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Sub-gaussian measures and associated semilinear problems

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Abstract. We study the existence, smoothing properties and the long time behaviour for a class of nonlinear Cauchy problems in infinite dimensions under the assumption of F-Sobolev inequalities.

1. Markovian semilinear Cauchy problems: Introduction

In the bulk of this work we consider the following formal Cauchy problem:

$$(\mathbb{MCP}) \qquad \begin{cases} \frac{\partial}{\partial t}u(t) &= Lu(t) + \lambda\, u(t)\, G\left(\frac{u^2(t)}{\mu(u(t)^2)}\right) \\ u(0) &= f, \end{cases}$$

where L is a (linear) Markov generator and G is a certain nonlinearity (vanishing at one), to be specified later, and μ is a probability measure. In the next paragraph we are going to explain what is needed to understand the meaning of this equation. Let us nevertheless note here that under our hypothesis, constants are global solutions of ($\mathbb{MCP}$) and positivity of the initial data results with positive solutions. This partially justifies to call it a *Markovian Cauchy Problem*.

Our analysis is carried out in suitable functional spaces involving a probability measure on an underlying metric space, for which the growth of the volume changes in a nonpolynomial way. This is necessary as we are in particular interested in infinite dimensional problems. In such situation the Sobolev inequality which provides a cornerstone for classical PDE analysis cannot be satisfied and we have to rely on weaker coercive inequalities which survive the infinite dimensional limit and are of the following form:

$$\mu\Big(g^2 F\Big(\frac{g^2}{\mu g^2}\Big)\Big) \leq c\,\mu\,|\nabla g|^2,$$

with a constant $c \in (0,\infty)$ independent of the function g and where the right hand side involves the quadratic form of the elliptic operator L. Inequalities of this type,

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called later on F-Sobolev inequalities, have been recently studied in [3], [4], [29] and [25] (see also references therein) for probability measures with tails decaying more slowly than the Gaussian ones but faster than exponentially.

In our setup the linear operator L is monotone in the usual sense, while the nonlinear part may work to an opposite effect. Our study determines how large the coupling constant $\lambda > 0$ can be, so that the system is still stable in the sense of existence, uniqueness, smoothing properties and the ergodic long time behaviour of a (weak) solution.

We note that the linear semigroup corresponding to L is hypercontractive in an appropriate family of Orlicz spaces; in fact, as shown in [3] (generalising the celebrated result of Gross [18]), such hypercontractivity is equivalent to F-Sobolev inequality. Under suitable conditions, we show that the nonlinear semigroup obtained as the solution of ($\mathbb{MCP}$) is C_0, positivity and unit preserving, and is also hypercontractive in the appropriate family of Orlicz spaces. The key ingredients in our programme are provided by the F-Sobolev inequality and the fact that the quantity on its left hand side has similar properties to the relative entropy.

In recent years, an extensive effort has been made to understand better the coercive inequalities in infinite dimensional functional spaces, (see, e.g., [19], [3], [29], [25] and references therein). This provides a basis and partial motivation to study nonlinear problems. One may hope that a study in this direction may in the future shed also some light on or provide a complementary systematic understanding for a class of problems in infinite dimensions for which some understanding was achieved in the past (as, e.g., problems from mathematical physics). This work is also partially motivated by [16] where certain preliminary results where obtained for the case when logarithmic Sobolev inequalities are true. We note also that non-local problems involving certain normalisation condition were extensively studied in a finite dimensional setup in connection, for example, with statistical mechanics (mean field equation), self-dual gauge theory, theory of electrolytes and thermistors, mathematical biology (chemotaxis) and others (see, e.g., [9], [6], [8] and references therein).

The organization of our paper is as follows. In Section 2 we introduce the general setting and describe in detail conditions imposed on the linear and nonlinear operators appearing in our problem.

In Section 3 we introduce some Young functions whose associated Orlicz spaces play a key role in our analysis of ($\mathbb{MCP}$). We also prove there some bounds involving Dirichlet forms and these Young functions.

In Section 4 we prove the existence and uniqueness of the weak solution of problem ($\mathbb{MCP}$) (weak solutions are formulated in terms of the Dirichlet form $(\mathcal{E}, \mathcal{D})$ associated to L). In short, our strategy is as follows. We first consider a mollified problem with initial data in $L_2(\mu)$ defined by smoothing the nonlinear part with the linear semigroup P_ε, $\varepsilon > 0$, generated by L. Then, under the assumption that the coupling constant $\lambda > 0$ is sufficiently small, we employ the F-Sobolev inequality to prove the existence and uniqueness of the mollified problem via a nonlinear iteration scheme. The estimates and technique developed there will help us later to remove the mollification and to demonstrate that in the limit $\varepsilon \to 0$ we obtain a unique solution of our original problem. The essential part of the analysis which allows us to arrive to that conclusion is based on the fact that for initial data from a suitable Orlicz space (dense in $L_2(\mu)$) the solution lives within a much finer space. Let us notice here that this approach may be performed when $(\mathcal{E}, \mathcal{D})$ is a general Dirichlet form.

In Section 5 we show that the solution of ($\mathbb{MCP}$) defines a C_0-semigroup which preserves positivity ($L_2(\mu)$ contractivity of this nonlinear semigroup was already proven in Section 4). Moreover, we demonstrate that the solution decays exponentially to a constant in $L_2(\mu)$ space and consequently the time average of the solution converges almost everywhere to that constant.

In Section 6 we prove that the semigroup is uniformly hypercontractive in certain family of Orlicz norms, i.e., hypercontractive in the corresponding metrics (as we are dealing with nonlinear semigroup, hypercontractivity in the norms is in general a weaker property).

In Section 7 we demonstrate that the coercive inequalities which formed a basis for our study hold true in a large class of infinite dimensional models.

In Section 8 we briefly consider the corresponding local problem (in which normalisation by mean value with the measure μ is not present). The analysis here is entirely based on smoothing properties of the linear semigroup generated by L which follows directly from corresponding F-Sobolev inequality. Therefore it allows us to consider essentially weaker nonlinearities than the ones considered earlier for ($\mathbb{MCP}$).

Finally, Appendix I collects all the definitions and properties about Orlicz spaces we need, while Appendix II contains an explicit example the reader may like to keep in mind while reading the paper.

2. General setting and main theorem

2.1. Linear part, coercive inequality, admissible nonlinearity

Condition (C0): *The linear operator L involved in ($\mathbb{MCP}$) is the infinitesimal generator of a $\mathcal{C}_0$ Markov semigroup $(P_t)_{t\geq 0}$ symmetric with respect to some tight probability measure μ on a separable metric space.*

It is well known that tightness of μ comes for free when the underlying space is a Polish space.

Everywhere in the paper we will use the notation $\mu f \equiv \mu(f) \equiv \int f d\mu$ to denote the integral w.r.t. μ of an integrable – or nonnegative measurable – function f .

Let us give some useful precisions at least for non specialists. We refer to [12] for a brief overview of fundamental notions, see also the nice introductory part of Section 4 in [31]. Let $\mathbb{M}$ be a separable metric space, let $\mathcal{B}_{\mathbb{M}}$ be its Borel σ-field and let μ be a tight probability measure on it. A densely defined unbounded operator L on $\mathcal{D}(L) \subset \mathbb{L}^2(\mu)$ satisfies condition (**C0**) provided it is a non-positive self-adjoint operator such that the associated symmetric $\mathcal{C}_0$ semigroup $(P_t)_{t\geq 0} \equiv (e^{t\,L})_{t\geq 0}$ of contractions on $\mathbb{L}^2(\mu)$ is Markovian, in the sense that it satisfies, for any $t \geq 0$,

(i) Positivity preserving: $P_t f \geq 0$ for any $f \geq 0$.

(ii) Contraction on $\mathbb{L}^\infty(\mu)$: $\|P_t f\|_\infty \leq \|f\|_\infty$.

(iii) Mass conservation: $P_t 1 = 1$.

In case of linear semigroups, naturally (iii)+(i) implies (ii). By duality, P_t may be extended to a contraction semigroup on $\mathbb{L}^1(\mu)$. From (i) and (ii), as well as from symmetry of P_t, one can get the following representation of P_t under our (quite weak) topological assumptions[1] on $\mathbb{M}$: there exist a measurable family of probability measures $p_t(x, dy)$ on $(\mathbb{M}, \mathcal{B}_\mathbb{M})$ such that, for any $t \geq 0$, any $f \in \mathbb{L}^1(\mu)$, and for μ almost every $x \in \mathbb{M}$,

$$P_t f(x) = \int_\mathbb{M} f(y)\, p_t(x, dy). \tag{2.1}$$

In finite dimension one often talks about the kernel as the density of $p_t(x, dy)$ with respect to some natural measure, while in infinite dimensions it may be difficult to have such reference measure and even more such densities (as we do not have ultracontractivity in our setup). As a consequence of (2.1), Jensen inequality holds: for any $f \in \mathbb{L}^1(\mu)$ and any nonnegative convex function Φ, $\Phi(P_t f) \leq P_t(\Phi(f))$.

Note also that from the symmetry of P_t and (iii) it follows that μ is an invariant measure, *i.e.*, $\mu(P_t f) = \mu(f)$ for any $f \in \mathbb{L}^1(\mu)$.

In the one to one correspondence between non-positive self-adjoint operators and symmetric (non-negative definite) closed forms given by

$$\mathcal{E}(u, v) = \mu\big(((-L)^{1/2}u)\,((-L)^{1/2}v)\big), \quad u, v \in \mathcal{D} \equiv \mathcal{D}((-L)^{1/2}),$$

the Beurling–Deny conditions show that Markov generators correspond to conservative[2] Dirichlet forms. Namely, those forms on which normal contractions[3] operate. See [17] for finite dimensional setting and under local compactness assumption on $\mathbb{M}$; see [7] and [27] for some infinite dimensional and/or non-symmetric setting and without topological assumptions. This provides a concrete way to construct symmetric $\mathcal{C}_0$ Markov semigroups: the semigroup is specified once we choose an appropriate domain to close a given closable Markovian form.

One may characterize the domain $\mathcal{D}$ of the Dirichlet form by means of spectral theory in the following way. For any $u \in \mathbb{L}^2(\mu)$, $t \in (0, \infty) \mapsto \frac{1}{t}\mu((u - P_t u)\, u)$ is non increasing and

$$\mathcal{D} = \Big\{ u \in \mathbb{L}^2(\mu) : \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t}\, \mathcal{E}_{P_t}(u, u) = \sup_{t > 0} \frac{1}{t}\, \mathcal{E}_{P_t}(u, u) < +\infty \Big\} \tag{2.2}$$

[1]Note that one can get rid of usual assumption of completeness of $\mathbb{M}$ provided the probability measure μ is tight: follow the approach of Proposition 3.1 in [2] and use disintegration $\bar{\mu}_t(dx, dy) = p_t(x, dy)\mu(dx)$ of the measure on $\mathbb{M} \times \mathbb{M}$ defined by $\bar{\mu}_t(A \times B) \equiv \int P_t(\mathbf{I}_A)\, \mathbf{I}_B\, d\mu$. For existence of such a disintegration, see [13] or [14], where the proof of Theorem 10.2.2 may be adapted to our setup.

[2]That is, μ is a probability measure, $1 \in \mathcal{D}$ and $\mathcal{E}(1, 1) = 0$.

[3]A function $\psi : \mathbb{R} \to \mathbb{R}$ is a normal contraction if $|\psi(x) - \psi(y)| \leq |x - y|$, $x, y \in \mathbb{R}$ and $\psi(0) = 0$. ψ operates on $\mathcal{E}$ provided, for any $u \in \mathcal{D}$, $\psi(u) \in \mathcal{D}$ and $\mathcal{E}(\psi(u), \psi(u)) \leq \mathcal{E}(u, u)$.

with $\mathcal{E}_{P_t}(u,u) \equiv \mu((u - P_t u)\, u)$ (see page 22 of [17], for example). But, by invariance property,

$$\mu\big((u - P_t u)\, u\big) = \frac{1}{2} \int_{\mathbb{M}} \mu(dx) \int_{\mathbb{M}} \big(u(x) - u(y)\big)^2 \, p_t(x, dy).$$

And one has the following representation formula: for any $u, v \in \mathcal{D}$,

$$\mathcal{E}(u,v) = \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{2t} \int_{\mathbb{M}} \mu(dx) \int_{\mathbb{M}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, p_t(x, dy). \tag{2.3}$$

When considering the Dirichlet form, we will write abusively

$$\mathcal{E}(f,f) \equiv \mu(|\nabla f|^2) = \int |\nabla f|^2 \, d\mu$$

for natural reasons (see Example 2.1 for instance).

Note finally that from classical semigroup theory, for any $t > 0$ and any $f \in \mathbb{L}^2(\mu)$, $P_t f$ belongs to $\mathcal{D}(L)$, is differentiable in $\mathbb{L}^2(\mu)$ and $v(t) \equiv P_t f$ is the strong solution (in $\mathbb{L}^2(\mu)$ sense) of

$$\begin{cases} \frac{\partial}{\partial t} v(t) &= Lv(t), \quad t > 0, \\ v(0) &= f. \end{cases} \tag{2.4}$$

The reader might keep in mind, as a guideline, the following basic example, that illustrates the general setting we have just presented.

Example 2.1. We adapt [31] to our setting. Let M be a complete connected Riemannian manifold (without boundary) with Riemannian volume dx (the problems we investigate are already of interest in $M = \mathbb{R}^n$). Let U be a smooth function on M such that $Z = \int_M e^{-U(x)} dx < \infty$ and let $\mu(dx) \equiv e^{-U(x)} dx / Z$. We denote by $\mathcal{C}_c^\infty(M)$ the space of compactly supported smooth functions on M. The operator $Lf = \Delta f - \nabla U \cdot \nabla f$, with f in the domain $\mathcal{C}_c^\infty(M) \subset \mathbb{L}^2(\mu)$, is a symmetric non positive operator (note that here Δ, ∇ and the scalar product $\cdot$ are relative to the Riemannian metric). This comes from the integration by parts formula: for $f, g \in \mathcal{C}_c^\infty(M)$,

$$\mathcal{E}(f,g) \equiv \int_M (-L) f(x)\, g(x) \mu(dx) = \int_M \nabla f(x) \cdot \nabla g(x)\, \mu(dx).$$

Hence, the form $\mathcal{E}$ is closable and its closure is associated to a self-adjoint operator still denoted by L with domain $\mathcal{D}(L) \subset \mathbb{L}^2(\mu)$ (which generates a symmetric $\mathcal{C}_0$ semigroup of contractions $(P_t)_{t \geq 0}$ on $\mathbb{L}^2(\mu)$). See [11], in particular Theorems 4.12 and 4.14. Positivity preserving and contraction of $\mathbb{L}^\infty(\mu)$ follow from Dirichlet form theory by stability of $\mathcal{D}$ by appropriate smooth approximations ϕ_ε of the unit contraction $(\cdot \wedge 1) \vee 0$, for which one easily checks that $\mathcal{E}(\phi_\varepsilon(f), \phi_\varepsilon(f)) \leq \mathcal{E}(f,f)$. As $(P_t)_{t \geq 0}$ solves (2.4), they also may be seen as consequences of the parabolic maximum principle. Note furthermore that regularity theory for the parabolic equations ensures that $P_t f \in \mathcal{C}^\infty(M)$ for any $f \in \mathbb{L}^1(\mu)$, from which existence of the kernels $p_t(x, dy)$ follows. As M is complete, the closure $\overset{\circ}{W}{}^{1,2}(\mu)$ of $\mathcal{C}_c^\infty(M)$ for the norm $\|f\|_{W^{1,2}} \equiv \big(\mu(f^2) + \mu(|\nabla f|^2)\big)^{1/2}$ coincides with the closure $W^{1,2}(\mu)$ of the space of $\mathcal{C}^\infty$ functions with finite norm $\|\cdot\|_{W^{1,2}}$ (see [20] for instance). Hence $\overset{\circ}{W}{}^{1,2}(\mu)$ and also $\mathcal{D}(L)$ contain all constant functions, and one has $L1 = 0$ and $P_t 1 = 1$.

In Subsection 7.1 we describe some more advanced examples where μ is a Gibbs measure in some infinite dimensional models coming from interacting spins systems. See also [21] for a wider class of examples including degenerate generators.

Coercive inequality as a constraint on the Dirichlet structure. The admissible nonlinearity in ($\mathbb{MCP}$) is specified by some regularity of the *Dirichlet structure* $(\mu, \mathcal{E})$, $\mathcal{E}$ being a Dirichlet form on $\mathbb{L}^2(\mu)$. Precisely, we introduce the following requirements.

We assume that $(\mu, \mathcal{E})$ satisfies an F-*Sobolev* inequality (a notion introduced by Wang [32]) for some function F we specify below. That is, there exists a constant $c_F \in (0, +\infty)$ such that

$$\textbf{(FS)} \qquad \int f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \le c_F \int |\nabla f|^2 \, d\mu$$

for any $f \in \mathcal{D}$ (or any sufficiently smooth function f). In this case we will use a shorthand notation $\mu \in \mathbf{FS}(c_F)$. See the forthcoming Section 2.3 for more comments on F-Sobolev inequalities.

In the case, $F(x) = \log(x)$, F-Sobolev inequality is the well known log-Sobolev (or Gross) inequality. Let us note here that the scope of this paper does not include directly log-Sobolev inequality: our approach to show existence of weak solutions of ($\mathbb{MCP}$) is based in particular on a regularity property proved in the forthcoming Lemma 4.2 and strong convergence of a mollified solution in an appropriate Hilbert space (see Theorem 4.10) which are not available when $F = \log$ due to the singularity at 0.

Let us introduce some conditions on the function F.

Condition (C1): *In all what follows $F : [0, \infty) \to \mathbb{R}$ denotes a non decreasing $\mathcal{C}^2$ function such that $F(1) = 0$. We assume that there exist constants $\theta \ge 1$ and $\bar{B} > 0$ such that*

$$\textbf{(C1)} \qquad \begin{cases} \text{(i) } F \text{ is concave on } [\theta, +\infty), \\ \text{(ii) } \forall x \ge 0\,, \quad xF'(x) \le \bar{B}. \end{cases}$$

Note that, as $F(1) = 0$, (**FS**) is a tight inequality in the restrictive sense that both sides are zero for constant functions.

Note also that when F satisfies condition (**C1**), the value $F(0) \le 0$ is well defined. Let us define $A \equiv -F(0) \ge 0$ so that

$$A = \max_{x \in [0,1]} |F(x)| = \max_{x \ge 0} -F(x). \tag{2.5}$$

Condition (C2): *$x\,F(x)$ is convex.*

Condition (C3): *There exists a constant $0 \le R < \infty$ such that*

$$F(a\,b) \le F(a) + F(b) + R \quad \textit{for any } a, b \in (0, \infty).$$

In Appendix II, we present a simple example of a μ-symmetric Markov generator for some μ with tails between Gaussian and exponential. Such μ satisfies an F-Sobolev inequality for some associated F satisfying conditions (**C1**) to (**C3**). The case when μ is a Gibbs measure with the prescribed tails is studied in Subsection 7.1.

Nonlinearity. The nonlinear part in equation ($\mathbb{MCP}$) is described by a function G. We assume that G is a perturbation of F which satisfies the following condition.

Condition (C4): *With F satisfying condition* (**C1**–**C3**), *we assume that*

$$G = F + \mathcal{J}$$

with a bounded $\mathcal{C}^1$ perturbation $\mathcal{J} : [0, \infty) \to \mathbb{R}$ such that $\sup x|\mathcal{J}'(x)| < \infty$.

Under these hypothesis, $\widetilde{B} \equiv \sup_{x \ge 0} x|G'(x)| < \infty$.

Note that $G(0)$ is well defined and G is Lipschitz at 0 (for a non-Lipschitz at 0 example, see [16]). When additionally

- $\mathcal{J} \le 0$ and $\mathcal{J}(1) = 0$

(so that $G \le F$ and $G(1) = 0$), we will say that G satisfies (**MC4**). Then constants are global solutions of the corresponding parabolic problem ($\mathbb{MCP}$) and we will see later that positivity of the initial data results with positive solutions.

2.2. Main theorem

We now state our main theorem. See Section 4 for definition of a weak solution.

Theorem 2.2. *Let L be a Markov generator like in condition* (**C0**). *Assume that the associated Dirichlet structure $(\mu, \mathcal{E})$ satisfies an F-Sobolev inequality with constant c_F with F satisfying conditions* (**C1**), (**C2**) *and* (**C3**). *Then, for any $\lambda \in [0, c_F^{-1})$, any function G satisfying condition* (**C4**) *and any $f \in \mathbb{L}^2(\mu)$, the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t} u &= Lu + \lambda\, u\, G\left(\frac{u^2}{\mu(u^2)}\right) \\ u(0) &= f \end{cases}$$

admits a unique weak solution on $[0, \infty)$.

