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# Nearly optimal interpolation of data in $C^2(\mathbb{R}^2)$. Part I

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**Abstract.** Given $\epsilon > 0$, we compute a function taking prescribed values at $N$ given points in $\mathbb{R}^2$, whose $C^2$-norm is within a factor $(1+\epsilon)$ of least possible. The computation takes $C(\epsilon)N \log N$ computer operations.

## 0. Introduction

### The problem

Our goal, here and in [4], is to interpolate data by a smooth function. We work in $C^m(\mathbb{R}^n)$, the space of real-valued functions $F$ whose derivatives up to order $m$ are continuous and bounded on $\mathbb{R}^n$. We fix a norm on $C^m(\mathbb{R}^n)$, e.g.,

(1) $$\| F \|_{C^m(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\le m} |\partial^\alpha F(x)|.$$

Let $f : E \to \mathbb{R}$ be a real-valued function on a finite set $E \subset \mathbb{R}^n$. An "interpolant" for $f$ is a function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on $E$. We define

(2) $$\| f \|_{C^m(E)} = \inf\{\| F \|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n),\ F = f \text{ on } E\}.$$

Elementary examples show that the inf in (2) needn't be a minimum. Given a real number $A > 1$, we say that $F \in C^m(\mathbb{R}^n)$ is an "A-optimal interpolant" for $f : E \to \mathbb{R}$, provided $F = f$ on $E$ and $\| F \|_{C^m(\mathbb{R}^n)} \le A \| f \|_{C^m(E)}$.

Our main problem is to compute an $A$-optimal interpolant for $f$, where $A$ is not too large.

To "compute" an interpolant, we provide an algorithm to be implemented on an (idealized) digital computer. We want to minimize the number of computer operations, and the size of the computer memory, needed to execute our algorithm.

In [7] and [8], Fefferman–Klartag gave an efficient algorithm to compute an $A$-optimal interpolant, where $A$ is a constant depending only on $m$ and $n$. Unfortunately, the constant $A$ arising from the algorithm in [7] and [8] is large, even for modest $m$ and $n$.

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Motivated by the hope of eventual practical applications, we therefore pose the following

(3) Sharp Interpolation Problem: Given a function $f : E \to \mathbb{R}$ on a finite set $E \subset \mathbb{R}^n$, and given $\epsilon > 0$, compute a $(1+\epsilon)$-optimal interpolant for $f$.

Our main result, here and in [4], is an algorithm to solve the above Sharp Interpolation Problem for the case of $C^2(\mathbb{R}^2)$. For sets $E$ consisting of $N$ points, our algorithm requires at most $C(\epsilon)N \log N$ computer operations, where $C(\epsilon)$ depends only on $\epsilon$, and on our choice of the norm on $C^2(\mathbb{R}^2)$.

This improves our previous result in [6] (specialized to $C^2(\mathbb{R}^2)$), which computes a $(1+\epsilon)$-optimal interpolant using $C(\epsilon)N^5(\log N)^2$ operations. The algorithm in [6] reduces matters to a linear programming problem of size $C(\epsilon)N$. Here and in [4], we instead reduce matters to $O(N)$ "little" linear programming problems, each of size $C(\epsilon)$.

The previous results of Fefferman–Klartag [7], [8] and Fefferman [6] were all based on "finiteness principles", which we explain below. The natural finiteness principle relevant to our Sharp Interpolation Problem fails for $C^2(\mathbb{R}^2)$. Nevertheless, we are able to give an efficient algorithm for this case.

## Notation

Fix $m, n \geq 1$. For $F \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we write $J_x(F)$ (the "jet" of $F$ at $x$) to denote the $m^{\text{th}}$ order Taylor polynomial of $F$ at $x$. Thus, $J_x(F)$ belongs to $\mathcal{P}$, the vector space of (real) $m^{\text{th}}$ degree polynomials on $\mathbb{R}^n$.

Now let $E \subset \mathbb{R}^n$ be a finite set. A "Whitney field" on $E$ is a family of polynomials

(4) $\vec{P} = (P^x)_{x \in E}$, indexed by the points of $E$, such that each $P^x$ belongs to $\mathcal{P}$. We write $\mathrm{Wh}(E)$ to denote the vector space of all Whitney fields on $E$.

For $F \in C^m(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ finite, the "jet" $J_E(F)$ of $F$ at $E$ is defined as

(5) $J_E(F) = (J_x(F))_{x \in E} \in Wh(E)$.

If $\vec{P}$ is a Whitney field as in (4), and if $S \subset E$, then the "restriction" $\vec{P}|_S$ is defined as

(6) $\vec{P}|_S = (P^x)_{x \in S}$, with the same $P^x$ as in (4).

We define a norm on Whitney fields $\vec{P} \in Wh(E)$, by setting

(7) $\| \vec{P} \|_{Wh(E)} = \inf\{\| F \|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n),\ J_E(F) = \vec{P}\}$.

If $\vec{P} = (P^x)_{x \in E} \in Wh(E)$ and $f : E \to \mathbb{R}$, then we say that $\vec{P}$ "agrees" with $f$ provided

(8) $(P^x)(x) = f(x)$ for each $x \in E$.

Similarly if $\vec{P} = (P^x)_{x \in E_1} \in Wh(E_1)$ and $f : E_2 \to \mathbb{R}$, then we say that $\vec{P}$ "agrees" with $f$ at a given point $\bar{x} \in E_1 \cap E_2$, provided

(9) $(P^{\bar{x}})(\bar{x}) = f(\bar{x})$.

### The $C^m$ norm

Our choice of the norm in (1) is somewhat arbitrary. We could just as well have defined, say,

(10) $\| F \|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sum_{|\alpha| \le m} |\partial^\alpha F(x)|$,

(11) $\| F \|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \left( \sum_{|\alpha| \le m} |\partial^\alpha F(x)|^2 \right)^{1/2}$, or

(12) $\| F \|_{C^m(\mathbb{R}^n)} = \sum_{|\alpha| \le m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$.

Each of these norms gives rise to a different Sharp Interpolation Problem (3). To allow freedom to pick our favorite $C^m$-norm, we suppose from now on that we are given a family of norms $| \cdot |_x$ on $\mathcal{P}$, parametrized by $x \in \mathbb{R}^n$.

For $\Omega \subset \mathbb{R}^n$, and for $F \in C^m_{loc}(\Omega)$, we then define

(13) $\| F \|_{C^m(\Omega)} = \sup_{x \in \Omega} |J_x(F)|_x$

For instance, we recover the $C^m$-norms (1) and (10)–(11) by taking

(14) $|P|_x = \max_{|\alpha| \le m} |\partial^\alpha P(x)|$,

(15) $|P|_x = \sum_{|\alpha| \le m} |\partial^\alpha P(x)|$, and

(16) $|P|_x = \left( \sum_{|\alpha| \le m} |\partial^\alpha P(x)|^2 \right)^{1/2}$,

respectively, for $P \in \mathcal{P}$.

The $C^m$-norm (12) is not given in the form (13); we do not consider it further. The norms $| \cdot |_x$ are assumed to satisfy two reasonable conditions, called the "Bounded Distortion Property" and "Approximate Translation-Invariance". These properties are given in Section 5. The norms (14), (15), (16) satisfy these two conditions.

From now on, whenever we mention the $C^m$-norm, we assume that the norm is defined by (13), or by its special case (1). This applies in particular to the definition of $\| \vec{P} \|_{Wh(E)}$ by (7).

### The computer

Our Sharp Interpolation Problem asks us to "compute" a function using a "computer". We suppose that our computer has standard von Neumann architecture [14]. We assume that each memory cell and each register is capable of holding an arbitrary real number. We suppose that the computer can perform elementary arithmetic operations on exact real numbers, without roundoff error. (The arithmetic operations include exponentials and logarithms, and the "greatest integer" function.) To perform a single arithmetic operation, or to read or write a single number to memory, costs us one unit of "work". See [12] and [17] for a more detailed discussion of this model of computation (and its pitfalls).

Our computer will have to acquire information on the family of norms $|\cdot|_x$ used to specify the $C^m$-norm in (13). We suppose that our computer has access to an Oracle. Given a point $x \in \mathbb{R}^n$ and a polynomial $P \in \mathcal{P}$, the Oracle returns the value of $|P|_x$, at a charge of one unit of "work". (This assumption can be weakened; see Section 5 below.) For the family of norms $|\cdot|_x$ given by (14), (15) or (16), an obvious algorithm serves as an Oracle.

### Computing a function

Our computer can only calculate finitely many real numbers. What does it mean to "compute a function" $F \in C^m(\mathbb{R}^n)$? As in [6], [8], we have in mind the following dialogue with the computer: First, we enter the data ($m, n, E, f, \epsilon$ for our Sharp Interpolation Problem). Next, the computer executes an algorithm, performing $W_1$ operations of "one-time work". After the one-time work is complete, the computer signals that it is ready to accept queries. A "query" consists of a point $x \in \mathbb{R}^n$. When we enter a query $x$, the computer responds by executing a "query algorithm", involving $W_Q$ operations (the "query work"), and then returning the values of $\partial^\alpha F(x)$ for $|\alpha| \leq m$. We may enter as many queries as we please. We insist that the function $F$ be uniquely determined once the computer signals that it is ready for queries. In particular, we disallow "adaptive algorithms", in which the function $F$ depends on our queries. We also disallow calls to the Oracle by the query algorithm.

The computer resources used to compute a function are the one-time work $W_1$, the query work $W_Q$, and the "storage" or "space" (i.e., the number of memory cells in the computer's random-access memory).

### The main result

After the above preparations, we are ready to state our main result. We work in $C^2(\mathbb{R}^2)$; thus $m = n = 2$ above.

**Theorem 1.** *Fix a norm on $C^2(\mathbb{R}^2)$ of the form (13). Suppose we are given $0 < \epsilon < \frac{1}{2}$ and $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^2$, $\#(E) = N$.*

*Then, with work $C(\epsilon) N \log N$ and storage $C(\epsilon)N$, we can compute a non-negative real number $|||f|||$ such that $(1+\epsilon)^{-1}\, |||f||| \leq \| f \|_{C^2(E)} \leq (1+\epsilon)\, |||f|||$.*

*Moreover, we can compute a $(1+\epsilon)$-optimal interpolant for $f$, using one-time work $C(\epsilon) N \log N$, query work $C \log(N/\epsilon)$, and storage $C(\epsilon)N$.*

*Here, $C$ depends only on our choice of the $C^2$-norm, and $C(\epsilon)$ depends only on $\epsilon$ and our choice of the $C^2$-norm.*

Most likely, the $N$-dependence in Theorem 1 is optimal. Our $C(\epsilon)$ depends superexponentially on $\epsilon$; we hope this can be improved.

**Previous work**

To place our main result in context, and to discuss its proof, we recall the previous work of Fefferman–Klartag [7], [8] and Fefferman [6].

**Theorem 2.** *Define the* $C^m(\mathbb{R}^n)$*-norm by* (1). *Suppose we are given* $f : E \to \mathbb{R}$, *with* $E \subset \mathbb{R}^n$, $\#(E) = N$.

*Then, using work at most* $CN \log N$ *and storage at most* $CN$, *we can compute a non-negative real number* $|||f|||$ *such that*

$$(17)\quad |||f||| \leq \| f \|_{C^m(E)} \leq A\,|||f|||.$$

*Moreover, we can compute an* $A$*-optimal interpolant for* $F$, *using one-time work at most* $CN \log N$, *query work at most* $C \log N$, *and storage at most* $CN$. *Here,* $A$ *and* $C$ *depend only on* $m$ *and* $n$.

Unfortunately, the constant $A$ arising from [7], [8] is large.

As an easier variant of our Sharp Interpolation Problem (3), we pose the following

Sharp Interpolation Problem for Whitney Fields: Given $\vec{P} \in Wh(E)$, and given $\epsilon > 0$, compute a function $F \in C^m(\mathbb{R}^n)$, such that

$$(18)\quad J_E(F) = \vec{P} \text{ and } \| F \|_{C^m(\mathbb{R}^n)} \leq (1+\epsilon)\, \| \vec{P} \|_{Wh(E)}.$$

If (18) holds, then we call $F$ a "$(1+\epsilon)$-optimal interpolant" for $\vec{P}$.

The following result answers the Sharp Interpolation Problem for Whitney Fields:

**Theorem 3.** *Fix a norm on* $C^m(\mathbb{R}^n)$ *of the form* (13). *Suppose we are given* $0 < \epsilon < \frac{1}{2}$ *and* $\vec{P} \in Wh(E)$, *with* $E \subset \mathbb{R}^n$, $\#(E) = N$.

