $\mathbb{R}^3$.

In particular, the strategy of using Bernstein-Sato polynomials to build the meromorphic continuation of the non isotropic Riesz potentials $\rho(x)^{z-\tilde{Q}}$ (here $\tilde{Q} = \frac{1}{2m_{d-1}} + \frac{1}{2m_{d-2}} + \ldots + \frac{1}{2m_1}$) works in the multidimensional case as well, with some additional computational difficulties in finding the zeros of the Bernstein-Sato polynomials [1].

Remark 3

As an example of the class of operators studied in this paper we exhibit the following operator, arising in the study of the $L^p - L^q$ boundedness of a double analytic family of fractional integrals along curves in the space (see [3] for the planar case).

Let ψ be a smooth function on $\mathbb{R}^2$, such that $\psi(u_1, -u_2) = \psi(u_1, u_2)$ for every $(u_1, u_2) \in \mathbb{R}^2$, $\psi \equiv 1$ on $B_{\frac{1}{2}}$ and $\psi \equiv 0$ outside B_1, with $0 \le \psi \le 1$ on $\mathbb{R}^2$ (here B_r, $r > 0$, denotes the non-isotropic ball in $\mathbb{R}^2$ defined by (5.6)).

Define an analytic family of distributions K_z^γ, for γ and z in $\mathbb{C}$, $\Re e \gamma \ge 0$, in the following way

$$(6.2) \qquad < K_z^\gamma, f > := \int < D_z(u_1, u_2), f(t, u_1 t^m, u_2 t^n) > |t|^\gamma \frac{dt}{t},$$

where D_z, $\Re e\, z > 0$, denotes the family of analytic distributions given by

$$< D_z, h > := < \psi(\cdot, \cdot)\, I^z(\cdot, \cdot)\, , h(\cdot + 1, \cdot + 1) >$$
$$= C_z \int_{\mathbb{R}^2} \rho(u_1 - 1, u_2 - 1)^{z-Q} \psi\,(u_1 - 1, u_2 - 1)\, h(u_1, u_2)\, du_1, du_2\,,$$

with $C_z := \frac{G(0)}{2\sigma(S)\Gamma(z)G(z)}$ and $h \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. It is straightforward to check that D_z may be extended to all $z \in \mathbb{C}$.

As a consequence of Proposition 2.6 we have

$$(6.3) \qquad\qquad\qquad < D_0, h > = h(1, 1).$$

We remark that, if $\Re e \gamma = 0$, then

$$< K_z^\gamma, f > := \lim_{\varepsilon \to 0} \int < D_z(u_1, u_2), f(t, u_1 t^m, u_2 t^n) > |t|^{i\rho+\varepsilon} \frac{dt}{t}\,,$$

where $\Im m \gamma = \rho$, for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Observe moreover that K_z^γ depends analytically on both γ and z.

At this point we may introduce the family of convolution operators with kernel K_z^γ defined by (6.2), that is

$$(S_z^\gamma f)\,(x_1, x_2, x_3) := (K_z^\gamma * f)\,(x_1, x_2, x_3)$$
$$= \int < D_z(u_1, u_2), f(x_1 - t, x_2 - u_1 t^m, x_3 - u_2 t^n) > |t|^\gamma\, \frac{dt}{t}.$$

We observe that, in the light of (6.3), we have

$$\left(S_0^\gamma f\right)(x_1, x_2, x_3) := C_0 \int_{\mathbb{R}^3} f(x_1 - t, x_2 - t^m, x_3 - t^n) > |t|^\gamma \frac{dt}{t},$$

that is, at the height $z = 0$ we recover the fractional integration operator along the curve $t \mapsto (t, t^m, t^n)$ in the space.

In a forthcoming paper [2] we shall give a complete picture of the characteristic set of the operator S_z^γ. A key step in the proof of that result is the fact that at the height $\Re e z = 0$ and for $\Re e \gamma = 0$ the kernel K_z^γ is a product kernel adapted to the curve $x_1 \mapsto (x_1^m, x_2^n)$. Thus Theorem 1.2 yields L^p-boundedness for the operator S_z^γ for $\Re e z = 0$ and $\Re e \gamma = 0$ for all $p \in (1, \infty)$.

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