One may hope that under suitable assumptions we could perform regularity theory in infinite dimensions, so that classical strong solutions exist. We do not investigate this problem here.

By changing F into F/c_F (and similarly with L and G), one can reduce the problem to the case when $c_F = 1$. Nevertheless, we state Theorem 2.2 as above because, in general, there are very few examples where the best constant c_F in F-Sobolev inequality can be explicitly computed. However, for simplicity, in all the sequel we will assume that F is chosen so that to have

$$\boxed{c_F \equiv \inf\{c > 0 : \mu \in \mathbf{FS}(c)\} = 1}$$

and we will write $\mu \in \mathbf{FS}$ without further mention on the best constant c_F.

2.3. Some properties of F-Sobolev functional

Let us discuss briefly the basic properties and the links between F-Sobolev inequalities and the usual Poincaré and log-Sobolev inequalities. With this aim, we consider (in this paragraph exclusively) more general F's than those satisfying (**C1**–**C3**), in particular we allow singularity at 0 and/or discontinuities.

As already mentioned, the choice $F = \log$ corresponds to the well known logarithmic Sobolev inequality [18]. On the other side, $F(x) = \mathbb{1}_{[2,\infty)}$ (see Remark 22 of [3]) corresponds to Poincaré inequality. In this paper we shall deal with intermediate inequalities corresponding to F behaving like $\log^\beta$, $\beta \in [0,1]$. Hence, in principle, the coercive inequality we will deal with is stronger than the Poincaré inequality, and weaker than the log-Sobolev inequality. However, to make this rigorous one possibility is to add some regularity assumption. Indeed, if F is $\mathcal{C}^2$ in a neighbourhood of 1, then the Poincaré inequality holds as soon as $2F'(1) + F''(1) \neq 0$, see Lemma 8 in [3]. On the other hand, the same conclusion holds if $F \geq c\mathbb{1}_{[2,\infty)}$ for some $c > 0$ by Remark 22 of [3].

Also, assuming that Poincaré inequality holds, only the behavior of F at infinity is relevant (see Lemma 21 of [3] for a result in this direction). This can be also explained using the Rothaus-type inequality (2.6) below.

However, note that in our setting the assumption $\mu \in \mathbf{FS}$ alone does not guarantee, a priori, that μ satisfies a Poincaré inequality.

In Section 8 an other type of coercive inequality, called Φ-Sobolev inequality, will be introduced and used. Since such inequalities are not relevant for the main part of this paper we refer the reader to Section 8 for comments on them.

Now we state two useful results.

Lemma 2.3 (Generalized Relative Entropy Inequality). *Suppose a function F satisfies condition* (**C1**). *Then there exists $B \in (0,\infty)$, such that, for any $x, y \geq 0$,*

$$xF(y) \leq xF(x) + By.$$

Therefore for any probability measure μ, and any $f, g \in \mathbb{L}^2(\mu)$,

$$\text{(GREI)} \qquad \int f^2 F\Big(\frac{g^2}{\mu(g^2)}\Big) d\mu \leq \int f^2\, F\Big(\frac{f^2}{\mu(f^2)}\Big) d\mu + B\mu(f^2).$$

Proof. Define $\delta(x,y) \equiv x\,(F(y) - F(x))$. In the case $x \geq y$, $\delta(x,y) \leq 0$ as F is non decreasing and so nothing has to be proved. So assume $x \leq y$. Now, if $x \geq \theta$, then

$$\delta(x,y) = x\frac{F(y) - F(x)}{y - x}(y - x) \leq xF'(x)y \leq \bar{B}y.$$

And if on the contrary $x \leq \theta$, $\delta(x,y) \leq \theta(F(y) - F(x))$. But, in the case $y \geq 1$, $F(y) - F(x) \leq F(y) + A \leq (K + A)y$ where A was defined in (2.5) and K is any constant such that $\forall \xi \geq 0$, $F(\xi) \leq K\xi$. Whereas, in the case $x \leq y \leq 1$, $F(y) - F(x) \leq \|F'\mathbb{1}_{[0,1]}\|_\infty\, y$. This ends the proof of the bound $xF(y) \leq xF(x) + By$ from which inequality (GREI) easily follows. □

The next result (due to Rothaus [30] for the log-Sobolev inequality) is usually used to tighten inequalities, using Poincaré inequality. Namely, by (2.6), if μ satisfies $\int f^2F\big(\frac{f^2}{\mu(f^2)}\big)d\mu \leq c\int|\nabla f|^2 d\mu + c'\mu(f^2)$ for some $c, c' > 0$ independent of $f \in \mathcal{D}$, and if μ satisfies also a Poincaré inequality, then $\mu \in \mathbf{FS}(c'')$ for some $c'' > 0$. It is probably possible to state a result involving a general set of norms, or semi-norms, for which the result below apply. We do not investigate this problem here.

Lemma 2.4 (see [19] or [3]). *Assume F satisfies condition* (**C1**). *Then*

$$\mu\Big(f^2F\Big(\frac{f^2}{\mu f^2}\Big)\Big) \leq \mu\bigg(\widetilde{f}^{\,2}F\Big(\frac{\widetilde{f}^{\,2}}{\mu\widetilde{f}^{\,2}}\Big)\bigg) + C\mu(\widetilde{f}^{\,2}) \tag{2.6}$$

where $\widetilde{f} = f - \mu f$ and $C \equiv 4\bar{B} + B$.

3. Dirichlet/Young bounds

A key point in our analysis of ($\mathbb{MCP}$) will be to obtain a regularity result in some Orlicz spaces. The associated Young functions and the Dirichlet form satisfy some bounds which we introduce now.

3.1. Young functions and Orlicz spaces

We refer to Appendix I for basics on Orlicz spaces and Young functions.

For any non decreasing C^2 function $F : [0,\infty) \to \mathbb{R}$ such that $x\,F(x)$ is convex and any $q \geq 0$, the function $\Upsilon_q(x) \equiv |x|\,e^{q\,F(|x|)}$ is a Young function so that $\Phi_q(x) \equiv \Upsilon_q(x^2)$ is a Nice Young function (in short, N-function as called in [28]). The associated Orlicz space satisfies $\mathbb{L}^{\Phi_q}(\mu) \subset \mathbb{L}^2(\mu)$ with continuous embedding.

We now present some properties of the family $(\Phi_q)_{q\geq 0}$ which will allow to get the Dirichlet/Young type bounds we mentioned before. It is assumed here without further mention that F satisfies conditions (**C1**) and (**C2**).

Even if we won't need this here, let us mention that an immediate consequence of the additional condition (**C3**) is the sub-multiplicativity property for the N-function Φ_q usually called Δ'-Condition:

$$\Phi_q(x\,y) \leq e^{Rq}\Phi_q(x)\,\Phi_q(y). \tag{3.1}$$

3.1.1. Some computations and a remark. The following simple computations will be useful in the sequel. For any $x \geq 0$, one has $\Upsilon_q'(x) = e^{q\,F(x)}\,(1 + qx\,F'(x))$,

$$\begin{aligned}\Upsilon_q''(x) &= q\,e^{q\,F(x)}\left(\{2\,F'(x) + x\,F''(x)\} + qx(F'(x))^2\right)\\ &= q\,e^{q\,F(x)}\left((xF(x))'' + qx(F'(x))^2\right)\end{aligned} \tag{3.2}$$

and

$$\Phi_q''(x) = 2\Upsilon_q'(x^2) + 4x^2\Upsilon_q''(x^2). \tag{3.3}$$

Remark 3.1. From the previous formulae, one easily gets that, in the case when F is bounded, $\Upsilon_q'(x)$ and $x\Upsilon_q''(x)$ are bounded, so that $\Phi_q''(x)$ is bounded as well. This will be another ingredient to get our regularity result by approximating F by truncated functions.

3.1.2. Differential inequality and Dirichlet/Young bounds. Let us define $\Psi_q(x) = \sqrt{\Phi_q(x)} = |x|e^{\frac{q}{2}F(x^2)}$. The following differential inequality holds:

$$\forall x \in \mathbb{R}, \quad \Phi_q''(x) \geq k_q\,\frac{\left(\Phi_q'(x)\right)^2}{4\Phi_q(x)} = k_q\,\left(\Psi_q'(x)\right)^2, \tag{3.4}$$

with the constant

$$k_q = 2/\left(1 + q\,\bar{B}\right). \tag{3.5}$$

For parity reasons, one may assume $x > 0$. Note that $\left(\Psi_q'(0)\right)^2$ makes sense. As Υ_q is convex and $\Phi_q(x) = \Upsilon_q(x^2)$, (3.3) gives

$$\Phi_q''(x) \geq 2\,\Upsilon_q'(x^2) = \Phi_q'(x)/x.$$

Thus the relation

$$x\,\Phi_q'(x) = 2\Phi_q(x)\big(1 + qx^2F'(x^2)\big) \leq 2\,\left(1 + q\,\bar{B}\right)\,\Phi_q(x) \tag{3.6}$$

leads to the announced differential inequality on Φ_q.

Remark 3.2. One can take $k_q = 2$ instead of (3.5) in inequality (3.4) provided $\Psi_q \equiv \sqrt{\Phi_q}$ is a convex function, which occurs for any $q \geq 0$, if and only if F satisfies the following additional **Condition** (**C2bis**): for any $x \geq 0$, $(xF(x))'' \geq \frac{1}{2}F'(x)$.

The differential inequality (3.4) leads to the following Dirichlet/Young bounds: for any $u \in \mathcal{D}$ and any $q \geq 0$, $\Psi_q(u) \in \mathcal{D}$ provided $\Phi_q'(u) \in \mathcal{D}$, and one has

$$\mathcal{E}(\Phi_q'(u), u) \geq k_q\mathcal{E}\left(\Psi_q(u), \Psi_q(u)\right). \tag{3.7}$$

This is a direct consequence of the following lemma.

Lemma 3.3. *Let L be as in condition* (**C0**) *and let $(\mathcal{E}, \mathcal{D})$ be the associated Dirichlet form. Let ξ and ζ be two absolutely continuous functions on $\mathbb{R}$ satisfying the differential inequality*

$$\xi' \geq c(\zeta')^2 \quad a.e., \tag{3.8}$$

for some $c > 0$. Then, for any $u \in \mathcal{D}$,

$$\xi(u) \in \mathcal{D} \Rightarrow \zeta(u) \in \mathcal{D}$$

and one has

$$\mathcal{E}(\xi(u), u) \geq c\,\mathcal{E}(\zeta(u), \zeta(u))\,.$$

Proof. The differential inequality (3.8) is equivalent to the following slope bound:

$$\frac{\xi(x) - \xi(y)}{x - y} \geq c\left(\frac{\zeta(x) - \zeta(y)}{x - y}\right)^2 \quad \text{for any } x \neq y. \tag{3.9}$$

Namely, (3.9) implies (3.8) at any point where the derivatives do exist. And the converse follows from Jensen inequality: for any $x < y$,

$$\left(\frac{\zeta(y) - \zeta(x)}{y - x}\right)^2 = \left(\frac{1}{y - x}\int_x^y \zeta'(s)\,ds\right)^2 \leq \frac{1}{y - x}\int_x^y (\zeta'(s))^2\,ds.$$

Let us first show that

$$u, \xi(u) \in \mathbb{L}^2(\mu) \Rightarrow \zeta(u) \in \mathbb{L}^2(\mu). \tag{3.10}$$

As μ is a probability measure, one may assume that $\xi(0) = \zeta(0) = 0$. Then (3.9) implies that, for any $x \in \mathbb{R}$,

$$c\,(\zeta(x))^2 \leq \xi(x)\,x = |\xi(x)|\,|x| \tag{3.11}$$

as $\xi(x)$ and x have the same sign. Hence, (3.10) follows from (3.11) and Cauchy–Schwarz inequality in $\mathbb{L}^2(\mu)$. Now, the claim of the lemma follows from the characterization (2.2) and the representation formula (2.3) as (3.9) can be rewritten as

$$\forall x, y \in \mathbb{R}, \qquad (\xi(x) - \xi(y))(x - y) \geq c(\zeta(x) - \zeta(y))^2.$$

4. Existence problem

To prove the existence of a weak solution for Cauchy problem ($\mathbb{MCP}$) we implement a constructive nonlinear approximation procedure. Usual Gelfand triple for Sobolev spaces on a domain $\Omega \subset \mathbb{R}^d$, that is,

$$\mathring{W}^{1,2}(\Omega) \subset \mathbb{L}^2(\Omega) \subset W^{-1,2}(\Omega)$$

(see [34]), has to be replaced by $\mathcal{D} \subset \mathbb{L}^2(\mu) \subset \mathcal{D}'$, where $\mathcal{D}$ is equipped with the domain Hilbert structure and $\mathcal{D}'$ is its topological dual space.

4.1. Weak solutions and preliminary regularity result

Given $T \in (0, \infty)$, define $\mathbb{H}_{T,+}(\mathcal{E}) \equiv \mathbb{L}^2([0,T], \mathcal{D})$ as a Banach space of (classes of) functions $v : [0,T] \times \mathbb{M} \to \mathbb{R}$, such that

$$||v||^2_{\mathbb{H}_{T,+}} \equiv \int_0^T ds\ \mu v^2 + \int_0^T ds\ \mu|\nabla v|^2 < \infty,$$

By $\mathbb{H}_{T,-}(\mathcal{E}) \equiv \mathbb{L}^2([0,T], \mathcal{D}')$, we will denote the dual space of $\mathbb{H}_{T,+}(\mathcal{E})$.

Let $\mathcal{A} : \mathbb{H}_{T,+}(\mathcal{E}) \to \mathbb{H}_{T,-}(\mathcal{E})$ be an abstract nonlinear operator. We say that a function $u \in \mathbb{H}_{T,+}(\mathcal{E})$ is a weak solution (on $[0,T]$) of the following Cauchy problem

$$\left\{ \begin{array}{rcl} \partial_t u &=& Lu + \mathcal{A}(u) \\ u_{|t=0} &=& f \end{array} \right. \tag{4.1}$$

with $f \in \mathbb{L}^2(\mu)$, if and only if, for any $v \in \mathcal{C}^\infty([0,T]; \mathcal{D})$ and any $t \in [0,T]$, we have

$$\begin{aligned} \int_0^t \mu\left(u(s)\partial_s v(s)\right) ds \ &= \mu u(t)v(t) - \mu f v(0) + \int_0^t \mu \nabla u(s) \cdot \nabla v(s) ds \\ &\quad - \int_0^t \langle \mathcal{A}(u)(s), v(s)\rangle_{\mathcal{D}',\mathcal{D}} ds, \end{aligned} \tag{4.2}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}',\mathcal{D}}$ stands for the duality bracket.

Note that condition (4.2) may be extended, by density, to any function $v \in \mathbb{L}^2\big([0,T], \mathcal{D}\big) \cap W^{1,2}\big((0,T), \mathbb{L}^2(\mu)\big)$.

Remark 4.1 (Time continuity in $\mathbb{L}^2(\mu)$ of weak solutions). A function $u \in \mathbb{L}^2\big([0,T], \mathcal{D}\big)$ satisfying (4.2) for any $v \in \mathcal{C}_0^\infty((0,T); \mathcal{D})$ admits a weak time derivative $\partial_t u$ in $\mathcal{D}'$ which belongs to $\mathbb{L}^2([0,T], \mathcal{D}')$, and so $u \in C\left([0,T], \mathbb{L}^2(\mu)\right)$ as shown below. So that (4.2) makes sense when $v \in \mathcal{C}^\infty([0,T]; \mathcal{D})$. From (4.2) at $t = 0$, it follows that $u(0) = f$.

To understand this, first note that L may be seen in a weak sense as an operator from $\mathcal{D}$ to $\mathcal{D}'$ by setting, for any $u, v \in \mathcal{D}$,

$$\langle (-L)u, v\rangle_{\mathcal{D}',\mathcal{D}} \equiv \mathcal{E}(u,v) = \mu(\nabla u \cdot \nabla v).$$

Applied to $v(s) = \Phi(s)\mathrm{v}$ for $\Phi(s) \in \mathcal{C}_0^\infty((0,T))$, $\mathrm{v} \in \mathcal{D}$ and $t = T$, (4.2) implies that

$$\left\langle \int_0^T u(s)\Phi'(s) ds, \mathrm{v} \right\rangle_{\mathcal{D}',\mathcal{D}} = \left\langle \int_0^T \{(-L)u(s) - \mathcal{A}(u)(s)\}\, \Phi(s) ds, \mathrm{v} \right\rangle_{\mathcal{D}',\mathcal{D}}$$

where $\nu \equiv \int_0^T u(s)\Phi'(s)ds \in \mathbb{L}^2(\mu)$ is considered as an element of $\mathcal{D}'$ via $\mathbb{L}^2(\mu)$ pairing: $\langle \nu, \mathrm{v}\rangle_{\mathcal{D}',\mathcal{D}} \equiv \mu(\nu\, \mathrm{v})$. This means that, in $\mathcal{D}'$, $u(\cdot)$ admits the weak time derivative $\partial_t u = Lu + \mathcal{A}(u)$ which belongs to $\mathbb{L}^2([0,T], \mathcal{D}')$ as $\|(-L)u(t)\|_{\mathcal{D}'} \leq \|u(t)\|_{\mathcal{D}}$. Deriving from this that $u \in C\left([0,T], \mathbb{L}^2(\mu)\right)$ may be found in Theorem 3, page 287, of [15].

Later on in this paper we discuss a situation when the operator $\mathcal{A}$ is given by $\mathcal{A}(u) \equiv \lambda\mathcal{V}(u)$ with a parameter $\lambda \in \mathbb{R}$ and

$$\mathcal{V}(u)(s) \equiv \mathbb{V}(u(s)) \equiv u(s)G(\sigma^2(u(s)))$$

where $\sigma(\mathrm{u}) \equiv \frac{\mathrm{u}}{(\mu(\mathrm{u}^2))^{1/2}}$ for $\mathrm{u} \in \mathbb{L}^2(\mu)$, $\mathrm{u} \neq 0$, and $\sigma(0) \equiv 0$.

In which sense this operator $\mathcal{A}$ maps $\mathbb{H}_{T,+}(\mathcal{E})$ to $\mathbb{H}_{T,-}(\mathcal{E})$ as before (so that one may consider weak solutions for Cauchy problem ($\mathbb{MCP}$) like in (4.2)) is made precise in the following basic regularity result.

