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Tree-like decompositions of simply connected domains

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Abstract. We show that any simply connected rectifiable domain Ω can be decomposed into Lipschitz crescents using only crosscuts of the domain and using total length bounded by a multiple of the length of $\partial\Omega$. In particular, this gives a new proof of a theorem of Peter Jones that such a domain can be decomposed into Lipschitz domains.

1. Introduction

Can every domain be efficiently decomposed into nice pieces? Of course, this depends on what the words "efficiently" and "nice" mean, but one possible answer was given by Peter Jones who proved in [10] that every simply connected plane domain Ω has a decomposition into Lipschitz domains $\{\Omega_k\}$, such that

$$\sum_k \ell(\partial\Omega_k) = O(\ell(\partial\Omega)).$$

However, Jones' proof is based on the conformal mapping from the disk onto Ω, so that the construction of these pieces might not be very efficient from a computational point of view. In this note we will give a simpler proof of a stronger result, replacing conformal maps by an object from computational geometry: the medial axis.

Theorem 1.1. *There is an $M < \infty$ so that every simply connected plane domain Ω has a collection of disjoint circular arc crosscuts $\Gamma = \cup\gamma_k$ with $\sum_k \ell(\gamma_k) \leq M\ell(\partial\Omega)$ and so that each connected component of $\Omega \setminus \Gamma$ is an M-Lipschitz crescent.*

A crosscut is a Jordan arc in Ω with distinct endpoints on $\partial\Omega$. Since Γ consists of crosscuts, the components of $\Omega \setminus \Gamma$ form the vertices of a tree under the obvious adjacency relation. This is analogous to the idea of taking a triangulation of a polygon using only vertices on the polygon, as opposed to allowing new vertices in

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the interior of the polygon. The pieces in our decomposition are not quite Lipschitz domains, but they satisfy a slightly weaker condition we call being a Lipschitz crescent (see Section 2) and are chord-arc with uniform bounds. A Lipschitz crescent is easily decomposed into Lipschitz domains with the correct length bounds, so Jones' theorem follows from our result. Moreover, each crosscut γ in our construction is a circular arc that lies in a disk D contained inside the domain, and γ lies within an M-neighborhood of the hyperbolic geodesic (for either D or Ω) with the same endpoints. Distinct crosscuts are uniformly separated in the hyperbolic metric of Ω. Finally, the construction is invariant under Möbius transformations of Ω.

The medial axis of a domain is the subset of points that are equidistant from two or more boundary points. For a simply connected plane domain, it is a real tree and for polygons it is a finite tree. Briefly, our construction works by taking a medial axis disk and moving it along arms of the medial axis until a certain angle between the moving disk and the starting disk becomes too large. Then we insert a crosscut and start the process again. This is similar in spirit to the construction in [10] that uses a stopping time based on the growth of the derivative of the conformal map $f : \mathbb{D} \to \Omega$. Computation of the medial axis is easier in many cases (e.g., linear time for n-gons) and Stephen Vavasis has suggested using tree-like decompositions in the numerical computation of conformal maps, so using conformal maps to construct the decompositions would be circular. Our result is also an illustration of the idea that results that are proven using conformal mapping can sometimes be obtained from constructions using the medial axis. This may be of interest since the medial axis makes sense for any domain, even in higher dimensions.

This paper is one of three related papers that were prompted by questions of Stephen Vavasis. He asked whether a tree-like decomposition into "nice" pieces always exists, and he conjectured that such a decomposition could be used to construct an approximately conformal map to the disk. He also suggested that tree-like decompositions could be used to give bounds for the L^2 norm of harmonic conjugation on $\partial\Omega$. In this note, we answer his first question affirmatively. In [3] we answer the second question by showing that a tree-like decomposition into uniformly chord-arc subdomains can be used to define a simple map $\partial\Omega \to \partial\mathbb{D}$ that has a uniformly quasiconformal extension to the interiors. In [4] we answer the third question by bounding the norm of harmonic conjugation using tree-like decompositions.

Jonas Azzam and Raanan Schul observed that the result in this paper proves that any connected, finite length set γ in the plane is a subset of a connected finite set Γ of comparable length that is M-quasi-convex. This means that any two points $x, y \in \Gamma$ can be joined by a path in Γ of length at most $M|x-y|$ for some M independent of x, y and Γ. To prove this we add the convex hull of γ to γ and apply the theorem to each bounded complementary component Ω_j. Let Γ be γ, it convex hull and all the crosscuts added to all the domains. The total length is $O(\ell(\gamma))$. Given any two points of $x, y \in \Gamma$, we connect them by the line segment $S = [x, y]$ and replace each component of $S \cap \Omega_j$ by an arc of $\partial\Omega_j$ with the same endpoints and comparable length, giving a curve in Γ connecting x to y with length comparable to $|x-y|$.

In some parts of the literature such a Γ is called an M-spanner. The result sketched in the previous paragraph also follows from Jones' decomposition in [10], and has been extended to higher dimensions by Assam and Schul in [2]. A related result for the plane was proven by Kenyon and Kenyon in [11] also using the medial axis: they construct a Γ that is a spanner for all pairs of points in γ, but not necessarily for points in Γ itself. I thank Azzam and Schul for pointing out this reference to me. I also thank the referee for many helpful comments and corrections.

In Section 2 we recall some definitions related to rectifiable domains and in Section 3 we discuss the medial axis and the medial axis flow and prove a length decreasing property of the flow. In Section 4 we describe how we define circular crosscuts of the domain. In Section 5 we define a distance on the medial axis using angles between medial axis disks and use it to partition the medial axis into subtrees. In Section 6 we partition Ω into pieces corresponding to these subtrees and show they are Lipschitz crescents. In Section 7 we prove a technical lemma and in Section 8 we complete the proof of Theorem 1.1 by showing the lengths of our crosscuts have the correct length sum. We give some final remarks in Section 9.

2. Background

Given a set E in the plane we define its 1-dimensional measure as

$$\ell(E) = \lim_{\delta \to 0} \inf \Big\{ \sum 2r_j : E \subset \cup B(x_j, r_j), r_j \leq \delta \Big\}$$

where the infimum is over all covers of E by open balls. We denote it by $\ell(E)$, since if E is a Jordan curve, this agrees with the usual notion of length using inscribed polygons. We say that a simply connected domain Ω has a rectifiable boundary if $\ell(\partial\Omega) < \infty$. In this case $\partial\Omega$ is locally connected and may be parameterized by a Lipschitz map from the unit circle that is at most 2-to-1 almost everywhere on the circle. It is also convenient to define

$$\tilde{\ell}(\partial\Omega) = \int_{\mathbb{T}} |f'(z)||dz|,$$

where f is a conformal map from the disk onto a simply connected domain Ω. For Jordan domains this equals $\ell(\partial\Omega)$ and in general $\tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega)$. This measures the length of the boundary "with multiplicity". For example, if $\Omega = \mathbb{D} \setminus [0,1)$, then $\ell(\partial\Omega) = 2\pi + 1$ and $\tilde{\ell}(\partial\Omega) = 2\pi + 2$ because $\tilde{\ell}$ counts the slit from both sides.