*Then, with work* $\exp(C/\epsilon) N \log N$ *and storage* $\exp(C/\epsilon)N$, *we can compute a non-negative real number* $|||\vec{P}|||$ *such that* $|||\vec{P}||| \leq \| \vec{P} \|_{Wh(E)} \leq (1+\epsilon)\, |||\vec{P}|||$.

*Moreover, we can compute a* $(1+\epsilon)$*-optimal interpolant for* $\vec{P}$, *using one-time work at most* $\exp(C/\epsilon) N \log N$, *query work at most* $C \log(N/\epsilon)$, *and storage at most* $\exp(C/\epsilon)N$.

*Here* $C$ *depends only on* $m, n$ *and our choice of* $C^m$*-norm.*

Using Theorem 3, we can reduce the Sharp Interpolation Problem (3) (for functions) to a linear programming problem of size $\exp(C/\epsilon)N$. This leads to the following preliminary result on (3); see [6].

**Theorem 4.** *Fix a norm on* $C^m(\mathbb{R}^n)$ *of the form* (13). *Suppose we are given* $0 < \epsilon < \frac{1}{2}$ *and* $f : E \to \mathbb{R}$, *with* $E \subset \mathbb{R}^n$, $\#(E) = N$.

*Then, with work* $\exp(C/\epsilon) N^5 (\log N)^2$ *and storage* $\exp(C/\epsilon) N^2$, *we can compute a non-negative real number* $|||f|||$ *such that* $|||f||| \leq \| f \|_{C^m(E)} \leq (1+\epsilon)\, |||f|||$.

*Moreover, we can compute a* $(1+\epsilon)$*-optimal interpolant for* $f$, *using one-time work at most* $\exp(C/\epsilon) N^5 (\log N)^2$, *query work at most* $C \log(N/\epsilon)$, *and storage at most* $\exp(C/\epsilon) N^2$. *Here,* $C$ *depends only on* $m, n$ *and our choice of the* $C^m$*-norm.*

Thus, for fixed $\epsilon$, the Sharp Interpolation Problem (3) can be solved in polynomial time.

### Finiteness principles

The ideas behind Theorems 2 and 3 start with the classic Whitney Extension Theorem [13], [18], [19], [20], which we state in the special case of finite sets $E$.

**Theorem 5** (Whitney). *Fix $m, n \geq 1$, and define the $C^m(\mathbb{R}^n)$-norm by (1). Let $\vec{P} = (P^x)_{x \in E} \in Wh(E)$, where $E \subset \mathbb{R}^n$ is finite. Assume the estimates: $|\partial^\alpha P^x(x)| \leq 1$ for $|\alpha| \leq m$, $x \in E$; and $|\partial^\alpha(P^x - P^y)(y)| \leq |x - y|^{m-|\alpha|}$ for $|\alpha| < m$, $x, y \in E$.*

*Then there exists $F \in C^m(\mathbb{R}^n)$ such that $\| F \|_{C^m(\mathbb{R}^n)} \leq C$ and $J_E(F) = \vec{P}$. Here, $C$ depends only on $m, n$.*

Whitney's theorem may be restated in the following equivalent form:

**Theorem 6.** *Fix $m, n \geq 1$, and define the $C^m(\mathbb{R}^n)$ norm by (1). Let $\vec{P} \in Wh(E)$, where $E \subset \mathbb{R}^n$ is finite. Suppose that $\| (\vec{P}|_S) \|_{Wh(S)} \leq 1$ for each subset $S \subset E$ containing at most two points.*

*Then $\| \vec{P} \|_{Wh(E)} \leq C$, where $C$ depends only on $m$ and $n$.*

Theorem 6 is the simplest case of a "finiteness principle".

The proof of Theorem 2 is based on the following deeper finiteness principle (see Brudnyi–Shvartsman [2], Fefferman [10], Bierstone–Milman [1], and Shvartsman [15]):

**Theorem 7.** *Fix $m, n \geq 1$, and define the $C^m(\mathbb{R}^n)$-norm by (1). Then there exist constants $k^\#$, $C$, depending only on $m, n$, such that the following holds:*

*Let $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^n$ finite. Suppose that $\| (f|_S) \|_{C^m(S)} \leq 1$ for each subset $S \subset E$ containing at most $k^\#$ points. Then $\| f \|_{C^m(E)} \leq C$.*

Similarly, the proof of Theorem 3 is based on the following finiteness principle, called the "$(1 + \epsilon)$-Whitney theorem" in [5], [6]:

**Theorem 8.** *Fix $m, n \geq 1$, and fix a $C^m(\mathbb{R}^n)$-norm of the form (13). Let $\vec{P} \in Wh(E)$, with $E \subset \mathbb{R}^n$ finite; and let $0 < \epsilon < \frac{1}{2}$. Suppose that $\| (\vec{P}|_S) \|_{Wh(S)} \leq 1$ for each subset $S \subset E$ containing at most $\exp(C/\epsilon)$ points.*

*Then $\| \vec{P} \|_{Wh(E)} \leq 1 + \epsilon$. Here, $C$ depends only on $m, n$ and our choice of the $C^m$-norm.*

We believe that the exponential $\exp(C/\epsilon)$ can be replaced by a power of $1/\epsilon$ in Theorems 3 and 8. A remarkable result of LeGruyer [11] suggests the possibility of dramatic further improvements.

LeGruyer's theorem pertains to the space $C^{1,1}(\mathbb{R}^n)$ of function $F$ whose gradients are Lipschitz 1. On $C^{1,1}(\mathbb{R}^n)$, we take the natural seminorm

$$\| F \| = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \left| \nabla F(x) - \nabla F(y) \right| / |x - y| .$$

(Here, $|\nabla F(x) - \nabla F(y)|$ and $|x-y|$ are defined in terms of the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$.)

For $x \in \mathbb{R}^n$ and $F \in C^{1,1}(\mathbb{R}^n)$, we write $j_x(F)$ to denote the first-degree Taylor polynomial of $F$ at $x$. Le Gruyer's theorem is as follows:

**Theorem 9** ([11]). *Let $E \subset \mathbb{R}^n$. For each $x \in E$, let $P_x$ be a given first-degree polynomial on $\mathbb{R}^n$. Then the following are equivalent:*

(A) *There exists a function $F \in C^{1,1}(\mathbb{R}^n)$ such that $\| F \| \leq 1$ and $j_x(F) = P_x$ for each $x \in E$.*

(B) $|P_x(z) - P_y(z)| \leq \frac{1}{2}(|z-x|^2 + |z-y|^2)$ *for all $x, y \in E$ and $z \in \mathbb{R}^n$.*

**Corollary.** *Suppose that, for any two points $x, y \in E$, there exists $F^{x,y} \in C^{1,1}(\mathbb{R}^n)$ such that $\| F^{x,y} \| \leq 1$, $j_x(F^{x,y}) = P_x$ and $j_y(F^{x,y}) = P_y$.*

*Then there exists $F \in C^{1,1}(\mathbb{R}^n)$ such that $\| F \| \leq 1$ and $j_x(F) = P_x$ for each $x \in E$.*

Thus, finiteness principles lie at the heart of all the above previous work on interpolation problems. In view of the above results, it is natural to make the following conjectures:

**Conjecture 1** (Finiteness Principle). *Fix $m, n \geq 1$, and fix a $C^m(\mathbb{R}^n)$-norm of the form* (13).

*Given $\epsilon > 0$, there exists a constant $k^\#(\epsilon)$, depending only on $\epsilon, m, n$ and our choice of the $C^m$-norm, such that the following holds:*

*Let $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^n$ finite. Assume that $\| (f|_S) \|_{C^m(S)} \leq 1$ for each subset $S \subset E$ containing at most $k^\#(\epsilon)$ points. Then $\| f \|_{C^m(E)} \leq 1 + \epsilon$.*

**Conjecture 2** (Sharp Interpolation Algorithm). *Fix $m, n \geq 1$, and fix a $C^m(\mathbb{R}^n)$-norm of the form* (13).

*Given $\epsilon > 0$, and given $f : E \to \mathbb{R}^n$, $\#(E) \leq N$, we can compute a $(1+\epsilon)$-optimal interpolant for $f$, using one-time work at most $C(\epsilon) N \log N$, query work at most $C \log(N/\epsilon)$, and storage at most $C(\epsilon)N$. Here, $C(\epsilon)$ depends only on $\epsilon, m, n$ and our choice of the $C^m$-norm; and $C$ depends only on $m, n$ and choice of the $C^m$-norm.*

Moreover, it is natural to guess that the proof of Conjecture 1 will lead to the algorithm promised in Conjecture 2.

Unfortunately, the facts are otherwise. A counterexample in Fefferman–Klartag [9] shows that Conjecture 1 fails, already for $C^2(\mathbb{R}^2)$. Nevertheless, we prove, here and in [4], that Conjecture 2 is correct for $C^2(\mathbb{R}^2)$; that is the content of Theorem 1. Perhaps Conjecture 2 holds for $C^m(\mathbb{R}^n)$ (any $m, n$). A proof will require substantial new ideas. We do not yet know what lies at the heart of the Sharp Interpolation Problem.

### Reduction of Theorem 1 to a main algorithm

To prove Theorem 1, we will present the following

(19) MAIN ALGORITHM: Fix a $C^2(\mathbb{R}^2)$-norm of the form (13).

Given $0 < \epsilon < \frac{1}{2}$, and given a function $f : E \longrightarrow \mathbb{R}$, with $E \subset \mathbb{R}^2$ and $\#(E) = N$, we produce one of the following two outcomes:

(20) Bad News: We guarantee that there exists no interpolant for f with $C^2(\mathbb{R}^2)$-norm less than 1.

(21) Good News: We guarantee that there exists an interpolant for f with $C^2(\mathbb{R}^2)$-norm less than $1+\epsilon$. Moreover, for one such interpolant F, we compute the jet

(22) $\vec{P} := J_E(F)$.

The work and storage used to produce one of these two outcomes are at most $C(\epsilon)N \log N$ and $C(\epsilon)N$, respectively. Here, $C(\epsilon)$ depends only on $\epsilon$ and our choice of the $C^2$-norm.

To prove Theorem 1 for a given $f : E \longrightarrow \mathbb{R}$, we first apply Theorem 2 to compute $\| f \|_{C^2(E)}$ up to a factor of C, where C depends only on our choice of the $C^m$-norm. Next, by repeatedly applying the above MAIN ALGORITHM to constant multiples of f, we compute $\| f \|_{C^2(E)}$ up to a factor of $(1+\epsilon)$. Without loss of generality, we may now suppose that $(1+\epsilon)^{-1} \leq \| f \|_{C^2(E)} < 1$. Another application of our MAIN ALGORITHM produces a Whitney field $\vec{P} \in Wh(E)$ such that $\vec{P}$ agrees with f and $\| \vec{P} \|_{Wh(E)} \leq 1+\epsilon$.

Applying Theorem 3, we compute a function $F \in C^2(\mathbb{R}^2)$, such that $J_E(F) = \vec{P}$ and $\| F \|_{C^2(\mathbb{R}^2)} \leq (1+\epsilon) \, \| \vec{P} \|_{Wh(E)}$. In particular, we have $\| F \|_{C^2(\mathbb{R}^2)} \leq (1+\epsilon)^2$, $F = f$ on E, and $\| f \|_{C^2(E)} \geq (1+\epsilon)^{-1}$. Thus, we have computed a $(1+\epsilon)^3$-optimal interpolant F, using one-time work, query work and storage as indicated in Theorem 1. We conclude that Theorem 1 reduces easily to the MAIN ALGORITHM (19), together with Theorems 2 and 3.

The rest of this Introduction sketches some of the ideas used in the MAIN ALGORITHM.

### Data structures

We will be working with convex polyhedra. A "convex polyhedron" in a finite-dimensional vector space V is a compact subset $K \subset V$ of the form

(23) $K = \{v \in V : \lambda_i(v) \geq \beta_i \text{ for } i = 1, \ldots, I\}$, where each $\lambda_i$ is a (real) linear functional on V, and each $\beta_i$ is a real number.

We say that K is "defined" by the "constraints" $\lambda_i(v) \geq \beta_i$ $(i = 1, 2, \ldots, I)$. Note that K may be empty, and that a single $K \subset V$ may be defined by many different lists of constraints.

We will work with squares $Q \subset \mathbb{R}^2$. We always suppose that the sides of Q are parallel to the coordinate axes. We write $\delta_Q$ to denote the sidelength of Q. For positive real numbers A, we write AQ to denote the square obtained by dilating Q about its center by a factor of A.