Lemma 4.2 (Regularity for the nonlinear operator). *Let, for any* $\mathrm{u} \in \mathcal{D}$, $\mathbb{V}(\mathrm{u}) \equiv \mathrm{u}G(\sigma^2(\mathrm{u}))$. *Suppose* $\mu \in$ **FS**. *Then, for any* $\mathrm{u}, g \in \mathcal{D}$, $g\mathbb{V}(\mathrm{u}) \in \mathbb{L}^1(\mu)$ *and there exists* $C \in (0,\infty)$ *such that*

$$|\mu\,(g\,\mathbb{V}(\mathrm{u}))| \leq C\ \|\mathrm{u}\|_{\mathcal{D}}\ \|g\|_{\mathcal{D}}$$

In particular, $\mathbb{V}(\mathrm{u}) \in \mathcal{D}'$ *when acting on* $\mathcal{D}$ *with* $\mathbb{L}^2$ *pairing*

$$\langle \mathbb{V}(\mathrm{u}), g\rangle_{\mathcal{D}',\mathcal{D}} \equiv \mu\,(g\,\mathbb{V}(\mathrm{u}))\,.$$

Moreover, the operator $\mathbb{V}\colon \mathrm{u} \in \mathcal{D} \mapsto \mathbb{V}(\mathrm{u}) \in \mathcal{D}'$ *is Lipschitz continuous. As a consequence, for any* $u \in \mathbb{H}_{T,+}(\mathcal{E})$, $\mathcal{V}(u) \in \mathbb{H}_{T,-}(\mathcal{E})$ *and* $\mathcal{V} : \mathbb{H}_{T,+}(\mathcal{E}) \longrightarrow \mathbb{H}_{T,-}(\mathcal{E})$ *is Lipschitz continuous.*

Proof. Suppose u and g in $\mathcal{D}$. Then one has, recalling $G = F + \mathcal{J}$ by condition (**C4**) and denoting $\chi_- \equiv \chi_{u^2<\mu(u^2)}$ and $\chi_+ \equiv \chi_{u^2\geq\mu(u^2)}$,

$$\begin{aligned}
\left|\mu\big(g\,\mathrm{u}G\left(\sigma^2(\mathrm{u})\right)\big)\right| &\leq \|\mathcal{J}\|_\infty\,\mu\,(|g|\,|\mathrm{u}|) + \mu\left(|g|\,|\mathrm{u}|\,\left|F\left(\sigma^2(\mathrm{u})\right)\right|\right)\\
&\leq \|\mathcal{J}\|_\infty\ \|\mathrm{u}\|_2\,\|g\|_2 + \mu\left(|g|\,|\mathrm{u}|(-F)\left(\sigma^2(\mathrm{u})\right)\chi_-\right) + \mu\left(|g|\,|\mathrm{u}|F\left(\sigma^2(\mathrm{u})\right)\chi_+\right)\\
&\leq (\|\mathcal{J}\|_\infty + A)\,\|\mathrm{u}\|_2\,\|g\|_2 + \left(\mu\left(g^2F\left(\sigma^2(\mathrm{u})\right)\chi_+\right)\right)^{1/2}\ \left(\mu\left(\mathrm{u}^2F\left(\sigma^2(\mathrm{u})\right)\chi_+\right)\right)^{1/2},
\end{aligned}$$

where we used that $F(x) \geq 0$ for $x \geq 1$, the definition (2.5) of A and the Cauchy–Schwarz inequality. Now,

$$\begin{aligned}
\mu\left(\mathrm{u}^2F\left(\sigma^2(\mathrm{u})\right)\chi_+\right) &= \mu\left(\mathrm{u}^2F\left(\sigma^2(\mathrm{u})\right)\right) + \mu\left(\mathrm{u}^2(-F)\left(\sigma^2(\mathrm{u})\right)\chi_-\right)\\
&\leq \mu\left(|\nabla \mathrm{u}|^2\right) + A\,\mu\left(\mathrm{u}^2\right) \leq \max(1,A)\ \|\mathrm{u}\|_{\mathcal{D}}^2
\end{aligned}$$

thanks to F-Sobolev inequality (**FS**). Similarly,

$$\begin{aligned}
\mu\left(g^2F\left(\sigma^2(\mathrm{u})\right)\chi_+\right) &\leq \mu\left(g^2F\left(\sigma^2(\mathrm{u})\right)\right) + A\mu\left(g^2\right)\\
&\leq \mu\left(g^2F\left(\sigma^2(g)\right)\right) + (A+B)\mu\left(g^2\right)\\
&\leq \mu\left(|\nabla g|^2\right) + (A+B)\mu\left(g^2\right) \leq \max(1, A+B)\,\|g\|_{\mathcal{D}}^2
\end{aligned}$$

thanks to (GREI) and another use of F-Sobolev inequality. So that finally

$$\left|\mu\left(g\,\mathrm{u}G\left(\sigma^2(\mathrm{u})\right)\right)\right| \leq (\|\mathcal{J}\|_\infty + A + \max(1, A+B))\ \|g\|_{\mathcal{D}}\ \|\mathrm{u}\|_{\mathcal{D}}\,.$$

Let us now turn to Lipschitz estimate. Suppose $\mathrm{v} \neq \mathrm{u}$ and g are still in $\mathcal{D}$. From the first part of the proof and $\mathbb{V}(0) = 0$, one may assume that $\mathrm{u} \neq 0$ and $\mathrm{v} \neq 0$. Let us set $\mathrm{u}_\alpha \equiv \alpha \mathrm{u} + (1-\alpha)\mathrm{v}$, $\alpha \in [0,1]$, and let $\mathrm{w} \equiv \mathrm{u} - \mathrm{v}$. Assume first that $\mathrm{u}_\alpha \neq 0$ for any α. Then we have

$$\left|\mu\left(g\left[\mathrm{u}G\left(\sigma^2(\mathrm{u})\right) - \mathrm{v}G\left(\sigma^2(\mathrm{v})\right)\right]\right)\right| \leq \int_0^1 d\alpha \left|\mu\left(g \frac{d}{d\alpha}\left[\mathrm{u}_\alpha G\left(\sigma^2(\mathrm{u}_\alpha)\right)\right]\right)\right|,$$

with $\frac{d}{d\alpha}\mathrm{u}_\alpha G\left(\sigma^2(\mathrm{u}_\alpha)\right)$ explicitly given by

$$\mathrm{w}G\big(\sigma^2(\mathrm{u}_\alpha)\big) + 2\,\sigma^2(\mathrm{u}_\alpha)\,G'\left(\sigma^2(\mathrm{u}_\alpha)\right)\,\mathrm{w} - 2\,\sigma^3(\mathrm{u}_\alpha)\,G'\left(\sigma^2(\mathrm{u}_\alpha)\right)\,\mu\big(\sigma(\mathrm{u}_\alpha)\,\mathrm{w}\big). \tag{4.3}$$

Since by our assumption $\sigma^2\left|G'(\sigma^2)\right| \leq \widetilde{B}$, we get

$$\begin{aligned} \left|\mu\left(g\left[\mathrm{u}G\left(\sigma^2(\mathrm{u})\right) - \mathrm{v}G\left(\sigma^2(\mathrm{v})\right)\right]\right)\right| &\leq \int_0^1 d\alpha \left|\mu\left(g\mathrm{w}G\left(\sigma^2(\mathrm{u}_\alpha)\right)\right)\right| \\ &+ 2\widetilde{B}\mu\left(|g|\,|\mathrm{w}|\right) + 2\,\widetilde{B}\int_0^1 d\alpha\,\mu\left(|g|\,|\sigma(\mathrm{u}_\alpha)|\right)\mu\left(|\sigma(\mathrm{u}_\alpha)|\,|\mathrm{w}|\right). \end{aligned} \tag{4.4}$$

Now, by similar arguments as above, one has

$$\left|\mu\left(g\,\mathrm{w}G\left(\sigma^2(\mathrm{u}_\alpha)\right)\right)\right| \leq \left(\|\mathcal{J}\|_\infty + A + \max(1, A+B)\right)\|g\|_{\mathcal{D}}\,\|\mathrm{w}\|_{\mathcal{D}}\,. \tag{4.5}$$

On the other hand, Cauchy–Schwarz inequality applied twice gives

$$\mu\left(|g|\,|\sigma(\mathrm{u}_\alpha)|\right)\mu\left(|\sigma(\mathrm{u}_\alpha)|\,|\mathrm{w}|\right) \leq \|\mathrm{w}\|_{\mathbb{L}^2}\,\|g\|_{\mathbb{L}^2},$$

so that finally

$$\left|\mu\left(g\left[\mathrm{u}G\left(\sigma^2(\mathrm{u})\right) - \mathrm{v}G\left(\sigma^2(\mathrm{v})\right)\right]\right)\right| \leq C\,\|g\|_{\mathcal{D}}\,\|\mathrm{w}\|_{\mathcal{D}}$$

with a constant $C = \|\mathcal{J}\|_\infty + A + \max(1, A+B) + 4\widetilde{B}$. We conclude by noting that, in the case when $\mathrm{u}_\alpha = 0$ for some $\alpha \in (0,1)$ –so that $\sigma(\mathrm{u}_\alpha)$ is singular–, one has $\sigma^2(\mathrm{u}) = \sigma^2(\mathrm{v})$ and (4.5) with u instead of u_α provides the corresponding estimate. □

4.2. Mollified problem

Given $f \in \mathbb{L}_2(\mu)$ and fixed parameters $\lambda \in \mathbb{R}$ and $\varepsilon \in (0,\infty)$, we define a sequence $u_n : \mathbb{R}^+ \times \mathbb{M} \to \mathbb{R}$, $n \in \mathbb{Z}^+$, such that u_0 is a unique solution of

$$\left\{ \begin{array}{rcl} \partial_t u_0 &=& Lu_0 \\ u_{0|t=0} &=& f \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} \partial_t u_{n+1} &=& Lu_{n+1} + \lambda P_\varepsilon \mathcal{V}(u_n) \\ u_{n+1|t=0} &=& f, \end{array} \right. \tag{$\mathbb{A}_\varepsilon$}$$

in the sense that

$$u_{n+1}(t) \equiv P_t f + \lambda \int_0^t ds\ P_{\varepsilon+t-s} \mathbb{V}(u_n(s)). \tag{4.6}$$

We would like to argue that, in the case when $\varepsilon = 0$, for any $T \in (0, \infty)$, $u_n \in \mathbb{H}_{T,+}$, then u_{n+1} is a weak solution of $(\mathbb{A}_0)$ on $[0, T]$ and $u_{n+1} \in \mathbb{H}_{T,+}(\mathcal{E})$ provided (**FS**) with a constant 1 is satisfied, if we take $\lambda \in [0, 1)$ and $T > 0$ sufficiently small. And that such sequence of solutions converges strongly to a weak solution of our problem in a corresponding small time interval. Unfortunately, when $\varepsilon = 0$, equation (4.6) has only a formal meaning as in general $\|P_{t-s}\|_{\mathcal{D}' \to \mathcal{D}} \leq C/(t-s)$ and not better. This is the reason why we have to consider weak solutions instead of strong solutions in $\mathbb{L}^2(\mu)$ as our solution will be a limit in time dependent Banach spaces of the approximated solution we get when introducing an additional smoothing by taking $\varepsilon > 0$. In this case $P_\varepsilon \mathbb{V}$ is a Lipschitz continuous operator from $\mathcal{D}$ to itself. This ensures that, for any $f \in \mathbb{L}^2(\mu)$ and for any $T > 0$, (4.6) determines a (unique) $u_{n+1} \in C([0, T], \mathbb{L}^2(\mu)) \cap \mathbb{H}_{T,+}(\mathcal{E})$. Moreover, for any $t > 0$, $u_{n+1}(t)$ belongs to the domain $\mathcal{D}(L)$ of L and is differentiable in $\mathbb{L}^2(\mu)$ with respect to t. So that differential equation $(\mathbb{A}_\varepsilon)$ holds in a strong sense (in $\mathbb{L}^2(\mu)$).

We now state a key technical lemma which will be useful many times later. We introduce notation specific to this lemma in order to adapt the result to different situations without confusion.

Lemma 4.3 (A priori estimates for strong and weak solutions). *Suppose $\mu \in$ **FS**. Let $0 \leq \lambda < 1$ and $T > 0$. Let $\bar{u}_i(t, x)$, $\bar{v}_i(t, x)$, $i = 0, 1$, be four functions in $\mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu))$.*

1) *Assume that, for $i = 0, 1$ and any $t \in (0, T]$, $\bar{u}_i(\cdot)$ is differentiable in $\mathbb{L}^2(\mu)$ at time t, $\bar{u}_i(t)$ belongs to the domain of L and is solution in a strong sense (that is in $\mathbb{L}^2(\mu)$) of*

$$(\mathbb{C}_\varepsilon) \qquad \begin{cases} \frac{\partial}{\partial t}\bar{u}_i(t) &= L\bar{u}_i(t) + \lambda P_{\varepsilon_i} \mathbb{V}(\bar{v}_i(t)), \quad t \in (0, T] \\ \bar{u}_i(0) &= \bar{f}_i \end{cases}$$

with $\varepsilon_i > 0$ and initial value $\bar{f}_i \in \mathbb{L}^2(\mu)$. Then, with $\bar{w} \equiv \bar{u}_1 - \bar{u}_0$ and $\bar{z} \equiv \bar{v}_1 - \bar{v}_0$ and for any $t \in (0, T]$, one has

$$\begin{aligned} &\mu\bar{w}^2(t) + (2-\lambda)\int_0^t ds\mu|\nabla \bar{w}(s)|^2 \leq \\ &\leq e^{\lambda a t}\left\{\mu(\bar{f}_1 - \bar{f}_0)^2 + \lambda a \int_0^t \mu\left(\bar{z}^2(s)\right) ds + \lambda \int_0^t \mu\left(|\nabla \bar{z}(s)|^2\right) ds\right\} \\ &\quad + e^{\lambda a t}\left\{2\lambda\left(\int_0^t ds\mu\left[(P_{\varepsilon_1} - P_{\varepsilon_0})\bar{w}(s)\right]^2\right)^{\frac{1}{2}}\left(\int_0^t ds\mu\big(\mathbb{V}(\bar{v}_0(s))^2\big)\right)^{\frac{1}{2}}\right\}, \end{aligned} \tag{4.7}$$

with a constant

$$a = \big(\|\mathcal{J}\|_\infty + 2A + B + 4\widetilde{B}\big). \tag{4.8}$$

2) *Assume now that $\bar{u}_i$ are two weak solutions on $[0,+\infty[$ of* (MCP) *with initial values $\bar{f}_i$, $i=0,1$. Then with $\bar{w}=\bar{u}_1-\bar{u}_0$, one has*

$$\mu\left(\bar{w}^2(t)\right)+2(1-\lambda)\int_0^t ds\mu|\nabla\bar{w}(s)|^2\leq e^{2\lambda at}\mu\left((\bar{f}_1-\bar{f}_0)^2\right) \tag{4.9}$$

with the constant a as specified before.

To get a priori estimates for weak solutions, we will use the following time regularization procedure. For any Banach space X, and any $v\in\mathbb{L}^2([0,T],X)$, the Steklov average

$$a_h(v)(t)=\begin{cases}\dfrac{1}{h}\displaystyle\int_t^{t+h}v(\tau)\,d\tau, & 0\leq t\leq T-h,\\ 0, & T-h<t\leq T\end{cases} \tag{4.10}$$

converges to v in $\mathbb{L}^2([0,T],X)$ when $h\to 0$. Moreover, provided $v\in C([0,T],X)$, $a_h(v)\in C^1([0,T-h],X)$, $\frac{d}{dt}a_h(v)(t)=\frac{1}{h}(v(t+h)-v(t))$ in X, and $a_h(v)(t)$ converges to $v(t)$ in X still as $h\to 0$. The space X will be $\mathbb{L}^2(\mu)$, $\mathcal{D}$ or $\mathcal{D}'$ depending on the context.

Proof of Lemma 4.3.

1) *The case of strong solutions:* One may assume that $\varepsilon_1\geq\varepsilon_0$. We note first that

$$\frac{1}{2}\frac{d}{dt}\mu(\bar{w}^2)=\mu(\bar{w}\partial_t\bar{w})=-\mu|\nabla\bar{w}|^2+\lambda\mu\big(\bar{w}\,\{P_{\varepsilon_1}(\mathbb{V}(\bar{v}_1))-P_{\varepsilon_0}(\mathbb{V}(\bar{v}_0))\}\big) \tag{4.11}$$

Using definition of P_{ε_i} on $\mathcal{D}'$ by duality, one has

$$\begin{aligned}\mu\big(\bar{w}\,\{P_{\varepsilon_1}(\mathbb{V}(\bar{v}_1))-P_{\varepsilon_0}(\mathbb{V}(\bar{v}_0))\}\big)&\\ &=\mu\left(\bar{w}\,P_{\varepsilon_1}\{\mathbb{V}(\bar{v}_1)-\mathbb{V}(\bar{v}_0)\}\right)+\mu\left(\bar{w}\left(P_{\varepsilon_1}-P_{\varepsilon_0}\right)\mathbb{V}(\bar{v}_0)\right)\\ &=\mu\left(P_{\varepsilon_1}\bar{w}\,\{\mathbb{V}(\bar{v}_1)-\mathbb{V}(\bar{v}_0)\}\right)+\mu\left(\left(P_{\varepsilon_1}-P_{\varepsilon_0}\right)\bar{w}\,\mathbb{V}(\bar{v}_0)\right).\end{aligned}$$

We deal with the first term as in the proof of Lemma 4.2, with linear interpolation $\bar{v}_\alpha\equiv\alpha\bar{v}_1+(1-\alpha)\bar{v}_0$, $0\leq\alpha\leq 1$. So we may transpose here inequality (4.4). Write $\widetilde{w}\equiv P_{\varepsilon_1}\bar{w}$, recall that $G=F+\mathcal{J}$ and use additionally $|xy|\leq\frac{1}{2}(x^2+y^2)$ and (GREI) to get

$$\begin{aligned}&\big|\mu\big(P_{\varepsilon_1}\bar{w}\,\{\mathbb{V}(\bar{v}_1)-\mathbb{V}(\bar{v}_0)\}\big)\big|\\ &\qquad\leq(\|\mathcal{J}\|_\infty+2\widetilde{B})\mu(|\widetilde{w}\bar{z}|)\\ &\qquad\quad+\int_0^1 d\alpha\left(\int\frac{1}{2}(\widetilde{w}^2+\bar{z}^2)\,|F|\left(\sigma^2(\bar{v}_\alpha)\right)d\mu+2\widetilde{B}\,\mu\left(|\widetilde{w}\,\sigma(\bar{v}_\alpha)|\right)\,\mu\left(|\bar{z}\,\sigma(\bar{v}_\alpha)|\right)\right)\\ &\qquad\leq\frac{1}{2}\left(\|\mathcal{J}\|_\infty+2A+B+4\widetilde{B}\right)\left(\mu\left(\widetilde{w}^2\right)+\mu\left(\bar{z}^2\right)\right)+\frac{1}{2}\int\widetilde{w}^2F\left(\sigma^2(\widetilde{w})\right)d\mu\\ &\qquad\quad+\frac{1}{2}\int\bar{z}^2F\left(\sigma^2(\bar{z})\right)d\mu.\end{aligned}$$

Hence, using (**FS**), the fact that $\mu\left(\widetilde{w}^2\right) \le \mu\left(\bar{w}^2\right)$ and $\mu\left(|\nabla\widetilde{w}|^2\right) \le \mu\left(|\nabla\bar{w}|^2\right)$ and returning to (4.11), we arrive at a differential inequality which (after integration with respect to time and taking into the account the time zero condition $\bar{w}(0) = \bar{f}_1 - \bar{f}_0$) leads to

$$\begin{aligned}\mu\left(\bar{w}^2(t)\right) &+ (2-\lambda)\int_0^t \mu\left(|\nabla\bar{w}(s)|^2\right) ds \\ &\le \mu(\bar{f}_1 - \bar{f}_0)^2 + \lambda a \int_0^t \mu\left(\bar{z}^2(s)\right) ds + \lambda \int_0^t \mu\left(|\nabla\bar{z}(s)|^2\right) ds \\ &\quad + 2\lambda \int_0^t \mu\left((P_{\varepsilon_1} - P_{\varepsilon_0})\left(\bar{w}(s)\right)\mathbb{V}(\bar{v}_0(s))\right) ds + \lambda a \int_0^t \mu\left(\bar{w}^2(s)\right) ds\end{aligned}$$

with a constant a defined in (4.8) for every $t \in [0,T]$. Inserting trivial bound $\mu\left(\bar{w}^2(s)\right) \le \mu\left(\bar{w}^2(s)\right) + (2-\lambda)\int_0^s \mu\left(|\nabla\bar{w}(r)|^2\right) dr$ and then using Gronwall type arguments, we get for any $t \in [0,T]$,

$$\begin{aligned}\mu\bar{w}^2(t) &+ (2-\lambda)\int_0^t ds\mu|\nabla\bar{w}(s)|^2 \\ &\le e^{\lambda a t}\left\{\mu(\bar{f}_1 - \bar{f}_0)^2 + \lambda a \int_0^t \mu\left(\bar{z}^2(s)\right) ds + \lambda \int_0^t \mu\left(|\nabla\bar{z}(s)|^2\right) ds\right\} \\ &\quad + e^{\lambda a t}\left\{2\lambda \int_0^t ds\mu\left[(P_{\varepsilon_1} - P_{\varepsilon_0})\left(\bar{w}(s)\right)\cdot\mathbb{V}(\bar{v}_0(s))\right]\right\}\end{aligned}$$

Finally we use Cauchy–Schwarz inequality to get (4.7).