A set is called regular (or sometimes Ahlfors-regular or Ahlfors–David regular) if there is a constant $M < \infty$ so that

$$\ell(E \cap B(x,r)) \leq Mr,$$

for every disk in the plane. If $E = \Gamma$ is a Jordan curve, we say it is chord-arc (or Lavrentiev) if there is a constant $C < \infty$ so that

$$\ell(\Gamma_{x,y}) \leq C|x-y|,$$

where Γ_{xy} is the arc between x and y (or the shortest arc in the case that Γ is a closed Jordan curve).

A real valued function is called M-Lipschitz if

$$|f(x) - f(y)| \leq M|x - y|$$

for all x, y in its domain. A curve Γ in the plane is called a M-Lipschitz graph if it is an isometric image (e.g., rotation and translation) of a set of the form

$$\{(x, f(x)) : a \leq x \leq b\}$$

where f is a M-Lipschitz function.

A bounded domain Ω in the plane is called a Lipschitz domain if every boundary point has a neighborhood U so that $\partial\Omega \cap U$ is a Lipschitz graph. We will say Ω is a Lipschitz crescent if there are $\epsilon > 0$ and $\theta \in (0, \frac{\pi}{2})$ so that $\partial\Omega$ consists of two arcs connecting -1 to $+1$; the first a circular arc in the upper half-plane than makes angle θ with the real line at ± 1 and the second is a Lipschitz graph for which the slopes are bounded above by $\theta - \epsilon$ and below by $-\epsilon$. We will also call any bounded Möbius image of such a domain a Lipschitz crescent.

It is a standard exercise to verify that any such domain Ω is a chord-arc curve (with a constant that depends only on ϵ and θ) and that any such domain can be decomposed into Lipschitz domains $\{\Omega_j\}$ so that $\sum \ell(\partial\Omega_j) = O(\ell(\partial\Omega))$. The point is that there are only two points where the domain fails to be Lipschitz: the vertices where the two boundary arcs meet. By removing circular crosscuts centered at these points with geometrically decaying radii, we can cut the domain into Lipschitz disks with uniformly bounded constants. See Section 9 for a discussion of why our construction does not give Lipschitz domains directly.

3. The medial axis

Suppose Ω is a simply connected planar domain. A medial axis disk is an open disk D in Ω so that $\partial D \cap \partial\Omega$ contains at least two points. The medial axis of Ω is the set of all centers of such disks. A point of the medial axis is called a vertex if the boundary of the corresponding disk hits $\partial\Omega$ in three or more points. A point that is not a vertex is called an interior edge point (and the corresponding disk hits ∂D in exactly two points). The medial axis is always a union of countably many rectifiable arcs (see [9]; also [5] and [8]). Fremlin shows in [9] that there is a $\lambda < \infty$, so that for each point z in the medial axis there is a ball $B(z, r)$ so that any point $w \in B(z, r)$ on the medial axis can be connected to z by a path in the medial axis of length $\leq \lambda r$.

If Ω is a polygon with n sides or if Ω is a union of n disks, then the medial axis is a finite tree with at most $O(n)$ vertices and whose edges are either straight lines or parabolic arcs (these only occur for polygons). For the basic properties of the medial axis, see [1], [6], [7], [12]. It is easy to see by a limiting argument that if we prove Theorem 1.1 for one of these special classes (with a uniform bound) then it follows for general simply connected domains with the same bound. Suppose $\{\Omega_n\}$ is a nested, increasing sequence of domains of one of these special types and that we have the desired decomposition for each, all with the same base point. We may also assume the boundary lengths are uniformly bounded. Passing to a subsequence we can assume the decomposition piece containing the base point converges in the Hausdorff metric to a Lipschitz crescent. Choose a longest crosscut on the boundary of this crescent (it exists since our decompositions have uniformly bounded lengths) and pass to a further subsequence so that the decomposition piece attached to the first one across the selected crosscut also converges. Now continue as above, and apply a diagonal argument; in the limit we obtain a tree of crescents with uniformly bounded lengths that decompose Ω. Therefore, in what follows, the reader may assume Ω has one of the special forms above (polygon or finite union of disks).

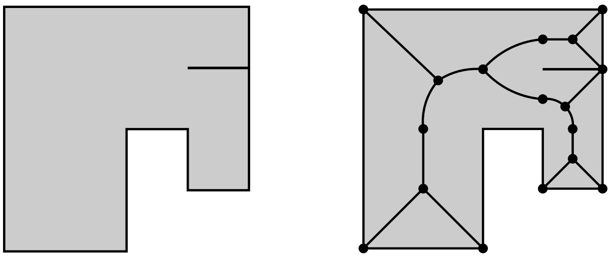

FIGURE 1: A polygon and its medial axis.

If Ω is a finite union of disks, then its medial axis is a finite tree and we can rewrite Ω as the union of disks $\{D_k\}$ corresponding to vertices of this tree and give this collection of disk the same tree structure. Choose one of these, D_0, as the root of the tree. Then each non-root disk has a parent disk (the one that is adjacent and closer to the root) and if we remove the parent from the disk, we are left with a crescent. Thus Ω may be written as the union of the root disk and a finite union of crescents. Each crescent is foliated by circular arcs orthogonal to each boundary arc, and by following the foliation lines we get a map from $\partial\Omega$ to ∂D_0. See Figure 2. Note that for such a domain, the map is piecewise Möbius, since it is a composition of elliptic Möbius transformations. We call this the medial axis flow from $\partial\Omega$ to ∂D_0.

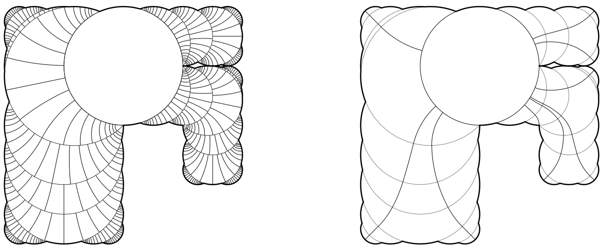

FIGURE 2: A finite union of disks written as a union of a root disk D_0 and several crescents. Each crescent is foliated by circular arcs orthogonal to the boundary and following the foliation gives a map from $\partial\Omega$ to ∂D_0.

The following is the length decreasing property of the medial axis flow:

Lemma 3.1. *If T is a connected subset of the medial axis, let Ω_T be the union of medial axis disks with center in T. Then $\tilde{\ell}(\partial\Omega_T) \leq \tilde{\ell}(\partial\Omega)$.*

Suppose that Ω is a finite union of disks. Let ι_T be the map from $\partial\Omega$ to $\partial\Omega_T$ defined by following the medial axis flow. As noted above, ι_T is a composition of elliptic Möbius transformations on each circular arc in $\partial\Omega$. Each of these transformations is associated to a crescent as described earlier. Note that the crescents that we use are always of the form $W = D_2 \setminus D_1$ and that we are mapping the edge $\partial W \cap \partial D_2$ to the edge $\partial W \cap \partial D_1$. Thus we only need to check that the length of the circular arc passing through $-1, 1$ and making angle θ with the real axis is an increasing function of θ, namely $\theta/(\pi \sin(\theta))$ (thanks to the referee for pointing out this simplification of my original argument).

The previous lemma says that the medial axis flow decreases the total length of the boundary. It is also true that the flow decreases the length of any subset of the boundary. We won't need this more precise version to prove out main result, but perhaps it is worth recording.

