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Quantization of Contact Manifolds

By

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Abstract

We show the existence of the stack of micro-differential modules on an arbitrary contact manifold, although we cannot expect the global existence of the ring of micro-differential operators.

§ 0. Inroduction

In [SKK], we defined the sheaf of micro-differential operators on the cotangent bundle and we associated a quantized contact transformation with a given contact transformation.

More precisely, for a complex manifold X, let us denote by $\mathscr{E}_X$ the ring of micro-differential operators regarded as a sheaf of rings on the projective cotangent bundle P^*X. Let X and Y be two manifolds with the same dimension. Let U_X and U_Y be open subsets of P^*X and P^*Y respectively, and let $f: U_X \to U_Y$ be a holomorphic map preserving the canonical 1-form. Then for any point $p \in U_X$ there exists an open neighborhood U of p and a $\mathbf{C}$-ring isomorphism φ: $f^{-1}\mathscr{E}_Y|_U \to \mathscr{E}_X|_U$. Such a φ is not unique, although with other extra data we can reduce the uniqueness of φ up to the inner automorphism by micro-differential operators with 1 as its principal symbol.

Now let us consider a contact manifold Z with $(2n+1)$ dimension. This means that Z is endowed with an invertible $\mathcal{O}_Z$-module $\mathcal{O}_Z(1)$ and a 1-form $\omega \in \Gamma(Z, \Omega_Z^1 \otimes \mathcal{O}_Z(1))$ such that $\omega \wedge (d\omega)^n$ is a generator of $\Omega_Z^{2n+1} \otimes \mathcal{O}_Z(2n+1)$. Here $\mathcal{O}_Z(k) = \mathcal{O}_Z(1)^{\otimes k}$.

The purpose of this paper is to show that we can naturally define a stack (a sheaf of categories) on Z that is locally isomorphic to the stack of modules over the ring of micro-differential operators.

Let us take an open covering $Z = \bigcup_{i \in I} U_i$ and contact embeddings $f_i: U_i \hookrightarrow P^*X_i$. Set $\mathscr{A}_i = f_i^{-1}((\Omega_{X_i}^n)^{\otimes 1/2} \otimes \mathscr{E}_{X_i} \otimes (\Omega_{X_i}^n)^{\otimes -1/2})$. Then, $\mathscr{A}_i$ is a sheaf of $\mathbf{C}$-rings on

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U_i endowed with an antiautomorphism $*$ such that $*^2 = 1$. The ring $\mathscr{A}_i$ has the order filtration $F(\mathscr{A}_i)$ such that $Gr_k^F \mathscr{A}_i = \mathscr{O}_Z(k)$.

Lemma 1. *Let $\mathscr{G}$ be the sheaf of automorphisms of $\mathscr{A}_i$ commuting with $*$. Then $\{P \in F_0(\mathscr{A}_i); P^*P = 1, \sigma_0(P) = 1\} \to \mathscr{G}$ given by $P \mapsto Ad(P)$ is bijective. Here σ_0 is the symbol map $F_0(\mathscr{A}_i) \to Gr_0^F \mathscr{A}_i = \mathscr{O}_Z$.*

Proof. For $\lambda \in \mathbf{C}$, let $\mathscr{E}(\lambda)$ be the sheaf of micro-differential operators of order $\lambda + \mathbf{Z}_{\leq 0}$. Then any automorphism φ of $\mathscr{E}_X$ is given by $Ad(P)$ for some λ and some invertible element P of $\mathscr{E}(\lambda)$. If φ commutes with $*$, then $Ad(P^*P) = \mathrm{id}$ and hence P^*P must be constant. Hence P is order 0 and we can normalize $P^*P = 1$ and $\sigma_0(P) = 1$ by dividing P by a suitable constant. Q.E.D.

Now, shrinking U_i if necessary, we may assume that there exists a $\mathbf{C}$-ring isomorphism $f_{ij} : \mathscr{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathscr{A}_i|_{U_{ij}}$ which commutes with $*$. Here we employed the notation

$$U_{i_0 i_1 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}.$$

Then $f_{ij} \circ f_{jk} : \mathscr{A}_k|_{U_{ijk}} \to \mathscr{A}_i|_{U_{ijk}}$ is not equal to $f_{ik}|_{U_{ik}}$ in general. Hence we cannot patch $\mathscr{A}_i$ together to get a ring globally defined on Z.