Let $0 < \epsilon < \frac{1}{2}$ be given. We will write $c, C, C'$, etc., to denote constants depending only on our choice of the $C^2$-norm; and we write $c(\epsilon)$, $C(\epsilon)$, etc., to denote constants depending only on $\epsilon$ and on our choice of the $C^2$-norm. These symbols may denote different constants in different occurrences.

As a first crude attempt to represent a function $F \in C^2(Q)$ in a computer memory, we fix an "$\epsilon^{100}$-net" $S \subset Q$, i.e., a finite subset $S \subset Q$ such that

(24) Any point $z \in Q$ satisfies $|z - z'| < \epsilon^{100}\delta_Q$ for some $z' \in S$, and

(25) $\#(S) \leq C\epsilon^{-200}$.

We then represent the function $F \in C^2(Q)$ to the computer, simply by keeping the Whitney field $J_S(F)$. This captures a lot of information about the behavior of $F$, but it misses fine details on lengthscales smaller than $\epsilon^{100}\delta_Q$. In particular, if $E \subset Q$, and if the distance between nearest neighbors in $E$ is smaller than $\epsilon^{100}\delta_Q$, then we cannot tell from $J_S(F)$ whether $F = f$ on $E$. Therefore, we will later introduce a more sophisticated data structure to represent $F \in C^2(Q)$.

### Tools from the proofs of previous results

Our main algorithm will make use of two tools from previous work. From ideas in the proof of Theorem 3, we obtain the following algorithm:

**Algorithm AUB** ("Approximate Unit Ball"): Fix a norm on $C^2(\mathbb{R}^2)$ of the form (13).

Given a square $Q$, a finite subset $S \subset Q$, and a positive number $\epsilon$, such that $\#(S) \leq C\epsilon^{-200}$, we compute a convex polyhedron $K_{AUB}(S, Q) \subset Wh(S)$, with the following properties:

(26) Let $F \in C^2(2Q)$ with norm $\leq 1$. Then $J_S(F) \in K_{AUB}(S, Q)$.

(27) Let $\vec{P} \in K_{AUB}(S, Q)$. Then there exists $F \in C^2(Q)$ with norm $\leq 1 + \epsilon$, such that $J_S(F) = \vec{P}$.

(28) The polyhedron $K_{AUB}(S, Q)$ is defined by at most $C(\epsilon)$ constraints.

The work and storage used to compute $K_{AUB}(S, Q)$ are at most $C(\epsilon)$.

When applying the above algorithm in this oversimplified introduction, we may blur the distinction between $2Q$ in (26) and $Q$ in (27). Clearly, Algorithm AUB gives us good control over the requirement that $\| F \|_{C^2} \leq 1 + O(\epsilon)$. On the other hand, so far we have no control over the requirement that $F = f$ on $E$.

Our second tool is a Calderón–Zygmund decomposition of $\mathbb{R}^2$, taken from our proof of Theorem 2 in [8] (specialized to $C^2(\mathbb{R}^2)$). That decomposition partitions $\mathbb{R}^2$ into Calderón–Zygmund squares $\{Q_\nu\}$ with sidelengths $\delta_{Q_\nu} \leq 1$, such that, for each $\nu$, $E \cap 3Q_\nu$ is contained in the graph of a function. More precisely, either

(29) $E \cap 3Q_\nu \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \varphi_\nu(x_1)\}$ or

(30) $E \cap 3Q_\nu \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \varphi_\nu(x_2)\}$,

where $\varphi_\nu$ in (29), (30) satisfies

(31) $|\varphi'_\nu| \le C$, $|\varphi''_\nu| \le C\delta_{Q_\nu}^{-1}$.

Moreover, for each $\nu$, there exist a base point $z_\nu \in 9Q_\nu$, and a convex polyhedron $\Gamma(z_\nu) \subset \mathcal{P}$, defined by at most $C$ constraints, with the following properties:

(32) Let $F \in C^2(\mathbb{R}^2)$ with norm $\le 1$. If $F = f$ on $E$, then $J_{z_\nu}(F) \in \Gamma(z_\nu)$.

(33) Let $P, P' \in \Gamma(z_\nu)$. Then $|\partial^\alpha(P - P')(z_\nu)| \le C\,\delta_{Q_\nu}^{2-|\alpha|}$ for $|\alpha| \le 2$.

When we look for an interpolant $F$, (32) shows that we may restrict attention to functions such that $J_{z_\nu}(F) \in \Gamma(z_\nu)$ for each $\nu$. Thanks to (33), this tells us a lot about $F$ on $9Q_\nu$, when $\delta_{Q_\nu}$ is small. When $\delta_{Q_\nu}$ isn't small, (32) and (33) give no useful information.

## The plan

Our main algorithm is based on the Calderón–Zygmund decomposition described in (29)–(33).

For each Calderón–Zygmund square $Q_\nu$, we pick an "$\epsilon^{100}$-net" $S_\nu \subset Q_\nu$, as in (24) and (25).

Step I: For each $\nu$, we compute a convex polyhedron $K_\nu \subset Wh(S_\nu)$, with the following properties:

(34) Let $F \in C^2(CQ_\nu)$, with norm at most 1. Assume that $F = f$ on $E \cap 3Q_\nu$, and that $J_{z_\nu}(F) \in \Gamma(z_\nu)$. Then $J_{S_\nu}(F) \in K_\nu$.

(35) Conversely, let $\vec{P} \in K_\nu$ be given. Then there exists $F \in C^2(9Q_\nu)$ with norm at most $1 + C\epsilon$, such that $F = f$ on $E \cap 3Q_\nu$, $J_{z_\nu}(F) \in \Gamma(z_\nu)$, and $J_{S_\nu}(F) = \vec{P}$. Moreover, we can compute the jet $J_{E\cap 3Q_\nu}(F)$ for one such $F$.

Thus, $K_\nu$ is analogous to $K_{AUB}(S_\nu, Q_\nu)$ in (26), (27), (28) with the crucial difference that $K_\nu$ takes into account the condition $F = f$ on $E \cap 3Q_\nu$.

Perhaps some of the $K_\nu$ are empty. In that case, we know from (34) that there exists no function $F \in C^2(\mathbb{R}^2)$ with norm $\le 1$ such that $F = f$ on $E$. We can then report Bad News and terminate the MAIN ALGORITHM. (See (19) and (20).)

Step II: Suppose all the $K_\nu$ are non-empty.

Using the polyhedra $K_\nu$, we attempt to patch together local interpolants $F_\nu \in C^2(9Q_\nu)$ into a global interpolant $F$ using a partition of unity. We hope that $F$ will have norm less than $1 + C\epsilon$ in $C^2(\mathbb{R}^2)$, because the $F_\nu$ fit together well. This may or may not be possible. If we cannot find local interpolants that fit together properly, then we report Bad News and terminate the MAIN ALGORITHM. (Again, see (19) and (20).) However, if we can find $F_\nu$ that fit together well, then we succeed in patching together the $F_\nu$ into a global interpolant $F$ with norm at most $1 + C\epsilon$ in $C^2(\mathbb{R}^2)$. In this favorable case, we can compute the jet $J_E(F)$ for our global interpolant $F$. Thus, we can report Good News, and terminate the MAIN ALGORITHM. (See (19) and (21).)

The purpose of this paper is to carry out Step I. In the sequel [4], we will carry out Step II, using ideas from the proof of Theorem 3 in [6], and complete our explanation of the MAIN ALGORITHM.

The rest of this introduction sketches some of the main ideas used in Step I, in the context of a simplified model problem. We put off all explanations of Step II until our later paper [4].

### A Model Problem

We prepare to introduce a model problem to illustrate our approach to Step I above. Fix a norm on $C^2(\mathbb{R}^2)$, of the form (13). Let $0 < \epsilon < \frac{1}{2}$ and $N > 1$ be given. We regard $\epsilon$ as small but fixed, while $N$ is arbitrarily large. We introduce the set $E \subset Q_0$, where

(36) $E = \{(\frac{1}{N}, 0), (\frac{2}{N}, 0), \ldots, (\frac{N-1}{N}, 0), (1, 0)\} \subset \mathbb{R}^2$ and

(37) $Q_0 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| \leq 2\}$.

Also, we fix an $\epsilon^{100}$-net

(38) $S_0 \subset Q_0$, as in (24), (25).

Suppose we are given a function

(39) $f : E \to \mathbb{R}$.

Our Model Problem is to compute a convex polyhedron $K_0 \subset Wh(S_0)$, defined by at most $C(\epsilon)$ constraints, and having the following properties:

(40) Let $F \in C^2(CQ_0)$ with norm less than 1. If $F = f$ on $E$, then $J_{S_0}(F) \in K_0$.

(41) Conversely, let $\vec{P} \in K_0$ be given. Then there exists a function $F \in C^2(Q_0)$ with norm at most $1 + C\epsilon$, such that $F = f$ on $E$ and $J_{S_0}(F) = \vec{P}$. Moreover, we can compute the jet $J_E(F)$ for one such $F$.

We want to compute the polyhedron $K_0$ with work at most $C(\epsilon)N \log N$ and storage at most $C(\epsilon)N$. Moreover, the computation of $J_E(F)$ in (41) should require work and storage at most $C(\epsilon)N$.

Clearly, the above Model Problem is close to Step I (see (29), (34), (35)) for the case of a Calderón–Zygmund square $Q_\nu$ of sidelength $\delta_{Q_\nu} = 1$. Note that the polyhedron $\Gamma(z_\nu)$ from (32) and (33) plays no rôle here, since our square $Q_0$ in (37) is not small.

In the following pages, we will explain some of the main ideas in the solution of the Model Problem. We hope this will lighten the task of understanding our treatment of Step I in the MAIN ALGORITHM.

**More notation**

We introduce additional notation to discuss our Model Problem. We start by fixing an interval $I_0$, such that

(42) $[0,1]$ is contained in the middle half of $I_0$, but $|I_0| \leq 100$. We will call $I_0$ the "starting interval".

The interval $I_0$, and all the intervals that may be obtained from $I_0$ by repeated bisection, will be called "dyadic intervals". The set of all dyadic intervals will be called the "dyadic grid" $\mathcal{G}(I_0)$. The set $T(I_0)$, consisting of all dyadic intervals I of length $|I| \geq \frac{1}{1024N}$, forms a binary tree under inclusion. The root of $T(I_0)$ is the interval $I_0$. The tree $T(I_0)$ consists of "leaves" and "internal nodes". Each leaf $I \in T(I_0)$ is a dyadic interval of length between $\frac{1}{1024N}$ and $\frac{1}{512N}$. Each internal node $I \in T(I_0)$ has two "children" in $T(I_0)$, namely the two dyadic intervals obtained by bisecting I.

For each $I \in T(I_0)$, we introduce the square $Q(I)$, with sidelength $50|I|$, centered at $(\bar{x}, 0) \in \mathbb{R}^2$, where $\bar{x}$ is the midpoint of I.

We note the following elementary properties of the squares $Q(I)$:

(43) $Q(I) \subset Q(I')$ whenever $I \subset I'$.

(44) $Q_0 \subset Q(I_0) \subset CQ_0$, with $Q_0$ as in (37).

(45) If $I \in T(I_0)$ is a leaf, then $Q(I)$ contains at most one point of $E$.

A leaf $I \in T(I_0)$ such that $Q(I)$ contains a point of $E$ will be said to be of "type C1". If $I \in T(I_0)$ is a leaf of type C1, then we write $z_!(I)$ to denote the one and only point of $E$ belonging to $Q(I)$.

For each $I \in T(I_0)$, we introduce an $\epsilon^{100}$-net

(46) $S(I) \subset Q(I)$, as in (24), (25).

We can easily pick the $S(I)$ so that

(47) $z_!(I) \in S(I)$, whenever $I \in T(I_0)$ is a leaf of type C1.

We introduce a $C^2$-partition of unity

(48) $1 = \sum_{I \in T(I_0)} \theta_I$ on $Q_0$; with each $\theta_I$ satisfying

(49) $\mathrm{supp}\ \theta_I \subset Q(I)$, and

(50) $|\partial^\alpha \theta_I| \leq C|I|^{-|\alpha|}$ for $|\alpha| \leq 2$,

(51) $\theta_I \geq 0$.

We can easily arrange for the $\theta_I$ to satisfy the following additional properties:

(52) $\mathrm{supp}\ \theta_I \cap \mathrm{supp}\ \theta_{I'} \neq \emptyset$ implies $c|I| \leq |I'| \leq C|I|$.

(53) Any given point $z \in Q_0$ belongs to $\mathrm{supp}\ \theta_I$ for at most $C$ distinct $I \in T(I_0)$.