Lemma 3.2. *Suppose Ω is a crescent that lies on one side of the line L passing through its two vertices. Let γ_1, γ_2 be the circular arcs in $\partial\Omega$ with γ_1 between γ_2 and L. If τ is the elliptic Möbius transformation fixing the two vertices and mapping γ_2 to γ_1 then $|\tau'(z)| \leq 1$ on γ_2.*

Proof. To see this suppose $\tau(z) = (az+b)/(cz+d)$ where $ad - bc = 1$ (which we can always assume by normalizing). Then a simple calculation shows $|\tau'(z)| < 1$ iff $|1/c| < |z + d/c|$. Note that $-d/c = \tau^{-1}(\infty)$. By normalizing by a Euclidean similarity, we may assume the vertices are 1 and -1 and the crescent lies in the upper half-plane. See Figure 3. Then $-d/c$ is on the negative imaginary axis and $|\tau'(z)| < 1$ outside a circle C centered at $-d/c$ passing through -1 and 1 (since the derivative of an elliptic transformation has modulus one at the two fixed points). Let γ be the arc of this circle between 1 and -1 that lies in the upper half-plane. We claim that γ_2, the upper edge of our crescent, lies above γ.

Suppose the elliptic transformation is a rotation by θ around the points $-1, 1$. Since γ_2 and its image are both in the upper half-plane, $\theta < \pi$. Therefore $-d/c$ lies on a circle that makes angle $\pi - \theta$ with the segment $[-1, 1]$. See Figure 3. Hence the isosceles triangle with base $[-1, 1]$ and vertex $-d/c$ has two base angles of $\psi = (\pi - \theta)/2$ and the circle C makes angle $\pi/2 - \psi = \theta/2$ with $[-1, 1]$. Since γ_1 lies in $\mathbb{H}$, γ_2 makes angle of at least θ with $[-1, 1]$ and hence lies above C. □

This implies the length decreasing property for finite unions of disks. Any polygon can be approximated by a finite union of disks by taking a union of disks over a finite set in the medial axis. As the points become denser in the medial axis these domains converge to the polygon and the medial axis maps converge to the medial axis map for the polygon. Since the length decreasing property is clearly preserved under limits, the medial axis flow is length decreasing for all polygons (even all simply connected domains). See Figure 4 for some examples of the medial axis flow in some polygons; the length decreasing property is readily apparent for the plotted flow lines.

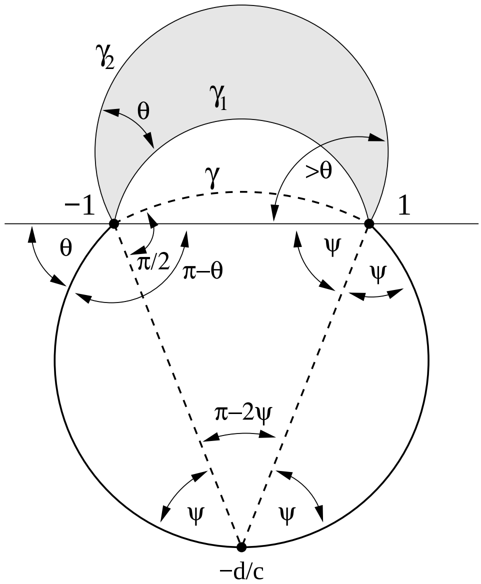

Figure 3: The setup in Lemma 3.2. We prove that γ_2 lies above γ by showing that γ makes angle $\theta/2$ with $[-1,1]$, but γ_2 makes angle $> \theta$ with the same segment.

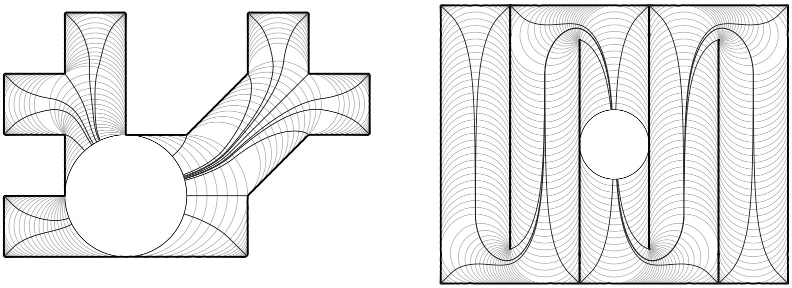

Figure 4: More examples of the medial axis flow in a polygon. We approximate the polygon by a union of a root disk and crescents and draw the orthogonal flow from the boundary to the root disk. This flow decreases length, even if we stop it when it hits a union of medial axis disks, rather than just one as shown. By passing to a limit we obtain a length decreasing map for any polygon.

4. Defining the crosscuts

Given a medial axis disk D of Ω let $C = C_D$ be the hyperbolic convex hull (in D) of $E = \partial D \cap \partial\Omega$. This is simply the region bounded by replacing each arc in $\partial D \setminus E$ by a circular arc in D with the same endpoints and perpendicular to ∂D. See Figure 5. One can easily check that these sets are disjoint for distinct medial axis disks. Thus there can be at most countably many with interior, i.e., there are at most a countably many vertices of the medial axis.

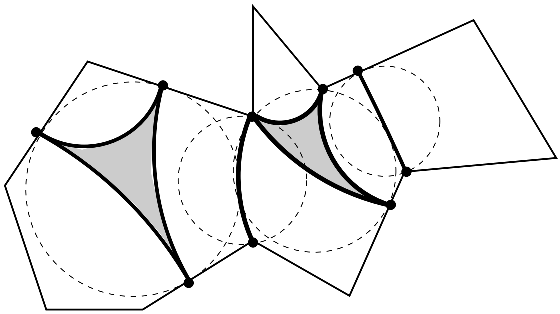

FIGURE 5: The convex hulls of $\partial D \cap \partial\Omega$ can either be arcs or hyperbolic polygons. Disjoint points of the medial axis give disjoint convex hulls (disjoint in Ω) they may have common points on the boundary).

Given a point p of the medial axis and an angle $0 < \theta < \pi/2$, we also define a "thickened" version of the convex hull of $\partial D_p \cap \partial\Omega$ adding a crescent of angle $\phi = \frac{\pi}{2} - \theta$ along each face (if the convex hull is an arc, we add a crescent along both sides). See Figure 6. Such a piece will be used as the root of our decomposition.

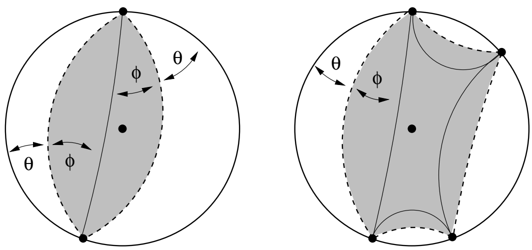

FIGURE 6: These show thickenings of the convex hull by adding crescents of angle ϕ along each boundary geodesic of the convex hull. If the root is not a vertex, then its medial axis disk meets $\partial\Omega$ in two points and the decomposition piece is a crescent with internal angle 2ϕ. If the root is a vertex, then the decomposition piece has several sides.

