On curvature and the bilinear multiplier problem

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Abstract. We provide sufficient normal curvature conditions on the boundary of a domain $D \subset \mathbb{R}^4$ to guarantee unboundedness of the bilinear Fourier multiplier operator T_D with symbol χ_D outside the local L^2 setting, *i.e.*, from $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \to L^{p'_3}(\mathbb{R}^2)$ with $\sum \frac{1}{p_j} = 1$ and $p_j < 2$ for some j. In particular, these curvature conditions are satisfied by any domain D that is locally strictly convex at a single boundary point.

1. Introduction

The celebrated ball multiplier theorem of C. Fefferman ([4]) states that the characteristic function of the unit ball B_d in $\mathbb{R}^d$, $d \geq 2$, is not a bounded Fourier multiplier on $L^p(\mathbb{R}^d)$ for $p \neq 2$. As an immediate corollary of the proof, one obtains the corresponding result with the ball replaced by any connected domain D in $\mathbb{R}^d$ whose boundary is a sufficiently smooth hypersurface with nonzero second fundamental form (or equivalently a nonzero principal curvature) at some point.

Interest has arisen in studying analogues of the ball multiplier question in the bilinear setting; namely, given a domain $D \subset \mathbb{R}^{2d}$, one may ask whether the bilinear Fourier multiplier $\mathrm{T}_D : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ given by

$$\mathrm{T}_D \left(f, g\right)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_D(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i (\xi + \eta) \cdot x} \, \mathrm{d}\xi \, \mathrm{d}\eta$$

extends to a bounded bilinear operator from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for suitable ranges of p, q and r; here χ_D denotes the characteristic function of D, and $\mathcal{S}$ denotes the space of Schwartz test functions. For dimension $d = 1$, the case of $D = B_2$ the unit disc of $\mathbb{R}^2$ was treated by Grafakos and Li, who showed in [6] that in fact T_{B_2} *is* a bounded operator from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$ in the local L^2 case (*i.e.*, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1$ with $p, q, r' \geq 2$). However, the status of the bilinear disc multiplier *outside* the local L^2 case remains unknown as of this writing, and for the majority of this paper we concern ourselves only with dimensions $d \geq 2$.

In the linear setting, Fefferman's theorem and the boundedness of the Hilbert transform give the following dichotomy: "Polyhedral" domains (with finitely many faces) yield bounded Fourier multipliers, while domains whose boundaries possess curvature (or simply a suitably rich collection of tangent hyperplanes) give rise to unbounded multipliers, as noted above. By contrast, the situation is less well-understood in the bilinear setting, largely because the boundedness properties of even the half-space multiplier operators $T_{P_{\vec{v}}}$ are not well-understood for $d \geq 2$; here $P_{\vec{v}} = \left\{\vec{\xi} \in \mathbb{R}^{2d} \mid \vec{\xi} \cdot \vec{v} > 0\right\}$ (Theorem 5.1 below provides an unboundedness result for $T_{P_{\vec{v}}}$ in a rather limited range of exponents). These operators are essentially higher-dimensional analogues of the bilinear Hilbert transform and are of independent interest (see Section 2; see also [3] for a discussion of related operators and connections to ergodic theory).

Nonetheless, in high dimensions the ideas of Fefferman's original argument have been successfully adapted to yield unboundedness results for bilinear Fourier multipliers associated to domains with boundary curvature. For $d \geq 2$ and B_{2d} the unit ball of $\mathbb{R}^{2d}$, Diestel and Grafakos ([2]) proved that $T_{B_{2d}}$ is not a bounded bilinear Fourier multiplier *outside* the local L^2 setting; in [8], Grafakos and Reguera generalized this result to replace the ball B_{2d} with a compact, strictly convex domain D whose boundary is a smooth hypersurface in $\mathbb{R}^{2d}$.

For both the statements and the proofs of our results, we will adopt a symmetric presentation in terms of trilinear forms rather than bilinear operators; this approach rids us of the inconvenience of dealing with duality, and more importantly it has the decided benefit of placing our curvature conditions below in a natural geometric setting. For now we restrict our attention to dimension $d = 2$; see Remark 2) of subsection 6.2 for a discussion of higher dimensions. To any bilinear operator $T : \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ we can associate a trilinear form Λ on $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$ defined by

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^2} T(f_1, f_2)(x) f_3(x) \, \mathrm{d}x.$$

For triples $\vec{p} = (p_1, p_2, p_3)$ with $1 \leq p_j \leq \infty$ for all j, the boundedness of T from $L^{p_1} \times L^{p_2}$ to $L^{p_3'}$ is equivalent to the boundedness of Λ on $L^{p_1} \times L^{p_2} \times L^{p_3}$:

$$\left|\Lambda(f_1, f_2, f_3)\right| \leq \|T\| \prod_{j=1}^{3} \|f_j\|_{p_j};$$

in this case, we say that the form Λ is of "type $\vec{p}$". The natural range of exponent triples $\vec{p}$ under consideration is given by demanding that the trivial form $\Lambda_0(f_1, f_2, f_3) := \int f_1 f_2 f_3$ be bounded; *viz.*, we consider only "homogeneous" $\vec{p}$ with $\sum \frac{1}{p_j} = 1$.

For the bilinear Fourier multiplier operators T_D as above, the associated trilinear forms are given by embedding D into $\mathbb{R}^6$ as follows: Let Γ be the subspace

$$\Gamma := \left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \xi_1 + \xi_2 + \xi_3 = 0\right\} \subset \mathbb{R}^6,$$

with $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \Gamma$ the obvious isomorphism given by

$$\Phi(\xi_1, \xi_2) = \big(\xi_1, \xi_2, -(\xi_1 + \xi_2)\big).$$

Then, for $D \subset \mathbb{R}^4$, the trilinear form associated to T_D is

$$\Lambda_{\Phi(D)}(f_1, f_2, f_3) = \iiint \delta(\xi_1 + \xi_2 + \xi_3)\chi_{\Phi(D)}(\xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} \widehat{f}_j(\xi_j) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

$$= \int_\Gamma \chi_{\Phi(D)} \, (\widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3) \, \mathrm{d}\sigma.$$

The measure $\mathrm{d}\sigma = \delta(\xi_1 + \xi_2 + \xi_3) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$ appearing here is simply the push-forward $\Phi_*(\mathrm{d}\xi_1 \mathrm{d}\xi_2)$ of Lebesgue measure on $\mathbb{R}^2 \times \mathbb{R}^2$, viewing Φ as a map into $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. In the sequel, we will always identify $\mathbb{R}^4$ with $\mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^6$ with $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, and by a "$j$-th coordinate slice" in $\mathbb{R}^6$ we mean a 4-plane of the form

$$\big\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^6 \mid \xi_j = \xi_0\big\}$$

for some fixed $\xi_0 \in \mathbb{R}^2$. Note that the intersection of Γ with any j-th coordinate slice is a 2-plane.