(54) Let $I \in T(I_0)$. If $E \cap \mathrm{supp}\ \theta_I \neq \emptyset$, then I is a leaf of type C1, and $E \cap \mathrm{supp}\ \theta_I = \{z_!(I)\}$.

Note that the functions $\theta_I$ are defined only on $Q_0$.

**Refined data structure**

To solve our model problem, we will have to understand functions $F \in C^2(Q_0)$ on lengthscales much smaller than $\epsilon^{100}\delta_{Q_0}$. Therefore, we will represent an interpolant $F$ in the computer, as a family of Whitney fields

(55) $\mathbb{P} = (\vec{P}_I)_{I \in T(I_0)}$, where $\vec{P}_I \in Wh(S(I))$ for each $I$.

To pass from a function $F \in C^2(Q(I_0))$ to a family of Whitney fields $\mathbb{P}$ as in (55), we simply define

(56) $\vec{P}_I = J_{S(I)}(F)$ for each $I \in T(I_0)$.

If $F = f$ on $E$, then $\mathbb{P}$ satisfies

(57) $\vec{P}_I$ agrees with $f$ at $z_!(I)$, for each leaf $I$ of type $C1$, since by definition, $z_!(I) \in E$.

Conversely, we want to pass from a family $\mathbb{P}$ as in (55), to a function $F \in C^2(Q_0)$. It's not immediately clear how to do that. Our plan is to define a "local function" $F_I \in C^2(Q(I))$ for each $I \in T(I_0)$, such that

(58) $J_{S(I)}(F_I) = \vec{P}_I$, and $\| F_I \|_{C^2(Q(I))} \leq 1 + C\epsilon$.

We will then patch together the $F_I$ by setting

(59) $F = \sum_{I \in T(I_0)} \theta_I F_I \in C^2(Q_0)$.

Thanks to Algorithm AUB, we understand well the problem of producing $F_I$ satisfying (58). We hope that the function $F$ in (59) satisfies

(60) $F = f$ on $E$, and

(61) $\| F \|_{C^2(Q)} \leq 1 + C'\epsilon$.

In fact, (60) holds provided our family of Whitney fields $\mathbb{P}$ satisfies (57). Indeed, if (57) holds, then $F_I = f$ at $z_!(I)$ for each leaf $I$ of type $C1$, as follows from (57) and (58). Equality (60) therefore follows from (54).

To prove (61), we would like our local functions $F_I$ to satisfy the strong consistency condition

(62) $|\partial^\alpha(F_I - F_{I'})| \leq C\epsilon|I|^{2-|\alpha|}$ on $\mathrm{supp}\,\theta_I \cap \mathrm{supp}\,\theta_{I'}$ for $|\alpha| \leq 1$, whenever $\mathrm{supp}\,\theta_I \cap \mathrm{supp}\,\theta_{I'} \neq \emptyset$.

If we can pick the $F_I$ to satisfy (62), then it is easy to prove the desired estimate (61).

Unfortunately, it's far from clear how to produce families of local functions $F_I$ that satisfy (62). Therefore, in place of (62), we will settle for the following weaker dyadic analogue:

(63) Let $I \in T(I_0)$, and let $I'$ be one of the two dyadic children of $I$. Then we have $|\partial^\alpha(F_I - F_{I'})| \le C\epsilon|I|^{2-|\alpha|}$ on $Q(I') \subset Q(I)$, for $|\alpha| \le 1$.

We will construct families of local functions $F_I$ satisfying (58) and (63); then we will see how the function $F$ in (59) behaves for such families of local functions. Finally, we will apply what we have learned to our Model Problem.

To construct $F_I$ satisfying (58) and (63), the key idea is to define and compute a certain convex polyhedron $K(I) \subset Wh(S(I))$ for each $I \in T(I_0)$.

We now explain the construction of the $K(I)$.

### The basic polyhedra

We construct a polyhedron $K(I) \subset Wh(S(I))$ for $I \in T(I_0)$, by bottom-up recursion in the tree $T(I_0)$. The recursion proceeds as follows:

In the base case, $I$ is a leaf in $T(I_0)$. We then define

(64) $K(I) = \{\vec{P} \in K_{AUB}(S(I),\ Q(I)) : \vec{P}$ agrees with $f$ at $z_!(I)\}$ if the leaf $I$ is of type C1, and

(65) $K(I) = K_{AUB}(S(I),\ Q(I))$ if the leaf $I$ isn't of type C1.

(For the polyhedron $K_{AUB}(\cdots)$, see Algorithm AUB above.)

For the induction step, suppose $I \in T(I_0)$ is an internal node with children $I_1, I_2$; and suppose we have already defined the convex polyhedra $K(I_1) \subset Wh(S(I_1))$ and $K(I_2) \subset Wh(S(I_2))$. We then define $K(I) \subset Wh(S(I))$, as follows: Let

(66) $S^+(I) = S(I) \cup S(I_1) \cup S(I_2) \subset Q(I)$.

Next, define

(67) $\widetilde{K}(I) = \{\vec{P}^+|_{S(I)} : \vec{P}^+ \in K_{AUB}(S^+(I), Q(I)), \vec{P}^+|_{S(I_1)} \in K(I_1), \vec{P}^+|_{S(I_2)} \in K(I_2)\}$.

Finally, we take $K(I)$ to be a polyhedron, defined by at most $C(\epsilon)$ constraints, and slightly larger than $\widetilde{K}(I)$. In particular, $K(I) \supset \widetilde{K}(I)$.

The purpose in passing from $\widetilde{K}(I)$ to $K(I)$ here, is to prevent the number of constraints defining $K(I)$ from growing rapidly as $I$ moves up the tree $T(I_0)$ in our bottom-up recursion. Such a rapid growth of the number of constraints would greatly increase the number of computer operations needed to carry out our algorithms.

However, to simplify the presentation, let us pretend in this introduction that we simply take

(68) $K(I) = \widetilde{K}(I)$.

This completes our description of the recursive definition of the polyhedra $K(I)$.

It is a routine task to follow the above recursive definition, and compute the polyhedra $K(I)$. From now on, we assume that they are known.

**Using the basic polyhedra**

Let us relate the polyhedra $K(I)$ to our previous discussion (55)–(63). Suppose first that we are given a function

(69) $F \in C^2(Q(I_0))$ with norm at most 1, such that $F = f$ on $E$.

Then an easy bottom-up recursion in the tree $T(I_0)$, using the defining properties of $K_{AUB}(\cdots)$, shows that

(70) $J_{S(I)}(F) \in K(I)$ for each $I \in T(I_0)$.

In particular, (69) implies

(71) $J_{S(I_0)}(F) \in K(I_0)$.

Conversely, suppose we are given a Whitney field

(72) $\vec{P}_0 \in K(I_0)$.

By top-down recursion in the tree $T(I_0)$, we will compute a Whitney field $\vec{P}_I \in K(I)$ for each $I \in T(I_0)$. The top-down recursion proceeds as follows:

In the base case, we have $I = I_0$, the root of the tree $T(I_0)$. In this case, we simply set

(73) $\vec{P}_{I_0} = \vec{P}_0 \in K(I_0)$.

For the induction step, let $I \in T(I_0)$ be an internal node, with children $I_1, I_2$. Suppose we have already computed $\vec{P}_I \in K(I)$. We will then compute $\vec{P}_{I_1} \in K(I_1)$ and $\vec{P}_{I_2} \in K(I_2)$. This will complete the top-down recursion. To produce $\vec{P}_{I_1}$ and $\vec{P}_{I_2}$, we recall that (we are pretending that) $K(I) = \tilde{K}(I)$, defined by (67). Since $\vec{P}_I \in K(I)$, it follows that there exists a Whitney field

(74) $\vec{P}^+ \in K_{AUB}(S^+(I), Q(I))$, such that

(75) $\vec{P}^+|_{S(I)} = \vec{P}_I$, $\vec{P}^+|_{S(I_1)} \in K(I_1)$, and $\vec{P}^+|_{S(I_2)} \in K(I_2)$.

Since the Whitney field $\vec{P}_I$ and the polyhedra $K_{AUB}(S^+(I), Q(I)), K(I_1), K(I_2)$ are known, we can compute a particular $\vec{P}^+$ satisfying (74) and (75), by routine linear programming. Once we have found such a $\vec{P}^+$, we set

(76) $\vec{P}_{I_1} = \vec{P}^+|_{S(I_1)}$ and $\vec{P}_{I_2} = \vec{P}^+|_{S(I_2)}$.

In particular, $\vec{P}_{I_1} \in K(I_1)$ and $\vec{P}_{I_2} \in K(I_2)$, thanks to (75). This completes our top-down recursion.

Thus, given $\vec{P}_0 \in K(I_0)$, we have computed $\vec{P}_I \in K(I)$ for all $I \in T(I_0)$; in particular $\vec{P}_{I_0} = \vec{P}_0$.

Next, for each $I \in T(I_0)$, we produce a "local function"

(77) $F_I \in C^2(Q(I))$, with norm $\leq 1 + C\epsilon$, such that $J_{S(I)}(F_I) = \vec{P}_I$.

To do so, suppose first that $I$ is a leaf of $T(I_0)$. Then, by definition (64), (65), $K(I) \subset K_{AUB}(S(I), Q(I))$. Since $\vec{P}_I \in K(I)$, the defining property (27) of $K_{AUB}(\cdots)$ gives us a function $F_I$ satisfying (77).

On the other hand, suppose $I$ is an internal node of $T(I_0)$, with children $I_1, I_2$. Then (74), (75), (76) hold for a Whitney field $\vec{P}^+$. From (74) and the defining property (27) of $K_{AUB}(\cdots)$, we obtain a function $F_I \in C^2(Q_I)$ with norm at most $1+\epsilon$, such that $J_{S^+(I)}(F_I) = \vec{P}^+$. Thanks to (75) and (76), this $F_I$ satisfies

(78) $J_{S(I)}(F_I) = \vec{P}_I,\ J_{S(I_1)}(F_I) = \vec{P}_{I_1}, J_{S(I_2)}(F_I) = \vec{P}_{I_2}$.

In particular, our $F_I$ satisfies (77). Thus, (77) holds in all cases.

Furthermore, our local functions $F_I$ do satisfy the "dyadic consistency condition" (63). To see this, we return to (78), and apply (77) to the intervals $I_1$ and $I_2$. Thus, we find that $J_{S(I_1)}(F_I) = J_{S(I_1)}(F_{I_1}) = \vec{P}_{I_1}$, and similarly for $I_2$. In particular,

(79) $J_{S(I_1)}(F_I - F_{I_1}) = 0$ and $J_{S(I_2)}(F_I - F_{I_2}) = 0$.

We recall from (77) that $\| F_{I_1} \|_{C^2(Q(I_1))} \leq 1+\epsilon$ and $\| F_I \|_{C^2(Q(I_1))} \leq \| F_I \|_{C^2(Q(I))} \leq 1+\epsilon$. Moreover, we have picked $S(I_1)$ so that any $z \in Q(I_1)$ satisfies $|z - z'| \leq \epsilon^{100}\delta_{Q(I_1)} = C\epsilon^{100}|I_1|$ for some $z' \in S(I_1)$.

In view of the above remarks and Taylor's theorem, (79) implies the estimate $|\partial^\alpha(F_I - F_{I_1})| \leq C\epsilon^{100}|I_1|^{2-|\alpha|}$ on $Q(I_1)$ for $|\alpha| \leq 1$, and similarly for $I_2$. This is stronger than the desired estimate (63), since we have here $\epsilon^{100}$ in place of $\epsilon$. In any event, we have proven (63).

Let us summarize the discussion so far.

(80) For each $I \in T(I_0)$, we have computed the convex polyhedron $K(I) \subset Wh(S(I))$.

(81) Suppose $F \in C^2(Q(I_0))$ with norm $\leq 1$, with $F = f$ on $E$. Then $J_{S(I_0)}(F) \in K(I_0)$.

(82) Conversely, given $\vec{P}_0 \in K(I_0)$, we can compute $\vec{P}_I \in K(I)$ for each $I \in T(I_0)$, with $\vec{P}_{I_0} = \vec{P}_0$. Moreover, for each $I \in T(I_0)$, we have defined a "local function" $F_I \in C^2(Q(I))$ with norm $\leq 1+\epsilon$, such that $J_{S(I)}(F_I) = \vec{P}_I$.

The $F_I$ in (82) satisfy the dyadic consistency condition

(83) $|\partial^\alpha(F_I - F_{I'})| \leq C\epsilon|I|^{2-|\alpha|}$ on $Q(I')$ for $|\alpha| \leq 1$, whenever $I'$ is a child of the internal node $I \in T(I_0)$.