For an interior edge point of the medial axis, the set E has exactly two points and C is the circular arc in D with endpoints E and perpendicular to ∂D. Given an angle $0 < \theta \leq \pi/2$ we will let γ_θ to be the circular arc with the same endpoints but making angle θ with ∂D. Thus $\gamma_{\pi/2} = C$. If $\theta \neq \pi/2$ then there is some ambiguity about which of two possible arcs we mean. However, if we choose a fixed basepoint p_0 on the medial axis, then for any distinct medial axis non-vertex point, the corresponding geodesic C divides Ω into two components only one of which hits p_0. We choose γ_θ to be in the other component, i.e., γ_θ "bends away" from p_0. See Figure 7.

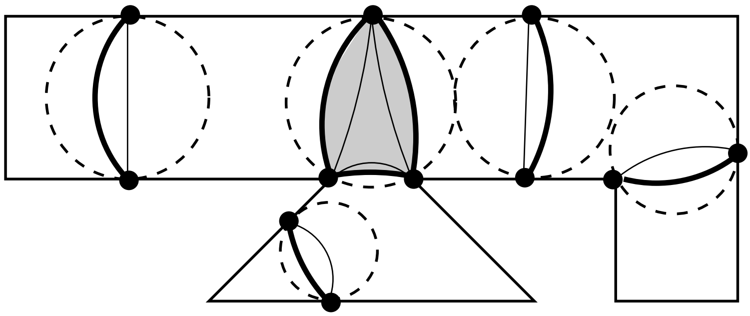

Figure 7: We cut the domains by circular crosscuts that have their endpoints in the set $\partial D \cap \partial\Omega$, where D is the medial axis disk corresponding to a chosen division point. The crosscut is not a geodesic in D but it makes a fixed angle with the geodesic and "bends away" from the root of the medial axis. In the picture the thin crosscuts are the hyperbolic geodesics and the thicker ones are the "bent" crosscuts.

If p is a vertex point in the medial axis, then the set $E = \partial D \cap \partial\Omega$ is a closed set with at least three points. If p is not the root, then $\partial D \setminus E = \cup_j I_j$ is a union of at least three (and possibly countably many) open intervals, and exactly one of these has the property that it is closest to the root p_0 in the sense that there is a crosscut with the same endpoints that separates the other intervals from the root in Ω. Let this special interval be denoted I_0. For each of the remaining intervals, I_j, define a circular crosscut on D with the same endpoints as I_j and making angle θ with I_j and bending away from the root as above. The union of these circular crosscuts will be the γ_θ corresponding to the vertex point p.

The general idea is as follows. Fix some angle $\theta \in [\frac{1}{4}\pi, \frac{3}{8}\pi]$, $\epsilon > 0$ and a root p of the medial axis. The root of our decomposition is the thickened convex hull corresponding to p. We will define a distance function on the medial axis and use it to partition the medial axis into subtrees of diameter $\simeq \epsilon$. For each division point between adjacent subtrees we insert the crosscut γ_θ described above, giving the other pieces of the decomposition. In the remaining sections we explain:

- How to partition the medial axis.
- Why each decomposition piece is a Lipschitz crescent.
- Why the total length of the crosscuts is $O(\ell(\partial\Omega))$.

5. How to choose the subtrees

The most natural distance on the medial axis might be the hyperbolic metric of Ω, but it turns out that using this to construct our decompositions pieces leads to "bad shapes". In order to get "nice shapes", we introduce a more complicated distance defined in terms of the angles. The first step is to define an angle between two circular arcs (which don't necessarily intersect each other).

Suppose I and J are circular arcs (or line segments). Apply a Möbius transformation σ to both arcs, chosen so that I is mapped to the real segment $[-1, 1]$. If I is already a line segment, we take σ linear. If I is an arc of circle C, then we take σ so that it maps the point q of C that is opposite the center of I to ∞. Then we define the angle between I and J at $w \in J$ to be the angle $\sigma(J)$ makes with the horizontal at $\sigma(w)$. The angle between I and J is the maximum angle $\sigma(J)$ makes with the horizontal. See Figure 8.

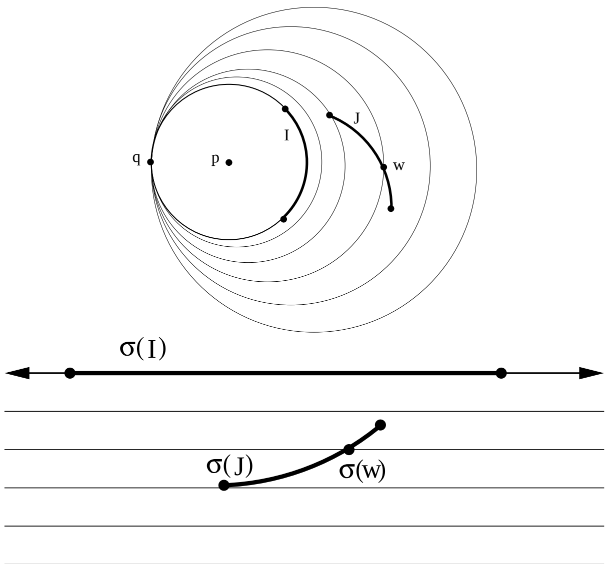

Figure 8: We define angles with respect to a family of circles tangent to D at a point opposite from I. If we map the disk to a half-plane this just becomes the angle between J and the horizontal.

Next we use this notion of angles to define a function on pairs of points z, w in the medial axis. Let D_z, D_w be the corresponding medial axis disks. Note that $\partial D_z \cap \Omega$ has at least two components, exactly one of which hits D_w or separates D_w from z. This arc is called the near arc of z with respect to w and its complement in ∂D_z is called the far arc of z with respect to w. Let I be the near arc of w with respect to z and let J be the far arc of z with respect to w (see Figure 9). Let $d(w, z)$ be the angle between I and J, as defined above and let $D(w, z) = \sup d(w, x)$ where the supremum is over all points x on the unique path in the medial axis between x and z. This is the function we will use to decompose the medial axis.

Although it is not important to our application, we should point out that the function $D(w, z)$ is not generally symmetric. The only information we use about w is the circle centered at w and the point q opposite the arc I. If we changed the domain so that this arc changed, but the circle and opposite point remained the same then the value of $D(w, z)$ would not change (assuming the changes did not effect the circle centered at z). However, it is easy to find an example of such a change that does alter the value of $D(z, w)$. Is there a suitable alternative that is a metric on the medial axis?

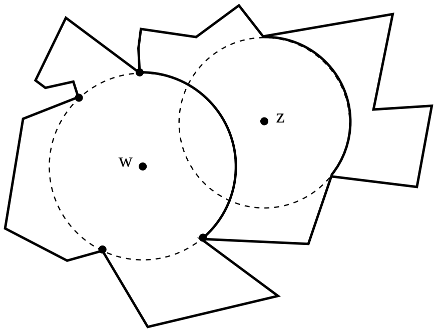

Figure 9: If w, z are in the medial axis $d(w, z)$ is the angle of the far arc of ∂D_z (shown as a solid arc in the figure) with respect to w compared to the near arc of ∂D_w with respect to z (also solid).