By Lemma 1, there exists $P_{ijk} \in \Gamma(U_{ijk}; F_0(\mathscr{A}_i))$ such that

$$f_{ij} \circ f_{jk} = Ad(P_{ijk}) \circ f_{ik} \text{ and } P_{ijk}^* P_{ijk} = 1,\ \sigma_0(P_{ijk}) = 1. \tag{0.1}$$

For $i, j, k, l \in I$, we have

$$(f_{ij} \circ f_{jk}) \circ f_{kl} = Ad(P_{ijk}) \circ f_{ik} \circ f_{kl} = Ad(P_{ijk} P_{ikl}) \circ f_{il}$$

and

$$f_{ij} \circ (f_{jk} \circ f_{kl}) = f_{ij} \circ Ad(P_{jkl}) \circ f_{jl} = Ad(f_{ij}(P_{jkl})) \circ f_{ij} \circ f_{jl} = Ad(f_{ij}(P_{jkl}) P_{ijl}) \circ f_{il}.$$

Hence by Lemma 1, we obtain

$$P_{ijk} P_{ikl} = f_{ij}(P_{jkl}) P_{ijl}. \tag{0.2}$$

This cocycle relation permits us to patch the categories of $\mathscr{A}_i$-modules to get a stack globally defined over Z.

§ 1. Stack

Let us recall the definition of a stack on a topological space X. A prestack $\mathscr{C}$ on X consists of following data:

(1.1) a category $\mathscr{C}(U)$ for any open subset U of X, (1.2) A functor $r_{VU}: \mathscr{C}(U) \to \mathscr{C}(V)$ for open subsets V and U with $V \subset U$.

(1.3) An isomorphism of functors $\theta_{WVU}: r_{WV} \circ r_{VU} \to r_{WU}$ for open subsets $W \subset V \subset U$.

They are assumed to satisfy the following axioms.

($PS1$) $r_{UU} = id$.

($PS2$) $\theta_{UUU} = id$.

($PS3$) For open subsets $U_1 \subset U_2 \subset U_3 \subset U_4$, the diagram

$$\begin{array}{ccc} r_{U_1U_2} \circ r_{U_2U_3} \circ r_{U_3U_4} & \xrightarrow{\theta_{U_2U_3U_4}} & r_{U_1U_2} \circ r_{U_2U_4} \\ \downarrow{\scriptstyle \theta_{U_1U_2U_3}} & & \downarrow{\scriptstyle \theta_{U_1U_2U_4}} \\ r_{U_1U_3} \circ r_{U_3U_4} & \xrightarrow{\theta_{U_1U_3U_4}} & r_{U_1U_4} \end{array}$$

commutes.

A prestack $\mathscr{C}$ is called a stack if it satisfies furthermore the following axioms.

($S1$) For any open subset U and $A, B \in Ob(\mathscr{C}(U))$, the presheaf on U

$$\mathscr{H}om(A, B): U \supset V \mapsto \mathrm{Hom}_{\mathscr{C}(V)}(r_{VU}(A), r_{VU}(B))$$

is a sheaf.

($S2$) Let $\{U_i\}$ be an open covering of an open set U, $A_i \in Ob(\mathscr{C}(U_i))$ and let $\varphi_{ij}: r_{U_{ij}U_j}(A_j) \to r_{U_{ij}U_i}(A_i)$ be an isomorphism. Assume the commutativity of the following diagram for any i, j, k:

$$\begin{array}{ccccc} r_{U_{ijk}U_{jk}} r_{U_{jk}U_k} A_k & \xrightarrow{\varphi_{jk}} & r_{U_{ijk}U_{jk}} r_{U_{jk}U_j} A_j & \xrightarrow{\theta_{U_{ijk}U_{jk}U_j}} & r_{U_{ijk}U_j} A_j \\ \downarrow{\scriptstyle \theta_{U_{ijk}U_{jk}U_k}} & & & & \wr\uparrow{\scriptstyle \theta_{U_{ijk}U_{ij}U_j}} \\ r_{U_{ijk}U_k} A_k & & & & r_{U_{ijk}U_{ij}} r_{U_{ij}U_j} A_j \\ \wr\uparrow{\scriptstyle \theta_{U_{ijk}U_{ik}U_k}} & & & & \downarrow{\scriptstyle \varphi_{ij}} \\ r_{U_{ijk}U_{ik}} r_{U_{ik}U_k} A_k & & & & r_{U_{ijk}U_{ij}} r_{U_{ij}U_i} A_i \\ \downarrow{\scriptstyle \varphi_{ik}} & & & & \downarrow{\scriptstyle \theta_{U_{ijk}U_{ij}U_i}} \\ r_{U_{ijk}U_{ik}} r_{U_{ik}U_i} A_i & & \xrightarrow{\theta_{U_{ijk}U_{ik}U_i}} & & r_{U_{ijk}U_i} A_i. \end{array}$$