Given $\vec{P}_0 \in K(I_0)$, we take $\vec{P}_I, F_I$ as in (82), (83), and define

(84) $F^{I_0} = \sum_{I \in T(I_0)} \theta_I F_I \in C^2(Q_0)$, as in (59).

(We have changed notation, by writing $F^{I_0}$ in place of $F$; this change will soon be useful.) We would be happy if $F^{I_0}$ satisfied

(85) $F^{I_0} = f$ on $E$, and

(86) $\| F^{I_0} \|_{C^2(Q_0)} \leq 1 + C\epsilon$.

This would provide a complete converse to (81); we would then have essentially solved our Model Problem.

Our $F^{I_0}$ is easily seen to satisfy (85). In fact, by definition (64), we see that $\vec{P}_I$ agrees with $f$ at $z_!(I)$ whenever $I$ is a leaf of type C1. Since $z_!(I) \in S(I)$ for such $I$, and since $J_{S(I)}(F_I) = \vec{P}_I$ always, it follows that $F_I = f$ at $z_!(I)$, for each leaf $I$ of type C1. Thanks to (54), it follows in turn that $F_I = f$ on $E \cap \operatorname{supp} \theta_I$ for any $I \in T(I_0)$, from which (85) follows by definition (84). Thus, our $F^{I_0}$ is an interpolant for $f$.

Unfortunately, we do not have the strong estimate (86) for $F^{I_0}$. To obtain that estimate, we would need the consistency condition (62), whereas we have achieved merely the weaker dyadic consistency (83). Accordingly, in place of (86), we obtain only the following weaker condition:

For a certain set Goodpoints$(I_0) \subset Q_0$ containing all but $C\epsilon$ of the area of $Q_0$, we find that

(87) $|J_z(F^{I_0})|_z \leq 1 + C\epsilon$ for all points $z \in$ Goodpoints$(I_0)$,

but only

(88) $|J_z(F^{I_0})|_z \leq C$ for all points $z \in Q_0 \setminus$ Goodpoints$(I_0)$.

We recall from the definition (13) of the $C^2$ norm that our desired estimate $\| F^{I_0} \|_{C^2(Q_0)} \leq 1 + C\epsilon$ amounts to saying that $|J_z(F)|_z \leq 1 + C\epsilon$ for *all* $z \in Q_0$.

The distinction between (87) and (88) is closely related to a familiar property of dyadic intervals, which we now recall.

Our dyadic intervals were defined by repeatedly bisecting a "starting interval" $I_0$. Let $I \subset I_0$ be a tiny subinterval, not necessarily dyadic. Typically, $I$ is contained in a dyadic interval not much bigger than $I$. However, if $I$ is very badly placed, it may happen that the only dyadic intervals containing $I$ are much larger than $I$. (In the worst case, when $I$ is centered at the midpoint of $I_0$, it is contained in no proper dyadic subinterval of $I_0$.)

Property (87) of $F^{I_0}$ follows straightforwardly from our dyadic consistency condition. Property (88) is a lot harder to prove. It is based on our ability to pick the local functions $F_I$ to satisfy a further dyadic consistency condition, involving $\frac{\partial}{\partial x_2} F_I(x_1, x_2)$ evaluated at the endpoints of the interval $I \times \{0\} \subset \mathbb{R}^2$. This consistency condition is relevant only for $C^2(\mathbb{R}^2)$, and has no useful analogue for, say, $C^2(\mathbb{R}^3)$. Many of the differences between the oversimplified discussion in this introduction and the unfortunate truth arise from the need to establish the additional dyadic consistency and obtain (88). For purposes of this introduction, let us take (88) for granted.

We have succeeded in computing an interpolant $F^{I_0} \in C^2(Q_0)$ that satisfies (85), (87), (88), but not (86). To overcome this obstacle, we consider an ensemble of $C(\epsilon)$ distinct starting intervals $I_0$. For each $I_0$ in our ensemble, we know how to compute an interpolant $F^{I_0} \in C^2(Q_0)$ that satisfies (85), (87), (88). We then define our interpolant $F$ to be the average of the interpolants $F^{I_0}$ over all $I_0$ in our ensemble. Since (85) holds for each $I_0$, we have

(89) $F = f$ on $E$,

so at least we have done no harm. Moreover, if we pick the ensemble of starting intervals correctly, then the following holds:

(90) For any fixed $z \in Q_0$, we have Goodpoints$(I_0) \ni z$ for all but at most $\epsilon$ percent of the starting intervals $I_0$ in our ensemble.

Thanks to (90), we can average our estimates (87), (88) over all $I_0$ in the ensemble, and we find that $|J_z(F)|_z \leq 1 + C'\epsilon$ for all $z \in Q_0$, i.e.,

(91) $\| F \|_{C^2(Q_0)} \leq 1 + C'\epsilon$.

Since $F$ satisfies (89) and (91), we have succeeded in constructing interpolants $F$ for $f$, having norm at most $1 + C'\epsilon$ in $C^2(Q_0)$. We were fortunate that (90) holds, even though each particular set Goodpoints$(I_0)$ omits part of $Q_0$.

At this point, we have not yet solved the Model Problem (see (36)–(41)), but we now have enough ideas to allow us to give the solution without real difficulty. Perhaps the time has come to end this long introduction. Let us begin our proof of Theorem 1.

We again warn the reader that our introduction is oversimplified. The correct discussion starts in the next section. In particular, we discard the notation and conventions of the introduction, and start afresh.

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## 1. Notation

(0) The label (i.j) denotes equation j in Section i. Within Section i, we abbreviate (i.j) to (j).

(1) If I is an interval and $A$ is a positive real number, then $A$I denotes the interval with the same center as I, and with $A$ times the length of I. If I is open, then so is $A$I. If I is closed, then so is $A$I; and similarly for half-closed intervals.

(2) We write |I| to denote the length of an interval I, and we write center (I) to denote the midpoint of I.

(3) $\mathcal{P}$ denotes the vector space of real-valued second-degree polynomials on $\mathbb{R}^2$.

(4) If $z \in \mathcal{R}^2$ and $F$ is locally $C^2$ in a neighborhood of $z$, then $J_z(F) \in \mathcal{P}$ denotes the second-degree Taylor polynomial of $F$ at $z$.

(5) We define a multiplication $\odot_z$ on $\mathcal{P}$ for each $z \in \mathbb{R}^2$, by stipulating that $J_z(FG) = J_z(F) \odot_z J_z(G)$ for $F, G \in C^2(\mathbb{R}^2)$.

If $S \subset \mathbb{R}^2$ is finite, then $\#(S)$ denotes the number of points of $S$, and:

(6) $Wh(S)$ denotes the vector space of all families $\vec{P} = (P^z)_{z \in S}$ of polynomials $P^z \in \mathcal{P}$, indexed by the points of $S$. We call such a family $\vec{P}$ a "Whitney field" on $S$.

(7) We write $J_S(F)$ for the Whitney field $(J_z(F))_{z \in S}$.

(8) If $\vec{P} = (P^z)_{z\in S} \in Wh(S)$, and if $z_0 \in S$, then $\mathsf{val}(\vec{P}, z_0)$ denotes the real number $(P^{z_0})(z_0)$.

We use $(x_1, x_2)$ as rectangular coordinates on $\mathbb{R}^2$.

(9) If $\vec{P} = (P^z)_{z\in S} \in Wh(S)$, and if $z_0 \in S$, then we write $\mathsf{val}(\partial_i \vec{P}, z_0)$ $(i = 1, 2)$ to denote the real number $\left(\frac{\partial P^{z_0}}{\partial x_i}\right)(z_0)$.

Similarly, we write $\mathsf{val}(\nabla \vec{P}, z_0)$ to denote the vector $(\nabla P^{z_0})(z_0) \in \mathbb{R}^2$.

Note that, for $\vec{P} = J_S(F)$, we have $\mathsf{val}(\vec{P}, z_0) = F(z_0)$, $\mathsf{val}(\partial_i \vec{P}, z_0) = \partial_i F(z_0)$, and $\mathsf{val}(\nabla \vec{P}, z_0) = \nabla F(z_0)$.

(10) A "square" in $\mathbb{R}^2$ is a product of intervals $Q = I \times J \subset \mathbb{R}^2$, with $|I| = |J|$. We write $\delta_Q$ to denote the side length of $Q$, and (for real $A > 0$) we write $AQ$ to denote the square $AI \times AJ$.

## 2. Conventions regarding constants

Within any given section, we may specify a (possibly empty) list of finitely many constants, which we will take to be the "boiler-plate constants" for that section. We then define a "controlled constant" to be a positive real number computed from the boiler-plate constants by applying an algorithm. (In particular, a controlled constant is uniquely determined by the boiler-plate constants.) We write $c, C, C'$, etc., to denote controlled constants. These symbols may denote different controlled constants in different occurrences.

More generally, suppose that, in addition to the boiler-plate constants, we are given real numbers (say $A, \eta$), and/or integer constants (say, $k$). Then we write $C(A, \eta, k)$, $C'(A, \eta, k)$, etc., to denote a positive real number computed by applying an algorithm whose inputs are $A, \eta, k$ and the boiler-plate constants. An expression such as $C(A, \eta, k)$ needn't denote the same quantity in different occurrences. We call $C(A, \eta, k)$ an "$(A, \eta, k)$-controlled constant".

The above notions may change from one section to the next, since each section will have its own list of boiler-plate constants.

Within any particular section, we may be given a positive number $\epsilon$. We always assume that $\epsilon$ is less than a small enough controlled constant. We refer to this assumption as the "small $\epsilon$ assumption".

## 3. Convex polyhedra

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, and let $V^*$ be its dual. By a "convex polyhedron" in $V$, we mean a *compact* convex $K \subset V$ of the form

(1) $K = \{v \in V : \lambda_i(v) \geq \beta_i \text{ for } i = 1, \dots, I\}$, where $\lambda_i \in V^*$ and $\beta_i \in \mathbb{R}$ for each $i$.

We allow the case $K = \emptyset$. We call the inequality $\lambda_i(v) \geq \beta_i$ a "constraint", and we say that $K \subset V$ is a "convex polyhedron defined by I constraints". To specify a convex polyhedron, we provide a list of constraints.

In this section, we state the algorithms used in later sections to manipulate convex polyhedra. These algorithms are well-known, so we omit explanations here. (See [14]). We recall our conventions regarding constants. We use no boiler plate constants in this section. Thus, e.g., $C(A, B)$ denotes a constant computable from $A$ and $B$.

**Algorithm CP1.** (Linear Programming): Given a convex polyhedron $K \subset V$ defined by I constraints, we compute a point $v \in K$, or guarantee that $K$ is empty. To do so, we use work and storage at most $C(I, \dim V)$.

**Algorithm CP2.** (Extreme Points): Given a convex polyhedron $K \subset V$ defined by I constraints, we compute the set $S$ of extreme points of $K$. We have $K =$ convex hull $(S)$ and $\#(S) \leq C(I, \dim V)$. The work and storage used to compute $S$ are at most $C(I, \dim V)$.

**Algorithm CP3.** (Convex Hull): Given a finite set $S \subset V$, we compute a convex polyhedron $K$, defined by at most $C(\#(S), \dim V)$ constraints, such that $K =$ convex hull $(S)$. To do so, we use work and storage at most $C(\#(S), \dim V)$.

**Algorithm CP4.** (Image under Linear Maps): Given a convex polyhedron $K \subset V$ defined by I constraints, and given a linear map $T : V \longrightarrow W$ of finite-dimensional vector spaces, we compute a convex polyhedron $K' \subset W$, defined by at most $C(I, \mathrm{dim} V, \dim W)$ constraints, such that $K' = T(K)$. The work and storage used to do so are at most $C(I, \dim V, \dim W)$.

**Algorithm CP5.** (Intersection): Given convex polyhedra $K_1, K_2 \subset V$, defined by $I_1, I_2$ constraints respectively, we compute the convex polyhedron $K_1 \cap K_2 \subset V$, defined by $I_1 + I_2$ constraints. The work and storage used to do so are at most $C(I_1, I_2, \dim V)$.

**Algorithm CP6.** (Convex Hull of Union I): Given convex polyhedra $K_1, K_2 \subset V$, defined by $I_1, I_2$ constraints respectively, we compute a convex polyhedron $K \subset V$, defined by at most $C(I_1, I_2, \dim V)$ constraints, such that $K =$ convex hull $(K_1 \cup K_2)$. To do so, we use work and storage at most $C(I_1, I_2, \dim V)$.