2) *The case of weak solutions:* in this case, we perform the computations with the Steklov average $a_h(\bar{w})(t)$ of $\bar{w}$ for any $h > 0$. Recall that $\bar{w} \in \mathcal{C}([0,T], \mathbb{L}^2(\mu)) \cap \mathbb{L}^2([0,T], \mathcal{D})$ for any $T > 0$. Hence, $a_h(\bar{w})(t)$ is differentiable with respect to t in $\mathbb{L}^2(\mu)$. Moreover, as h goes to 0, $a_h(\bar{w})(t) \to \bar{w}(t)$ in $\mathbb{L}^2(\mu)$ for any t, and $a_h(\bar{w})$ converges to $\bar{w}$ in $\mathbb{L}^2([0,T], \mathcal{D})$. Using the definition of a weak solution (with the constant test function $a_h(\bar{w})(s) \in \mathcal{D}$ on the interval $[s, s+h]$), we get

$$\begin{aligned}\frac{1}{2}\mu\left(a_h(\bar{w})(t)\right)^2 &= \frac{1}{2}\mu\left(a_h(\bar{w})(0)\right)^2 + \int_0^t ds\,\mu\left[a_h(\bar{w})(s)\frac{1}{h}\left(\bar{w}(s+h) - \bar{w}(s)\right)\right] \\ &= \frac{1}{2}\mu\left(a_h(\bar{w})(0)\right)^2 \\ &\quad + \int_0^t \frac{ds}{h}\int_s^{s+h} d\tau\left[-\mathcal{E}\left(a_h(\bar{w})(s), \bar{w}(\tau)\right) + \lambda\langle(\mathcal{V}(\bar{u}_1) - \mathcal{V}(\bar{u}_0))(\tau), a_h(\bar{w})(s)\rangle_{\mathcal{D}',\mathcal{D}}\right] \\ &= \frac{1}{2}\mu\left(a_h(\bar{w})(0)\right)^2 \\ &\quad + \int_0^t ds\left[-\mathcal{E}\Big[a_h(\bar{w})(s), a_h(\bar{w})(s)\Big] + \lambda\Big\langle a_h\big[\mathcal{V}(\bar{u}_1) - \mathcal{V}(\bar{u}_0)\big](s), a_h(\bar{w})(s)\Big\rangle_{\mathcal{D}',\mathcal{D}}\right].\end{aligned}$$

We can pass to the limit with $h \to 0$, which yields

$$\frac{1}{2}\mu\left[\bar{w}(t)\right]^2 = \frac{1}{2}\mu[\bar{f}_1 - \bar{f}_0]^2 - \int_0^t ds\,\mathcal{E}[\bar{w}(s), \bar{w}(s)] + \lambda\int_0^t ds\mu\left[\bar{w}(s)\left[\mathcal{V}(\bar{u}_1) - \mathcal{V}(\bar{u}_0)\right]\right].$$

And the remaining is similar to proof of point 1) with the nuance that $\bar{u}_i = \bar{v}_i$. After linear interpolation and an appropriate use of Gronwall lemma, one gets (4.9). □

Using that, we show the following uniform boundedness property:

Proposition 4.4 (Uniform bound in $C([0,T],\mathbb{L}^2) \cap \mathbb{H}_{T,+}(\mathcal{E})$). *Suppose $\mu \in \mathbf{FS}(1)$. Fix $\lambda \in [0,1)$, $\varepsilon > 0$ and $f \in \mathbb{L}^2(\mu)$. Let u_n be the recursive solution of the mollified problem $(\mathbb{A}_\varepsilon)$. Then for any $T \in (0,\infty)$ such that*

$$\eta_T \equiv \left\{ \frac{\lambda}{2-\lambda} + \lambda(||\mathcal{J}||_\infty + 2A)T \right\} e^{\lambda DT} < 1$$

where $D \equiv 2A+B+||\mathcal{J}||_\infty$, we have, for any $n \in \mathbb{N}$,

$$\sup_{0\le t\le T} \left(\mu u_n^2(t) + (2-\lambda)\int_0^t \mu|\nabla u_n|^2(s)ds \right) \le \frac{e^{\lambda DT}}{1-\eta_T}\, \mu(f^2). \tag{4.12}$$

Hence we have in particular

$$||u_n||^2_{C([0,T],\mathbb{L}^2)} + ||u_n||^2_{\mathbb{H}_{T,+}} \le \frac{(T+2)e^{\lambda DT}}{1-\eta_T}\, \mu(f^2),$$

with the right hand sides independent of $\varepsilon > 0$.

Proof. One may adapt the proof of Lemma 4.3 to the present situation to get, for any $t > 0$ and with $D \equiv 2A+B+||\mathcal{J}||_\infty$,

$$\mu\left(u_n^2(t)\right) + (2-\lambda)\int_0^t \mu\left(|\nabla u_n(s)|^2\right) ds$$
$$\le e^{\lambda Dt} \cdot \left\{ \mu(f^2) + \lambda(||\mathcal{J}||_\infty + 2A)\int_0^t \mu\left(u_{n-1}^2(s)\right) ds + \lambda\int_0^t \mu\left(|\nabla u_{n-1}(s)|^2\right) ds \right\},$$

for any $t \in \mathbb{R}^+$. Setting

$$\mathcal{Z}_n(t) \equiv \mu\left(u_n^2(t)\right) + (2-\lambda)\int_0^t \mu\left(|\nabla u_n(s)|^2\right) ds \tag{4.13}$$

we can see that the following inductive inequality is true

$$\mathcal{Z}_n(t) \le \mu(f^2)e^{\lambda Dt} + \frac{\lambda}{2-\lambda}e^{\lambda Dt}\mathcal{Z}_{n-1}(t) + \lambda\left(||\mathcal{J}||_\infty + 2A\right)e^{\lambda Dt}\int_0^t \mathcal{Z}_{n-1}(s)ds.$$

Using this for all $t \in [0,T]$, with $Z_n \equiv Z_n(T) \equiv \sup_{t\in[0,T]} \mathcal{Z}_n(t)$, we obtain

$$Z_n \le \mu(f^2)e^{\lambda DT} + \eta_T\, Z_{n-1} \tag{4.14}$$

with

$$\eta_T \equiv \left\{ \frac{\lambda}{2-\lambda} + \lambda(||\mathcal{J}||_\infty + 2A)T \right\} e^{\lambda DT}.$$

Assuming that $0 < \lambda < 1$, $\eta_T \in (0,1)$ for all $T \in (0,\infty)$ small enough. In this case (4.14) can be iterated to obtain the following bound uniform in $n \in \mathbb{N}$ as well as $\varepsilon > 0$

$$Z_n \leq \frac{e^{\lambda DT}}{1-\eta_T}\, \mu(f^2) \tag{4.15}$$

as $Z_0 \equiv Z_0(u_0) \leq \mu(f^2)$. The proof is complete. □

Proposition 4.5 (Convergence scheme in $C([0,T],\mathbb{L}^2) \cap \mathbb{H}_{T,+}(\mathcal{E})$). *Suppose that $\mu \in$ **FS** and let $\lambda \in [0,1)$. For $T \in (0,\infty)$, let*

$$\eta_*(T) \equiv \left[\lambda a T + \frac{\lambda}{2-\lambda}\right] e^{\lambda a T}, \tag{4.16}$$

where $a = (\|\mathcal{J}\|_\infty + 2A + B + 4\widetilde{B})$. Let $T_0 \in (0,\infty)$ be small enough so that $\eta_*(T_0) < 1$. Then, for any $0 < T \leq T_0$, the function $w_n \equiv u_{n+1} - u_n$, satisfies the following bound:

$$\sup_{t\in[0,T]} \left(\mu\left(w_n^2(t)\right) + (2-\lambda)\int_0^t \mu\left(|\nabla w_n(s)|^2\right) ds\right) \leq C(\eta_*(T))^n \mu f^2,$$

with a constant $C \in (0,\infty)$ independent of $\varepsilon > 0$ and $T \leq T_0$. As a consequence,

$$\|u_{n+1}-u_n\|^2_{C([0,T],\mathbb{L}^2)} + \|u_{n+1}-u_n\|^2_{\mathbb{H}_{T,+}} \leq C\,(T+2)\,\mu(f^2)\,(\eta_*(T))^{n-1}$$

uniformly in $\varepsilon > 0$.

Proof. Take $\varepsilon_0 = \varepsilon_1 = \varepsilon$, $\bar{u}_1 = u_{n+1}$, $\bar{u}_0 = u_n$, $\bar{v}_1 = u_n$ and $\bar{v}_0 = u_{n-1}$ and mainly $f_0 = f_1 = f$. Set $w_n \equiv u_{n+1} - u_n$. Applying Lemma 4.3 gives

$$\mu\left(w_n^2(t)\right) + (2-\lambda)\int_0^t \mu\left(|\nabla w_n(s)|^2\right) ds \\ \leq e^{\lambda a t}\left(\lambda a \int_0^t \mu\left(w_{n-1}^2(s)\right) ds + \lambda \int_0^t \mu\left(|\nabla w_{n-1}(s)|^2\right) ds\right).$$

Replacing u_n by w_n in the definition of $\mathcal{Z}_n(t)$ (given in (4.13)), one then carry on the same outline as in the proof of the Uniform bound Proposition 4.4. This leads to the following inductive bound:

$$\sup_{t\in[0,T]} \mathcal{Z}_n(t) \leq \eta_*(T) \sup_{t\in[0,T]} \mathcal{Z}_{n-1}(t)\,, \tag{4.17}$$

with $\eta_*(T)$ defined in (4.16). If $0 < \lambda < 1$, then there exists $T_0 > 0$ (independent of $\varepsilon > 0$ and of the initial condition f) such that $\eta_*(T_0) \in (0,1)$. In this situation, using the uniform bound of Proposition 4.4, we arrive at

$$\sup_{t\in[0,T]} \mathcal{Z}_n(t) \leq C\mu(f^2)\,(\eta_*(T))^n$$

with a constant $C = C(T_0) \in (0,\infty)$ independent of $\varepsilon > 0$. As a consequence we conclude that there exists $T \in (0,\infty)$, independent of $\varepsilon > 0$ and of the initial value $f \in \mathbb{L}^2(\mu)$, such that the sequence $(u_n(t))_{n\in\mathbb{N}}$, $t \in [0,T]$, converges in $\mathbb{H}_{T,+}(\mathcal{E}) \cap C\big([0,T],\mathbb{L}^2(\mu)\big)$ uniformly in $\varepsilon > 0$. □

Proposition 4.6 (Uniqueness for Mollified Problem). *Assume $\mu \in \mathbf{FS}(1)$ and let $\lambda \in [0,1)$, $\varepsilon > 0$ and $f \in \mathbb{L}^2(\mu)$. Then, for any $T > 0$, there exists at most one weak solution on $[0,T]$ of the mollified Cauchy problem*

$$(\mathbb{C}_\varepsilon) \qquad \begin{cases} \partial_t u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda P_\varepsilon \mathcal{V}(u^{(\varepsilon)}), \\ u^{(\varepsilon)}{}_{|t=0} &= f. \end{cases}$$

Proof. Assume there are two distinct weak solutions $u^{(\varepsilon)}$ and $v^{(\varepsilon)}$ on $[0,T]$ with the same initial value f. Let $w = u^{(\varepsilon)} - v^{(\varepsilon)}$. Noting that $w(0) = 0$ and using the a priori estimate (4.9) of Lemma 4.3, that we get also for $\varepsilon > 0$, one gets

$$\mu\left(\big(w(t)\big)^2\right) + 2(1-\lambda)\int_0^t \mu\left(|\nabla w(s)|^2\right) ds \le 0.$$

This contradicts our assumption that two distinct weak solutions exist. □

Theorem 4.7 (Solution of the Mollified Problem). *Suppose $\mu \in \mathbf{FS}$, and let $\lambda \in [0,1)$ and $\varepsilon > 0$. For $T \in (0,\infty)$, define $\eta_*(T)$ as in (4.16) and choose $T_0 \in (0,\infty)$ such that $\eta^*(T_0) < 1$. Then*

1) *The function $u^{(\varepsilon)} \equiv \lim_{n\to\infty} u_n^{(\varepsilon)}$, with the limit taken in the space $\mathbb{H}_{T_0,+}(\mu) \cap C\big([0,T_0],\mathbb{L}^2(\mu)\big)$, is a unique weak solution on $[0,T_0]$ of the Mollified Cauchy problem*

$$(\mathbb{C}_\varepsilon) \qquad \begin{cases} \partial_t u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda P_\varepsilon \mathcal{V}(u^{(\varepsilon)}), \\ u^{(\varepsilon)}{}_{|t=0} &= f. \end{cases}$$

2) *The later solution is indeed a unique global (i.e. on $[0,\infty)$) strong solution of problem $(\mathbb{C}_\varepsilon)$.*

3) *Moreover, for any $t \ge 0$, one has the following estimate:*

$$\mu\big(u^{(\varepsilon)}(t)\big)^2 + 2(1-\lambda)\int_0^t ds\, \mu|\nabla u^{(\varepsilon)}(s)|^2 \le e^{2\lambda a t}\,\mu\big(f^2\big), \tag{4.18}$$

with the right hand side independent on ε.

Remark 4.8. As follows from Proposition 4.4 (uniform bound), $||u^{(\varepsilon)}||_{\mathbb{H}_{T,+}}$ is uniformly bounded in $\varepsilon > 0$.

Proof of Theorem 4.7. By definition of $u^{(\varepsilon)}$ and by completness of the space $C\big([0,T],\mathbb{L}^2(\mu)\big)$, we have $u^{(\varepsilon)} \in C\big([0,T],\mathbb{L}^2(\mu)\big)$. Fix a test function $v \in C^\infty\big([0,T],\mathcal{D}\big) \subset \mathbb{H}_{T,+}(\mathcal{E})$. First, for any $t \in (0,T]$,

$$\int_0^t ds \mu\left(v(s)P_\varepsilon\mathcal{V}(u_n^{(\varepsilon)})(s)\right) \longrightarrow \int_0^t ds \mu\left(v(s)P_\varepsilon\mathcal{V}(u^{(\varepsilon)})(s)\right) \text{ as } n \text{ goes to } \infty. \tag{4.19}$$

Indeed, from Lemma 4.2, it follows that $P_\varepsilon \mathcal{V} : \mathbb{H}_{T,+} \to \mathbb{H}_{T,+}$ is Lipschitz continuous. In particular,

$$P_\varepsilon \mathcal{V}\big(u^{(\varepsilon)}\big) = \mathbb{H}_{T,+} - \lim_{n\to\infty} P_\varepsilon \mathcal{V}\big(u_n^{(\varepsilon)}\big)$$

(with short hand notation for limit in space $\mathbb{H}_{T,+}(\mathcal{E})$). And so

$$P_\varepsilon \mathcal{V}\big(u^{(\varepsilon)}\big) = \mathbb{H}_{T,-} - \lim_{n\to\infty} P_\varepsilon \mathcal{V}\big(u_n^{(\varepsilon)}\big)$$

when acting on $\mathbb{H}_{T,+}(\mathcal{E})$ with $\mathbb{L}^2(\mu)$-type pairing. Thus (4.19) follows.

Recall that by classical arguments $L : \mathbb{H}_{T,+}(\mathcal{E}) \to \mathbb{H}_{T,-}(\mathcal{E})$ acting by

$$\langle Lu, v\rangle_{\mathbb{H}_{T,-},\mathbb{H}_{T,+}} \equiv -\int_0^T \mu\left(\nabla u(s)\cdot \nabla v(s)\right) ds$$

is continuous so that

$$\int_0^t \mu\left(\nabla u_n^{(\varepsilon)}(s)\cdot \nabla v(s)\right) ds \longrightarrow \int_0^t \mu\left(\nabla u^{(\varepsilon)}(s)\cdot \nabla v(s)\right) ds.$$

Convergence of $u_n^{(\varepsilon)}$ to $u^{(\varepsilon)}$ in $\mathbb{L}^2\big([0,T],\mathbb{L}^2(\mu)\big)$ leads to

$$\int_0^t \mu\left(u_n^{(\varepsilon)}(s)\,\partial_s v(s)\right) ds \longrightarrow \int_0^t \mu\left(u^{(\varepsilon)}(s)\,\partial_s v(s)\right) ds$$

whereas the convergence in $C\big([0,T],\mathbb{L}^2(\mu)\big)$ ensures that $\mu\big(u_n^{(\varepsilon)}(t)\,v(t)\big)$ goes to $\mu\left(u^{(\varepsilon)}(t)\,v(t)\right)$ and $u^{(\varepsilon)}(0) = f$. This completes the proof that $u^{(\varepsilon)}$ is a weak solution of ($\mathbb{C}_\varepsilon$). Uniqueness of the solution was proved in Proposition 4.6.

The weak solution $u^{(\varepsilon)}$ is in fact global. This follows from the fact that the time $T_0 > 0$ in the foregoing does not depend on initial condition f. A posteriori, by Lemma 4.2, it follows that

$$u^{(\varepsilon)}(t) = P_t f + \lambda \int_0^t ds P_{t-s+\varepsilon}\mathcal{V}(u^{(\varepsilon)})(s)$$

and so $u^{(\varepsilon)}$ is a strong solution, as the left hand side belongs to the domain of L and is differentiable in $\mathbb{L}^2(\mu)$ with respect to time at any $t > 0$.

The last estimate is again a suitably adapted version of Lemma 4.3. □

Φ-bounds. In this section, we investigate regularity for mollified solutions in the Orlicz space $\mathbb{L}^{\Phi_q}(\mu)$ provided the initial value also belongs to this space. See Section 3.1 for definition of Φ_q.

Theorem 4.9. *Suppose $\mu \in$ **FS** and conditions* (**C0**) *to* (**C4**) *are satisfied. Let $q \in (0,\infty)$ be fixed. Suppose that $f \in \mathbb{L}^{\Phi_q}(\mu)$ and $\lambda \in (0,(1+q\bar{B})^{-2})$. Fix $\varepsilon > 0$. Then the weak solution $u^{(\varepsilon)}(t)$ of the mollified Cauchy problem*

$$\begin{cases} \frac{\partial u^{(\varepsilon)}(t)}{\partial t} & = Lu^{(\varepsilon)}(t) + \lambda P_\varepsilon \left[u^{(\varepsilon)}(t) G(\sigma^2(u^{(\varepsilon)}(t)))\right], \\ u^{(\varepsilon)}(0) & = f, \end{cases}$$

satisfies the following bound:

$$\mu\big(\Phi_q(u^{(\varepsilon)}(t))\big) + 2\,C(q,\lambda)\int_0^t ds\mu|\nabla\sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 \le e^{\tilde{a}\lambda t}\mu\left(\Phi_q(f)\right)$$

with some constants

$$C(q,\lambda) = \big((1+q\bar{B})^{-1} - \lambda(1+q\bar{B})\big) > 0 \quad \textit{and} \quad \tilde{a} = 2(1+q\bar{B})\,(2A+B+\|\mathcal{J}\|_\infty).$$

Proof of Theorem 4.9. Recall that $u^{(\varepsilon)}(t)$ is differentiable in $\mathbb{L}^2(\mu)$ for any $t>0$ and is a strong solution of the considered Cauchy problem. We first justify our computations in the case when we replace F by a bounded function in the definition of Φ_q and then get the claimed result by approximation.