Choose a root p of the medial axis. This point our first subtree. Removing it breaks the medial axis into connected components. Suppose $\epsilon > 0$ is fixed. For each connected component, we take all the points z so that $D(p, z) \leq \epsilon$. By the definition of D, this is a connected set. At each step, we choose connected components of points whose D-distance from what we have already chosen is $\leq \epsilon$. This partitions the medial axis into countably many subtrees. Each subtree has leaves that are either (1) leaves of the medial axis or (2) interior points of the medial axis. In former case we do nothing and in the latter case we divide Ω using the corresponding γ_θ (which is a single circular arc unless z is a vertex of the medial axis, when it is a collection of such arcs). The idea is illustrated in Figure 10. Figure 11 shows why we take $\theta < \pi/2$; if $\theta = \pi/2$ then cusps may form in certain situations.

It is useful to note that for a fixed w, $D(w, z)$ is a uniformly Lipschitz function of z with respect to the hyperbolic metric. Consider the normalized situation, when the circle centered at w has been mapped to the upper half-plane and z is located in the lower half-plane. If I is the far arc of z with respect to w and we move z to a nearby point (in the hyperbolic metric) z', then the far arc for z' is in the crescent formed by removing the medial axis disk centered at z from the one centered at z'. For z' δ close to z, the far arc for z' is within $O(\delta)$ of parallel to the far arc for z, so its angle with respect to w can only increase by $O(\delta)$. Since D is defined in a way to make it increasing as we move along the median axis, this upper bound implies D is Lipschitz (without the supremum in the definition of D, it could jump down discontinuously at vertices of the medial axis). The Lipschitz property means that at each stage of our construction our subtrees cover a fixed hyperbolic neighborhood of the previous step, and this implies that eventually we cover the whole medial axis (the medial axis is path connected).

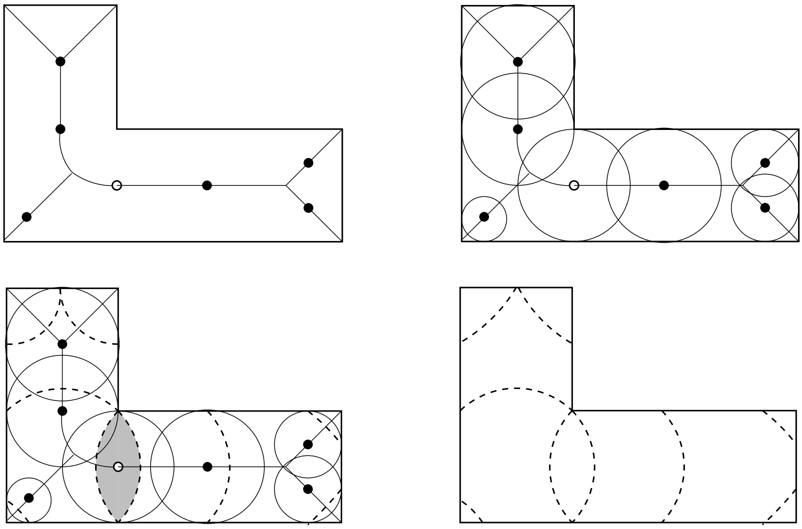

Figure 10: The top left shows a polygon, its medial axis and a collection of points that divide the medial axis into subtrees. The one chosen as root has a white interior. On the top right we show the medial axis disks corresponding to these points. On the lower left are the "bent" geodesics we use for crosscuts. Note the shaded region that is a thickened convex hull and is the decomposition piece corresponding to the root. On the lower right we erase the medial axis and disks to show just the resulting decompositions.

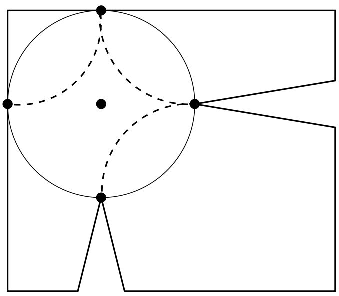

Figure 11: Here is a situation when we want to avoid using crosscuts that are perpendicular to the corresponding medial axis circles. If a leaf of the subtree is a vertex of the medial axis, then we may add crosscuts to two bottom arcs that are adjacent and then the resulting piece will have a cusp and the crosscuts will not be uniformly separated in the hyperbolic metric.

6. Decomposition pieces are Lipschitz crescents

In this section we will show that the decomposition pieces $\{W_n\}$ we have constructed are Lipschitz crescents. This is easy to see for the root piece, so we will only deal with non-root pieces. Each such piece corresponds to a subtree of the medial axis that is rooted at the point p_n closest to the root of the medial axis. Lipschitz crescents are invariant under Möbius transformations, so it suffices to show W_n is such a crescent after we normalize by mapping the medial axis disk D_n centered at p_n to the upper half-plane. After this normalization, W_n looks like the region illustrated in Figure 12.

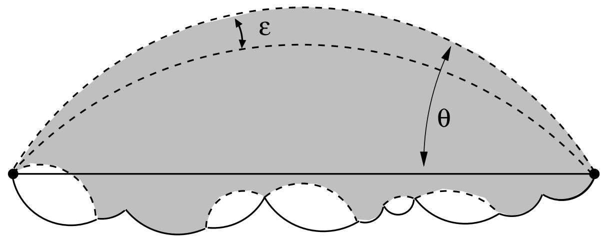

Figure 12: Here is a typical decomposition piece where we have normalized the medial axis disk of the root of the subtree by sending it to the upper half-plane. The large dashed arc at the top is the crosscut corresponding to the the root of the subtree. Along the bottom are arcs corresponding to the other leaves of the subtree (dashed if the leaf is an interior point of the medial axis, solid if it is also a leaf of the medial axis). The bottom arc makes an angle with the horizontal that is bounded between $-\epsilon$ and $\theta - \epsilon$. The top and bottom arcs of W_n are separated by a crescent of fixed angle ϵ.

When we normalize, we see that the each piece has a top edge that is the circular crosscut corresponding to the root of the subtree. We claim that the lower edge of W_n is a Lipschitz graph. This is certainly true on the circular arc crosscuts corresponding to stopping points (because we stopped the first time the angle $= \epsilon$ anywhere on the arc). On the other hand, at any other points of the lower edge correspond to paths on the medial axis where we never stopped, and such a point is the tip of a cone with sides of angle ϵ (with the horizontal). See Figure 13. When we add in the crosscuts γ_θ to get the decomposition piece W, the crosscuts form an angle between $-\epsilon$ and $\theta - \epsilon$ with the horizontal.

Finally, we only have to check that the top and bottom edges are separated by a crescent of angle ϵ. See Figure 12. Let C be the crescent of angle ϵ with endpoints ± 1 and top edge equal to the top edge of W. We claim that $C \subset W$, i.e., the bottom edge of W does not hit this crescent. By taking a finite approximation of the medial axis we can assume the lower edge of W is a finite union of circular arcs and each is either an arc of $\partial\Omega$, or a circular crosscut that was added at a leaf of the subtree. If the arc is a boundary arc of Ω then it lies in the lower half-plane and so does not hit C (assuming $\epsilon < \theta$). If the arc corresponds to an interior point of the medial axis, then there is a point in this arc where it makes angle ϵ with the horizontal. This point must be an endpoint of the arc, and then the corresponding arc γ_θ makes angle $\theta - \epsilon$ with the horizontal at this point. If this arc crosses into the upper half-plane, then at the points where it crosses, it must make angle $\leq \theta - \epsilon$ with the horizontal. Thus it lies beneath the circular arc from -1 to 1 that makes angle $\theta - \epsilon$ with the real axis, as claimed. From this it is clear that the normalized domain W is a Lipschitz crescent, as desired. By definition, any bounded Möbius image is also a Lipschitz crescent, so the unnormalized piece is as well.