Then there exist an object A of $\mathscr{C}(U)$ and a family of isomorphisms $\phi_i: r_{U_iU}(A) \xrightarrow{\sim} A_i$ such that

$$\begin{array}{ccccc} r_{U_{ij},U_j} r_{U_j,U} A & \xrightarrow{\theta_{U_{ij},U_j,U}} & r_{U_{ij},U_j} A & \xleftarrow[\sim]{\theta_{U_{ij},U_i,U}} & r_{U_{ij},U_i} r_{U_i,U} A \\ \downarrow \phi_j & & & & \downarrow \phi_i \\ r_{U_{ij},U_j} A_j & & \xrightarrow{\varphi_{ij}} & & r_{U_{ij},U_i} A_i \end{array}$$

commutes.

For an open subset U of X, we can define the restriction $\mathscr{C}|_U$ to U, which is a stack on U.

For two stacks $\mathscr{C}_1$, $\mathscr{C}_2$ on X, we can define the notion of functors from $\mathscr{C}_1$ to $\mathscr{C}_2$ and for two functors f, g from $\mathscr{C}_1$ to $\mathscr{C}_2$, we can define the notion of morphisms from f to g. We call a functor $f\colon \mathscr{C}_1 \to \mathscr{C}_2$ an equivalence if there exists a functor $g\colon \mathscr{C}_2 \to \mathscr{C}_1$ such that $f \circ g$ and $g \circ f$ are isomorphic to the identity respectively.

§ 2. Patching of Stacks

Let $\{U_i\}$ be an open covering of X and $\mathscr{C}_i$ a stack on U_i. Let $\varphi_{ij}\colon \mathscr{C}_j|_{U_{ij}} \to \mathscr{C}_i|_{U_{ij}}$ be an equivalence of stacks. Let $\phi_{ijk}\colon \varphi_{ij} \circ \varphi_{jk} \xrightarrow{\sim} \varphi_{ik}$ be an isomorphism of functors from $\mathscr{C}_k|_{U_{ijk}}$ to $\mathscr{C}_i|_{U_{ijk}}$. Assume that

For any i, j, k, l the diagram

(*PC*)
$$\begin{array}{ccc} \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} & \xrightarrow{\phi_{jkl}} & \varphi_{ij} \circ \varphi_{jl} \\ \downarrow \phi_{ijk} & & \downarrow \phi_{ijl} \\ \varphi_{ik} \circ \varphi_{kl} & \xrightarrow{\phi_{ikl}} & \varphi_{il} \end{array}$$

commutes.

Then there exists a stack $\mathscr{C}$ and an equivalence $\mathscr{C}|_{U_i} \to \mathscr{C}_i$ satisfying the plausible compatibility conditions. Such a $\mathscr{C}$ is unique up to equivalence.

§ 3. Patching of Stacks of Modules

In this paper, a ring means a (not necessarily commutative) ring with 1. Let $\{U_i\}$ be an open covering of X and let $\mathscr{A}_i$ be a sheaf of rings on U_i. Assume that there is given a ring isomorphism $f_{ij}\colon \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ and $a_{ijk} \in \Gamma(U_{ijk}; \mathscr{A}_i^{\times})$ such that

(*C*1) $$f_{ij} \circ f_{jk} = Ad(a_{ijk}) f_{ik} \quad \text{in} \quad \mathrm{Hom}(\mathscr{A}_k|_{U_{ijk}}, \mathscr{A}_i|_{U_{ijk}})$$

and

(C2) $$a_{ijk}a_{ikl}=f_{ij}(a_{jkl})a_{ijl} \text{ in } \Gamma(U_{ijk};\mathscr{A}_i^{\times}).$$

Here $\mathscr{A}_i^{\times}$ denotes the sheaf of invertible sections of $\mathscr{A}_i$.
Note that if the $\mathscr{A}_i$ are commutative, then $\{f_{ij}\}$ satisfies the chain conditions and hence we can define the globally defined ring $\mathscr{A}$ such that $\mathscr{A}|_{U_i}\simeq\mathscr{A}_i$. In a non-commutative case, we cannot construct such an $\mathscr{A}$ in general, but we can construct a stack locally isomorphic to the stack of $\mathscr{A}_i$-modules.