The case of bounded F: Let us consider $\widetilde{F}:[0,\infty)\to\mathbb{R}$ a bounded function satisfying conditions (**C1**) (with constant $\bar{B}$) and define $\widetilde{\Phi}_q(x) = \widetilde{\Upsilon}_q(x^2) = x^2 e^{q\widetilde{F}(x^2)}$, whereas, in the definition of $u^{(\varepsilon)}$, G is still a perturbation of F. Then, $\mu\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(t))\big)$ is finite for any t, is differentiable and we have

$$\frac{d}{dt}\mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(t))\right) = \mu\big(\widetilde{\Phi}'_q(u^{(\varepsilon)}(t))\frac{\partial}{\partial t}u^{(\varepsilon)}(t)\big).$$

Indeed, first, $\widetilde{\Phi}_q(x) \le e^{q\|\widetilde{F}\|_\infty}x^2$ so that $\widetilde{\Phi}_q(u^{(\varepsilon)}(t))$ is integrable w.r.t. μ. Moreover, recall from Remark 3.1 that, as $\widetilde{F}$ is bounded, $\widetilde{\Upsilon}'_q(x)$ and $\widetilde{\Phi}''_q(x)$ are bounded. One has

$$\Delta_{s,t} \equiv \frac{\widetilde{\Phi}_q(u^{(\varepsilon)}(s)) - \widetilde{\Phi}_q(u^{(\varepsilon)}(t))}{s-t} = \frac{u^{(\varepsilon)}(s)-u^{(\varepsilon)}(t)}{s-t}\int_0^1 \widetilde{\Phi}'_q(u^{s,t}_\alpha)\,d\alpha \tag{4.20}$$

with $u^{s,t}_\alpha \equiv \alpha u^{(\varepsilon)}(s) + (1-\alpha)\,u^{(\varepsilon)}(t)$. As s goes to t, on the one hand, $\frac{u^{(\varepsilon)}(s)-u^{(\varepsilon)}(t)}{s-t}$ converges in $\mathbb{L}^2(\mu)$ to $\frac{\partial}{\partial t}u^{(\varepsilon)}(t)$ and, on the other hand,

$$\begin{aligned}\left|\int_0^1 \widetilde{\Phi}'_q(u^{s,t}_\alpha)\,d\alpha - \widetilde{\Phi}'_q(u^{(\varepsilon)}(t))\right| &\le \int_0^1 \left|\widetilde{\Phi}'_q(u^{s,t}_\alpha) - \widetilde{\Phi}'_q(u^{(\varepsilon)}(t))\right|\,d\alpha \\ &\le \frac{\kappa}{2}\left|u^{(\varepsilon)}(s)-u^{(\varepsilon)}(t)\right|\end{aligned}$$

goes to 0 in $\mathbb{L}^2(\mu)$. Here $\kappa \equiv \sup_x \widetilde{\Phi}''_q(x)$. So that equation (4.20) proves that

$$\lim_{s\to t}\Delta_{s,t} = \frac{\partial}{\partial t}u^{(\varepsilon)}(t)\,\widetilde{\Phi}'_q(u^{(\varepsilon)}(t))$$

in $\mathbb{L}^1(\mu)$. Hence, one has

$$\begin{aligned}&\frac{d}{dt}\mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(t))\right) \\ &\qquad = \mu\left(\widetilde{\Phi}'_q(u^{(\varepsilon)}(t))\left\{Lu^{(\varepsilon)}(t) + \lambda P_\varepsilon\left(u^{(\varepsilon)}(t)G(\sigma^2(u^{(\varepsilon)}(t)))\right)\right\}\right).\end{aligned} \tag{4.21}$$

This formula shows that this time derivative belongs to $\mathbb{L}^1([0,T])$. This follows from $u^{(\varepsilon)} \in \mathbb{L}^2([0,T],\mathcal{D})$ and the following bounds. One has $\widetilde{\Phi}'_q(u) \in \mathcal{D}$ and

$$\mathcal{E}(\widetilde{\Phi}'_q(u^{(\varepsilon)}(t)), u^{(\varepsilon)}(t)) \le \kappa\, \mathcal{E}(u^{(\varepsilon)}(t), u^{(\varepsilon)}(t))$$

with $\kappa \equiv \sup_x \widetilde{\Phi}''_q(x)$, whereas

$$\begin{aligned}\left|\mu\left(\widetilde{\Phi}'_q(u^{(\varepsilon)}(t)) P_\varepsilon\left(u^{(\varepsilon)}(t) G(\sigma^2(u^{(\varepsilon)}(t)))\right)\right)\right| &\le \|\widetilde{\Phi}'_q(u^{(\varepsilon)}(t))\|_2 \|P_\varepsilon \mathbb{V}(u^{(\varepsilon)}(t))\|_2\\ &\le \frac{\kappa c}{\sqrt{\varepsilon}} \|\mathbb{V}(u^{(\varepsilon)}(t))\|_{\mathcal{D}'} \|u^{(\varepsilon)}(t))\|_2\\ &\le \frac{\kappa c'}{\sqrt{\varepsilon}} \|u^{(\varepsilon)}(t)\|^2_{\mathcal{D}}\end{aligned}$$

thanks to Lemma 4.2. Integrating (4.21) with respect to time between $\delta > 0$ and t and letting δ go to 0 – use $|\widetilde{\Phi}_q(u^{(\varepsilon)}(\delta)) - \widetilde{\Phi}_q(f)| \le \big(\sup_x |\widetilde{\Upsilon}'_q(x)|\big)\, |(u^{(\varepsilon)}(\delta))^2 - f^2|$, and convergence in $\mathbb{L}^2$ of $u^{(\varepsilon)}(\delta)$ to f – after simple rearrangements one arrives at the following inequality:

$$\begin{aligned}\mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(t))\right) \le \mu\left(\widetilde{\Phi}_q(f)\right) &+ \int_0^t ds\, \mu\left(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))\, L u^{(\varepsilon)}(s)\right)\\ &+ \lambda \int_0^t ds\, \mu\left(\left|P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) u^{(\varepsilon)}(s)\right|\, |F|(\sigma^2(u^{(\varepsilon)}(s))))\right)\\ &+ \lambda \|\mathcal{J}\|_\infty \int_0^t ds\, \mu\left(\left|P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) u^{(\varepsilon)}(s)\right|\right).\end{aligned}$$

First, from the Dirichlet/Young bound (3.7),

$$\mu\left(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))\, L u^{(\varepsilon)}(s)\right) \le -k_q\, \mu\Big|\nabla \sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\,\Big|^2.$$

Next, we note that by Young inequality and Jensen inequality for the semigroup, we have

$$\begin{aligned}|P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) \cdot u^{(\varepsilon)}(s)| &\le \widetilde{\Phi}^*_q(P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s)))) + \widetilde{\Phi}_q(u^{(\varepsilon)}(s))\\ &\le P_\varepsilon \widetilde{\Phi}^*_q(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) + \widetilde{\Phi}_q(u^{(\varepsilon)}(s))\end{aligned}$$

with $\widetilde{\Phi}^*_q(y) = \sup_{x\in\mathbb{R}} xy - \widetilde{\Phi}_q(x)$, the conjugate of $\widetilde{\Phi}_q$. Since $\widetilde{\Phi}^*_q(\widetilde{\Phi}'_q(x)) = x\widetilde{\Phi}'_q(x) - \widetilde{\Phi}_q(x)$, thanks to (3.6), we have $\widetilde{\Phi}^*_q(\widetilde{\Phi}'_q(x)) \le (1 + 2q\bar{B})\widetilde{\Phi}_q(x)$. Hence,

$$|P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) \cdot u^{(\varepsilon)}(s)| \le (1 + 2q\bar{B}) P_\varepsilon \widetilde{\Phi}_q(u^{(\varepsilon)}(s)) + \widetilde{\Phi}_q(u^{(\varepsilon)}(s)).$$

Using this, $|F| = F + 2F_- \le F + 2A$ with A defined in (2.5), then (GREI) twice and at last invariance property for P_ε w.r.t. μ, we have on the one hand,

$$\mu\left(\left|P_\varepsilon(\widetilde{\Phi}'_q(u^{(\varepsilon)}(s))) u^{(\varepsilon)}(s)\right|\right) \le 2\left(1 + q\bar{B}\right) \mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(s)\right)$$

and

$$
\begin{aligned}
\mu\Big(\big| \, P_\varepsilon\big(\widetilde{\Phi}_q'(u^{(\varepsilon)}(s))\big)u^{(\varepsilon)}(s)\big||F|(\sigma^2(u^{(\varepsilon)}(s)))\Big) \\
&\le \mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))|F|(\sigma^2(u^{(\varepsilon)}(s)))\right) \\
&\quad + (1+2q\bar{B})\ \mu\left(P_\varepsilon\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big)|F|(\sigma^2(u^{(\varepsilon)}(s)))\right) \\
&\le \mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))F(\sigma^2(u^{(\varepsilon)}(s)))\right) + 4A\left(1+q\bar{B}\right)\mu\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(s)\right) \\
&\quad + (1+2q\bar{B})\mu\left(P_\varepsilon\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big)F(\sigma^2(u^{(\varepsilon)}(s)))\right) \\
&\le \mu\Big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))F\Big(\sigma^2\Big(\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\Big)\Big)\Big) + 2(2A+B)\left(1+q\bar{B}\right)\mu\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big) \\
&\quad + (1+2q\bar{B})\ \mu\Big(P_\varepsilon\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big)F\Big(\sigma^2\Big(\sqrt{P_\varepsilon\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big)}\ \Big)\Big)\Big)
\end{aligned}
$$

on the other hand. Since $xF(x)$ is convex by condition (**C2**), use Jensen inequality for the measure P_ε and then invariance property to get

$$
\begin{aligned}
&\mu\Big(P_\varepsilon\left(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\right)F\Big(\sigma^2\Big(\sqrt{P_\varepsilon\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big)}\ \Big)\Big)\Big) \\
&\qquad\qquad \le \mu\Big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))F\Big(\sigma^2\Big(\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\Big)\Big)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mu\Big(\,\widetilde{\Phi}_q'(u^{(\varepsilon)}(s))\,Lu^{(\varepsilon)}(s)\Big) &+ \lambda\,\|\mathcal{J}\|_\infty\ \mu\Big(\big|\widetilde{\Phi}_q'(u^{(\varepsilon)}(s))P_\varepsilon u^{(\varepsilon)}(s)\big|\Big) \\
&+ \lambda\mu\Big(\big|\widetilde{\Phi}_q'(u^{(\varepsilon)}(s))P_\varepsilon u^{(\varepsilon)}(s)\big||F|(\sigma^2(u^{(\varepsilon)}(s)))\Big) \\
&\le -k_q\,\mu\big|\nabla\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\big|^2 + 2\lambda(1+q\bar{B})\,\mu\Big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))F\Big(\sigma^2\Big(\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\ \Big)\Big)\Big) \\
&+ 2\lambda(1+q\bar{B})\ (2A{+}B+\|\mathcal{J}\|_\infty)\ \mu\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big).
\end{aligned}
$$

With the use of (FS) inequality, the last can be bounded by

$$
-2\left(\frac{k_q}{2}-\tilde{\lambda}\right)\mu\big|\nabla\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\big|^2 + 2\tilde{\lambda}\ (2A{+}B+\|\mathcal{J}\|_\infty)\ \mu\big(\widetilde{\Phi}_q(u^{(\varepsilon)}(s))\big),
$$

where $\tilde{\lambda} \equiv \lambda(1{+}q\bar{B})$. Combining all the above we arrive at the following inequality:

$$
\begin{aligned}
\mathcal{Z}_\varepsilon(t) \equiv \mu\widetilde{\Phi}_q(u^{(\varepsilon)}(t)) + 2\left(\frac{k_q}{2}-\tilde{\lambda}\right)&\int_0^t ds\,\mu\big|\nabla\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\big|^2 \\
&\le \mu\,\widetilde{\Phi}_q(f) + 2\tilde{\lambda}\ (2A{+}B+\|\mathcal{J}\|_\infty)\int_0^t ds\,\mu\,\widetilde{\Phi}_q(u^{(\varepsilon)}(s)) \\
&\le \mu\,\widetilde{\Phi}_q(f) + 2\tilde{\lambda}\ (2A{+}B+\|\mathcal{J}\|_\infty)\int_0^t \mathcal{Z}_\varepsilon(s)ds,
\end{aligned}
$$

provided $\tilde{\lambda} < \frac{k_q}{2}$. As here $k_q = 2/\left(1+q\,\bar{B}\right)$ we get the announced constraint on λ.

Now Gronwall type inequality leads to the following bound:

$$\mu\widetilde{\Phi}_q(u^{(\varepsilon)}(t)) + 2\left(\frac{k_q}{2} - \tilde{\lambda}\right)\int_0^t ds\, \mu\big|\nabla\sqrt{\widetilde{\Phi}_q(u^{(\varepsilon)}(s))}\big|^2 \leq e^{2\tilde{\lambda}(2A+B+\|\mathcal{J}\|_\infty)t}\mu\,\widetilde{\Phi}_q(f).$$

From bounded to unbounded F. Assume F satisfies (**C2**). For any $b \gg 1$, let F_b be a non decreasing C^2 *bounded* truncated function of F such that

1) $F_b = F$ on $[0,b]$ and F_b is concave on $[\theta,\infty)$,
2) $F_b' \leq F'$,
3) $F_b(x)$ satisfies (**C2**).

One may construct such a function in the following way. Noting that $F(x)$ satisfies (**C2**) if and only if $F'(x) = g(x)/x^2$ for a non decreasing function g, we define $F_b(x) = \int_1^x g_b(s)\frac{ds}{s^2}$ where $g_b(x) = g(b) + \int_b^x g'(s)\,\psi(\frac{s}{b}-1)ds$ with $\psi(x) = 1 - \int_0^x \phi(s)ds$ and $\phi \in \mathcal{C}_c^\infty((0,1))$ such that $\int_{\mathbb{R}}\phi(s)\,ds = 1$.

The first part of the proof applied to $\widetilde{F} = F_b$ and $\widetilde{\Phi}_{b,q}(x) \equiv x^2 e^{qF_b(x^2)}$ ensures that

$$\begin{aligned}\mu\,\widetilde{\Phi}_{b,q}(u^{(\varepsilon)}(t)) &+ 2\left(\frac{k_q}{2} - \tilde{\lambda}\right)\int_0^t ds\,\mu\big|\nabla\sqrt{\widetilde{\Phi}_{b,q}(u^{(\varepsilon)}(s))}\big|^2\\ &\leq e^{2\tilde{\lambda}(2A+B+\|\mathcal{J}\|_\infty)t}\mu\,\widetilde{\Phi}_{b,q}(f) \leq e^{2\tilde{\lambda}(2A+B+\|\mathcal{J}\|_\infty)t}\mu\,\Phi_q(f).\end{aligned} \tag{4.22}$$

Recall that, for any $v \in \mathbb{L}^2(\mu)$, $\mathcal{E}^{(\tau)}(v,v) \equiv \frac{1}{\tau}\mu\big((v - P_\tau v)v\big)$ is non decreasing as $\tau \downarrow 0$, $\tau > 0$, and

$$\begin{cases}\mathcal{D} = \left\{v \in \mathbb{L}^2(\mu) : \lim_{\substack{\tau\to 0\\ \tau>0}}\mathcal{E}^{(\tau)}(v,v) < \infty\right\},\\ \mathcal{E}(v,v) = \lim_{\tau\downarrow 0}\mathcal{E}^{(\tau)}(v,v), \quad v \in \mathcal{D}.\end{cases}$$

So that making use of monotone convergence theorem, Lebesgue dominated convergence theorem and Fatou lemma leads to the result when b goes to ∞ in (4.22). □

Removing the smoothing.

Theorem 4.10 (Convergence in $\mathbb{H}_{T,+}\cap C([0,T],\mathbb{L}^2(\mu))$ when $\varepsilon \to 0$). *Let F and G satisfy conditions* (**C1**) *to* (**C4**). *Assume $(\mu,\mathcal{E}) \in$* (**FS**) *and* (**C0**) *is satisfied. For a fixed $\lambda \in [0,1)$, let $u^{(\varepsilon)}(t)$ denote the solution on $[0,\infty)$ of the approximated Cauchy problem*

$$\begin{cases}\frac{\partial}{\partial t}u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda P_\varepsilon\mathcal{V}(u^{(\varepsilon)})\\ u^{(\varepsilon)}(0) &= f\end{cases} \tag{$\mathbb{C}_\varepsilon$}$$

with $\varepsilon > 0$. Assume that initial value $f \in L^\infty(\mu)$.

Then, for any $T \in (0,+\infty)$, when $\varepsilon \to 0$, the solutions $u^{(\varepsilon)}$ converge in the Banach space $\mathbb{H}_{T,+}(\mu) \cap C([0,T],\mathbb{L}^2(\mu))$.

Proof. For $\varepsilon > \varepsilon' > 0$ define $w \equiv w_{\varepsilon,\varepsilon'} \equiv u^{(\varepsilon)} - u^{(\varepsilon')}$. A suitable use of Lemma 4.3 leads to

$$\mu\, w^2(t) + 2(1-\lambda)\int_0^t ds\, \mu|\nabla w(s)|^2 \le 2\lambda\, e^{2\lambda a t}\Big(\int_0^t ds\, \mu\, [(P_\varepsilon - P_{\varepsilon'})\, w(s)]^2\Big)^{1/2}\Big(\int_0^t ds\, \mu \mathbb{V}^2(u^{(\varepsilon')}(s))\Big)^{1/2}, \tag{4.23}$$

where a is defined in (4.8). First we note that, since $w(s)$ belongs to the domain of L,

$$\int_0^t ds\mu\, [(P_\varepsilon - P_{\varepsilon'})\, w(s)]^2 = \int_0^t ds\mu \left[(-L)^{\frac14} w(s)\, (P_\varepsilon - P_{\varepsilon'})^2\, (-L)^{-\frac14} w(s)\right]$$
$$\le \Big(\int_0^t ds\mu\big((-L)^{\frac14} w(s)\big)^2\Big)^{1/2}\cdot\Big(\int_0^t ds\mu\big((P_\varepsilon - P_{\varepsilon'})^2(-L)^{-\frac14} w(s)\big)^2\Big)^{1/2}.$$

Next we observe that (using the symmetry of L, spectral theory and $\sqrt{\xi} \le 1+\xi$)

$$\Big(\int_0^t ds\mu\big((-L)^{\frac14} w(s)\big)^2\Big)^{1/2} \le \Big(\int_0^t ds\, \big[\mu(w^2(s)) + \mu|\nabla w(s)|^2\big]\Big)^{1/2} \le \|w\|_{\mathbb{H}_{T,+}}$$

But $\|w\|_{\mathbb{H}_{T,+}} \le 2C(T)\,(\mu f^2)^{\frac12}$ with some constant $C(T) \in (0,\infty)$ independent on $\varepsilon, \varepsilon'$. From (4.18), one can choose

$$C(T) = \Big(T + \frac{1}{2(1-\lambda)}\Big)^{1/2} e^{\lambda a T}.$$

Once again by spectral theory, denoting by $\nu_{w(s)}$ the spectral measure associated to $w(s)$ (and $-L$), we have

$$\mu\left((P_\varepsilon - P_{\varepsilon'})^2\, (-L)^{-\frac14} w(s)\right)^2 = \int_0^\infty e^{-4\varepsilon'\eta}(e^{-(\varepsilon-\varepsilon')\eta} - 1)^4 \eta^{-\frac12}\nu_{w(s)}(d\eta)$$

which we bound by

$$\sup_{\eta>0}\big(e^{-4\varepsilon'\eta}(e^{-(\varepsilon-\varepsilon')\eta} - 1)^4\cdot\eta^{-1}\big)\int_0^\infty \eta^{\frac12}\nu_{w(s)}(d\eta) \le (\varepsilon-\varepsilon')\int_0^\infty \eta^{\frac12}\nu_{w(s)}(d\eta)$$
$$\le (\varepsilon-\varepsilon')\,\big(\mu w^2(s) + \mu|\nabla w(s)|^2\big).$$

To bound the supremum we notice that in the case when $(\varepsilon-\varepsilon')\eta \le 1$, we have $|e^{-(\varepsilon-\varepsilon')\eta} - 1|^4\cdot\eta^{-1} \le |(\varepsilon-\varepsilon')\eta|^4/\eta \le (\varepsilon-\varepsilon')$, while for $(\varepsilon-\varepsilon')\eta \ge 1$, we have $(e^{-4\varepsilon'\eta}(e^{-(\varepsilon-\varepsilon')\eta} - 1)^4\cdot\eta^{-1}) \le \eta^{-1} \le (\varepsilon-\varepsilon')$. Hence we obtain the following bound:

$$\Big(\int_0^t ds\mu\big((P_\varepsilon - P_{\varepsilon'})^2\,(-L)^{-\frac14} w(s)\big)^2\Big)^{1/2} \le 2\,C(T)\,(\mu f^2)^{\frac12}\,(\varepsilon-\varepsilon')^{\frac12}.$$

Combining the above estimates we arrive at the following bound:

$$\Big(\int_0^t ds\mu\left[(P_\varepsilon - P_{\varepsilon'})\,w(s)\right]^2\Big)^{1/2} \leq 2\,C(T)\,(\mu f^2)^{1/2}\,(\varepsilon-\varepsilon')^{1/4}. \tag{4.24}$$

Hence, coming back to (4.23), we have proved that $(u^{(\epsilon)})_{\varepsilon>0}$ is Cauchy in the space $\mathbb{H}_{T,+}(\mu)\cap C([0,T],\mathbb{L}^2(\mu))$ as ε goes to 0 provided we can bound

$$\Big(\int_0^T ds\,\mu\big(\mathbb{V}^2(u^{(\varepsilon')}(s))\big)\Big)^{1/2}$$

uniformly in ε'. This is the aim of Lemma 4.11 below, or more precisely of its Corollary 4.12. The proof is complete. □

Lemma 4.11. *For F satisfying conditions* (**C1**) *to* (**C3**) *and $F(+\infty)=+\infty$, let $\Upsilon_q(x)=|x|e^{qF(|x|)}$, $q>0$, and $\Upsilon_q^*(y)=\sup_{x\in\mathbb{R}}\left[|xy|-\Upsilon_q(x)\right]$. Then, there exists $C_q\in(0,\infty)$ such that*

$$\Upsilon_q^*\left(F^2(z)\right)\leq C_q(1+z), \quad \forall z\geq 0. \tag{4.25}$$

Corollary 4.12. *Let $u^{(\varepsilon)}$ be as in Theorem 4.10, for an initial condition $f\in\mathbb{L}^\infty(\mu)$ and a coupling constant $\lambda\in[0,1)$. Let $q>0$ be small enough such that $\lambda<(1+q\bar{B})^{-2}$ and let $T>0$ be fixed. Then, for any $\varepsilon>0$ and any $t\in[0,T]$,*

$$\Big(\int_0^t ds\mu\mathbb{V}^2(u^{(\varepsilon)}(s))\Big)^{1/2} \leq A_{q,T}\,\left(\mu(\Phi_q(f))+\mu(f^2)+1\right)^{1/2}$$

with some constant $A_{q,T}\in(1,\infty)$ which is independent of ε.