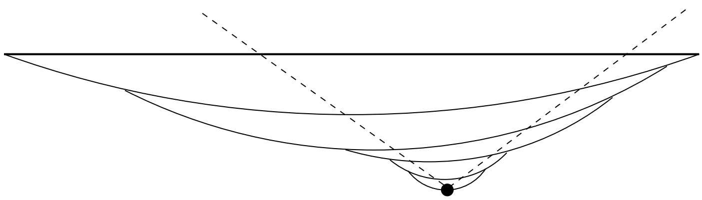

FIGURE 13: At a boundary point of the normalized piece Where we never stop, consider the union of medial axis disks corresponding to the path leading to this point. Each bottom edge of each corresponding disk makes angle $\leq \epsilon$ with the horizontal and hence the point is the vertex of a cone in the piece with sides of angle ϵ. Thus the bottom edge of W is a Lipschitz graph.

7. A length estimate

Suppose we have a decomposition of the medial axis into subtrees $\{T_n\}$ where T_0 is the "root" and consists of a single point. Let Γ denote the collection of circular arc crosscuts corresponding to this partition of the medial axis and let $\Omega \setminus \Gamma$ be the corresponding decomposition of Ω.

For any n, we let Ω_n be the union of all medial axis disks centered in T_n. For each n we let W_n be the component of $\Omega \setminus \Gamma$ that is inside Ω_n. These are the pieces of our decomposition. We let γ_n denote the crosscut that separates W_n from its parent. If $n \neq 0$, let $U_n = \Omega_n \setminus \Omega_{n^*}$ where n^* denotes the index of the parent domain of Ω_n (i.e., Ω_{n^*} separates Ω_n from Ω_0). ∂U_n has one circular arc edge in common with its parent U_{n^*}. We will call this the top edge of U_n, and denote it by τ_n. Note that τ_n and γ_n are both circular arcs with the same endpoints, but that they bound a crescent with interior angle θ. Because the angles used to define the crosscuts in Γ have angles bounded between $\frac{1}{4}\pi$ and $\frac{3}{8}\pi$, the length of the crosscut γ_n dividing W_n from its parent is at most a uniform multiple of the distance between its endpoints. This distance, in turn, is at most the length of τ_n (possibly much shorter in some cases), i.e.,

$$\ell(\gamma_n) = O(\ell(\tau_n)). \tag{7.1}$$

Note that $\tau_0 = \emptyset$ since the root piece has no top edge. See Figure 14. The domain U_n is introduced because it will be easier to estimate its boundary length, and then use this to control the boundary lengths of our decomposition pieces, W_n.

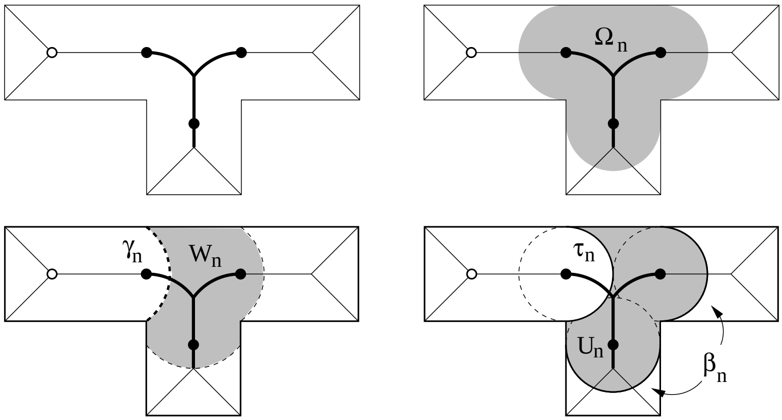

Figure 14: The upper left shows a domain and its medial axis. The selected root is shown as a white dot. A subtree T_n has been indicated by the darker edges. The top right shows the union of medial axis disks for this subtree; this is Ω_n. The lower left shows the corresponding decomposition piece W_n. It is bounded by arcs that deviate from the hyperbolic geodesic with the same endpoints by a fixed angle. Note that they bend away from the root. The lower right shows U_n.

The remainder of the boundary, $\partial U_n \setminus \tau_n$, is called the "bottom" edge of U_n and will be denoted β_n. This is a Jordan arc. By the length decreasing property of the medial axis we have

$$\ell(\tau_n) \leq \ell(\beta_n), \tag{7.2}$$

since β_n is mapped to τ_n by the medial axis flow that collapses T_n onto the point it shares with its parent.

Fix some $\delta > 0$. We say that U_n is *boundary-like* if

$$\ell(\beta_n \cap \partial\Omega) \geq \delta\ell(\beta_n), \tag{7.3}$$

and is *interior-like* otherwise. Since $\ell(\tau_n) \leq \ell(\beta_n))$, in boundary-like pieces we also have

$$\ell(\beta_n \cap \partial\Omega) \geq \delta\ell(\tau_n). \tag{7.4}$$

We will bound the length of Γ using:

Lemma 7.1. *Suppose that every non-root, interior-like subdomain U_n satisfies*

$$\ell(\beta_n) \geq (1+\delta)\ell(\tau_n). \tag{7.5}$$

Then $\sum \ell(\gamma_n) = \ell(\Gamma) \leq \frac{C}{\delta}\tilde{\ell}(\partial\Omega)$ for some absolute $C < \infty$.

Proof. Assume that the subdomains $\{W_n\}$ are indexed so that $n^* < n$ for all n (i.e., every subdomain comes somewhere after its parent in the list). Let $V_n = \cup_{k\leq n}\Omega_n$. Then $V_0 \subset V_1 \subset \cdots \cup_n V_n = \Omega$ and $\tilde{\ell}(\partial V_0) \leq \tilde{\ell}(\partial V_1) \leq \cdots \leq \tilde{\ell}(\partial\Omega)$ by Lemma 3.1, because each domain is mapped to the previous one by a medial axis flow.