Let $\mathrm{Mod}(\mathscr{A}_i)$ be the stack of left $\mathscr{A}_i$-modules on U_i. In order to patch $\mathrm{Mod}(\mathscr{A}_i)$ together, let us apply the result in § 2.
For $M\in\mathrm{Mod}(\mathscr{A}_j)$, let $\varphi_{ij}(M)$ be the $\mathscr{A}_i$-module with a sheaf isomorphism $\alpha_{ij}\colon M\to\varphi_{ij}(M)$ such that

$$a\alpha_{ij}(u)=\alpha_{ij}(f_{ji}(a)u) \quad\text{for } a\in\mathscr{A}_i \text{ and } u\in M.$$

This defines the functor $\varphi_{ij}\colon \mathrm{Mod}(\mathscr{A}_j)|_{U_{ij}}\to\mathrm{Mod}(\mathscr{A}_i)|_{U_{ij}}$.
Let us define an isomorphism of functors

$$\psi_{ijk}\colon \varphi_{ij}\circ\varphi_{jk}\to\varphi_{ik}$$

as follows. For $M\in\mathrm{Mod}(\mathscr{A}_k)|_{U_{ijk}}$, we define

$$\psi_{ijk}(M)\colon \varphi_{ij}\circ\varphi_{jk}(M)\to\varphi_{ik}(M)$$

by $\alpha_{ij}\alpha_{jk}(u)\mapsto\alpha_{ik}(a_{kji}^{-1}u)$ for $u\in M$. Let us check that $\psi_{ijk}(M)$ is $\mathscr{A}_i$-linear. For $a\in\mathscr{A}_i$ and $u\in M$, we have

$$a\alpha_{ij}\alpha_{jk}(u)=\alpha_{ij}(f_{ji}(a)\alpha_{jk}(u))=\alpha_{ij}\alpha_{jk}(f_{kj}f_{ji}(a)u)=\alpha_{ij}\alpha_{jk}(\alpha_{kji}f_{ki}(a)a_{kji}^{-1}u).$$

Hence we obtain

$$\psi_{ijk}(M)(a\alpha_{ij}\alpha_{jk}(u))=\alpha_{ik}(f_{ki}(a)a_{kji}^{-1}u)=a\alpha_{ik}(a_{kji}^{-1}u)=a\psi_{ijk}(M)(\alpha_{ij}\alpha_{jk}(u)).$$

Thus, $\psi_{ijk}(M)$ is $\mathscr{A}_i$-linear and hence ψ_{ijk} is a well-defined morphism of functors.

Next, we shall check the chain condition (PC). The composition $\psi_{ikl}\circ\psi_{ijk}$ is calculated as follows:

$$\begin{aligned}\psi_{ikl}\psi_{ijk}(\alpha_{ij}\alpha_{jk}\alpha_{kl}(u))&=\psi_{ikl}\alpha_{ik}(a_{kji}^{-1}\alpha_{kl}(u))=\psi_{ikl}\alpha_{ik}\alpha_{kl}(f_{lk}(a_{kji}^{-1})u)\\&=\alpha_{il}(a_{lki}^{-1}f_{lk}(a_{kji}^{-1})u).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi_{ijl}\psi_{jkl}(\alpha_{ij}\alpha_{jk}\alpha_{kl}(u))&=\psi_{ijl}\alpha_{ij}(\psi_{jkl}(\alpha_{jk}\alpha_{kl}(u)))=\psi_{ijl}\alpha_{ij}\alpha_{jl}(a_{lkj}^{-1}u)\\&=\alpha_{il}(a_{lji}^{-1}a_{lkj}^{-1}u).\end{aligned}$$

Then $\psi_{ikl}\circ\psi_{ijk}=\psi_{ijl}\circ\psi_{jkl}$ follows from (C2).

By the arguments in § 2, we can patch $\mathrm{Mod}(\mathscr{A}_i)$ together and we obtain a globally defined stack.

Now, we can apply this result to the situation in § 1, and we obtain

Theorem 2. *For any contact manifold Z, we can define canonically a stack* Mod (Z) *on Z, which is locally equivalent to the stack of* $\mathscr{E}_X$*-modules.*

We call an object L of Mod (Z) invertible if it is locally isomorphic to $\mathscr{A}_\iota$. If there is an invertible object L, then $\mathscr{A}=\mathrm{End}(L)$ is a sheaf of rings locally isomorphic to the sheaf of micro-differential operators and Mod (Z) is equivalent to $\mathrm{Mod}(\mathscr{A})$. Hence the existence of a globally defined ring of micro-differential operators is equivalent to the existence of an invertible object.