Proof of Lemma 4.11. We start with a bound on $\Upsilon_q^*(y)$ (which is finite for any y as $F(+\infty)=+\infty$). Let $y\geq 0$ such that $\Upsilon_q^*(y)>0$ (so that, in particular, $y>e^{qF(0)}$). Note that $xy-\Upsilon_q(x)=x\left(y-e^{qF(x)}\right)\leq 0$ for any $x\geq 0$ such that $F(x)\geq\frac{1}{q}\log y$, or equivalently $x\geq F^-(\frac{1}{q}\log y)$, where $F^-(u)\equiv\inf\{a\geq 0\colon u\leq F(a)\}$ for $u\in[F(0),+\infty)$ is the generalized inverse of F. Hence,

$$\Upsilon_q^*(y)=\sup_{0\leq x\leq F^-(\frac{1}{q}\log y)}\left[xy-\Upsilon_q(x)\right]\leq y\,F^-\Big(\frac{1}{q}\log y\Big). \tag{4.26}$$

We now turn to the bound (4.25). We only have to deal with the large values of z. From (4.26), $\Upsilon_q^*(F^2(z))\leq F^2(z)\,F^-(\frac{2}{q}\log F(z))$. On the one hand, $F^2(z)\leq\bar{B}^2\log^2 z\leq C(1+z)^{1/2}$ from condition (**C1**). On the other hand, for any fixed q, let A such that, for any $a\geq A$, $\frac{2}{q}\log(2a+R)\leq a$ with R as defined by condition (**C3**) and choose z large enough such that $F((1+z)^{1/2})\geq A$. Then, making use of (**C3**),

$$\frac{2}{q}\log F(z)\leq\frac{2}{q}\log\Big(2F((1+z)^{1/2})+R\Big)\leq F((1+z)^{1/2}).$$

Which is equivalent to $F^-(\frac{2}{q}\log F(z))\leq(1+z)^{1/2}$. Hence, (4.25) holds with $C_q=C$ for large values of z. The proof is complete. □

Proof of Corollary 4.12. We will use the Φ_q-bounds of Theorem 4.9 available here since $f \in \mathbb{L}^\infty(\mu) \subset \mathbb{L}^{\Phi_q}(\mu)$. So choose $q > 0$ small enough so that $\lambda < (1+q\bar{B})^{-2}$. Recall that $\Phi_q(x) = \Upsilon_q(x^2)$ where $\Upsilon_q(x) = |x|e^{qF(|x|)}$ is a Young function.

As $G = F + \mathcal{J}$ and by Young's inequality we have

$$\begin{aligned}(4.27)\quad \mu\left(\mathbb{V}^2(u^{(\varepsilon)}(s))\right) &= \mu\left[(u^{(\varepsilon)}(s))^2\, G^2(\sigma^2(u^{(\varepsilon)}(s)))\right] \\ &\le 2\,\|\mathcal{J}\|_\infty^2\, \mu((u^{(\varepsilon)}(s))^2) + 2\,\mu\Upsilon_q((u^{(\varepsilon)}(s))^2) + 2\,\mu\Upsilon_q^*(F^2(\sigma^2(u^{(\varepsilon)}(s)))).\end{aligned}$$

Using the uniform bound (4.18) and with C_q as in Lemma 4.11, one gets, for $s \in [0,T]$,

$$\begin{aligned}\mu\left(\mathbb{V}^2(u^{(\varepsilon)}(s))\right) &\le K_T\,\mu(f^2) + 2\mu\Phi_q(u^{(\varepsilon)}(s)) + 2C_q\mu\left(\sigma^2(u^{(\varepsilon)}(s)) + 1\right) \\ &\le 2\mu\Phi_q(u^{(\varepsilon)}(s)) + (4C_q + K_T\mu(f^2))\end{aligned}$$

with $K_T \equiv 2\,\|\mathcal{J}\|_\infty^2 e^{2\lambda a T}$. Hence, using the Φ-bound, we arrive at

$$\begin{aligned}\left(\int_0^t ds\mu\mathbb{V}^2(u^{(\varepsilon)}(s))\right)^{1/2} &\le \left(\int_0^t ds\left[2e^{\tilde{a}\lambda s}\mu\Phi_q(f) + (4C_q + K_T\mu(f^2))\right]\right)^{1/2} \\ &\le A_{q,T}\ \left(\mu(\Phi_q(f)) + \mu(f^2) + 1\right)^{1/2},\end{aligned}$$

with some constant $A_{q,T} \in (1,\infty)$ which is independent of ε. □

Global existence and uniqueness for (MCP). In this section, we complete the proof of our main theorem.

Proof of Theorem 2.2. Recall that we reduced the problem to the case $c_F = 1$. First, we mimic arguments given in the proof of Proposition 4.6 to get uniqueness for weak solutions on any interval $[0,T]$.

Let us turn to the proof of the existence on $[0,T]$. Choose $q > 0$ small enough so that $\lambda < (1+q\bar{B})^{-2}$. Then, provided the initial value $f \in \mathbb{L}^\infty(\mu)$, we can use Theorem 4.10 to exhibit a function $u \in \mathbb{H}_{T,+}(\mu) \cap C([0,T],\mathbb{L}^2(\mu))$ such that

$$\|u^{(\varepsilon)} - u\|_{\mathbb{H}_{T,+}(\mu)} + \sup_{t\in[0,T]} \|u^{(\varepsilon)}(t) - u(t)\|_{\mathbb{L}^2(\mu)} \longrightarrow 0$$

when ε goes to 0. Thus, by Lemma 4.2, one has

$$\|\mathcal{V}(u^{(\varepsilon)}) - \mathcal{V}(u)\|_{\mathbb{H}_{T,-}(\mu)} \longrightarrow 0.$$

Hence, for any $v \in \mathbb{H}_{T,+}(\mu)$ and $t \in [0,T]$,

$$\begin{aligned}&\int_0^t ds\left\langle P_\varepsilon\mathcal{V}(u^{(\varepsilon)})(s), v(s)\right\rangle_{\mathcal{D}',\mathcal{D}} \\ &\qquad = \int_0^t ds\left\langle \mathcal{V}(u^{(\varepsilon)})(s), P_\varepsilon v(s)\right\rangle_{\mathcal{D}',\mathcal{D}} \longrightarrow \int_0^t ds\,\langle\mathcal{V}(u)(s), v(s)\rangle_{\mathcal{D}',\mathcal{D}}\end{aligned}$$

as additionally $P_\varepsilon v$ goes to v in $\mathbb{H}_{T,+}(\mu)$. This together with other arguments developed in the proof of Theorem 4.7 shows that u is a weak solution on $[0,T]$ of problem (MCP). Hence, we are done provided the initial value $f \in \mathbb{L}^\infty(\mu)$.

Now, by a priori estimate (4.9) of Lemma 4.3, weak solutions of ($\mathbb{MCP}$) on $[0,T]$ are Lipschitz continuous w.r.t. initial value $f \in \mathbb{L}^2(\mu)$ (with values in $\mathbb{H}_{T,+}(\mu) \cap C([0,T], \mathbb{L}^2(\mu))$). Hence, if $f \in \mathbb{L}^2(\mu)$, $(f_n)_n \to f$ in $\mathbb{L}^2(\mu)$ with $(f_n)_{n\in\mathbb{N}} \subset \mathbb{L}^\infty(\mu)$ and u_n are the corresponding solutions of ($\mathbb{MCP}$) with initial value f_n, then $(u_n)_n$ is Cauchy in $\mathbb{H}_{T,+}(\mu) \cap C([0,T], \mathbb{L}^2(\mu))$. The limit u of $(u_n)_n$ is then a weak solution on $[0,T]$ of ($\mathbb{MCP}$) with $u(0) = f$ by arguments developed in the beginning of this proof. □

5. Properties of solutions of ($\mathbb{MCP}$)

5.1. Positivity preserving

We will make use that, for any Dirichlet form $\mathcal{E}$ and any $u \in \mathcal{D}$, $\mathcal{E}(u^+, u^+) \le \mathcal{E}(u, u^+)$ where $u^+ = \max(0, u)$. Indeed, this is equivalent to $\mathcal{E}(u^-, u^+) \le 0$, with $u^- = \max(0, -u)$, which easily follows from $\mathcal{E}(|u|, |u|) \le \mathcal{E}(u, u)$.

Proposition 5.1 (Positivity). *Assume that* $\mu \in \mathbf{FS}(1)$ *and that* G *satisfies* (**MC4**). *Then, for any* $\lambda \in [0,1)$, *any solution* $u(t)$ *of* ($\mathbb{MCP}$) *with initial value* $f \ge 0$ *satisfies* $u(t) \ge 0$ *for any* $t \ge 0$.

Proof. Let $u(t)$ be a weak solution of ($\mathbb{MCP}$) with initial value $f \ge 0$. We will prove that, μ-a.s., $u^-(t) = 0$. For that, we first consider the Steklov average $a_h(u)(t)$ and its negative part $a_h^-(u)(t) \equiv \max(0, -a_h(u)(t))$. Note that $a_h^-(u)(t) \to u^-(t)$ in $\mathbb{L}^2(\mu)$ and $a_h^-(u) \to u^-$ in $\mathbb{L}^2([0,T], \mathcal{D})$, whereas, in $W^{1,2}((0,T), \mathbb{L}^2(\mu))$,

$$\partial_s a_h^-(u)(s) = -\partial_s a_h(u)(s) \chi_{\{a_h(u)(s) \le 0\}} = -\frac{1}{h}(u(s+h) - u(s)) \chi_{\{a_h(u)(s) \le 0\}}$$

where χ denotes the indicator function. Hence, using the definition of a weak solution (with the constant test function $a_h^-(u)(s) \in \mathcal{D}$), we get

$$\begin{aligned}
\frac{1}{2}\mu\left(a_h^-(u)(t)\right)^2 &= \frac{1}{2}\mu\left(a_h^-(u)(0)\right)^2 + \frac{1}{2}\int_0^t ds\, \partial_s \mu\left(a_h^-(u)(s)\right)^2 \\
&= \frac{1}{2}\mu\left(a_h^-(u)(0)\right)^2 - \int_0^t ds\, \mu\left(a_h^-(u)(s)\frac{1}{h}\left(u(s+h) - u(s)\right)\right) \\
&= \frac{1}{2}\mu\left(a_h^-(u)(0)\right)^2 \\
&\quad + \int_0^t ds \frac{1}{h}\int_s^{s+h} d\tau \left[\mathcal{E}\left(a_h^-(u)(s), u(\tau)\right) - \lambda\langle \mathcal{V}(u)(\tau), a_h^-(u)(s)\rangle_{\mathcal{D}',\mathcal{D}}\right] \\
&= \frac{1}{2}\mu\left(a_h^-(u)(0)\right)^2 \\
&\quad + \int_0^t ds \Big[\mathcal{E}\big(a_h^-(u)(s), a_h(u)(s)\big) - \lambda\big\langle a_h\big(\mathcal{V}(u)\big)(s), a_h^-(u)(s)\big\rangle_{\mathcal{D}',\mathcal{D}}\Big].
\end{aligned}$$

We can pass to the limit with $h \to 0$ which yields (as $\mu\left((f^-)^2\right) = 0$)

$$\begin{aligned}
\frac{1}{2}\mu\left(u^-(t)\right)^2 &= \int_0^t ds\mathcal{E}(u^-(s), u(s)) - \lambda \int_0^t ds\mu\Big((u^-(s))u(s)G\Big(\frac{(u(s))^2}{||u(s)||_2^2}\Big)\Big) \\
&= -\int_0^t ds\mathcal{E}((-u)^+(s), (-u)(s)) \\
&\quad + \lambda \int_0^t ds\mu\Big(((-u)^+(s))\,(-u)(s)G\Big(\frac{(-u(s))^2}{||-u(s)||_2^2}\Big)\Big) \\
&= -\int_0^t ds\mathcal{E}((-u)^+(s), (-u)(s)) + \lambda \int_0^t ds\mu\Big(((-u)^+(s))^2 G\Big(\frac{((-u)^+(s))^2}{||-u(s)||_2^2}\Big)\Big) \\
&\leq -\int_0^t ds\mathcal{E}(u^-(s), u^-(s)) + \lambda \int_0^t ds\mu\Big((u^-(s))^2 F\Big(\frac{(u^-(s))^2}{||u^-(s)||_2^2}\Big)\Big) \leq 0
\end{aligned}$$

provided $\lambda < 1$ thanks to the F-Sobolev inequality. Note that we used $G \leq F$ and monotonicity of F. Hence,

$$\mu\left((u^-(t))^2\right) \leq 0.$$

The proof is complete. □

5.2. Further properties

For simplicity we set $\hat{u}^2(t) \equiv \sigma^2(u(t)) \equiv u^2(t)/\mu(u^2(t))$.

Theorem 5.2 (Exponential decay in $\mathbb{L}^2$). *Assume that $\mu \in$ **FS**(1) and G satisfies (**MC4**). Suppose also that μ satisfies the following spectral gap inequality:*

$$m\mu\left(g - \mu g\right)^2 \leq \mu|\nabla g|^2,$$

*with $m \in (0, \infty)$ independent of $g \in \mathcal{D}$. Choose $\lambda \in (0, (1 + (b/m))^{-1})$ where $b \equiv \left(\|\mathcal{J}\|_\infty + B + 4\widetilde{B}\right)$ with $B > 0$ as in the generalized relative entropy inequality, and $\widetilde{B}$ and $\mathcal{J}$ as in condition (**C4**). Then, the solution $u(t)$ of the problem (MCP) with initial data $f \in \mathbb{L}^2(\mu)$ satisfies for any $t \geq 0$,*

$$\mu\left((u(t) - \mu(u(t)))^2\right) \leq e^{-2Mt}\mu\left((f - \mu(f))^2\right) \tag{5.1}$$

with $M \equiv m - \lambda(m + b) > 0$.

Recall that, under condition (**C1**), F-Sobolev inequality does not necessarily imply spectral gap inequality.

Proof. Set $w(t) = u(t) - \mu(u(t))$ and recall

$$w_h(t) \equiv \frac{1}{h}\int_t^{t+h} w(\tau)d\tau$$

(and similarly for $u(t)$) so that $w_h(t) = u_h(t) - \mu(u_h(t))$.

Since $w_h(t)$ is differentiable and $\mu(w_h(t)) = 0$ for any t, we have

$$\frac{1}{2}\frac{d}{dt}\left(e^{2Mt}\mu\left((w_h(t))^2\right)\right) - Me^{2Mt}\mu\left((w_h(t))^2\right)$$
$$= e^{2Mt}\mu(w_h(t)\frac{1}{h}(u(t+h) - u(t)))$$
$$= e^{2Mt}\frac{1}{h}\int_t^{t+h} d\tau\Big\{ - \mu\left(\nabla w_h(t)\nabla u(\tau)\right) + \lambda \int w_h(t)u(\tau)G(\hat{u}^2(\tau))d\mu\Big\}.$$

Integrating from 0 to t, and passing to the limit with $h \to 0$, we arrive at

$$(5.2)\quad \frac{e^{2Mt}}{2}\mu(w(t))^2 = \frac{1}{2}\mu(f-\mu f)^2 + \int_0^t dse^{2Ms}\left\{-\mu|\nabla u(s)|^2 + \lambda\mu\left[w(s)u(s)G(\hat{u}^2(s))\right] + M\mu(w^2(s))\right\}.$$

Now, as G vanishes at one, we have

$$u(s)G(\hat{u}^2(s)) = \int_0^1 d\alpha \frac{d}{d\alpha}\left[u_{[\alpha]}(s)G(\hat{u}^2_{[\alpha]}(s))\right]$$

with $u_{[\alpha]}(s) \equiv \alpha u(s) + (1-\alpha)\mu(u(s))$ and $\hat{u}^2_{[\alpha]}(s) \equiv (u_{[\alpha]}(s))^2/\mu(u_{[\alpha]}(s))^2$. Evaluating this derivative as in (4.3), one gets

$$\begin{aligned}\mu\left[w(s)u(s)G(\hat{u}^2(s))\right] &= \int_0^1 d\alpha\,\mu\left(w^2(s)\left[G(\hat{u}^2_{[\alpha]}(s)) + 2\hat{u}^2_{[\alpha]}(s)G'(\hat{u}^2_{[\alpha]}(s))\right]\right)\\ &\quad - 2\int_0^1 d\alpha\,\mu\left[\hat{u}^3_{[\alpha]}(s)G'(\hat{u}^2_{[\alpha]}(s))w(s)\right]\,\mu\left(\hat{u}_{[\alpha]}(s)w(s)\right)\\ &\le \int_0^1 d\alpha\,\Big\{\mu\left(w^2(s)F(\hat{u}^2_{[\alpha]}(s))\right) + 2\widetilde{B}\left[\mu\left(\hat{u}_{[\alpha]}(s)w(s)\right)\right]^2\\ &\quad + \left(\|\mathcal{J}\|_\infty + 2\widetilde{B}\right)\mu(w^2(s))\Big\}\\ &\le \mu\left(w^2(s)F(\hat{w}^2(s))\right) + b\,\mu(w^2(s)),\end{aligned}$$

with $b = \|\mathcal{J}\|_\infty + 4\widetilde{B} + B$, by arguments we already detailed. Coming back to (5.2) and applying F-Sobolev inequality, we obtain (as the Dirichlet form is conservative)

$$\mu(w(t))^2 = e^{-2Mt}\mu(f-\mu f)^2 + 2\int_0^t dse^{-2M(t-s)}\left\{-(1-\lambda)\mu|\nabla u(s)|^2 + (\lambda b + M)\,\mu(w^2(s))\right\}$$

If $m \in (0,\infty)$ is the best constant in the following Poincaré inequality

$$m\cdot\mu(g-\mu g)^2 \le \mu|\nabla g|^2,$$

for any g in the domain of the form, then we get

$$\mu(w(t))^2 \le e^{-2Mt}\mu(f-\mu f)^2 - 2\int_0^t dse^{-2M(t-s)}\left\{\left[m(1-\lambda) - M - \lambda b)\right]\mu w^2(s)\right\}.$$

Thus, if $\lambda \in [0, \{1 + (b/m)\}^{-1})$, then $M = m(1-\lambda) - \lambda b > 0$ and we obtain

$$\mu(w(t))^2 \leq e^{-2Mt}\mu(f - \mu f)^2.$$

□

Proposition 5.3. *Under the assumptions of the previous theorem, for any initial value $f \in \mathbb{L}^2(\mu)$, $\mu(u(t))$ converges exponentially fast to a quantity we denote by $\mathbb{S}_\infty(f) \in \mathbb{R}$: there exists a constant $K' \in (0, \infty)$ such that, for any $t \geq 0$,*

$$|\mu(u(t)) - \mathbb{S}_\infty(f)| \leq K' e^{-Mt} \|f - \mu(f)\|_2.$$

Consequently, there exists a constant $K'' \in (0, \infty)$ such that

$$\|u(t) - \mathbb{S}_\infty(f)\|_2 \leq K'' e^{-Mt} \|f - \mu(f)\|_2.$$

Note that trivially $\mathbb{S}_\infty(f)$ coincides with the *nonlinear parabolic transfer operator* given by the $\mathbb{L}^2(\mu)$ limit $\lim_{T\to\infty} \frac{1}{T}\int_0^T u(s)ds$, whose existence in some abstract (continuous w.r.t. initial value) nonlinear Markov contraction semigroups setting would be worth studying.