**Algorithm CP7.** (Convex Hull of Union II): Given convex polyhedra $K_1, K_2 \subset V$ defined by $I_1, I_2$ constraints respectively, and given a point $v \in$ convex hull $(K_1 \cup K_2)$, we compute points $v_1 \in K_1$, $v_2 \in K_2$ and a number $t \in [0, 1]$ such that $v = tv_1 + (1-t)v_2$. The work and storage used to do so are at most $C(I_1, I_2, \dim V)$.

**Algorithm CP8.** (Minkowski sum): Given convex polyhedra $K_1, K_2 \subset V$ defined by $I_1, I_2$ constraints respectively, we compute a convex polyhedron $K \subset V$, defined by at most $C(I_1, I_2, \dim V)$ constraints, such that $K = \{v_1 + v_2 : v_1 \in K_1, v_2 \in K_2\}$. The work and storage used to do so are at most $C(I_1, I_2, \dim V)$.

**Algorithm CP9.** (Inverse Image): Given finite-dimensional vector spaces $V$ and $W_1, \ldots, W_L$ of dimensions $D$ and $D_1, \ldots, D_L$, respectively; given convex polyhedra $K_1 \subset W_1$, $K_2 \subset W_2, \ldots, K_L \subset W_L$, defined by $I_1, I_2, \ldots, I_L$ constraints, respectively; and given linear maps $T_1 : V \longrightarrow W_1, T_2 : V \longrightarrow W_2, \ldots, T_L : V \longrightarrow W_L$, such that $T_1^{-1}(K_1) \cap T_2^{-1}(K_2) \cap \cdots \cap T_L^{-1}(K_L) \subset V$ is compact; we compute a convex polyhedron $K \subset V$, defined by at most $C(I_1, \ldots, I_L, D_1, \ldots D_L, D)$ constraints, such that $K = T_1^{-1}(K_1) \cap T_2^{-1}(K_2) \cap \cdots \cap T_L^{-1}(K_L)$. The work and storage used to compute $K$ are at most $C(I_1, \ldots, I_L, D_1, \ldots, D_L, D)$.

The above algorithms may be implemented straightforwardly. Several of them require many computer operations, except in small cases. In later sections, we will use them without mentioning them explicitly.

## 4. Dyadic grids

For any given real number $t$, the "dyadic grid" $\mathcal{G}_t$ consists of all intervals of the form $[t + m \cdot 2^\ell, t + (m+1) \cdot 2^\ell)$, where $m, \ell \in \mathbb{Z}$.

Once we fix a dyadic grid $\mathcal{G}_t$, an interval $I \in \mathcal{G}_t$ will be called "dyadic". Each dyadic interval $I$ is partitioned into two dyadic subintervals $I_1, I_2$ of length $|I_1| = |I_2| = \frac{1}{2}|I|$. We call $I_1$ and $I_2$ the "dyadic children" of $I$. If $I_1$ lies to the left of $I_2$, then we call $I_1, I_2$ respectively, the "left dyadic child" and the "right dyadic child" of $I$. Also, each dyadic interval $I$ is contained in one and only one dyadic interval of length $2|I|$. We call that interval the "dyadic parent" of $I$, and denote it by $I^+$. Thus, each dyadic interval has one dyadic parent and two dyadic children. The above notions depend on the choice of the grid $\mathcal{G}_t$, i.e., on the number $t$.

Later on, we will encounter a collection $\mathcal{J}$ of dyadic intervals, having the following property for fixed $x \in \mathbb{R}$, $\delta > 0$:

Each $I \in \mathcal{J}$ satisfies $|I| \leq \delta$ and $3I \ni x$.

We hope that all the $I \in \mathcal{J}$ are contained in a single dyadic interval $\hat{I}$, with $|\hat{I}|$ not much bigger than $\delta$. Therefore, we make the following

**Definition.** Let $t, x \in \mathbb{R}$, and let $k_0, \ell$ be integers, with $k_0 > 0$. Let $\hat{I}$ be the interval of length $2^{k_0+\ell}$ in $\mathcal{G}_t$, such that $\hat{I} \ni x$. Then we say that $(x, \ell)$ is "$k_0$-regular" for $\mathcal{G}_t$, provided the following holds:

(1) Every $I \in \mathcal{G}_t$ such that $|I| \leq 2^\ell$ and $3I \ni x$ satisfies $I \subset \hat{I}$.

The following simple result shows that $(x, \ell)$ is often $k_0$-regular for $\mathcal{G}_t$.

**Lemma DG1.** *Let* $x, t \in \mathbb{R}$ *and* $k_0, \ell \in \mathbb{Z}$, *with* $k_0 > 0$. *Suppose* $(x, \ell)$ *is not* $k_0$*-regular for* $\mathcal{G}_t$. *Then* $\mathsf{dist}(x - t, 2^{\ell+k_0}\mathbb{Z}) \leq 2^{\ell+1}$.

*Proof.* Let $I \in \mathcal{G}_t$ satisfy $|I| \leq 2^\ell$, $3I \ni x$, $I \not\subset \hat{I}$, where $\hat{I} \in \mathcal{G}_t$ satisfies $|\hat{I}| = 2^{\ell+k_0}$, $\hat{I} \ni x$. Let $y \in I \setminus \hat{I}$. Since $x \in 3I$ and $|I| \leq 2^\ell$, we have $y \in I \subset [x - 2|I|, x + 2|I|] \subset [x - 2^{\ell+1}, x + 2^{\ell+1}]$. Thus, $x, y \in [x - 2^{\ell+1}, x + 2^{\ell+1}]$, with $x \in \hat{I}$ and $y \notin \hat{I}$. Consequently, $[x - 2^{\ell+1}, x + 2^{\ell+1}]$ contains an endpoint of $\hat{I}$. However, the endpoints of $\hat{I}$ belong to $t + 2^{\ell+k_0}\mathbb{Z}$. Therefore, $\mathsf{dist}(x, t + 2^{\ell+k_0}\mathbb{Z}) \leq 2^{\ell+1}$. □

To exploit Lemma DG1, we will average a function $F_t$ over an ensemble of dyadic grids $\mathcal{G}_t$. This will be done much later, but we now introduce the relevant ensemble of dyadic grids.

**Definition.** Let $k_{\max}$, $k_0$ be integers, with $k_0 > 0$. We write $\mathsf{Per}(k_0)$ to denote the set of all maps $\sigma : \mathbb{Z} \to \{0, 1\}$ such that $\sigma(j + k_0) = \sigma(j)$ for all $j \in \mathbb{Z}$. For each $\sigma \in \mathsf{Per}(k_0)$, we then define

$$t(\sigma; k_{\max}) = \sum_{j<k_{\max}} \sigma(j) \cdot 2^j \in \mathbb{R}.$$

Thus, the $\sigma(j)$ $(j < k_{\max})$ are the digits in the binary expansion of $t(\sigma; k_{\max})$. We define

$$T(k_0, k_{\max}) = \{t(\sigma; k_{\max}) : \sigma \in \mathsf{Per}(k_0)\} \subset \mathbb{R}.$$

By thinking about the digits in binary expansions, one sees easily that

(2) $T(k_0, k_{\max}) = \left\{ \frac{2^{k_{\max}}}{2^{k_0}-1} \cdot m : m = 0, 1, \dots, 2^{k_0} - 1 \right\}$. In particular

(3) $T(k_0, k_{\max}) \subset [0, 2^{k_{\max}}]$, and

(4) $\#(T(k_0, k_{\max})) = 2^{k_0}$.

Later on, when we average functions $F_t$ over an ensemble of dyadic grids, the grids in question will be the $\mathcal{G}_{t_0+t}$ for all $t \in T(k_0, k_{\max})$; here, $t_0$, $k_0$, $k_{\max}$ are fixed. The following simple result will be useful.

**Lemma DG2**: *Let* $k_0, k_{\max}, \ell \in \mathbb{Z}$, *with* $k_0 > 0$ *and* $\ell \le k_{\max} - k_0$. *Let* $z \in \mathbb{R}$. *Then* $\mathsf{dist}(z - t, 2^{\ell+k_0}\mathbb{Z}) \le 2^{\ell+1}$ *for at most 100 distinct* $t \in T(k_0, k_{\max})$.

*Proof.* For each $\sigma \in \mathsf{Per}(k_0)$, we write

$$\begin{aligned} t(\sigma; k_{\max}) &= \sum_{j<k_{\max}} \sigma(j) \cdot 2^j \\ &= \sum_{j<\ell} \sigma(j) \cdot 2^j + \sum_{\ell \le j < \ell+k_0} \sigma(j) \cdot 2^j + \sum_{\ell+k_0 \le j < k_{\max}} \sigma(j) \cdot 2^j \\ &\equiv t_{\ell o}(\sigma) + t_{\mathrm{med}}(\sigma) + t_{hi}(\sigma). \end{aligned}$$

Note that $0 \le t_{\ell o}(\sigma) \le 2^\ell$ and $t_{hi}(\sigma) \in 2^{\ell+k_0}\mathbb{Z}$. Therefore, for any given $\sigma$, if

$$\mathsf{dist}(z - t(\sigma; k_{\max}), 2^{\ell+k_0}\mathbb{Z}) \le 2^{\ell+1}, \quad \text{then} \quad \mathsf{dist}(z - t_{\mathrm{med}}(\sigma), 2^{\ell+k_0}\mathbb{Z}) \le 3 \cdot 2^\ell,$$

i.e.,

(5) $\mathsf{dist}(2^{-\ell}z - 2^{-\ell}\, t_{\mathrm{med}}(\sigma), 2^{k_0}\mathbb{Z}) \le 3$.

Hence, Lemma DG2 will follow, if we can show that (5) holds for at most 100 distinct $\sigma \in \mathsf{Per}(k_0)$. However, by thinking about binary digits, one sees that the map

$$\sigma \mapsto 2^{-\ell} \cdot t_{\mathrm{med}}(\sigma)$$

is a one-to-one correspondence between $\mathsf{Per}(k_0)$ and $\{0, 1, \dots, 2^{k_0} - 1\}$. Thus, Lemma DG2 holds because $\mathsf{dist}(2^{-\ell}z - m, 2^{k_0}\mathbb{Z}) \le 3$ for at most 100 of the integers $m = 0, 1, \dots, 2^{k_0} - 1$. □

Combining Lemmas DG1 and DG2, we obtain the following

**Lemma DG3.** *Let* $k_0$, $k_{\max}, \ell \in \mathbb{Z}$, *with* $k_0 > 0$ *and* $\ell \le k_{\max} - k_0$. *Let* $x, t_0 \in \mathbb{R}$. *Then there are at most* $100$ *distinct* $t \in T(k_0, k_{\max})$ *such that* $(x, \ell)$ *is not* $k_0$*-regular for* $\mathcal{G}_{t_0+t}$.

## 5. $C^2$ norms

For each $z \in \mathbb{R}^2$, we suppose we are given a norm $|\cdot|_z$ on $\mathcal{P}$. We assume that these norms satisfy the following

**Bounded Distortion Property**

(1) $c_0 \cdot \max_{|\alpha|\le 2} |\partial^\alpha P(z)| \le |P|_z \le C_0 \cdot \max_{|\alpha|\le 2} |\partial^\alpha P(z)|$ for all $P \in \mathcal{P}$, $z \in \mathbb{R}^2$.

**Approximate Translation-Invariance Property**

(2) $|P|_{z+h} \le \exp(C_1|h|) \cdot |P|_z$ for all $P \in \mathcal{P}$ and $z, h \in \mathbb{R}^2$.

Given $0 < \eta \le 1$ and $P \in \mathcal{P}$, $z \in \mathbb{R}^2$, we assume that an $\mathsf{Oracle}$ produces a number $\mathcal{N}(P, z, \eta)$ such that

(3) $(1+\eta)^{-1} \cdot \mathcal{N}(P, z, \eta) \le |P|_z \le (1+\eta) \cdot \mathcal{N}(P, z, \eta)$.

To compute a single $\mathcal{N}(P, z, \eta)$, the $\mathsf{Oracle}$ charges us "work"

(4) $\exp(C_2/\eta)$.

If $\Omega \subset \mathbb{R}^2$ is open, and if $F \in C^2_{\mathrm{loc}}(\Omega)$ is a real-valued function on $\Omega$, then we define

(5) $\| F \|_{C^2(\Omega)} := \sup_{z\in\Omega} |J_z(F)|_z$.

The following lemma, a slight variant of results in [6], concerns "gentle partitions of unity".

**Lemma GPU.** *Let* $A_1, \dots, A_4, \epsilon, \delta_z$ *be positive real numbers, with*

(6) $\delta_z \le 1$.