Our main result is the following:

Main Theorem. *Let $\widetilde{D}$ be a domain in $\Gamma \subset \mathbb{R}^6$ such that $\partial\widetilde{D} \cap U$ is a smooth, connected (three-dimensional) hypersurface for some open neighborhood $U \subset \Gamma$. Suppose that for some $j \in \{1, 2, 3\}$ the intersection of $\partial\widetilde{D} \cap U$ with some j-th coordinate slice is a plane curve of nonzero curvature. Then the trilinear form $\Lambda_{\widetilde{D}}$ fails to be of type $\vec{p} = (p_1, p_2, p_3)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $1 < p_1, p_2, p_3 < \infty$, and $p_i < 2$ for some $i \neq j$.*

Of course, this can be translated to a direct statement about bilinear multiplier operators associated to domains $D \subset \mathbb{R}^4$ by applying the theorem to $\Phi(D)$. First and second coordinate slices in Γ correspond to their natural analogues in $\mathbb{R}^4$, while third coordinate slices correspond to 2-planes of the form $\{(\xi_1, \xi_2) \in \mathbb{R}^4 \mid \xi_1 + \xi_2 = \text{constant}\}$. Since any strictly convex set D is easily seen to satisfy all three of the given curvature conditions, we obtain the following generalization of the Grafakos–Reguera result:

Corollary 1.1. *Let D be a domain in $\mathbb{R}^4$ whose boundary ∂D is smooth in some neighborhood $U \subset \mathbb{R}^4$, and suppose that either D or $\mathbb{R}^4 \setminus D$ is strictly convex in this neighborhood. Then for $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ with exactly one of p_1, p_2, p_3 less than 2, T_D does not extend to a bounded bilinear Fourier multiplier from $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2)$ to $L^{p_3'}(\mathbb{R}^2)$.*

The Main Theorem above can actually be stated more generally; namely, under essentially the same hypotheses on $\widetilde{D}$ one can also prove unboundedness of the operator T_D outside the interior of the "Banach triangle" (*i.e.*, from $L^{p_1} \times L^{p_2}$ to $L^{p_3'}$ with $p_3' \leq 1$, so that $p_3 = \infty$ or $p_3 < 0$). However, a symmetric statement

in terms of trilinear forms presents some difficulties, as the "type $\vec{p}$" formalism breaks down outside the Banach triangle; to avoid introducing potentially confusing technicalities at this point, we defer the statement of the general result to Section 5 below.

The idea of the proof of the Main Theorem is quite simple; heuristically speaking, our approach is simply to apply Fefferman's original argument on the appropriate coordinate slice. More specifically, assuming boundedness of $\Lambda_{\widetilde{D}}$, we first obtain a square function estimate for a family of trilinear forms associated to a family of half-spaces in Γ. To complete the proof, we produce a Besicovitch set-based counterexample to this estimate.

At this point, given the ease with which one can apply Fefferman's argument for the ball to more general domains in the linear setting, the reader may be skeptical as to the necessity of any further discussion once one has established the unboundedness of the ball bilinear multiplier. The key feature distinguishing the bilinear multiplier problem from the linear one here is a marked decrease in symmetry with respect to actions on the underlying Euclidean space. More specifically, the class of linear L^p-Fourier multipliers on $\mathbb{R}^d$ is invariant under the natural action of the isometry group $O_d(\mathbb{R}) \ltimes \mathbb{R}^d$ of $\mathbb{R}^d$ (and in fact under the full affine group $GL_d(\mathbb{R}) \ltimes \mathbb{R}^d$); however, the class of *bilinear* $L^p \times L^q \to L^r$-Fourier multipliers on $\mathbb{R}^d$ is not invariant under the usual rotation action of $SO_{2d}(\mathbb{R})$.[1] In fact, it is precisely this absence of SO_{2d}-invariance that prevents the proofs in [2] and [8] from extending easily to more general domains with curvature. To wit, though neither proof genuinely requires surjectivity of the Gauss map $N : \partial D \to S^{2d-1}$ (which is guaranteed by compactness and strict convexity), both arguments rely on the presence of a suitable collection of "projectively diagonal" normal vectors of the form $(v, \lambda v) \in \mathbb{R}^d \times \mathbb{R}^d$ (see Theorem 4.1 below). With such an approach, the ball appears to be a less generic example in the bilinear setting than in the linear one; in order to obtain a result treating more general domains that are merely strictly convex in a neighborhood of some arbitrary boundary point, one should avoid appealing to the full wealth of normal directions available on the sphere.

One might expect an "ideal" bilinear analogue of Fefferman's theorem (phrased in terms of operators rather than forms) to state that T_D is unbounded for any $D \subset \mathbb{R}^4$ whose boundary has some nonzero principal curvature at a point. However, the aforementioned loss of symmetry makes such an analogy impossible:

Proposition 1.2. *There exist domains $\widetilde{D} \subset \Gamma$ with smooth boundary such that $\partial\widetilde{D}$ has nontrivial second fundamental form at some point while $\Lambda_{\widetilde{D}}$ is of type (p_1, p_2, p_3) whenever $1 < p_1, p_2, p_3 < \infty$.*

Examples of such $\widetilde{D}$ are easily given by certain cylinder sets: A result of Muscalu (*viz.* Theorem 2.1.1 of [9]) gives the existence of domains $D_0 \subset \mathbb{R}^2$ with nontrivial boundary curvature for which the bilinear operators T_{D_0} are bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3'}(\mathbb{R})$ for all triples (p_1, p_2, p_3) as in the proposition.

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[1]This fact is well-known but seems to be folkloric; it can readily be observed by rotating the symbols of suitable operators falling under the scope of Lemma 1 of [6]. We provide yet another illustrative example in the discussion following Proposition 1.2 below.

From any such D_0, we construct the domain

$$D = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid (\xi_1, \xi_3) \in D_0 \right\}$$

(here we have of course broken with our convention of always writing $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$). Then it is easy to check that

$$\mathrm{T}_D \left(f_1, f_2 \right)(x_1, x_2) = \mathrm{T}_{D_0} \left(f_1(\,\cdot\,, x_2), \, f_2(\,\cdot\,, x_2) \right)(x_1),$$

so that boundedness of T_{D_0} gives the desired boundedness of T_D and the usual associated trilinear form $\Lambda_{\Phi(D)}$. These "degenerate" examples of course illustrate the aforementioned anisotropy of the bilinear setting; their curvature is restricted to planes of the form $\{\vec{\xi} \in \mathbb{R}^4 \mid (\xi_2, \xi_4) = (a, b)\}$ for fixed $(a, b) \in \mathbb{R}^2$, and they can be suitably rotated to fall under the scope of the Main Theorem.

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2. Notations and preliminaries

As in the linear case treated by Fefferman, the key feature obstructing boundedness of T_D (or of $\Lambda_{\Phi(D)}$) is the fact that ∂D (or $\partial \Phi(D)$) possesses many suitable tangent hyperplanes; we pause now to establish some notation and isolate the geometric properties we will exploit. As noted above, we identify $\mathbb{R}^6$ with $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$; symbols such as $\vec{\xi}$ and $\vec{v}$ will denote points and vectors in $\mathbb{R}^6$, while ξ and v will denote points and vectors in $\mathbb{R}^2$. Families of points or vectors will be indexed by superscripts, so that, for example,

$$\vec{\xi}^n = \left(\xi_1^n, \xi_2^n, \xi_3^n \right).$$

For two quantities A and B, we take $A \lesssim B$ to mean $A \leq cB$ for some constant c; when necessary, dependence of implied constants on certain parameters will be denoted by subscripts on "$\lesssim$."

Consider a domain $\widetilde{D} \subset \Gamma$ as in the Main Theorem, and suppose for the moment that $\partial \widetilde{D}$ has curvature in a *first* coordinate slice in $\mathbb{R}^6$. Then, for some small choice of $\theta_0 > 0$, we can take a continuum $\{\vec{\xi}^\theta \mid -\theta_0 \leq \theta \leq \theta_0\}$ of points on $\partial \widetilde{D}$ with the following properties:

- $\vec{\xi}^\theta = \left(\xi_1^0, \xi_2^\theta, \xi_3^\theta \right)$ for all θ.