Proof. We first prove the convergence of $\mu(u(t))$. As in the previous proof, let $u_{[\alpha]}(t) = \alpha u(t) + (1-\alpha)\mu(u(t))$, $w(t) = u(t) - \mu(u(t))$ and $u_h(t) = \frac{1}{h}\int_t^{t+h} u(\tau)d\tau$. We have

$$\begin{aligned}\partial_t \mu(u_h(t)) &= \frac{1}{h}\mu\left(u(t+h) - u(t)\right) = \lambda\frac{1}{h}\int_t^{t+h} ds\mu\left(u(s)G\left(\hat{u}(s)^2\right)\right) \\ &= \frac{1}{h}\int_t^{t+h} ds\mu\left(\lambda\int_0^1 \frac{d}{d\alpha}\left(u_{[\alpha]}(s)G\left(\hat{u}^2_{[\alpha]}(s)\right)\right)d\alpha\right).\end{aligned}$$

with

$$\begin{aligned}\mu\Big(\frac{d}{d\alpha}\left(u_{[\alpha]}(s)G\left(\hat{u}^2_{[\alpha]}(s)\right)\right)\Big) = &\int w(s)G\left(\hat{u}^2_{[\alpha]}(s)\right)d\mu + 2\int w(s)\hat{u}^2_{[\alpha]}(s)G'\left(\hat{u}^2_{[\alpha]}(s)\right)d\mu \\ &- 2\int \hat{u}^2_{[\alpha]}(s)G'\left(\hat{u}^2_{[\alpha]}(s)\right)\frac{u_{[\alpha]}(s)\mu(u_{[\alpha]}(s)w(s))}{\mu(u^2_{[\alpha]}(s))}d\mu.\end{aligned}$$

It follows from condition (**C4**) that $\widetilde{B} \equiv \sup|xG'(x)| < \infty$ and that $|G(x)| \leq C + x^{\frac{1}{2}}$ with some constant $C \in (0, \infty)$. Hence, using Hölder's inequality, we get

$$\left|\int \frac{d}{d\alpha}\left(u_{[\alpha]}(s)G\left(\hat{u}^2_{[\alpha]}(s)\right)\right)d\mu\right| \leq (C + 1 + 4\widetilde{B})\left(\mu w^2(s)\right)^{\frac{1}{2}}.$$

Combining our considerations, we obtain

$$|\partial_t\mu(u_h(t))| \leq \lambda(C + 1 + 4\widetilde{B})\frac{1}{h}\int_t^{t+h} ds\left(\mu w(s)^2\right)^{\frac{1}{2}}.$$

Now using the bound of Theorem 5.2 gives (uniformly in $h > 0$)

$$|\partial_t\mu(u_h(t))| \leq \lambda(C + 1 + 4\widetilde{B})e^{-Mt}(\mu(f - \mu f)^2)^{\frac{1}{2}}.$$

Thus, if $T \geq t$, one gets

$$|\mu(u_h(T)) - \mu(u_h(t))| \leq \lambda(C + 1 + 4\widetilde{B})\|f - \mu(f)\|_2 \int_t^\infty e^{-Ms} ds$$

so that, after passing to the limit $h \to 0$,

$$|\mu(u(T)) - \mu(u(t))| \leq \frac{e^{-Mt}}{M}\lambda(C + 1 + 4\widetilde{B})\|f - \mu(f)\|_2.$$

Hence, $(\mu(u(t)))_{t\geq 0}$ is Cauchy as t goes to ∞. Letting T go to infinity proves the first part of the proposition with $K' = \frac{\lambda}{M}(C + 1 + 4\widetilde{B})$.

The second part follows from the following inequality:

$$\mu\left((u(t) - \mathbb{S}_\infty(f))^2\right) \leq 2\mu\left((u(t) - \mu(u(t)))^2\right) + 2|\mu(u(t)) - \mathbb{S}_\infty(f)|^2,$$

the previous bound and Theorem 5.2. □

6. Uniform hypercontractivity

In [3] it is shown that in case of linear diffusion operators, the corresponding semigroup is hypercontractive in some family of Orlicz spaces. Moreover, this fact is equivalent to F-Sobolev inequality, generalizing Gross' Theorem. In this section we show that similar smoothing properties hold true in our setting. Note that notation sometimes differs slightly with other sections.

Define for any $r \geq 0$, $\tau_r(x) := x^2 e^{rF(x^2)}$ and assume that there exists a constant $k > 0$ such that for all $r \geq 0$: $\tau_r''\tau_r \geq \frac{k}{4}{\tau_r'}^2$.

In particular if we consider the function F defined in Appendix II, thanks to Lemma 37 of [3], we have for any $r \geq 0$,

$$(\tau_r^{(\alpha)})''\tau_r^{(\alpha)} \geq \frac{3 - 2(2-\alpha)/(\alpha\log(\theta))}{4}{(\tau_r^{(\alpha)})'}^2. \tag{6.1}$$

Suppose $\lambda \in [0, \min(1, k/2))$. Let $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 non-decreasing function satisfying $-k + \left[\lambda 2(1 + q(t)\bar{B}) + q'(t)\right] \leq 0$ and $q(0) = 0$. In particular, one may choose $q(t) \equiv \frac{\eta}{2\lambda\bar{B}}(1 - e^{-2\lambda\bar{B}t})$ with $0 \leq \eta \leq k - 2\lambda$. We set $\Phi_t := \tau_{q(t)}$, $t \geq 0$. We have $x^2 \equiv \Phi_0(x) \leq e^{Aq(\infty)}\Phi_t(x)$ and consequently $||f||_2 \leq e^{Aq(\infty)}||f||_{\Phi_t}$. Note that, under our hypothesis, $q(t)$ is bounded.

Theorem 6.1. *Assume conditions* (**C0**)–(**C4**). *Assume that* $\mu \in$ **FS** *and that* $\lambda \in [0, \min(1, k/2)]$. *Then, any solutions* $u(t)$ *and* $v(t)$ *of* ($\mathbb{MCP}$) *with initial data* $f \in \mathbb{L}^2(\mu)$ *and* $g \in \mathbb{L}^2(\mu)$, *respectively, satisfy, for all* $t \geq 0$,

$$\|u(t)\|_{\Phi_t} \leq \exp\left\{\lambda(B + 2A + \|\mathcal{J}\|_\infty)\int_0^t ds\,(1 + q(s)\bar{B})\right\}\|f\|_2 \tag{6.2}$$

and

$$\|u(t) - v(t)\|_{\Phi_t} \leq C_{u,v}(t)\|f - g\|_2,$$

where $C_{u,v}(t)$ *is given by*

$$C_{u,v}(t) = \exp\left\{\lambda\int_0^t ds\Big[2\widetilde{B}\Big(1+2\int_0^1 d\alpha\frac{\|u_{[\alpha]}(s)\|_{\Phi_s}}{\|u_{[\alpha]}(s)\|_2}\Big)+B+2A+\|\mathcal{J}\|_\infty)\Big](1+q(s)\bar{B})\right\}$$

with $u_{[\alpha]}(s) \equiv \alpha u(s) + (1-\alpha)v(s)$.

Remark 6.2. The constant $C_{u,v}$ can be slightly simplified with more effort if we assume $f, g \geq 0$, using the first part of the theorem and the positivity preservation of the solution.

Proof. The proof is standard and relies basically on Gross' arguments [18]. We give the main step of the proof, some computations and details are left to the reader. Let $u(t)$ and $v(t)$ be a solution of the Cauchy problem with smooth initial data f and g, respectively. The desired hypercontractivity once proven for the case of bounded smooth initial data, can later be extended to the general case. Let $w(t) \equiv u(t) - v(t)$. Let $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a general non-decreasing function with $q(0) = 0$ and consider first $N_h(t) = \|w_h(t)\|_{\tau_{q(t)}}$, where $w_h(t) \equiv \frac{1}{h}\int_t^{t+h} ds w(s)$. For simplicity, we set $T(x,q) = \tau_q(x)$. Then by definition of the Luxemburg norm, we have

$$\int T\left(\sigma_t(w_h(t)), q(t)\right) d\mu = 1, \quad \forall t \geq 0,$$

where $\sigma_t(w_h(t)) \equiv \frac{w_h(t)}{N_h(t)}$. If $N_h'(t) \leq 0$, there is nothing to prove. In case when $N_h'(t) \geq 0$, using convexity of T, by differentiation of the latter, passing to the limit $h \to 0$, we arrive at the following inequality with $N \equiv N(t) \equiv N_{\Phi_t}(w(t))$:

$$\begin{aligned}(6.3)\quad \frac{2N'}{N} \leq &-\int \frac{|\nabla w(t)|^2}{N^2}\partial_{11}T\left(\sigma_t(w(t)), q(t)\right) d\mu \\ &+ \lambda\int \frac{1}{N}\left[u(t)G\big(\sigma_0^2(u(t))\big) - v(t)G\big(\sigma_0^2(v(t))\big)\right]\partial_1 T\left(\sigma_t(w(t)), q(t)\right) d\mu \\ &+ q'(t)\int \partial_2 T\left(\sigma_t(w(t)), q(t)\right) d\mu\end{aligned}$$

For $\alpha \in [0,1]$, set $u_{[\alpha]} \equiv u_{[\alpha]}(t) \equiv \alpha u(t) + (1-\alpha)v(t)$. Using this interpolation we can estimate the second term as follows:

$$\begin{aligned}(6.4)\quad &\int \frac{1}{N}\left[u(t)G\big(\sigma_0^2(u(t))\big) - v(t)G\big(\sigma_0^2(v(t))\big)\right]\partial_1 T d\mu \\ &\leq \int_0^1 d\alpha\int d\mu\Big\{\left[F\big(\sigma_0^2(u_{[\alpha]}(t))\big) + 2A+\|\mathcal{J}\|_\infty\right]|\sigma_t(w(t))|\,\partial_1 T \\ &\quad + \frac{2}{N}\Big[\sigma_0^2(u_{[\alpha]}(t))G'\big(\sigma_0^2(u_{[\alpha]}(t))\big)\Big(w(t) - \frac{u_{[\alpha]}(t)}{||u_{[\alpha]}(t)||_2}\frac{d}{d\alpha}||u_{[\alpha]}(t)||_2\Big)\partial_1 T\Big]\Big\} \\ &\leq (\mathrm{I}) + (\mathrm{II}),\end{aligned}$$

where we wrote $\partial_1 T$ for $\partial_1 T\left(\sigma_t(w(t)), q(t)\right)$.

Note that, by our assumption we have $\sup|xG'(x)| \equiv \widetilde{B} < \infty$. Since

$$|\partial_1 T(x,q)| = 2|x|(1+qx^2F'(x^2))e^{qF(x^2)} \le 2(1+q\bar{B})|x|e^{qF(x^2)},$$

after several computations, one obtains

$$\text{(II)} \le 4\widetilde{B}(1+q(t)\bar{B})\Big(1+2\int_0^1 \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2}d\alpha\Big). \tag{6.5}$$

The first term (I) on the right hand side of (6.4) can be bounded by using $x\partial_1 T(x,q) = 2T(x,q) + 2qx^2F'(x^2)T(x,q) \le 2(1+q\bar{B})T(x,q)$, the generalized relative entropy inequality and $\mu \in \mathbf{FS}$, by

$$\text{(I)} \le 2(1+q(t)\bar{B})\int d\mu \left|\nabla\sqrt{T\left(\sigma_t(w(t)),q(t)\right)}\right|^2 + 2(B+\|\mathcal{J}\|_\infty)(1+q(t)\bar{B}). \tag{6.6}$$

Winding up (6.3)–(6.6), we get

$$\begin{aligned}\frac{2N'}{N} \le &-\int \frac{|\nabla w(t)|^2}{N^2}\partial_{11}T\left(\sigma_t(w(t)),q(t)\right)d\mu \\ &+\lambda 2(1+q(t)\bar{B})\int d\mu\left|\nabla\sqrt{T\left(\sigma_t(w(t)),q(t)\right)}\right|^2 \\ &+2\lambda(2\widetilde{B}\zeta_{u,v}(t)+B+2A+\|\mathcal{J}\|_\infty)(1+q(t)\bar{B}) \\ &+q'(t)\int \partial_2 T\left(\sigma_t(w(t)),q(t)\right)d\mu\end{aligned} \tag{6.7}$$

where we wrote

$$\zeta_{u,v}(t) = 1+2\int_0^1 \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2}d\alpha.$$

Next, under our assumption on τ_q and using (**FS**), we have

$$\int \partial_2 T(g,q(t))d\mu \le \int T(g,q(t))F(T(g,q(t)))d\mu \le \int |\nabla\sqrt{T(g,q(t))}|^2 d\mu. \tag{6.8}$$

Thus

$$\begin{aligned}\frac{2N'}{N} \le &\left(-k+\left[\lambda 2(1+q(t)\bar{B})+q'(t)\right]\right)\int |\nabla\sqrt{T(\sigma_t(w(t)),q(t))}|^2 d\mu \\ &+2\lambda(2\widetilde{B}\zeta_{u,v}(t)+B+2A+\|\mathcal{J}\|_\infty)(1+q(t)\bar{B})\end{aligned} \tag{6.9}$$

Choosing $q(t)$ such that $-k+c_{_F}\left[\lambda 2(1+q(t)\bar{B})+q'(t)\right] \le 0$, we get

$$\frac{2N'}{N} \le 2\lambda\tilde{a}(t)(1+q(t)\bar{B})$$

with $\tilde{a}(t) \equiv (2\widetilde{B}\zeta_{u,v}(t)+B+2A+\|\mathcal{J}\|_\infty)$. And this for any t such that $N'(t) \ge 0$. Thus by integration we arrive at the following bound:

$$\|u(t)-v(t)\|_{\Phi_t} \le \exp\left\{\lambda\int_0^t ds\tilde{a}(s)\left(1+q(s)\bar{B}\right)\right\}\|f-g\|_2,$$

which ends the proof of the metric type hypercontractivity. As for the hypercontractivity for the norm (6.2), the proof is simpler and may be developed by similar arguments from (6.3) by taking $w = u$ and $v = 0$. □

7. Functional inequalities for Gibbs measures

7.1. Gibbs measures on infinite product of manifolds and generalized Sobolev space

In this section we introduce the general infinite space we will consider. Let $\mathbb{M} = \prod_{i\in\mathcal{R}} M_i$ be an infinite product of Riemannian manifolds (M_i, g_i), where $\mathcal{R}$ is a countable set (an infinite graph). Given $z \in M_i$ and $\mathbf{x} = (x_i)_{i\in\mathcal{R}} \in \mathbb{M}$ we define $z \bullet_i \mathbf{x} \equiv \{(z \bullet_i \mathbf{x})_k \equiv \delta_{ik} z + (1-\delta_{ik})x_k : k \in \mathcal{R}\}$. We say that a function f on $\mathbb{M}$ is cylindrically smooth if f is localized on some finite subset $\Lambda \subset \mathcal{R}$ (that is f depends only on the coordinates in Λ) and is smooth when considered as a function on $M_\Lambda = \prod_{i\in\Lambda} M_i$. We denote by $\mathcal{C}$ the space of compactly supported cylindrically smooth functions. For $f \in \mathcal{C}$, we consider the following quadratic operator, called the square field operator,

$$|\nabla f|^2 = \sum_{i\in\mathcal{R}} |\nabla_i f|_i^2$$

where for each site $i \in \mathcal{R}$, $|\nabla_i f|_i(\mathbf{x}) \equiv |\nabla_i f_i(\cdot|\mathbf{x})|(x_i)$ is the length of the usual gradient ∇_i for the metric g_i at x_i of the function $M_i \ni z \mapsto f_i(\cdot|\mathbf{x})(z) \equiv f(\{z \bullet_i \mathbf{x}\})$.

Let μ be a probability measure on $\mathbb{M}$. For $f \in \mathcal{C}$, $\mu(|\nabla f|^2)$ makes sense. Actually, provided μ is a Gibbs measure, this can be defined on a wider class of functions on $\mathbb{M}$ generalizing the Sobolev space $W^{1,2}$.

Briefly speaking, a Gibbs measure is defined as follows. A specification is a family $\mu_\Lambda^\xi(dx_\Lambda)$, Λ finite subset of $\mathcal{R}$ and $\xi \in \mathbb{M}$, of absolutely continuous probability kernels on $M_\Lambda \equiv \prod_{i\in\Lambda} M_i$, that we extend to kernels E_Λ^ξ on $\mathbb{M}$ by taking product with $\otimes_{i\notin\Lambda}\delta_{\xi_i}$. These kernels are supposed to satisfy compatibility conditions (see [19] and references therein) making them possible candidates for being versions of laws (w.r.t. a probability measure μ on $\mathbb{M}$) conditionally to $\pi_{\mathcal{R}\setminus\Lambda}(\xi) \equiv (\xi_i)_{i\in\mathcal{R}\setminus\Lambda}$. Measures μ on $\mathbb{M}$ for which this holds are called Gibbs measures and can be multiple in general. They are characterized by the Dobrushin–Landford–Ruelle (DLR) conditions $\mu = \mu E_\Lambda^{\cdot}$ (when acting on bounded measurable functions).

Let μ be a fixed Gibbs measure. The generalized Sobolev space $W^{1,2}(\mu)$ can be defined as the space of functions $f \in \mathbb{L}^2(\mu)$ such that, for any $i \in \mathcal{R}$ and any $\xi \in \mathbb{M}$ μ a.e., $|\nabla_i f_i(\cdot|\xi)|$ in the sense of distributions in M_i belongs to $\mathbb{L}^2(M_i, \mu_{\{i\}}^\xi)$ and one has

$$\mu(|\nabla f|^2) \equiv \int_{\mathbb{M}} \sum_{i\in\mathcal{R}} \mu_{\{i\}}^\xi \left(|\nabla_i f_i(\cdot|\xi)|^2\right) \mu(d\xi) < \infty.$$

The notation $\mu(|\nabla f|^2)$ is not completely formal as this coincides with the similar quantity for cylindrically smooth compactly supported functions.

If we denote by $\mathring{W}^{1,2}(\mu)$ the closure of $\mathcal{C}$ for the following norm $(\mu(f^2) + \mu(|\nabla f|^2))^{1/2}$, then $(\mu(|\nabla f|^2), \mathring{W}^{1,2}(\mu))$ is a local Dirichlet form. For Gibbs measures with subgaussian tails, we will consider later a Dirichlet form $(\mathcal{E}, \mathcal{D})$ which coincides with the form $(\mu(|\nabla f|^2), \mathring{W}^{1,2}(\mu))$ on $\mathcal{C}$ and which satisfies an F-Sobolev inequality.

7.2. *F*-Sobolev and Orlicz–Sobolev inequalities

In this section we describe briefly the functional inequalities like F-Sobolev inequality (introduced in Section 2) and Orlicz–Sobolev inequality (we introduce below) for a class of non-product Gibbs measures in infinite dimensions; for more details we refer to [3] and [29].

We first introduce the Orlicz–Sobolev inequality. We say that $(\mu, \mathcal{E})$ satisfies an Orlicz–Sobolev inequality if, for any $f \in \mathcal{D}$,

$$\|(f-\mu(f))^2\|_\Phi \le c_\Phi \int |\nabla f|^2 d\mu$$

for some constant $c_\Phi > 0$ and some N-function Φ.

Since Φ is a N-function, Orlicz–Sobolev always implies Poincaré inequality. Moreover, under some specific and technical assumptions on F and Φ and combining various results based on the capacity-measure approach introduced in[4] [4] (namely Theorems 18, 20 and 22, and Lemma 19 of [3]; Theorem 1 of [29]), it is possible to prove that F-Sobolev inequalities and Orlicz–Sobolev inequalities are equivalent (up to constant) when $F(x) = x/\Phi^{-1}(x)$.

In order to prove that such inequalities hold for Gibbs measures, we will have to make use of a third family of inequalities we may call generalized Beckner inequality[5]. It is given in (7.1). Again, under specific assumptions such inequalities are equivalent to F-Sobolev inequality and Orlicz–Sobolev inequality. Furthermore, by construction, they imply Poincaré inequality.

Next theorem explains how the generalized Beckner inequality implies F-Sobolev and Orlicz–Sobolev inequalities.

Theorem 7.1. *Let $T : [0,1] \to \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Denote by C_T the optimal constant such that the Dirichlet structure $(\mu, \mathcal{E})$ satisfies for every $f \in \mathcal{D}$*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - (\int |f|^p d\mu)^{\frac{2}{p}}}{T(2-p)} \le C_T \int |\nabla f|^2 d\mu. \tag{7.1}$$

(i) *Let Φ be a N-function and fix a constant $k \in (0, +\infty)$ such that for any function f with $f^2 \in \mathbb{L}_\Phi(\mu)$, $||\mu(f)^2||_\Phi \le k||f^2||_\Phi$ (see (9.3) in Appendix* I*). Assume that there exists $c_1 > 0$ such that*

$$c_1\, x\, T\Big(\frac{1}{\log(1+x)}\Big) \le \Phi^{-1}(x), \quad \forall x > 2.$$

[4]The notion of (electrostatic) capacity goes back to Maz'ja [26]. In [4] the authors introduce a slightly different notion of capacity of a set with respect to a probability measure (in probability spaces the usual electrostatic capacity is always 0). This turns out to be appropriate in the study of functional inequalities in probability spaces. See Section 5.2 of [3] for a short introduction of this notion.