By (7.1) it is enough to show that

$$\sum_k \ell(\tau_k) \leq \frac{C}{\delta}\,\ell(\partial\Omega).$$

We will prove this by induction. Our hypothesis will be

$$\sum_{k\leq n} \ell(\tau_k) \leq \frac{1}{\delta}\,[\tilde{\ell}(\partial V_n) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)]. \tag{7.6}$$

This will suffice since $\tilde{\ell}(\partial V_n) \leq \tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega)$ by Lemma 3.1, and

$$\sum_{k\leq n} \ell(\beta_k \cap \partial\Omega) \leq \tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega).$$

The induction hypothesis (7.6) is trivial for $n = 0$ since the left hand side is zero (recall that $\tau_0 = \emptyset$). Assume the hypothesis for n and consider $n+1$. If U_{n+1} is boundary-like, then by the induction hypothesis, (7.4) and Lemma 3.1 we have,

$$\begin{aligned}
\sum_{k\leq n+1} \ell(\tau_k) &\leq \ell(\tau_{n+1}) + \sum_{k\leq n} \ell(\tau_k) \\
&\leq \frac{1}{\delta}\,\ell(\beta_{n+1} \cap \partial\Omega) + \frac{1}{\delta}[\tilde{\ell}(\partial V_n) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)] \\
&\leq \frac{1}{\delta}\,[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n+1} \ell(\beta_k \cap \partial\Omega)],
\end{aligned}$$

as desired. If U_{n+1} is interior-like, then by (7.5),

$$\ell(\tau_{n+1}) \leq \frac{1}{1+\delta}\ell(\beta_{n+1}) = \frac{1}{1+\delta}(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n) + \ell(\tau_n)),$$

so

$$\Big(1 - \frac{1}{1+\delta}\Big)\,\ell(\tau_{n+1}) \leq \frac{1}{1+\delta}\,\big(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n)\big),$$

which gives

$$\ell(\tau_{n+1}) \leq \frac{1}{\delta}\,\big(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n)\big).$$

Hence

$$\begin{aligned}\sum_{k\leq n+1} \ell(\tau_k) &\leq \ell(\tau_{n+1}) + \sum_{k\leq n} \ell(\tau_k)\\ &\leq \frac{1}{\delta}\big(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n)\big) + \frac{1}{\delta}[\tilde{\ell}(\partial V_n) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)]\\ &\leq \frac{1}{\delta}\Big[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)\Big]\\ &\leq \frac{1}{\delta}\Big[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n+1} \ell(\beta_k \cap \partial\Omega)\Big].\end{aligned}$$

Thus in either case, the induction hypothesis is verified and the lemma is proven. □

8. Decomposition pieces have the correct length bounds

To finish the proof of Theorem 1.1, it is now enough to check that (7.5) is satisfied for the non-root, interior-like pieces. We first need some simple geometric facts.

Lemma 8.1. *Suppose γ is a circular arc contained in the upper half-plane and contains at least one point where the tangent makes angle $\geq \epsilon > 0$ with the horizontal. Then it makes an angle $\geq \epsilon/2$ with the horizontal on at least $1/3$ of its length.*

Proof. If γ is a line segment, there is nothing to do since the slope is constant. Otherwise γ is an arc of a proper circle and subtends some angle θ with respect to the center of this circle. There are two arcs of this circle where the tangent makes angle $\leq \epsilon/2$ with the horizontal and each subtends angle ϵ with respect to the center of the circle. Each is separated by arcs of angle measure $\epsilon/2$ from the arcs of the circle where the angle to the horizontal is $\geq \epsilon$. Thus γ either has angle $\geq \epsilon/2$ on its whole length or it contains a subarc of angle measure $\geq \epsilon/2$ on which it makes angle $\geq \epsilon/2$ with the horizontal. This arc must account for at least 1/3 of its length, so we are done. □

Lemma 8.2. *Suppose γ is a circular arc contained in the upper half-plane and contains at least one point where the tangent makes angle $\geq \epsilon > 0$ with the horizontal. Then $\ell(\gamma) \geq (1 + c\epsilon^2)\ell(I)$ where I is the vertical projection of γ onto the real line and $c > 0$ is a fixed constant.*

Proof. Let $\gamma' \subset \gamma$ be the subarc where γ makes an angle of at least $\epsilon/2$ with the horizontal and let I' be its projection. Then $\ell(I') \geq \ell(I)/3$, so

$$\ell(\gamma) = \ell(\gamma \setminus \gamma') + \ell(\gamma') \geq \ell(I \setminus I') + \ell(I')/\cos(\epsilon^2) \geq \ell(I)(1 + c\epsilon^2).$$

□

Lemma 8.3. *Suppose $0 < \epsilon \leq 1/2$ and let $S = [-1,1] \times [0,\epsilon]$ and suppose γ is a circular arc in the upper half-plane, that is part of a circle centered in the lower half-plane and that connects the two short sides of S within S. Then γ makes an angle of at most 2ϵ with any horizontal line.*

Proof. Its not hard to see that the worst case is when γ connects -1 to $+1$ passing through $i\epsilon$. If r is the radius of the circle containing this arc, and θ is the angle subtended by half this arc, then (see Figure 15);

$$r^2 = 1^2 + (r-\epsilon)^2, \quad \frac{1}{r} = \frac{2\epsilon}{1+\epsilon^2},$$

$$\theta \leq \tan\theta \leq \frac{1}{r} \leq \frac{2\epsilon}{1+\epsilon^2} \leq 2\epsilon.$$

□

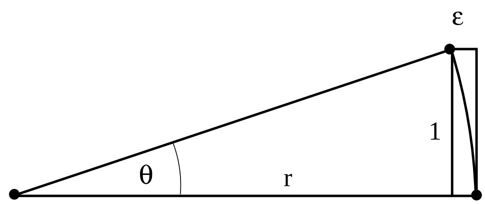

FIGURE 15: Definitions for proof of Lemma 8.3.

Lemma 8.4. *Suppose $R > 1$ and $0 < \eta < 1$ are given. Suppose σ is a Möbius transformation that fixes both -1 and 1 and maps 0 to $-iR$ (i.e., σ is an elliptic rotation around ± 1 with angle $\leq \pi/2$). Let γ be a circular arc in the rectangle $[-1/R, 1/R] \times [-\eta/R, 0]$ that makes an angle $\geq \epsilon$ with the horizontal at some point and whose σ-image is in the lower half-plane. Let I be the vertical projection of γ onto the real line. Then there is a $c > 0$, so that $\ell(\sigma(\gamma)) \geq (1 + c\epsilon^2)\ell(\sigma(I))$.*

Proof. The Möbius transformation σ is of the form

$$\sigma(z) = \frac{z+\mu}{\mu z + 1},$$

where $\mu = -iR$. The derivative is $\tau'(z) = \frac{1-\mu^2}{(\mu z+1)^2}$, so

$$|\tau'(z)| = \frac{1+R^2}{R^2} \frac{1}{|z-(-i/R)|}.$$

Thus $|\tau'(x-iy)|$ is an increasing function of $y \in [0, 1/R)$ for any $x \in [-1,1]$. In particular, if γ is as in the lemma, $z = x - iy \in \gamma$, then $|\tau'(z)| > |\tau'(x)|$. Since γ makes an angle $\geq \epsilon/2$ with the horizontal along at least a third of its length, we get

$$\ell(\tau(\gamma)) \geq (1 + c\epsilon^2)\ell(\tau(I)).$$

□

Now we show (7.5) is satisfied for the non-root, interior-like pieces U. First we do this assuming that the corresponding decomposition piece W has been normalized as in Figure 12(the medial axis disk of its root is the upper half-plane) and later we will verify the estimate for any Möbius image of W. Let β be the bottom edge of U and let τ be its top edge (which is a segment on the real line). By assumption, a fixed fraction δ of the length of β consists of interior arcs of Ω, and each of these arcs has a point where the angle with the horizontal is $\geq \epsilon$. By Lemma 8.2 this implies the length of the arc is strictly longer than its vertical projection onto τ by a factor depending only on ϵ, δ. Thus a normalized U satisfies (7.5).