- Let $\vec{v}^\theta = (v_1^\theta, v_2^\theta, v_3^\theta)$ denote a normal vector to $\partial \widetilde{D}$ at the point $\vec{\xi}^\theta$. ($\partial \widetilde{D}$ is a three-dimensional submanifold of the four-dimensional subspace $\Gamma \subset \mathbb{R}^6$; we define our normal vectors in this context, and all normal vectors are of

course chosen consistently with the orientation of $\partial \widetilde{D}$.) Elementary linear algebra shows that the projection of $\vec{v}^\theta$ to the 2-plane $\{(0, v, -v)\} \subset \Gamma$ is

$$\frac{1}{\sqrt{2}} \left(0, w^\theta, -w^\theta\right) := \frac{1}{\sqrt{2}} \left(0, v_2^\theta - v_3^\theta, v_3^\theta - v_2^\theta\right),$$

and we normalize $\vec{v}^\theta$ so that $|w^\theta| = 1$ for all θ. By the curvature condition on $\partial \widetilde{D}$, we can arrange that

$$w^\theta = \gamma_\theta w^0,$$

where $\gamma_\theta \in \mathrm{SO}_2(\mathbb{R})$ denotes rotation by the angle θ.

This discussion may seem a bit cumbersome; the salient point here is that, by the standard "Perron tree" construction (see *e.g.* [10]), the collection

$$\left\{ w^\theta = v_2^\theta - v_3^\theta \mid -\theta_0 \le \theta \le \theta_0 \right\}$$

"yields Besicovitch sets" in $\mathbb{R}^2$ in the following sense:

Definition 2.1. A family $\mathcal{F}$ of unit vectors in $\mathbb{R}^2$ *yields Besicovitch sets* if for every $\varepsilon > 0$ there is a set $K_\varepsilon \subset \mathbb{R}^2$ such that:

1. $K_\varepsilon = \bigcup_{n=1}^N R_n$ for some N depending on ε, where each R_n is a rectangle of dimensions $1 \times \frac{1}{N}$. The length-1 sides of each R_n point in the direction of some $v_n \in \mathcal{F}$.

2. $|K_\varepsilon| < \varepsilon$.

3. The rectangles R_n', $1 \le n \le N$, are disjoint, where R_n' is obtained by translating R_n by the vector $-2v_n$. (Since each v_n is a unit vector, R_n and R_n' are "reaches" of one another, in the terminology of [10].)

4. There is a fixed compact set K^* independent of ε such that $K_\varepsilon \subset K^*$.

In general, if $\widetilde{D}$ is as in the Main Theorem with curvature in a j_0-th coordinate slice, then $\widetilde{D}$ enjoys the following property:

Property 2.2. Given $\varepsilon > 0$, there is a Besicovitch set $K_\varepsilon = \bigcup_1^N R_n \subset K^*$ as above and a sequence of points $\vec{\xi}^1, \ldots, \vec{\xi}^N \in \partial \widetilde{D}$ such that:

- $\vec{\xi}^n = \left(\xi_1^n, \xi_2^n, \xi_3^n\right)$ with $\xi_{j_0}^n = \xi_0$ for all n.

- For all n, there is a normal vector $\vec{v}^n = \left(v_1^n, v_2^n, v_3^n\right)$ to $\partial \widetilde{D}$ at $\vec{\xi}^n$ such that the length-1 side of the rectangle R_n is parallel to the vector

$$w_{j_0}^n := v_{\sigma(j_0)}^n - v_{\sigma^2(j_0)}^n \in \mathbb{R}^2,$$

where σ is the cycle $(1\ 2\ 3)$ in the permutation group S_3, and $|w_{j_0}^n| = 1$.

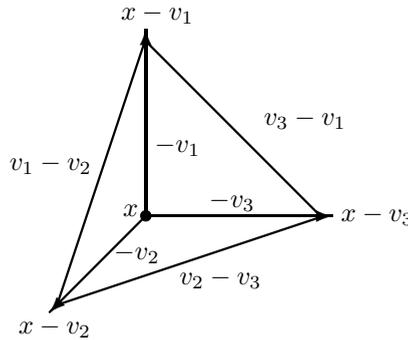- The vectors $\vec{v}^n$ all lie in a compact set A^*, which is independent of ε.

FIGURE 1: "Configuration triangle".

Finally, for $\vec{v} \in \Gamma$, consider a half-space $P_{\vec{v}} = \{\vec{\xi} \in \Gamma \mid \vec{\xi} \cdot \vec{v} > 0\}$ with associated trilinear form

$$\Lambda_{P_{\vec{v}}}(f_1, f_2, f_3) = \iiint \delta(\xi_1 + \xi_2 + \xi_3) \, \chi_{P_{\vec{v}}}(\vec{\xi}) \prod_j \widehat{f}_j(\xi_j) \, d\vec{\xi}$$

as above. It is a matter of routine to show that the trilinear form

$$\widetilde{\Lambda}_{\vec{v}}(f_1, f_2, f_3) := \int_{\mathbb{R}} \int_{\mathbb{R}^2} \prod_j f_j(x - tv_j) \, dx \, \frac{dt}{t}$$

is a linear combination of $\Lambda_{P_{\vec{v}}}$ and the pointwise-product trilinear form; here the integral in t is taken in the principal value sense. The forms $\widetilde{\Lambda}_{\vec{v}}$ are of course parameterized by the vectors $\vec{v} \in \Gamma$; it will also prove useful to view them as parameterized by the triangles (or similarity classes of triangles) in $\mathbb{R}^2$ determined by the vectors $v_j - v_{\sigma(j)}$ (see Figure 1). The bilinear operator associated to the form $\Lambda_{P_{\vec{v}}}$ is the half-space Fourier multiplier $T_{P_{\vec{w}}}$ on $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$ mentioned in the introduction, where $\vec{w} = \Phi^*(\vec{v}) \in \mathbb{R}^4$. Similarly, the bilinear operator associated to $\widetilde{\Lambda}_{\vec{v}}$ is given by

$$S_{\vec{w}}(f_1, f_2)(x) := \text{p. v.} \int_{\mathbb{R}} f_1(x - tw_1) f_2(x - tw_2) \, \frac{dt}{t};$$

as mentioned above, $S_{\vec{w}}$ may be viewed as a two-dimensional variant of a bilinear Hilbert transform.

3. Square function estimates

The results of this section are direct analogues of the lemma of Y. Meyer used in Fefferman's proof (Lemma 1 of [4]), and their proofs are essentially identical to that of the latter. We begin with a domain $\widetilde{D} \subset \Gamma$ and a sequence of points $\vec{\xi}^n \in \partial \widetilde{D}$ at which $\partial \widetilde{D}$ has normal vectors $\vec{v}^n$. Let Λ_n denote the trilinear form $\Lambda_{P_{\vec{v}^n}}$ associated to the half-space $P_n := P_{\vec{v}^n}$, and set $\widetilde{\Lambda}_n := \widetilde{\Lambda}_{\vec{v}^n}$ as in Section 2 above. As usual, the

main idea is that, by the translation- and dilation-invariance of multiplier norms, boundedness of the trilinear form $\Lambda_{\widetilde{D}}$ will yield strong uniform bounds (in fact ℓ^2 vector-valued bounds) for the forms Λ_n or $\widetilde{\Lambda}_n$.