[5]Beckner [5] introduced such inequalities for $T(x) = x$ which corresponds to Gaussian measures. Later Latała and Oleszkiewicz [23] studied the case $T(x) = x^\beta$, $\beta \in [0,1]$ (see also [4]). The general case was studied in [3] and [33]

Then, every $f \in \mathcal{D}$ satisfies

$$\left\|(f-\mu(f))^2\right\|_\Phi \le \frac{48(1+k)C_T}{c_1}\int |\nabla f|^2 d\mu.$$

(ii) *Let $F\colon [0,+\infty) \to [0,+\infty)$ be a non-decreasing function. Assume that $F(x) = 0$ if $x \le \theta$ for some $\theta > 2$ and that there exists a constant c such that $F(\theta y/2) \le c/T(1/\log(1+y))$ for any $y \ge \theta$.*

Then, for every $f \in \mathcal{D}$ one has

$$\int f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big) d\mu \le 3c\, C_T \Big(\frac{\theta}{\sqrt{\theta}-\sqrt{2}}\Big)^2 \int |\nabla f|^2 d\nu.$$

Proof. The proof of (i) can be found in Corollary 8 of [29]. The proof of (ii) follows from a combination of Theorem 9, Lemma 8 and Theorem 20 of [3]. Note in both references the results are given in $\mathbb{R}^n$. The generalization to our setting is straight forward. □

In the rest of this section we consider the following infinite dimensional models on a space $\Omega \equiv \mathbb{R}^{\mathbb{Z}^d} \equiv \{\omega = (\omega_i \in \mathbb{R})_{i\in\mathbb{Z}^d}\}$.

Let $\mathcal{U}_i \equiv \mathcal{U}_i(\omega_i)$, $i \in \mathbb{Z}$, be smooth convex functions such that

$$0 < \inf_{i\in\mathbb{Z}^d} \int e^{-\mathcal{U}_i(x)} dx \le \sup_{i\in\mathbb{Z}^d} \int e^{-\mathcal{U}_i(x)} dx < \infty.$$

Let $\mathcal{I} \equiv \{\mathcal{I}_X\}$, $X \Subset \mathbb{Z}^d$, $|X| \ge 1$, be a collection of smooth bounded cylinder functions, (dependent only on $\omega_X \equiv (\omega_i : i \in X)$, respectively), and such that

$$\|\mathcal{I}\|_{u,2} \equiv \sup_{i\in\mathbb{Z}^d} \Bigg(\sum_{\substack{X\Subset\mathbb{Z}^d\\ X\ni i}} \Big\{ \|\mathcal{I}_X\|_u + \sum_{j\in\mathbb{Z}^d} \Big[\|\nabla_j \mathcal{I}_X\|_u + \|\nabla_j\nabla_i \Phi_X\|_u \Big] \Big\} \Bigg) < \infty \tag{7.2}$$

where $\|\cdot\|_u$ denotes the uniform norm and $X \Subset \mathbb{Z}^d$ means that X is a finite subset of $\mathbb{Z}^d$. For $\Lambda \Subset \mathbb{Z}^d$, setting

$$U_\Lambda \equiv \sum_{i\in\Lambda} \mathcal{U}_i(\omega_i) + \sum_{X\cap\Lambda\neq\emptyset} \mathcal{I}_X(\omega_X)$$

we define

$$E^\omega_\Lambda(f) \equiv \frac{\int e^{-U_\Lambda(\widetilde\omega\circ_\Lambda\omega)} f(\widetilde\omega\circ_\Lambda\omega) d\widetilde\omega_\Lambda}{\int e^{-U_\Lambda(\widetilde\omega\circ_\Lambda\omega)} d\widetilde\omega_\Lambda}$$

where

$$(\widetilde\omega\circ_\Lambda\omega)_i \equiv \begin{cases} \widetilde\omega_i & if \quad i\in\Lambda \\ \omega_i & if \quad i\in\Lambda^c \end{cases}$$

A measure μ is called a Gibbs measure on Ω for local specification $\{E_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ if and only if for any integrable function f one has

$$\mu(E^{\cdot}_\Lambda f) = \mu(f) \quad \text{for all } \Lambda \Subset \mathbb{Z}^d.$$

For any $\Lambda \subset \mathbb{Z}^d$ and $i \in \Lambda$ we have

$$E_\Lambda(fL_ig) \equiv -E_\Lambda \nabla_i f \cdot \nabla_i g$$

for any functions f and g for which both sides make sense. Consider operators L_i such that

$$L_i f = e^{U_i} \mathrm{div}_i \left(e^{-U_i} \nabla_i f\right) = \Delta_i f - \nabla_i U_i \cdot \nabla_i f$$

where div_i and ∇_i are with respect to ω_i and $U_i \equiv U_{\{i\}}$.

We introduce the following Markov generator:

$$L \equiv \sum_{i \in \mathbb{Z}^d} L_i \,, \tag{7.3}$$

which is well defined on a domain including all smooth cylinder functions. Consequently we have

$$-\mu(fLg) = \sum_{i \in \mathbb{Z}^d} \mu(\nabla_i f \cdot \nabla_i g)$$

and if $P_t f \equiv e^{tL} f \equiv f_t$ is the corresponding Markov semigroup, we also have

$$\mu(fP_t g) = \mu(gP_t f).$$

For a construction of the semigroup $(P_t)_{t \geq 0}$ in the space of bounded continuous functions we refer to [19], (see also [36], [35], [22], [10], and references therein).

We note that in the present setup one has

$$|\nabla z|_2^2 \equiv \frac{1}{2}(Lz^2 - 2zLz) = \sum_i |\nabla_i z|^2$$

and the generator L has the following diffusion property (or chain rule): for any (localized) smooth vector functions $f = (f_1, \ldots, f_\nu)$ on Ω ($\nu \in \mathbb{N}$) and any smooth function Ψ on $\mathbb{R}^\nu$,

$$L\Psi(f_1, \ldots, f_\nu) = \sum_{k=1}^{\nu} \partial_k \Psi(f) L f_k + \sum_{k,l=1}^{\nu} \partial_{k,l}^2 \Psi(f) L \nabla f_k \cdot \nabla f_l.$$

In the above described setup we have the following results.

Theorem 7.2 ([29], [19]). *Fix $\alpha \in (1, 2)$. Assume that $\mathcal{I}_X = 0$ for all X and that for all $i \in \mathbb{N}$, $\mathcal{U}_i(x) = \mathcal{U}_\alpha(x)$ where $\mathcal{U}_\alpha$ is the following $\mathcal{C}^2$ function:*

$$\mathcal{U}_\alpha(x) = \begin{cases} |x|^\alpha & \text{for} \quad |x| > 1 \\ \frac{\alpha(\alpha-2)}{8} x^4 + \frac{\alpha(4-\alpha)}{4} x^2 + (1 - \frac{3}{4}\alpha + \frac{1}{8}\alpha^2) & \text{for} \quad |x| \leq 1. \end{cases}$$

Define the corresponding Gibbs measure μ_α (product in this case). Then, there exists a constant C_α such that, for any function f in the domain of the Dirichlet form,

$$\sup_{p \in (1,2)} \frac{\int f^2 d\mu_\alpha - (\int |f|^p d\mu\alpha)^{\frac{2}{p}}}{(2-p)^{2(1-\frac{1}{\alpha})}} \leq C_\alpha \int |\nabla f|^2 d\mu_\alpha.$$

Moreover if $\{\mathcal{I}_X\}$ *is such that* $||\mathcal{I}||_{u,2}$ *is sufficiently small* (*and if for all* $i \in \mathbb{N}$, $\mathcal{U}_i(x) = \mathcal{U}_\alpha(x)$ *as above*), *then the same results* (*with appropriate constants*) *remain true for the corresponding Gibbs measures* (*non product in this case*).

Note that the special choice of $\mathcal{U}_\alpha$ near the origin is not important. Any other smooth version of $|x|^\alpha$ would do the job.

We are now in position to give examples of Gibbs measures satisfying a F-Sobolev inequality and an Orlicz–Sobolev inequality.

Theorem 7.3 ([29]). *Fix* $\alpha \in (1,2)$ *and set* $\beta = 2(1-\frac{1}{\alpha})$. *Consider the function* F_α *defined in Appendix* II *and* $\Phi_\beta(x) = |x| \log(1+|x|)^\beta$. *Under the assumption and notations of Theorem* 7.2, *there exists a constant* D_α *such that any function* f *in the domain of the Dirichlet form satisfies*

$$\left\|(f-\mu_\alpha(f))^2\right\|_{\Phi_\beta} \le D_\alpha \int |\nabla f|^2 d\mu_\alpha$$

and

$$\int f^2 F_\alpha\Big(\frac{f^2}{\mu_\alpha(f^2)}\Big) d\mu_\alpha \le D_\alpha \int |\nabla f|^2 d\mu_\alpha .$$

Moreover if $\{\mathcal{I}_X\}$ *is such that* $||\mathcal{I}||_{u,2}$ *is sufficiently small* (*and if for all* $i \in \mathbb{N}$, $\mathcal{U}_i(x) = \mathcal{U}_\alpha(x)$ *as in Theorem* 7.2), *then the same results* (*with appropriate constants*) *remain true for the corresponding Gibbs measures.*

Proof. Set $T(x) = |x|^\beta$. It is not difficult (see [29]) to show that for any $x > 2$,

$$xT\Big(\frac{1}{\log(1+x)}\Big) \le \Phi_\beta^{-1}(x).$$

On the other hand, thanks to Remark 9.3, $||\mu_\alpha(f)^2||_{\Phi_\beta} \le e||f^2||_{\Phi_\beta}$. Consider a smoothed cylinder function f. We can apply Theorems 7.1 and 7.2 to get the result for the Orlicz–Sobolev inequality. A density argument ends the proof.

Recall the definition of θ defined in Appendix II. It is easy to prove that

$$F(\theta y/2) \le \frac{c}{T\left(\frac{1}{\log(1+y)}\right)}$$

for $y \ge \theta$. Hence we can apply Theorem 7.1 for smooth cylinder functions. The result follows by density. □

8. Local problems and Orlicz–Sobolev inequality

In this section, we mention some results on local semilinear problems (*i.e.* problems with non linearities $\mathbf{V}(u(t,x))$ whose value at point $x \in \mathbb{M}$ only depends on $u(t,x)$, contrary to ($\mathbb{MCP}$) for subGaussian measures in infinite dimensions. The analysis is based on smoothing properties which follow from Orlicz–Sobolev inequality. Proofs are easily obtained from the abstract setting presented in [16] (and references therein) and are omitted.

Proposition 8.1 (Smoothing via Orlicz–Sobolev). *Let Υ be a Young function. Assume that the associated Dirichlet structure $(\mu, \mathcal{E})$ satisfies the Orlicz–Sobolev inequality: for any f smooth enough,*

$$\left\|(f - \mu(f))^2\right\|_\Upsilon \leq C_\Upsilon \int |\nabla f|^2 d\mu,$$

for some constant C_Υ independent on f. Set $\Phi(x) := \Upsilon(x^2)$. Then, for any $t > 0$, the μ-symmetric Markov semigroup P_t associated to $(\mu, \mathcal{E})$ maps $\mathbb{L}^2(\mu)$ to $\mathbb{L}^\Phi(\mu)$ and, for any $T \in (0, \infty)$, any $t \in (0, T)$,

$$\|P_t f\|_\Phi \leq \frac{C_T}{\sqrt{t}} \|f\|_2$$

with

$$C_T^2 = \frac{C_\Upsilon}{e} + 2T\|\mathbb{1}\|_\Upsilon.$$

Proof. Adapt the proof of Theorem 4.3 in [16]. □

In the previous section we obtained Orlicz–Sobolev inequality for a class of Gibbs measures for $\Upsilon_\beta(x) \equiv |x| \log(1 + |x|)^\beta$. This allows to get some *continuous control of the norm* for some associated local nonlinearity as explained in the following proposition.

Proposition 8.2. *Let $\Upsilon_\beta = |x| \log(1 + |x|)^\beta$, $\beta \in (0, 1)$ and $\Phi_\beta = \Upsilon_\beta(x^2)$. Set $V_\beta = \sqrt{\Phi_\beta}$. Then, for any $f \in \mathbb{L}_{\Phi_\beta}(\mu)$, any $\beta \in (0, 1)$,*

$$\|V_\beta(f)\|_2 \leq W_\beta\left(\|f\|_{\Phi_\beta}\right)$$

for $W_\beta(x) = x + V_\beta(x) = x + x \log(1 + x^2)^{\frac{\beta}{2}}$.

The last two propositions as well as Theorem 7.3 are the ingredients to prove the following theorem. We refer to [16] for the definition of integral solutions.

Theorem 8.3. *Let $\Upsilon_\beta = |x| \log(1 + |x|)^\beta$, $\beta \in (0, 1)$ and $\Phi_\beta = \Upsilon_\beta(x^2)$. Set $V_\beta = \sqrt{\Phi_\beta}$. Let μ_α be the Gibbs measure defined in Theorem 7.2 on $\mathbb{R}^{\mathbb{Z}^d}$ and L be the Markov generator (7.3). Then, for any $\beta \in (0, 1)$, for any $f \in \mathbb{L}_{\Phi_\beta}(\mu_\alpha)$, the Cauchy problem*

$$\begin{cases} \partial_t u = Lu + \mathbf{V}(u) \\ u(0) = f \end{cases} \tag{8.1}$$

with $\mathbf{V}(u) = V_\beta \circ u$ acting by composition with V_β, admits a unique integral solution $u(t)$ on $[0, \infty)$. Consequently, there exists a nonlinear $\mathcal{C}^0$ semigroup $(S_t)_{t \geq 0}$ on $\mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$ such that for any $f \in \mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$, $u(t) = S_t f$.

9. Appendix I: Young functions and Orlicz spaces

In this section we collect some results on Orlicz spaces. We refer the reader to [28] for demonstrations and complements.

Definition 9.1 (Young function). A function $\Phi : \mathbb{R} \to [0, \infty]$ is a *Young function* if it is convex, even, such that $\Phi(0) = 0$, and $\lim_{x \to +\infty} \Phi(x) = +\infty$.

The Legendre transform Φ^* of Φ defined by

$$\Phi^*(y) = \sup_{x \geq 0} \{x|y| - \Phi(x)\}$$

is a lower semicontinuous Young function. It is called the *complementary function* or *conjugate* of Φ.

Among the Young functions, we will consider those continuous with finite values such that $\Phi(x)/x \to \infty$ as $x \to \infty$ (for stability reasons w.r.t. duality). When additionally $\Phi(x) = 0 \Leftrightarrow x = 0$ and $\Phi'(0_+) = 0$, Φ is called a *N-function* (using the notation of [28]).

For any lower semicontinuous Young function Φ (in particular if Φ has finite values), the conjugate of Φ^* is Φ. The pair (Φ, Ψ) is said to be a *complementary pair* if $\Psi = \Phi^*$ (or equivalently $\Phi = \Psi^*$). When $\Phi(1) + \Phi^*(1) = 1$, the pair (Φ, Φ^*) is said to be *normalized.* The conjugate of an N-function is an N-function.

We say that a Young function Φ satisfies the Δ_2 condition, if for some B and all $x \geq 0$, $\Phi(2x) \leq B\Phi(x)$.

The simplest example of N-function is $\Phi(x) = \frac{|x|^p}{p}$, $p > 1$, in which case, $\Phi^*(x) = \frac{|x|^q}{q}$, with $1/p + 1/q = 1$. The function $\Phi(x) = |x|^\alpha \ln(1 + |x|)^\beta$ is also a Young function for $\alpha \geq 1$ and $\beta \geq 0$ and an N-function when $\alpha > 1$ or $\beta > 0$.

Now let $(\mathcal{X}, \mu)$ be a measurable space, and Φ a Young function. The space

$$\mathbb{L}_\Phi(\mu) = \left\{ f : \mathcal{X} \to \mathbb{R} \text{ measurable}; \exists \alpha > 0, \int_{\mathcal{X}} \Phi(\alpha f) < +\infty \right\}$$

is called the *Orlicz space* associated to Φ. When $\Phi(x) = |x|^p$, then $\mathbb{L}_\Phi(\mu)$ is the standard Lebesgue space $\mathbb{L}_p(\mu)$.

We introduce the following Luxembourg norm, which gives to $\mathbb{L}_\Phi(\mu)$ a structure of Banach space,

$$\|f\|_\Phi = \inf \left\{ \lambda > 0; \int_{\mathcal{X}} \Phi\Big(\frac{f}{\lambda}\Big) d\mu \leq 1 \right\}.$$

Note that we changed the notation with respect to [28].

Comparison of norms

In what follows, we will often have to compare Orlicz norms associated to different Young functions. Let us notice that any Young function Φ satisfies $|x| = O\left(\Phi(x)\right)$ as x goes to ∞. It leads to the following lemma:

Lemma 9.2. *Any Orlicz space may be continuously embedded in $\mathbb{L}_1$. More precisely, let M and τ in $(0,\infty)$ such that $|x| \le \tau\,\Phi(x)$ for any $|x| \ge M$. Then, for any $f \in \mathbb{L}_\Phi$,*

$$\|f\|_1 \le (M+\tau)\,\|f\|_\Phi. \tag{9.1}$$

Consequently, if Φ and Ψ are two Young functions satisfying, for some constants $A, B \ge 0$, $\Phi(x) \le A|x| + B\Psi(x)$, then

$$\|f\|_\Phi \le \max\left(1, A\|\mathrm{Id}\|_{\mathbb{L}_\Psi \to \mathbb{L}_1} + B\right)\|f\|_\Psi. \tag{9.2}$$

Remark 9.3. When $\Phi(x)/x \to \infty$ as $x \to \infty$, we may choose $\tau = 1$ or any other positive constant. We get in particular the estimate

$$\left\|\mu(f)^2\right\|_\Phi \le (M+1)\,\|\mathbb{1}\|_\Phi \left\|f^2\right\|_\Phi, \tag{9.3}$$

where M is such that $|x| \le \Phi(x)$ for any $|x| \ge M$. For any constant C, $\|C\|_\Phi = C\|\mathbb{1}\|_\Phi$ trivially.

Proof of Lemma 9.2. Let $f \in \mathbb{L}_\Phi(\mu)$. By homogeneity, we may assume that $\|f\|_\Phi = 1$. Then $\int \Phi(f)\,d\mu \le 1$ and so

$$\begin{aligned}\int |f|\,d\mu &= \int_{\{|f|\le M\}} |f|\,d\mu + \int_{\{|f|\ge M\}} |f|\,d\mu \\ &\le M\mu\left(|f| \le M\right) + \tau \int_{\{|f|\ge M\}} \Phi(f)\,d\mu \le M + \tau.\end{aligned}$$

To get bound (9.2), assume now that $\|f\|_\Psi = 1$ and hence $\int \Psi(f)\,d\mu \le 1$ as well. For any $\lambda \ge 1$,

$$\int \Phi\left(f/\lambda\right)d\mu \le \frac{A}{\lambda}\|f\|_1 + B\int \Psi(f/\lambda)\,d\mu \le \frac{A}{\lambda}\|\mathrm{Id}\|_{\mathbb{L}_\Psi\to\mathbb{L}_1}\|f\|_\Psi + \frac{B}{\lambda}\int \Psi(f)d\mu \le 1$$

provided $\lambda \ge A\,\|\mathrm{Id}\|_{\mathbb{L}_\Psi\to\mathbb{L}_1} + B$. Note that for the second inequality we used convexity of Ψ. □

10. Appendix II: Example

We introduce here a prototype of function F which satisfies conditions (**C1**) to (**C3**) of Section 2. Fix $\theta > 2$, $\alpha \in (1,2]$ and consider a function $F_\alpha : \mathbb{R}^+ \to \mathbb{R}$

$$x \quad \mapsto \quad F_\alpha(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\theta], \\ (\log(x))^\beta - (\log\theta)^\beta & \text{if} \quad x \ge \theta. \end{cases}$$

where $\beta \equiv 2(1-\frac{1}{\alpha}) \in (0,1)$. Note that F_α is continuous, but not $\mathcal{C}^2$. To deal with differentiability at $x = \theta$ we introduce a $\mathcal{C}^\infty$ non-negative function g with compact support in $[-1,0]$ and such that $\int g(y)dy = 1$. For $\varepsilon > 0$, define $g_\varepsilon(x) = \frac{1}{\varepsilon}g(\frac{x}{\varepsilon})$.

Then $F(x) \equiv F_\alpha * g_\varepsilon(x) := \int F_\alpha(x-y)g_\varepsilon(y)dy$ is a $\mathcal{C}^\infty$ function vanishing on $[0,\theta-\varepsilon]$. Let us stress that the particular regularization of the function F above is not important. Many other regularizations would do the job.

After some rather standard computations left to the reader, one can check that F satisfies Condition (**C1**), (**C2**) and (**C3**) with $\bar{B}=1$ and $R=(\log\theta)^\beta$ provided that $\theta \geq e^{2(1-\beta)}$.

Finally, we remark that for F_α and the measure $d\mu_\alpha \equiv \exp\{-|x|^\alpha\}dx/Z_\alpha$, the inequality (**FS**) is true [4]. Hence, after some computations left to the reader, we conclude that corresponding coercive inequality is satisfied also with the function F (possibly with a different constant).

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