§ 4. Sheaf of Microfunctions

Let Z be a contact manifold and let $Z_{\mathbf{R}}$ be a real analytic submanifold such that Z is a complexification of $Z_{\mathbf{R}}$. Let $\bar{Z}$ be the complex conjugate of Z. By shrinking Z if necessary, we may assume that there is an isomorphism of complex manifolds $\bar{Z}\to Z$ that is set-theoretically the identity on $Z_{\mathbf{R}}$. Assume that and $\mathscr{O}_Z(1)$ has a complex conjugation and $\sqrt{-1}\,\omega$ is invariant by the complex conjugation. Let Λ_x be the set of oriented Lagrangian vector subspaces in $T_x(Z_{\mathbf{R}})$. Then $\Lambda=\cup\Lambda_x$ is a fiber bundle over $Z_{\mathbf{R}}$. Let $\pi:\Lambda\to Z_{\mathbf{R}}$ be the projection.

Since $\pi_1(\Lambda_x)\cong\mathbf{Z}$, there is a canonical double covering $p:\tilde{\Lambda}\to\Lambda\times_{Z_{\mathbf{R}}}\Lambda$ over $\Lambda\times_{Z_{\mathbf{R}}}\Lambda$ with a canonical map $i:\Lambda\to\tilde{\Lambda}$ such that $p\circ i$ is the diagonal embedding.

Let p_1 and p_2 be the first and the second projection from $\Lambda\times_{Z_{\mathbf{R}}}\Lambda$ onto Λ. Let σ be the covering automorphism of $p:\tilde{\Lambda}\to\Lambda\times_{Z_{\mathbf{R}}}\Lambda$ and let L be the subsheaf of $p_*\mathbf{C}_{\tilde{\Lambda}}$ consisting of sections s such that $\sigma^*s=-s$. Then L is locally isomorphic to $\mathbf{C}_{\Lambda\times_{Z_{\mathbf{R}}}\Lambda}$ and $i^{-1}L$ is canonically isomorphic to $\mathbf{C}_\Lambda$. Let $\mathscr{C}$ be the stack on $Z_{\mathbf{R}}$ defined by: for any open subset U of $Z_{\mathbf{R}}$, $\mathscr{C}(U)=\{(F,\varphi)\,;\,F$ is a sheaf on $\pi^{-1}(U)$ and φ is an automorphism $p_2^{-1}F\otimes L\simeq p_1^{-1}F$ such that $i^{-1}\varphi:F\to F$ is equal to the identity$\}$.

Then $\mathscr{C}$ is a stack locally equivalent to the stack of sheaves on $Z_{\mathbf{R}}$.

We can define the stack $\mathscr{C}\otimes\mathrm{Mod}(Z)$ over $Z_{\mathbf{R}}$ in an obvious way. Then for $M\in\mathrm{Mod}(Z)$ and $F\in\mathscr{C}\otimes\mathrm{Mod}(Z)$, $\mathscr{H}om(M,F)$ belongs to $\mathscr{C}$.

Now, we have

Proposition 3. *We can define canonically an object* $\mathscr{C}_{Z_{\mathbf{R}}}$ *of* $\mathscr{C}\otimes\mathrm{Mod}(Z)$, *which is locally isomorphic to the sheaf of microfunctions.*

§ 5. Regular Holonomic Systems

Since the notion of regular holonomic $\mathscr{E}$-modules is invariant by the quantized contact transformations, we can define the notion of regular holonomic systems for objects in $\mathrm{Mod}(Z)$. The subcategory $\mathrm{Reg}(Z)$ of regular holonomic systems in Mod(Z) forms a full abelian subcategory of Mod(Z).

Let Λ be a Lagrangian submanifold of Z. Then $(\Omega_\Lambda^{\dim\Lambda})^{\otimes 1/2}$ defines the stack $\mathscr{C}_\Lambda$ of twisted sheaves (cf. e.g. [K1]). The stack $\mathscr{C}_\Lambda$ is locally isomorphic to the stack of sheaves on Λ and it contains $(\Omega_\Lambda^{\dim\Lambda})^{\otimes 1/2}$ as an object. Then we have the following proposition, which is a translation of Theorem (10.3) [K2]

Proposition 4. *The category of regular holonomic systems with support in Λ is equivalent to the category of locally constant objects in $\mathscr{C}_\Lambda$.*

Here a locally constant object L in $\mathscr{C}_\Lambda$ is an object in $\mathscr{C}_\Lambda$ locally isomorphic to a constant sheaf of finite rank.

§ 6. Discussion

We know by the Riemann-Hilbert correspondence, the category of perverse sheaves is equivalent to the category of regular holonomic D_X-modules. We can ask what is the stack of "perverse sheaves on Z", which is equivalent to the stack Reg(Z) of regular holonomic systems on Z.

Another question is : we defined Mod(Z) for a contact manifold Z. Is there an analogue of Mod(Z) on any Poisson manifold Z?

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