Lemma 3.1. *Let $0 < p_1, p_2, p_3 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, and suppose*

$$|\Lambda_{\widetilde{D}}(f_1, f_2, f_2)| \lesssim \prod_{j=1}^{3} \|f_j\|_{p_j}$$

for all measurable functions f_1, f_2, f_3 on $\mathbb{R}^2$. Let $\vec{\xi}^{\,n}$ and $\widetilde{\Lambda}_n$ be as given above. Then[2]:

(a) *For all sequences of measurable functions f_1^n, f_2^n, f_3^n on $\mathbb{R}^2$ we have the estimate*

$$(3.1) \qquad \sum_n |\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)| \lesssim_{p_1,p_2,p_3} \prod_{j=1}^{3} \left\| \left(\sum_n |f_j^n|^2 \right)^{1/2} \right\|_{p_j}.$$

(b) *Suppose further that for some $j_0 \in \{1, 2, 3\}$ we have $\xi_{j_0}^n = \xi_0$ for all n, and let f_1^n, f_2^n, and f_3^n be sequences of measurable functions such that $f_{j_0}^n = f_0$ for all n. Then we have the estimate*

$$(3.2) \qquad \sum_n |\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)| \lesssim_{p_1,p_2,p_3} \|f_0\|_{p_{j_0}} \prod_{j \neq j_0} \left\| \left(\sum_n |f_j^n|^2 \right)^{1/2} \right\|_{p_j}.$$

The point of part (b) of this lemma is that we can afford to replace one of the sequences $(f_j^n)_n \in L^{p_j}(\mathbb{R}^2, \ell^2)$ from part (a) with a single function $f_0 \in L^{p_j}(\mathbb{R}^2, \mathbb{C})$, at the expense of requiring an extra condition on the boundary points $\vec{\xi}^{\,n}$; this burden will account for the coordinate-slice restriction in the curvature conditions of the Main Theorem.

Proof. For $r > 0$, let $\widetilde{D}_r$ denote the r-dilate $\{r\vec{\xi} \mid \vec{\xi} \in \widetilde{D}\}$ of $\widetilde{D}$, and set $\widetilde{D}_{r,n} = \widetilde{D}_r - r\vec{\xi}^{\,n}$, so that

$$\chi_{\widetilde{D}_{r,n}} \longrightarrow \chi_{P_n}$$

pointwise almost everywhere on Γ as $r \to \infty$. Then by dominated convergence we have

$$\Lambda_n(f_1, f_2, f_3) = \lim_{r \to \infty} \Lambda_{\widetilde{D}_{r,n}}(f_1, f_2, f_3) = \lim_{r \to \infty} \Lambda_{\widetilde{D}_r}(\mathrm{M}_{r\xi_1^n} f_1, \mathrm{M}_{r\xi_2^n} f_2, \mathrm{M}_{r\xi_3^n} f_3)$$

for all Schwartz functions f_1, f_2 and f_3, where M_ξ denotes the modulation operator defined by $\mathrm{M}_\xi f(x) = e^{2\pi i \xi \cdot x} f(x)$.

Now, since we assume boundedness of the trilinear form $\Lambda_{\widetilde{D}}$, the forms $\Lambda_{\widetilde{D}_r}$ are *uniformly* bounded on $L^{p_1} \times L^{p_2} \times L^{p_3}$, due to the dilation-invariance of Fourier

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[2]Part (a) of this lemma was originally proved in [2] and [8]. We do not use it in the proof of the Main Theorem; however, see the discussion after the proof in Section 4.

multiplier operator norms. Summing in n and appealing to Theorem[3] 6 of [7] (together with a straightforward application of duality), we obtain

$$\sum_n |\Lambda_n(f_1^n, f_2^n, f_3^n)| = \lim_{r \to \infty} \sum_n |\Lambda_{\widetilde{D}_r}(\mathrm{M}_{r\xi_1^n} f_1^n, \mathrm{M}_{r\xi_2^n} f_2^n, \mathrm{M}_{r\xi_3^n} f_3^n)|$$

$$\lesssim_{p_1,p_2,p_3} \lim_{r \to \infty} \prod_{j=1}^3 \Big\| \Big(\sum_k |\mathrm{M}_{r\xi_j^n} f_j^n|^2 \Big)^{1/2} \Big\|_{p_j} = \prod_{j=1}^3 \Big\| \Big(\sum_n |f_j^n|^2 \Big)^{1/2} \Big\|_{p_j}$$

for all sequences $f_1^n, f_2^n, f_3^n \in \mathcal{S}(\mathbb{R}^2)$; this is the estimate used in [2] and [8]. Part (a) of the lemma follows immediately, since $\widetilde{\Lambda}_n$ is a linear combination of Λ_n and the pointwise-product trilinear form.

We now turn to part (b). Fix $j_0 \in \{1, 2, 3\}$ such that the points $\vec{\xi}^n \in \partial\widetilde{D}$ above satisfy $\xi_{j_0}^n = \xi_0$ for all n, and consider three sequences f_1^n, f_2^n, f_3^n as before, with the additional caveat that $f_{j_0}^n = f_0$ for all n. Then for each fixed r, one of the arguments of

$$\Lambda_{\widetilde{D}_r}\big(\mathrm{M}_{r\xi_1^n} f_1^n, \mathrm{M}_{r\xi_2^n} f_2^n, \mathrm{M}_{r\xi_3^n} f_3^n \big)$$

is constant in n, so we may view this expression as a *bilinear* form in the other two arguments; let us denote it as $\overline{\Lambda}_r(\mathrm{M}_{r\xi_j^n} f_j^n, \mathrm{M}_{r\xi_k^n} f_k^n)$, where $j \neq j_0 \neq k$, and notice that the form $\overline{\Lambda}_r$ satisfies the estimate

$$\big| \overline{\Lambda}_r(g, h) \big| \lesssim \|f_0\|_{p_{j_0}} \|g\|_{p_j} \|h\|_{p_k}$$

uniformly in r, due to the original boundedness assumption on $\Lambda_{\widetilde{D}}$. Appealing to square function estimates for *linear* operators[4] in lieu of the Grafakos–Martell estimate used above, we may proceed as before to obtain

$$\sum_n |\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)| \lesssim_{p_1,p_2,p_3} \|f_0\|_{p_{j_0}} \prod_{j \neq j_0} \Big\| \Big(\sum_n |f_j^n|^2 \Big)^{1/2} \Big\|_{p_j}$$

as desired; the term $\|f_0\|_{p_{j_0}}$ appears via the aforementioned boundedness estimate for the bilinear forms $\overline{\Lambda}_r$. □

4. Proof of the Main Theorem

Let $\widetilde{D}$ be a domain in Γ with curvature in a j_0-th coordinate slice, and suppose toward a contradiction that we have the estimate

$$\big| \Lambda_{\widetilde{D}}(f_1, f_2, f_3) \big| \lesssim \prod_{j=1}^3 \|f_j\|_{p_j}.$$

---

[3]This theorem states that if $\mathrm{T} : L^{p_1}(X) \times L^{p_2}(X) \to L^{p_3'}(X)$ is bounded, then the natural vector-valued extension of T maps $L^{p_1}(X, \ell^2(\mathbb{N})) \times L^{p_2}(X, \ell^2(\mathbb{N}))$ to $L^{p_3'}(X, \ell^2(\mathbb{N} \times \mathbb{N}))$ continuously. This is a natural analogue of the classical square function estimate for linear operators used in [4]; like its linear predecessor, its proof is based on Khintchine's inequality.