A general U' is simply a Möbius image of a normalized U. Since we may ignore Euclidean similarities, we can assume this transformation σ is of the from in Lemma 8.4 and the top edge of U is $\tau = [-1, 1]$. This defines R. Choose $\eta \leq \frac{1}{3}\epsilon$. Then $\Gamma = \sigma(\tau)$ is a circular arc in the lower half-plane with endpoints ± 1. The image of β also has endpoints ± 1 and lies in the lower half-plane outside Γ. See Figure 16.

Define three adjacent rectangles

$$R_0 = [-1/R, 1/R] \times [-\eta/R, 0],$$

$$R_1 = [-3/R, -1/R] \times [-\eta/R, 0],$$

$$R_2 = [1/R, 3/R] \times [-\eta/R, 0].$$

The σ images of these are circular arc quadrilaterals of diameter $\simeq R$ that lie between Γ and Γ_η. See Figure 16.

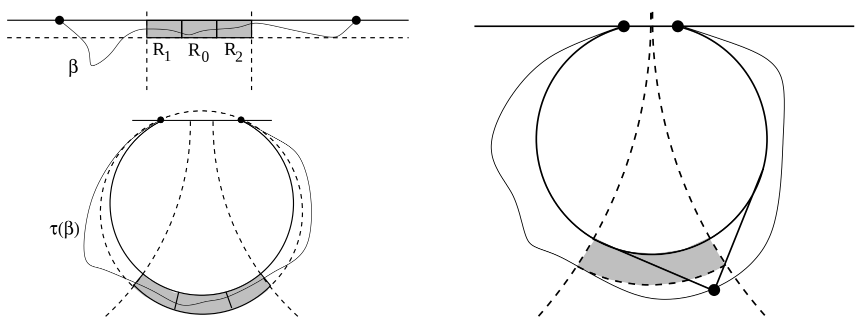

Figure 16: Three small rectangles map to three quadrilaterals with diameter comparable to R. If β contains a point below $R_0 \cup R_1 \cup R_2$, then its image contains a point at least distance $C\eta R$ from Γ. Then the length of $\tau(\beta)$ is at least the minimum length of a path in the lower halfplane connecting the endpoints of Γ and containing the point.

As the curve β goes from -1 to $+1$, it crosses from the vertical line $x = -3/R$ to the line $x = 3/R$. Either it stays entirely inside $R_0 \cup R_1 \cup R_2$ or it does not. If it does not, then β contains a point w between these lines but below the rectangles. Thus $\sigma(\beta)$ contains a point w that is at least distance $C\eta R$ from Γ. Thus the length of $\sigma(\beta)$ is at least the length of the shortest path connecting ± 1 in the lower half-plane, outside Γ and containing w. This is at least $(1+\mu)\ell(\Gamma)$, for some fixed $\mu > 0$ that depends only on η (hence only on ϵ). See the right side of Figure 16.

Otherwise β lies entirely inside the union of the three rectangles. We may also assume β consists of a finite number of circular arcs. Suppose one of these arcs crosses R_1. Then by Lemma 8.3 it must make angle $\leq 2\pi\eta$ with the horizontal along its whole length. If $2\pi\eta < \epsilon$ then such an arc must correspond to a leaf of the medial axis since it does not satisfy the stopping rule we used to partition the medial axis. But the image of this arc under τ has length $\simeq R \simeq \ell(\Gamma) \geq \delta\ell(\Gamma)$ if δ is small enough, which contradicts the assumption that the piece U is interior-like. Thus no arc of β crosses R_1. Similarly, no arc crosses R_2. Thus any arc in β that hits R_0 cannot leave $R_0 \cup R_1 \cup R_2$.

Consider the union of the stopped arcs in β that hit R_0. First suppose that at most half the length of β in R_0 consists of these stopped arcs. Then just as in the previous paragraph the σ image of the complement of these arcs has length $\simeq R$ and we deduce that the piece is boundary-like, not interior-like. Thus at least half the length of β in R_0 must be from stopped arcs, so by Lemma 8.4, we are done. This proves the desired estimate and completes the proof of Theorem 1.1.

9. Further remarks

The Lipschitz crescents described in the previous proof are built using two boundary arcs that are Lipschitz graphs, so why aren't the domains themselves always Lipschitz domains? There are two things that can go wrong.

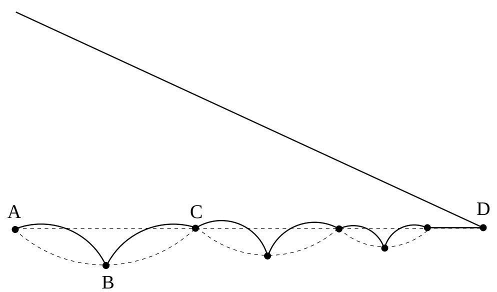

FIGURE 17: This shows that that the Lipschitz crescent we construct need not be a Lipschitz domain near a vertex.

The first problem is illustrated in Figure 17. Suppose that the point labeled D is the right endpoint of $I_0 = [-1, 1]$ and the upper edge of W makes angle θ with the real axis at D. Suppose that near D, the boundary of Ω_0 looks like the dashed curves in Figure 17, each of these is a crescent of angle ϵ. We get ∂W_0 by taking circular arcs that have the same endpoints and that make angle θ with these arcs. Thus the solid arc from points A to B is an arc in ∂W_0 and it makes angle $\theta - \epsilon$ with the horizontal at A and angle θ at B. Thus there can be a sequence of points converging to D where the upper and lower boundaries of the crescent have the same slope. Thus there is no neighborhood of D in which ∂W_0 is a Lipschitz graph.

The second problem is that even if W is Lipschitz with a uniform constant, a Möbius image of it need not be. Consider the case when W is a crescent with interior angle $< \pi/2$ and bottom edge equal to $[-1, 1]$. By applying an elliptic transformation fixing ± 1 and rotating by an angle slightly less than π we can map the bottom edge to a circular arc of very large diameter (as large as we want) while the top edge limits on a circular arc of fixed size. See Figure 18.

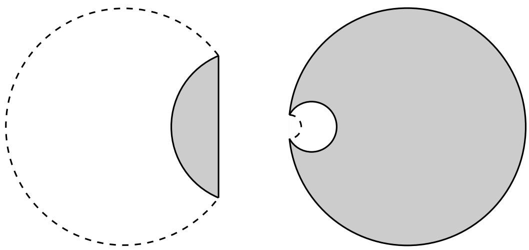

Figure 18: On the left is a Lipschitz crescent that is also a Lipschitz domain and on the right is a Möbius image that is not a Lipschitz domain(but it is a Lipschitz crescent by definition).

Clearly, the large crescent is not star-shaped since there is no interior point that sees both vertices. While it is true that every boundary point of the big crescent has some neighborhood in which the boundary is a Lipschitz graph, these neighborhoods can't have diameter comparable to the diameter of the whole domain (otherwise a neighborhood of one vertex would contain the other vertex). Thus we can't expect to get Lipschitz domains with uniform bounds unless we give up the Möbius invariance of the construction.

Is it always possible to find a tree-like decomposition of a simply connected domain into Lipschitz domains so that $\ell(\Gamma) = O(\ell(\partial\Omega))$? If so, is it possible to do this using only circular arc crosscuts? Straight line crosscuts?

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