*Let* $z \in \mathbb{R}^2$, *let* $U$ *be an open neighborhood of* $z$, *and let* $\theta_\nu, F_\nu \in C^2_{\mathrm{loc}}(U)$ *for* $\nu = 1, 2, \dots, \nu_{\max}$. *Assume the following:*

(7) $|J_z(F_\nu)|_z \le 1 + A_1\epsilon$ *for each* $\nu$.

(8) *At most* $A_2$ *of the jets* $J_z(\theta_\nu)$ *are nonzero.*

(9) $\sum_\nu \theta_\nu = 1$ *on* $U$, *and each* $\theta_\nu$ *is non-negative on* $U$.

*Assume also that either*

(a) $|\partial^\alpha \theta_\nu(z)| \le \epsilon A_3 \delta_z^{-|\alpha|}$ *for* $0 < |\alpha| \le 2$, $1 \le \nu \le \nu_{\max}$; *and* $|\partial^\alpha(F_\nu - F_{\nu'})(z)| \le A_4 \delta_z^{2-|\alpha|}$ *for* $|\alpha| \le 2$, *whenever* $z \in \operatorname{supp}\theta_\nu \cap \operatorname{supp}\theta_{\nu'}$; *or*

(b) $|\partial^\alpha \theta_\nu(z)| \leq A_3 \delta_z^{-|\alpha|}$ *for* $0 < |\alpha| \leq 2$, $1 \leq \nu \leq \nu_{\max}$; *and* $|\partial^\alpha (F_\nu - F_{\nu'})(z)| \leq \epsilon A_4 \delta_z^{2-|\alpha|}$ *for* $|\alpha| \leq 1$, *whenever* $z \in \mathsf{supp}\, \theta_\nu \cap \mathsf{supp}\, \theta_{\nu'}$.

*Then* $F = \sum_\nu \theta_\nu F_\nu$ *satisfies* $|J_z(F)|_z \leq 1 + A\epsilon$, *where* $A$ *may be computed from* $A_1 \ldots A_4$ *and the constant* $C_0$ *in the* Bounded Distortion Property. *More precisely,* $A = A_1 + C\, C_0 A_2 A_3 A_4$, *where* $C$ *is a universal constant.*

*Proof.* Pick $\nu_0$ such that $z \in \mathsf{supp}\, \theta_{\nu_0}$, and note that

$$(10)\quad J_z(F) = \sum_\nu \theta_\nu(z) J_z(F_\nu) + \sum_\nu [J_z(\theta_\nu \cdot [F_\nu - F_{\nu_0}]) - \theta_\nu(z) \cdot J_z(F_\nu - F_{\nu_0})].$$

Since $|\cdot|_z$ is a norm, $\theta_\nu(z) \geq 0$, and $\sum_\nu \theta_\nu(z) = 1$, we know that

$$(11)\quad \Big|\sum_\nu \theta_\nu(z) J_z(F_\nu)\Big|_z \leq \sum_\nu \theta_\nu(z) |J_z(F_\nu)|_z \leq \max_\nu |J_z(F_\nu)|_z \leq 1 + A_1 \epsilon\,.$$

Moreover, for $|\alpha| \leq 2$, and for any $\nu$ such that $z \in \mathsf{supp}\, \theta_\nu$, we have

$$\partial^\alpha \{J_z(\theta_\nu \cdot [F_\nu - F_{\nu_0}]) - \theta_\nu(z) \cdot J_z(F_\nu - F_{\nu_0})\}(z) = \\ = \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \alpha' \neq 0}} c(\alpha', \alpha'') \cdot \partial^{\alpha'} \theta_\nu(z) \cdot \partial^{\alpha''} (F_\nu - F_{\nu_0})(z),$$

where the $c(\alpha', \alpha'')$ are harmless coefficients. On the right-hand side here, we have $0 < |\alpha'| \leq 2$ and $|\alpha''| \leq 1$. Hence, assuming either hypothesis (a) or (b) of Lemma GPU, we see that

$$|\partial^\alpha \{J_z(\theta_\nu \cdot [F_\nu - F_{\nu_0}]) - \theta_\nu(z) \cdot J_z(F_\nu - F_{\nu_0})\}(z)| \leq \\ \leq \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \alpha' \neq 0}} C A_3 A_4 \epsilon \delta_z^{2 - |\alpha'| - |\alpha''|} \leq C' A_3 A_4 \epsilon \delta_z^{2 - |\alpha|} \leq C' A_3 A_4 \epsilon$$

where $C, C'$ are universal constants.

This holds for all $|\alpha| \leq 2$. Hence, by the Bounded Distortion Property,

$$|J_z(\theta_\nu \cdot [F_\nu - F_{\nu_0}]) - \theta_\nu(z) \cdot J_z(F_\nu - F_{\nu_0})|_z \leq C' C_0 A_3 A_4 \epsilon$$

whenever $z \in \mathsf{supp}\, \theta_\nu$. Summing over $\nu$, and recalling that there are at most $A_2$ nonzero summands, we conclude that

$$(12)\quad \Big|\sum_\nu [J_z(\theta_\nu \cdot [F_\nu - F_{\nu_0}]) - \theta_\nu(z) \cdot J_z(F_\nu - F_{\nu_0})]\Big|_z \leq C' C_0 A_2 A_3 A_4 \epsilon,$$

with $C_0$ as in (1), and with $C'$ a universal constant. Putting (11) and (12) into (10), we learn that

$$|J_z(F)|_z \leq 1 + A_1 \epsilon + C' C_0 A_2 A_3 A_4 \epsilon\,,$$

completing the proof of Lemma GPU. □

## 6. Approximate unit balls

In this section, we suppose we are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ $(z \in \mathbb{R}^2)$, together with an $\mathsf{Oracle}$, as in Section 5. The boiler-plate constants in this section are the constants called $c_0, C_0, C_1, C_2$ in that section.

Also, in this section, we work with a positive number $\epsilon$, assumed to satisfy the "small $\epsilon$ assumption" as explained in Section 2.

We will be concerned here with the following notion:

**Definition.** Let $\epsilon > 0$, let $Q \subset \mathbb{R}^2$ be an open square, let $S \subset Q$ be non-empty and finite, and let $L$ be a positive integer. We say that a convex polyhedron $K \subset Wh(S)$ belongs to $AUB(\epsilon, S, Q, L)$ ($K$ is an "approximate unit ball") if the following hold:

(1) $K$ is defined by at most $L$ constraints.

(2) Let $F \in C^2(2Q)$ with norm $\leq 1$. Then $J_S(F) \in K$.

(3) Let $\vec{P} \in K$. Then there exists $F \in C^2(Q)$ with norm $\leq 1 + \epsilon$, such that $J_S(F) = \vec{P}$.

The goal of this section is to compute a $K \in AUB(\epsilon, S, Q, L)$ (for suitable $L$), given any $\epsilon, S, Q$. The ideas needed to do so are contained in [6], but unfortunately, we cannot simply quote. For completeness, we provide details here. We use the following basic result:

**Smoothing Lemma.** *Let $\epsilon > 0$ (satisfying the "small $\epsilon$ assumption"), and let $Q \subset \mathbb{R}^2$ be an open square with sidelength $\delta_Q \leq 1$. Let $0 < \eta < \epsilon^2 \exp(-\frac{1}{\epsilon})$ be given.*

*Let $S \subset Q$, and assume that $|z - z'| > 2\eta \exp\left(\frac{1}{\epsilon}\right) \delta_Q$ for any $z, z' \in S$ distinct. Let $F \in C^2((1+\eta)Q)$, with norm $\leq 1$.*

*Then there exists $F^\# \in C^3(Q)$ with norm $\leq 1 + C\epsilon$ in $C^2(Q)$, such that*

(4) $J_s(F^\#) = J_s(F)$, and

(5) $|\partial^\alpha F^\#| \leq C\eta^{-2}\delta_Q^{-1}$ on $Q$, for $|\alpha| = 3$.

*Sketch of Proof*: Our present Smoothing Lemma is just the special case $m = n = 2$ of Lemma 12.2 in [6], in which balls are replaced by squares. The proofs of Lemmas 12.1 and 12.2 in [6] carry over to the present case, without difficulty. □

Also, directly from [6] , we have the following algorithm:

**Algorithm AUB0.** Given $\epsilon > 0$ and $z \in \mathbb{R}^2$, we compute a convex polyhedron $K_z \subset \mathcal{P}$, with the following properties:

(6) Any $P \in \mathcal{P}$ such that $|P|_z \leq 1$ belongs to $K_z$.

(7) Any $P \in K_z$ satisfies $|P|_z \leq 1 + \epsilon$.

(8) $K_z$ is defined by at most $C(\epsilon)$ constraints.

The work and storage used to compute $K_z$ are at most $C(\epsilon)$.

*Explanation.* See [6].

We now begin the work of computing approximate unit balls. We start with special cases, and build up to the general case.

**Algorithm AUB1.** Suppose we are given $\epsilon > 0$, $0 < \eta < \epsilon^2 \exp\left(\frac{-1}{\epsilon}\right)$, $Q \subset \mathbb{R}^2$ an open square, with sidelength $\delta_Q \leq 1$, and $S \subset Q$ non-empty and finite.

Assume that $|z - z'| > 2\eta \exp\left(\frac{1}{\epsilon}\right) \delta_Q$ for any $z, z' \in S$ distinct. Then we compute $K \in \mathcal{AUB}(C\epsilon, S, Q, L)$ where $L = C(\epsilon, \eta)$.

The work and storage used to compute $K$ are at most $C(\epsilon, \eta)$.

*Explanation.* We can trivially compute a "fine net" $S^+ \subset Q$, such that

(9) $S^+ \supset S$;

(10) Any $z \in Q$ satisfies $|z - z'| < \eta^{20} \delta_Q$ for some $z' \in S^+$; and

(11) $\#(S^+) \leq C(\epsilon, \eta)$.

For each $z \in S^+$, we apply Algorithm AUB0 to produce a convex polyhedron $K_z \subset \mathcal{P}$, satisfying (6), (7), (8).

We now define $K^+ \subset Wh(S^+)$ to be the set of all $\vec{P}^+ = (P^{+,z})_{z \in S^+} \in Wh(S^+)$ satisfying:

(12) $P^{+,z} \in (1 + C\epsilon)K_z$ for each $z \in S^+$; and

(13) $|\partial^\alpha (P^{+,z} - P^{+,z'})(z)| \leq C\eta^{-2}\delta_Q^{-1} \cdot |z - z'|^{3-|\alpha|}$ for $|\alpha| \leq 2$, $z, z' \in S^+$ distinct.

The controlled constant $C$ in (12), (13) will be picked in a moment. Note that

(14) $K^+ \subset Wh(S^+)$ is a convex polyhedron, defined by at most $C(\epsilon, \eta)$ constraints.

We check the following properties of $K^+$:

(15) Let $F \in C^2(2Q)$ with norm $\leq 1$. Then there exists $F^\# \in C^2(Q)$, such that $J_S(F^\#) = J_S(F)$ and $J_{S^+}(F^\#) \in K^+$.

(16) Let $\vec{P}^+ \in K^+$. Then there exists $F \in C^2(Q)$ with norm $\leq 1 + C\epsilon$, such that $J_{S^+}(F) = \vec{P}^+$.

To check (15) we apply the Smoothing Lemma. Thus, there exists $F^\# \in C^3(Q)$ with norm $\leq 1 + C\epsilon$ in $C^2(Q)$, such that (5) holds, and $J_S(F^\#) = J_S(F)$. We check that $J_{S^+}(F^\#) \in K^+$; this will complete the proof of (15).

Since $\| F^\# \|_{C^2(Q)} \leq 1 + C\epsilon$, we have $|J_z(F^\#)|_z \leq 1 + C\epsilon$ for $z \in S^+$.

Hence, (6) yields

(17) $J_z(F^\#) \in (1 + C\epsilon)K_z$ for each $z \in S^+$.

Also, (5) and Taylor's theorem yield the estimate

(18) $|\partial^\alpha \{J_z(F^\#) - J_{z'}(F^\#)\}(z)| \leq C'\eta^{-2}\delta_Q^{-1}|z - z'|^{3-|\alpha|}$ for $|\alpha| \leq 2$, $z, z' \in S^+$ distinct.

If the constant $C$ in (12), (13) is larger than the constants $C, C'$ in (17), (18), then we obtain

(19) $J_{S^+}(F^\#) = (J_z(F^\#))_{z \in S^+} \in K^+$.

We now pick $C$ in (12), (13) as just explained. This proves (19), thus proving (15) as well.

Next, we check (16). Let $\vec{P}^+ = (P^{+,z})_{z \in S^+} \in K^+$. From (7), (12) and the Bounded Distortion Property, we find that $|\partial^\alpha P^{+,z}(z)| \leq C \leq C\eta^{-2}\delta_Q^{-1}$ for $|\alpha| \leq 2$, $z \in S^+$.