[4]Cf. the previous footnote; we of course apply the dualized version of the classical square function estimate to each of the bilinear forms $\overline{\Lambda}_r$.

The goal now is to establish sufficient lower bounds for $\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)$ to contradict part (b) of Lemma 3.1; of course, this entails a suitable choice of the points $\vec{\xi}^n \in \partial\widetilde{D}$ that define $\widetilde{\Lambda}_n$, as well as a suitable choice of the functions f_j^n. As we resign ourselves to following Fefferman's approach, we will eventually take the f_j^n to be characteristic functions of aptly chosen rectangles in $\mathbb{R}^2$, and the lower bounds on $\widetilde{\Lambda}_n$ will just be pointwise lower bounds for the operators T_{P_n} in disguise.

Recall that for the moment we work within the Banach triangle outside the local L^2 setting; namely, we seek to prove the unboundedness of $\Lambda_{\widetilde{D}}$ on $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \times L^{p_3}(\mathbb{R}^2)$, with $\sum \frac{1}{p_j} = 1$ and $p_i < 2$ for exactly one $i \neq j_0$.[5] The key insight of Fefferman is that one can exploit Besicovitch sets to make the right-hand side of the square function estimate (3.2) arbitrarily small while keeping the left-hand side large; achieving this is only slightly more involved in our setting than in the linear. Indeed, let K_ε be a Besicovitch set as in Definition 2.1, with $K_\varepsilon = \bigcup_1^N R_n$. Then Hölder's inequality yields

$$(4.1) \qquad \left\|\left(\sum_n |\chi_{R_n}|^2\right)^{1/2}\right\|_{p_i} < \varepsilon^{\frac{2-p_i}{2p_i}},$$

which can be made arbitrarily small by decreasing ε, since $p_i < 2$. To exploit this estimate, we use the fact that $\partial\widetilde{D}$ has curvature in a j_0-th coordinate slice; taking any $\varepsilon > 0$, we have points $\vec{\xi}^1, \ldots, \vec{\xi}^N$, normal vectors $\vec{v}^1, \ldots, \vec{v}^N$, and a Besicovitch set $K_\varepsilon = \bigcup_1^N R_n$ provided by Property 2.2 from Section 2. In light of estimate (4.1) we set $f_i^n = \chi_{R_n}$, so that part (b) of Lemma 3.1 gives

$$(4.2) \qquad \sum_{n=1}^N \left|\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)\right| \lesssim \varepsilon^{\frac{2-p_i}{2p_i}} \|f_0\|_{p_{j_0}} \left\|\left(\sum_{n=1}^N |f_k^n|^2\right)^{1/2}\right\|_{p_k}$$

for all sequences of functions f_k^n and $f_{j_0}^n$ such that $f_{j_0}^n = f_0$ for all n; here $k \in \{1, 2, 3\}$ is the remaining index $j_0 \neq k \neq i$.

We now produce f_0 and f_k^n that will essentially maximize the left-hand side of (4.2). In fact, we will set $f_k^n = \chi_{R_n'}$, where R_n' is the reach[6] of R_n as given in Definition 2.1, and $f_0 = \chi_Q$ for some large rectangle Q to be determined. In this setting we have

$$\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \prod_{j=1}^3 f_j^n(x - tv_j^n)\, dx\, \frac{dt}{t}$$

$$= \iint \chi_{R_n}(x - tv_i^n)\, \chi_{R_n'}(x - tv_k^n)\, \chi_Q(x - tv_{j_0}^n)\, dx\, \frac{dt}{t}$$

$$(4.3) \qquad = \int_{\mathbb{R}} \int_{R_n} \chi_{R_n'}\big(x - t(v_k^n - v_i^n)\big)\, \chi_Q\big(x - t(v_{j_0}^n - v_i^n)\big)\, dx\, \frac{dt}{t}.$$

---

[5] For the remainder of the proof we suppress all dependence of constants on the exponents p_j.

[6] Of course, there are two choices of reach depending on our choice of orientation; we will always choose the one that is obviously expedient.
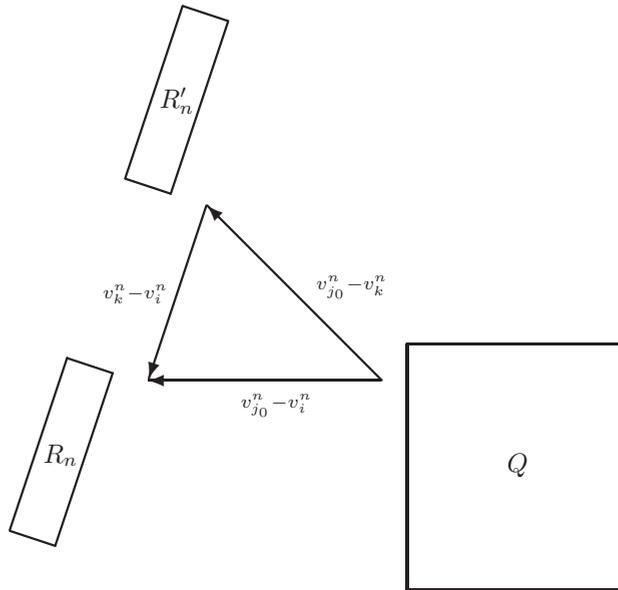
FIGURE 2: Estimating $\widetilde{\Lambda}_n$ on rectangles.

Note here that the vector $v_k^n - v_i^n$ is equal to $\pm(v_{\sigma(j_0)}^n - v_{\sigma^2(j_0)}^n)$, so that R_n is parallel to this vector with $R_n' = R_n - 2(v_k^n - v_i^n)$ by Property 2.2. Also note that $\vec{v}^n$ is chosen by design so that $|v_k^n - v_i^n| = 1$; here and in what follows, it may be helpful to think in terms of the configuration triangles of Section 2 (see Figure 2). Given these observations, it is clear from (4.3) that if we choose Q to be a large enough rectangle we have

$$(4.4) \qquad \left|\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)\right| \; \gtrsim \; |R_n| \; = \; \frac{1}{N}.$$

Moreover, recall that we have the provisos $K_\varepsilon \subset K^*$ for all ε and some fixed compact set K^*, and $\vec{v}^n \in A^*$ for some compact A^* which is again independent of ε. Thus we may in fact choose such a Q *independently* of n and ε; the important fact here is that we can set $f_{j_0}^n = f_0 := \chi_Q$ for all n and ensure

$$\|f_0\|_{p_{j_0}} \lesssim 1$$

independently of ε. If the preceding discussion seems lacking in motivation, the reader may note that in the heuristic limiting case of $Q = \mathbb{R}^2$, the bilinear forms

$$\Lambda_n'(f, g) := \widetilde{\Lambda}_n(f, g, \chi_{\mathbb{R}^2})$$

correspond to the directional Hilbert transforms (or linear half-space multipliers) used in Fefferman's original argument. (Note also that in the case $p_{j_0} = \infty$ this

observation can be used trivially to deduce unboundedness from Fefferman's proof.) Finally, we observe that since the rectangles R'_n are *disjoint*, we also have

$$\left\| \left(\sum_{n=1}^{N} |f_k^n|^2 \right)^{1/2} \right\|_{p_k} = \left\| \sum_{n=1}^{N} \chi_{R'_n} \right\|_{p_k} = 1.$$

Thus, combining estimates (4.2) and (4.4), we obtain

$$1 \lesssim \sum_{n=1}^{N} |\widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n)| \lesssim \varepsilon^{\frac{2-p_i}{2p_i}};$$

this renders our original boundedness assumption absurd and completes the proof of the Main Theorem.   □

For the sake of contrast, we now briefly summarize the approach of [2] and [8], which actually yields the following:

Theorem 4.1. *Let $\widetilde{D}$ be a domain in Γ such that $\partial\widetilde{D}$ possesses a family of normal vectors*

$$\left\{ \vec{v}^{\,\theta} = \left(v_\theta, \lambda_\theta v_\theta, -(1+\lambda_\theta) v_\theta \right) \mid \theta \in \mathcal{I} \right\} \subset \Gamma$$

for some index set $\mathcal{I}$, where the collection $\{v_\theta \mid \theta \in \mathcal{I}\} \subset \mathbb{R}^2$ yields Besicovitch sets as in Definition 2.1, and $\lambda_\theta \sim 1$ for all $\theta \in \mathcal{I}$. Then the trilinear form $\Lambda_{\widetilde{D}}$ is not of type $\vec{p} = (p_1, p_2, p_3)$ whenever $\sum \frac{1}{p_j} = 1$ and $p_i < 2$ for some $i \in \{1, 2, 3\}$.

This approach also follows Fefferman's argument, using part (a) of Lemma 3.1 where we have used part (b). Note that appealing to part (a) allows one to eliminate any restriction on the points of $\partial\widetilde{D}$ at which the normal vectors $\vec{v}^{\,\theta}$ occur; in exchange, use of the square function estimate (3.1) forces a rather stringent condition on the normal vectors themselves (*viz.*, all three of the component vectors of $\vec{v}^{\,\theta}$ must be parallel in $\mathbb{R}^2$). To prove the theorem, as above one chooses appropriate normal vectors $\vec{v}^{\,n} = \vec{v}^{\,\theta_n}$ associated to some Besicovitch set $K_\varepsilon = \bigcup_1^N R_n$ and considers the forms $\widetilde{\Lambda}_n := \widetilde{\Lambda}_{\vec{v}^{\,n}}$. Again setting $f_i^n = \chi_{R_n}$, part (a) of Lemma 3.1 then gives

$$(4.5) \qquad \sum_{n=1}^{N} \left| \widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n) \right| \lesssim \varepsilon^{\frac{2-p_i}{2p_i}} \prod_{j \neq i} \left\| \left(\sum_{n=1}^{N} |f_j^n|^2 \right)^{1/2} \right\|_{p_j}.$$

At this point, if one wishes to follow Fefferman by setting $f_j^n = \chi_{Q_j^n}$ for some rectangles Q_j^n, any productive use of estimate (4.5) clearly prohibits one from taking $|Q_j^n| \gtrsim 1$. However, since the component vectors v_j^n of $\vec{v}^{\,n}$ are all parallel, the configuration triangle for $\widetilde{\Lambda}_n$ is degenerate (*i.e.*, all vertices are collinear); thus one may choose Q_j^n to be appropriate "reaches" of R_n for *both* $j \neq i$ and still obtain

$$\left| \widetilde{\Lambda}_n(f_1^n, f_2^n, f_3^n) \right| \gtrsim \frac{1}{N}$$
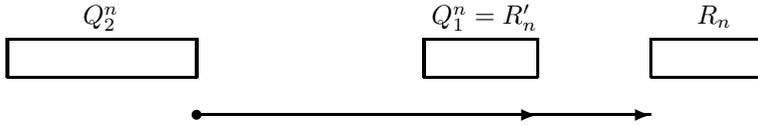
FIGURE 3: Degenerate configuration triangle.

as above. See Figure 3, which should be contrasted with Figure 2. Since we were able to choose $f_j^n = \chi_{Q_j^n}$ with $|Q_j^n| \sim 1/N$ and $\{Q_j^n\}_n$ disjoint for each $j \neq i$, the right-hand side of (4.5) is controlled by $\varepsilon^{\frac{2-p_i}{2p_i}}$, and we obtain a contradiction as above.

In fact, an examination of the geometric considerations in the proof shows that an approach toward contradicting part (a) of Lemma 3.1 essentially necessitates the use of degenerate configuration triangles, provided one insists on exploiting Besicovitch sets and taking the f_j^n to be characteristic functions of rectangles. Thus, with such an approach one cannot dispense with the restriction on the normal vectors appearing in Theorem 4.1; in particular, one cannot treat generic strictly convex domains in Γ.

5. Unboundedness on the border of and outside the Banach triangle

Our arguments thus far have been restricted to exponent triples $\vec{p} = (p_1, p_2, p_3)$ in the interior of the "Banach triangle" with $1 < p_1, p_2, p_3 < \infty$; as mentioned in the introduction, one can also obtain unboundedness results for $\vec{p}$ outside this range. However, to phrase such a general result in terms of trilinear forms, the notion of "type $\vec{p}$" is unsuitable; in fact, it is not hard to see that for $\vec{p}$ outside the Banach triangle the *only* trilinear form of type $\vec{p}$ is the 0 form (see *e.g.* Chapter 3 of [11]). On the other hand, it is perfectly reasonable (*i.e.* nontrivial) to investigate the boundedness of bilinear operators from $L^p \times L^q$ to L^r with $\frac{1}{2} \leq r \leq 1$ (so that $r' \leq -1$ or $r' = \infty$), and one would hope to be able to treat such questions symmetrically. One way of dealing with this state of affairs is to replace the notion of type $\vec{p}$ with that of "generalized restricted type $\vec{p}$", which we review below (once again, the reader may consult [11] for a more detailed treatment of this formalism).[7] Nonetheless, the reader will notice that we are forced to abandon the symmetric framework of trilinear forms when treating the boundary of the Banach triangle, where $p_j = \infty$ for some j.

Let us call an exponent triple $\vec{p} = (p_1, p_2, p_3)$ *admissible* if $|p_k| \geq 1$ for all k, $p_j \leq -1$ for at most one j, and $\sum_{k=1}^{3} p_k^{-1} = 1$; the p_k are of course allowed to be

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[7]Our notation is "reciprocal" to that of [11]; in the notation therein, our "generalized restricted type $\vec{p}$" corresponds to generalized restricted type $\alpha = \left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \right)$.

infinite. For such $\vec{p}$, we say the form Λ is of *generalized restricted type* $\vec{p}$ if for all triples (E_1, E_2, E_3) of measurable subsets of $\mathbb{R}^2$ there exists a subset $\widetilde{E}_j \subset E_j$ with $|\widetilde{E}_j| \geq \frac{1}{2}|E_j|$ for which we have the estimate

$$\left|\Lambda(f_1, f_2, f_3)\right| \lesssim \prod_{k=1}^{3} |E_k|^{1/p_k}$$

whenever $|f_k| \leq \chi_{E_k}$ and moreover $|f_j| \leq \chi_{\widetilde{E}_j}$. (If in fact $p_k \geq 1$ for all k, then the exceptional index j may be chosen at will.) Of course, inside the Banach triangle generalized restricted type $\vec{p}$ is implied by type $\vec{p}$; a generalized restricted type estimate for a form Λ gives a restricted weak-type estimate for an appropriate bilinear dual of the operator associated to Λ.

Finally, we introduce some geometric terminology. We say that a vector $\vec{v} \in \Gamma$ is *degenerate* if $\vec{v} = (\lambda_1 v, \lambda_2 v, \lambda_3 v)$ for some $v \in S^1$ and some scalars $\lambda_j \in \mathbb{R}$. Furthermore, we say that $\vec{v}$ is *strongly degenerate* if $\lambda_j = 0$ for some j. A domain $\widetilde{D} \subset \Gamma$ is called *(strongly) degenerate* if *every* normal vector to $\partial\widetilde{D}$ is (strongly) degenerate. In the following discussion we will omit any consideration of strongly degenerate domains; if $\partial\widetilde{D}$ is smooth such domains are given by particular cylinder sets, and the boundedness properties of their associated forms (or operators) fall within the purview of the linear theory.

The following theorem shows that, in sufficiently nondegenerate cases, the boundary curvature of a domain is actually *irrelevant* to the boundedness properties of the associated trilinear form or bilinear operator.[8]

Theorem 5.1. *Let* $P_{\vec{v}} = \left\{\vec{\xi} \in \Gamma \,|\, \vec{\xi} \cdot \vec{v} > 0\right\}$ *be a nondegenerate half-space in* Γ, *and let* $\vec{p} = (p_1, p_2, p_3)$ *be an admissible triple.*

(a) *If* $p_i \leq -1$ *for some* i, *the trilinear form* $\Lambda_{P_{\vec{v}}}$ *is not of generalized restricted type* $\vec{p}$.

(b) *If* $p_i = \infty$ *for some* i, *the bilinear multiplier operator* $\mathrm{T}_{\Phi^{-1}(P_{\vec{v}})}$ *associated to the standard preimage of* $P_{\vec{v}}$ *in* $\mathbb{R}^4$ *is unbounded from* $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2)$ *to* $L^{p_3'}(\mathbb{R}^2)$.

Proof. As usual, we will prove the equivalent statements for the trilinear form $\widetilde{\Lambda}_{\vec{v}}$ or the bilinear operator $\mathrm{S}_{\vec{w}}$ as in Section 2. To prove part (a), we proceed as in Figure 4. For $j \neq i \neq k$, set f_j to be the characteristic function of a rectangle R of width ε and length 1 oriented parallel to $v_j - v_k$, and let f_k be the characteristic function of its reach R' (as above, we normalize $\vec{v}$ so that $|v_j - v_k| = 1$). Let f_i be the characteristic function of a cube Q contained in the intersection of the strips $R + \mathbb{R} \cdot (v_j - v_i)$ and $R' + \mathbb{R} \cdot (v_k - v_i)$; choose Q to have measure comparable to 1. Computation yields

$$\left|\widetilde{\Lambda}_{\vec{v}}(f_1, f_2, f_3)\right| \gtrsim_{\vec{v}} \varepsilon,$$

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[8]It has been pointed out to the author by C. Thiele and C. Demeter that this result has been established independently as folklore, with essentially the same proof; the author is also indebted to C. Thiele for suggesting the argument used to prove part (b) of the theorem.
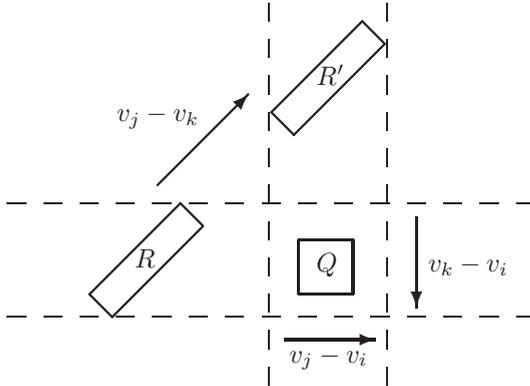
FIGURE 4: Failure of generalized restricted type for $\Lambda_{P_{\vec{v}}}$ outside the Banach triangle.

while

$$|R|^{1/p_j} \, |R'|^{1/p_k} \, |Q|^{1/p_i} \sim \varepsilon^{\frac{1}{p_j} + \frac{1}{p_k}} = \varepsilon^{1/p_i'}.$$

Both estimates continue to hold after the excision of any half-measure set from the cube Q, and sending ε to zero violates the generalized restricted type estimate since $\frac{1}{p_i'} > 1$.

This counterexample can be slightly modified to prove part (b). Since the nondegeneracy condition on $P_{\vec{v}}$ is preserved under duality (*i.e.*, permutation of the three coordinates in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$), we may assume that $p_3 = \infty$; thus it suffices to prove that $S_{\vec{w}}$ is unbounded from $L^p \times L^{p'}$ to L^1, where $\vec{w} = \Phi^*(\vec{v}) = (v_1 - v_3 \, , \, v_2 - v_3)$. As in the counterexample above, we take $f_1 = \chi_R$, where R is a rectangle of length 1 and width ε oriented parallel to $w_1 - w_2 = v_1 - v_2$; however, instead of obtaining f_2 from the reach of R, we simply set $f_2 = f_1 = \chi_R$. Computation then yields

$$\| \, S_{\vec{w}}(f_1, f_2) \|_1 \gtrsim_{\vec{w}} -\varepsilon \log \varepsilon,$$

while $\|f_1\|_p \, \|f_2\|_{p'} = \varepsilon$, and again sending ε to zero yields unboundedness. $\qquad\square$

One should note that the logarithmic divergence in the proof of part (b) occurs because $S_{\vec{w}}(\chi_R, \chi_R)$ is large on a region approaching the boundary of the rectangle R; thus if we study the expression

$$\widetilde{\Lambda}_{\vec{v}}(\chi_R, \chi_R, \chi_E) = \int_{\mathbb{R}^2} S_{\vec{w}}(\chi_R, \chi_R) \, \chi_E$$

for any measurable set E, we are not at liberty to delete an *arbitrary* half-measure subset of E and still obtain this divergence. Therefore, this counterexample cannot violate generalized restricted type $\vec{p}$ estimates for $\vec{p}$ on the boundary of the Banach

triangle; this state of affairs could be viewed as analogous to the fact that the Hilbert transform is unbounded on L^1 but is in fact of weak type $(1,1)$ (though we are of course making no claims of any such weak-type bounds in the present bilinear setting).

The reader should also note that the nondegeneracy assumption is not merely an artifact of the proof; indeed, if $P_{\vec{v}}$ is degenerate, the operator $\mathrm{S}_{\Phi^*(\vec{v})}$ inherits its boundedness properties from those of a one-dimensional bilinear Hilbert transform.

With this discussion completed, we can finally state the Main Theorem in full generality:

Main Theorem (General version). *Let $\widetilde{D}$ be a domain in $\Gamma \subset \mathbb{R}^6$ as in the Main Theorem above, with nontrivial curvature in a j-th coordinate slice for some $j \in \{1,2,3\}$. Assume further that $\widetilde{D}$ is not strongly degenerate. Then for admissible triples $\vec{p} = (p_1, p_2, p_3)$, the trilinear form $\Lambda_{\widetilde{D}}$ fails to be of generalized restricted type $\vec{p}$ whenever:*

- *$p_i < 2$ for some $i \neq j$,*

- *$p_i \leq -1$ for some i (i.e., $\vec{p}$ lies outside the Banach triangle).*

If $D = \Phi^{-1}(\widetilde{D})$ is the standard preimage of $\widetilde{D}$ in $\mathbb{R}^4$, then the operator T_D is unbounded from $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \to L^{p_3'}(\mathbb{R}^2)$ whenever $p_i = \infty$ for some i (i.e., whenever $\vec{p}$ lies on the border of the Banach triangle) and additionally $p_j \neq 2$ for all j. If $\widetilde{D}$ is further assumed to be nondegenerate, then the restriction $p_j \neq 2$ can be removed.

Proof. To treat the first case, one can observe that in the interior of the Banach triangle generalized restricted type $\vec{p}$ estimates imply imply "restricted type $\vec{p}$" estimates, and, since our counterexamples were constructed from characteristic functions, our proofs thus far can be applied (cf. Lemma 3.6 of [11]). Thus, we need only concern ourselves with $\vec{p}$ on the border of or outside the Banach triangle. If $\widetilde{D}$ is nondegenerate, one simply combines Theorem 5.1 with the usual dilation- and translation-invariance of bilinear multiplier norms to obtain the desired unboundedness or failure of generalized restricted type.

If $\widetilde{D}$ is degenerate, however, we need to exploit the curvature of $\partial\widetilde{D}$. Note that, since $\widetilde{D}$ is assumed not to be strongly degenerate, degeneracy of $\widetilde{D}$ and the coordinate-slice curvature hypothesis imply that $\widetilde{D}$ must in fact satisfy the hypotheses of Theorem 4.1. For $\vec{p}$ outside the Banach triangle, the proof of Theorem 4.1 carries over after the excision of half-measure sets wherever necessary, and one obtains the failure of generalized restricted type $\vec{p}$ for $\Lambda_{\widetilde{D}}$. If $\vec{p}$ lies on the boundary of the Banach triangle, by duality it suffices to disprove $L^p \times L^{p'} \to L^1$ bounds on T_D for any $\widetilde{D}$ satisfying the hypotheses of Theorem 4.1; note that these hypotheses are again symmetric under permutation of coordinates. If $1 \neq p \neq \infty$, this can be accomplished by following the methods of [2] or [8] (*i.e.*, the proof of Theorem 4.1 phrased in terms of bilinear operators). Finally, if $p = 1$ or $p = \infty$, we use dilations and translations of D to pass to a half-space multiplier T_P; from this point, we simply consider either $\mathrm{T}_P(1, \cdot)$ or $\mathrm{T}_P(\cdot, 1)$ and invoke the unboundedness of the Hilbert transform on L^1.   □
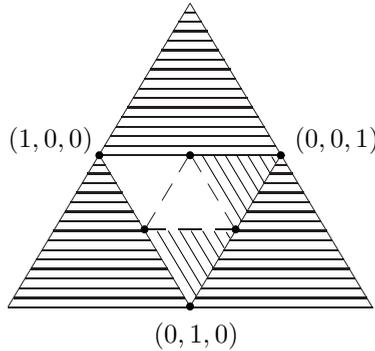
FIGURE 5: Type diagram of points $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right)$.

In summary, the presently known range of unboundedness of T_D or $\Lambda_{\widetilde{D}}$ for nondegenerate $\widetilde{D} \subset \Gamma$ with boundary curvature in a first coordinate slice is given by the shaded region of the type diagram in Figure 5; of course, the corresponding ranges for the other two slice curvature conditions are given by rotations of this diagram. If $\widetilde{D}$ is merely assumed not to be strongly degenerate, we are forced to omit the vertices of the local L^2 region; we reiterate that the status of the bilinear multiplier problem in the unshaded regions is completely unknown at present.

6. Open directions and remarks

6.1. Open directions

More exotic domains. Even in view of Proposition 1.2, the curvature conditions of the Main Theorem may seem somewhat *ad hoc*. There exist less degenerate domains $\widetilde{D} \subset \Gamma$ whose boundaries have nontrivial principal curvature at a point but are locally flat in the three coordinate directions of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$; for an example, consider the domain $\widetilde{D}_1 = \Phi(D_1)$, with

$$D_1 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid \xi_4 > \xi_1\xi_3 + \xi_1^2 \right\}.$$

This domain falls outside the scope of both the techniques of this paper and those of [2] and [8].

Comparison of the methods. It should be noted here that while Corollary 1.1 generalizes Theorem 1 of [8], the *methods* of this paper do not appear to be strictly stronger than those of Grafakos *et al.* In short, their methods require the availability of rather specific normal vectors, but there is no restriction on the boundary points at which these normal vectors occur as in our slice conditions; see Theorem 4.1. It is easy to construct examples of domains satisfying the hypotheses of our Main Theorem but failing those of Theorem 4.1 (cf. Corollary 1.1); examples

treatable by Theorem 4.1 but not the Main Theorem seem less trivial to produce. For instance, elementary algebraic arguments show that one cannot find such an example $\widetilde{D} = \Phi(D)$ with ∂D a quadratic subvariety of $\mathbb{R}^4$; however, the argument seems particular to the quadratic setting, and it could perhaps be interesting to find such examples in general.

Untreated ranges of exponents. Finally, of course there remains the question of the exact range of L^p spaces for which one should expect unboundedness results. An obvious problem is to consider a domain $\widetilde{D}$ satisfying exactly one of the coordinate-slice curvature conditions and address the omitted triangle lying outside local L^2 but within the Banach triangle (see Figure 5). This region seems beyond the reach of the rather standard methods used in this paper; in short, one needs to exploit the small area of Besicovitch sets by measuring the appropriate square function in L^{p_j} with $p_j < 2$, but if one only has curvature in a j-th coordinate slice there is no guarantee that the constituent rectangles will interact productively with their reaches under the application of $\Lambda_{\widetilde{D}}$.

No nontrivial result is currently known regarding the high-dimension ($d \geq 2$) bilinear multiplier problem for domains in the local L^2 case, and once again it seems that significantly different techniques should be used to treat this range of L^p spaces.

6.2. Remarks

1) Of course, as in Theorem 4.1, the local smoothness and curvature assumptions of the Main Theorem are not necessary *per se*. One need only guarantee that a collection of normal vectors to $\partial\widetilde{D}$ occurring in a coordinate slice yields Besicovitch sets as in Property 2.2; for a characterization of closely related collections, see the paper [1] of Bateman. It should be noted, however, that the "Ω admits Kakeya sets" condition appearing therein is weaker than the conditions of Definition 2.1; specifically, disjointness of the reaches R'_n is not required for a set of directions to admit Kakeya sets in the sense of [1].[9] This potential lack of disjointness is problematic when one attempts to control the right-hand side of estimate 4.2, and thus at least at first glance the sets of directions characterized in [1] are not quite suitable for our purposes.

2) Using (the proof of) the multilinear version of De Leeuw's Theorem proved in [2], one can readily derive analogues of the Main Theorem for multipliers given by domains $\widetilde{D}$ in higher-dimensional spaces, with $\Gamma = \Gamma_2$ replaced by

$$\Gamma_d := \big\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \mid \xi_1 + \xi_2 + \xi_3 = 0 \big\}.$$

However, the curvature conditions arising in this setting are markedly clumsier. As in the Main Theorem, the intersection of some neighborhood in $\partial\widetilde{D}$ with some 2-plane of a prescribed form must be a plane curve of nonzero curvature; the permissible such 2-planes are dictated by our conditions and De Leeuw's Theorem.

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[9]The author thanks M. Bateman for calling his attention to this distinction.

Of course, due to the abundance of nontrivial normal curvature guaranteed by strict convexity, Corollary 1.1 holds as stated with $\mathbb{R}^2$ replaced by $\mathbb{R}^d$ and $\mathbb{R}^4$ replaced by $\mathbb{R}^{2d}$.

3) In the same vein, one can generalize the Main Theorem to a statement about k-linear operators (or $(k+1)$-linear forms); again, the arising curvature conditions are obtained by slicing a domain in $\mathbb{R}^{kd}$ (or its appropriate embedding into $\mathbb{R}^{(k+1)d}$) by prescribed 2-planes. As in the previous remark, Corollary 1.1 continues to hold in the general k-linear setting.

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