Together with (13) and the classical Whitney extension theorem for finite sets, this shows that there exists $F \in C^3(\mathbb{R}^3)$ such that

(20) $|\partial^\alpha F| \leq C\eta^{-2}\delta_Q^{-1}$ on $\mathbb{R}^2$ for $|\alpha| = 3$, and

(21) $J_z(F) = P^{+,z}$ for each $z \in S^+$.

(Recall that $J_z(F)$ denotes the second degree Taylor polynomial, even though $F \in C^3(\mathbb{R}^2)$.)

We restrict $F$ to $Q$, and we check that

(22) $\| F \|_{C^2(Q)} \leq 1 + C\epsilon$.

In fact, let $z \in Q$ be given, and let $z' \in S^+$ be as in (10). Then (10), (20) and Taylor's theorem tell us that $|\partial^\alpha \{J_z(F) - J_{z'}(F)\}(z)| \leq C\eta^{-2}\delta_Q^{-1} \cdot (\eta^{20}\delta_Q)^{3-|\alpha|} \leq C\eta^{18}$ for $|\alpha| \leq 2$.

Hence, by the Bounded Distortion Property,

(23) $|J_z(F) - J_{z'}(F)|_z \leq C'\eta^{18} < \epsilon$, since $0 < \eta < \epsilon^2 \exp\left(\frac{-1}{\epsilon}\right)$.

Also, since $\vec{P}^+ \in K^+$ and $z' \in S^+$, (7), (12) and (21) yield $|J_{z'}(F)|_{z'} \leq 1 + C\epsilon$. Hence, by Approximate Translation-Invariance, we have

(24) $|J_{z'}(F)|_z \leq \exp(C|z - z'|) \cdot |J_{z'}(F)|_{z'} \leq \exp(C\eta^{20}\delta_Q) \cdot (1 + C\epsilon) \leq 1 + C'\epsilon$, since $\delta_Q \leq 1$ and $0 < \eta < \epsilon^2 \exp\left(\frac{-1}{\epsilon}\right)$.

Combining (23) and (24), we find that $|J_z(F)|_z \leq 1 + C''\epsilon$. This holds for arbitrary $z \in Q$, hence (22) holds. Thus, $F$ satisfies (21) and (22), completing the proof of (16).

We now define

(25) $K = \{\vec{P}^+|_S : \vec{P}^+ \in K^+\} \subset Wh(S)$.

By (11) and (14),

(26) $K \subset Wh(S)$ is convex polyhedron, defined by at most $C(\epsilon, \eta)$ constraints. Moreover, we can compute $K$ from $K^+$ using work and storage at most $C(\epsilon, \eta)$.

Comparing (25) with (15) and (16), we learn the following:

(27) Let $F \in C^2(2Q)$ with norm $\leq 1$. Then $J_S(F) \in K$.

(28) Let $\vec{P} \in K$. Then there exists $F \in C^2(Q)$ with norm $\leq 1 + C\epsilon$, such that $J_S(F) = \vec{P}$.

By (26), (27), (28), we have $K \in \mathcal{AUB}(C\epsilon, S, Q, L)$, with $L = C(\epsilon, \eta)$. The reader may check easily that the work and storage used to compute $K$ as above are at most $C(\epsilon, \eta)$. This completes our explanation of Algorithm AUB1.

Note that Algorithm AUB1 applies when $S = \{z_0\}$ is a singleton; we can just take

$$\eta = \frac{1}{2}\epsilon^2 \exp\Big(-\frac{1}{\epsilon}\Big).$$

Thus, we obtain $K \in \mathcal{AUB}(C\epsilon, S, Q, L)$ with $L = C(\epsilon)$, using work and storage at most $C(\epsilon)$.

The following algorithm will allow us to "glue together" two finite sets $S_1$ and $S_2$.

**Algorithm AUB2.** Suppose we are given real numbers $\epsilon > 0$, $r > 0$; open squares $Q_1, Q_2 \subset \mathbb{R}^2$; finite sets $S_1 \subset Q_1$ and $S_2 \subset Q_2$; a point $z_0 \in S_1 \cap S_2$; positive integers $L_1, L_2$; and approximate unit balls $K_1 \in \mathcal{AUB}(\epsilon, S_1, Q_1, L_1)$ and $K_2 \in \mathcal{AUB}(\epsilon, S_2, Q_2, L_2)$. We make the following assumptions:

(29) $\delta_{Q_2} \leq 1$.

(30) $2Q_1 \subset 2Q_2$.

(31) $S_1 \subset Q_2 \cap B(z_0, r)$.

(32) $S_2 \cap B(z_0, \exp\big(\frac{1}{\epsilon}\big)\, r) = \{z_0\}$.

(33) $Q_2 \cap B(z_0, \exp\big(\frac{1}{\epsilon}\big)\, r) \subset Q_1$.

(Here, $B(z_0, r)$ denotes an open disc in $\mathbb{R}^2$.) Then we compute

$$K \in \mathcal{AUB}(C\epsilon, S_1 \cup S_2, Q_2, L_1 + L_2).$$

The work and storage used to do so are at most $C(\epsilon, \#(S_1), \#(S_2), L_1, L_2)$.

*Explanation*: We set $K = \{\vec{P} \in \mathcal{Wh}(S_1 \cup S_2) : \vec{P}|_{S_1} \in K_1 \text{ and } \vec{P}|_{S_2} \in K_2\}$. Thus, $K$ is a convex polyhedron, defined by at most $L_1 + L_2$ constraints.

The work and storage used to compute $K$ are clearly as promised.

To show that $K \in \mathcal{AUB}(C\epsilon, S_1 \cup S_2, Q_2, L_1 + L_2)$, we must establish the following:

(34) Let $F \in C^2(2Q_2)$ with norm $\leq 1$. Then $J_{S_1 \cup S_2}(F) \in K$.

(35) Let $\vec{P} \in K$. Then there exists $F \in C^2(Q_2)$ with norm $\leq 1 + C\epsilon$, such that $J_{S_1 \cup S_2}(F) = \vec{P}$.

To prove (34), we just note that $J_{S_2}(F) \in K_2$ since $K_2 \in \mathcal{AUB}(\epsilon, S_2, Q_2, L_2)$; and $J_{S_1}(F) \in K_1$ since $2Q_1 \subset 2Q_2$ and $K_1 \in \mathcal{AUB}(\epsilon, S_1, Q_1, L_1)$. Thus, (34) holds trivially. Our task is to prove (35). Let $\vec{P} = (P^z)_{z \in S_1 \cup S_2} \in K$. By definition, $\vec{P}|_{S_1} \in K_1$ and $\vec{P}|_{S_2} \in K_2$. Since $K_i \in \mathcal{AUB}(\epsilon, S_i, Q_i, L_i)$ for $i = 1, 2$, there exist functions

(36) $F_1 \in C^2(Q_1)$ and $F_2 \in C^2(Q_2)$, both having norm $\leq 1 + \epsilon$, such that

(37) $J_{S_1}(F_1) = \vec{P}|_{S_1}$ and $J_{S_2}(F_2) = \vec{P}|_{S_2}$.

In particular, $J_{z_0}(F_1) = J_{z_0}(F_2)$ since $z_0 \in S_1 \cap S_2$. Hence, by Taylor's theorem,

(38) $|\partial^\alpha (F_1 - F_2)(z)| \le C|z - z_0|^{2-|\alpha|}$ for $|\alpha| \le 1$, $z \in Q_1 \cap Q_2$.

We take a partition of unity

(39) $1 = \theta_1 + \theta_2$ on $\mathbb{R}^2$, with $\theta_i \in C^2(\mathbb{R}^2)$, $\theta_i \ge 0$ on $\mathbb{R}^2$;

(40) $|\partial^\alpha \theta_i(z)| \le C\epsilon|z - z_0|^{-|\alpha|}$ for $0 < |\alpha| \le 2$, $z \in \mathbb{R}^2$ $(i = 1, 2)$; and

(41) $\mathsf{supp}\, \theta_1 \subset B(z_0, \exp\left(\frac{1}{\epsilon}\right) r)$, $\mathsf{supp}\, \theta_2 \subset \mathbb{R}^2 \setminus B(z_0, r)$.

We can achieve (39), (40), (41) by taking $\theta_1$, $\theta_2$ to be functions of $\epsilon \ln\left(\frac{|z-z_0|}{r}\right)$; details are omitted.

We now define

(42) $F(z) = \theta_1(z)F_1(z) + \theta_2(z)F_2(z)$ for $z \in Q_2$.

This makes sense, because, for $z \in \mathsf{supp}\, \theta_1 \cap Q_2$, we have $z \in B(z_0, \exp\left(\frac{1}{\epsilon}\right) r) \cap Q_2 \subset Q_1$ (see (33)), hence $F_1(z)$ is defined.

We estimate $|J_z(F)|_z$ for $z \in Q_2$.

In a small neighborhood of $z_0$, we have $\theta_1 = 1$, $\theta_2 = 0$. Hence, $J_{z_0}(F) = J_{z_0}(F_1)$. Since $\| F_1 \|_{C^2(Q_1)} \le 1 + \epsilon$, it follows that

(43) $|J_{z_0}(F)|_{z_0} \le 1 + \epsilon$.

For $z \in Q_2 \setminus \{z_0\}$, we use Lemma GPU with $\delta_z = c|z - z_0|$; note that $\delta_z \le 1$, as required for Lemma GPU, since $z$, $z_0 \in Q_2$ and $\delta_{Q_2} \le 1$. We have

$$|J_z(F_1)|_z \le 1 + \epsilon \quad \text{if } z \in Q_2 \cap \mathsf{supp}\, \theta_1 \text{ (see (33), (41)), and}$$
$$|J_z(F_2)|_z \le 1 + \epsilon \quad \text{(see (36)),}$$

as well as (39), (40). Hence, Lemma GPU applies, and it tells us that

$$|J_z(F)|_z \le 1 + C\epsilon \quad \text{for } z \in Q_2 \setminus \{z_0\}.$$

Together with (43), this yields

(44) $\| F \|_{C^2(Q_2)} \le 1 + C\epsilon$.

Next, we check that $J_{S_1 \cup S_2}(F) = \vec{P}$.

First, suppose $z \in S_1$. By (31) and (41), $z \notin \mathsf{supp}\, \theta_2$, hence $J_z(F) = J_z(F_1) = P^z$, thanks to (37).

On the other hand, suppose $z \in S_2 \setminus S_1 \subseteq S_2 \setminus \{z_0\}$. Then (32) and (41) show that $z \notin \mathsf{supp}\, \theta_1$, hence $J_z(F) = J_z(F_2) = P^z$, again thanks to (37).

Thus, $J_z(F) = P^z$ for all $z \in S_1 \cup S_2$, i.e., $J_{S_1 \cup S_2}(F) = \vec{P}$. Together with (44), this completes the proof of (35), as well as our explanation of Algorithm AUB2.

As a first application of Algorithm AUB2, we sharpen Algorithm AUB1 as follows:

**Algorithm AUB3.** Suppose we are given $\epsilon, S, Q, \eta$, with $S \subset Q$, $0 < \eta < \epsilon^2 \exp\left(-\frac{1}{\epsilon}\right)$, and $\delta_Q \leq 1$. Suppose $|z - z'| \geq \eta \operatorname{diam}(S)$ for any $z, z' \in S$ distinct. Then we compute $K \in AUB(C\epsilon, S, Q, L)$, with $L \leq C(\epsilon, \eta)$. The work and storage used to do so are at most $C(\epsilon, \eta)$.

*Explanation*: If $\operatorname{diam}(S) \geq \exp\left(\frac{-10}{\epsilon}\right)\delta_Q$, then we may apply Algorithm AUB1, with $\eta$ replaced by $\eta' = \frac{1}{2}\eta \exp\left(\frac{-12}{\epsilon}\right)$. Hence, we may suppose that $\operatorname{diam}(S) \leq \exp\left(\frac{-10}{\epsilon}\right)\delta_Q$.

Fix $z_0 \in S$, and let $r = 2\operatorname{diam}(S)$. We can trivially compute an open square $Q_1 \subset Q$ such that $S \subset Q_1$, $\delta_{Q_1} \leq 10 \exp\left(\frac{1}{\epsilon}\right) r$, and $Q \cap B(z_0, \exp\left(\frac{1}{\epsilon}\right) r) \subset Q_1$. We set $Q_2 = Q$, $S_1 = S, S_2 = \{z_0\}$. Note that

$$(45) \quad |z - z'| > c\eta \exp\left(\frac{-1}{\epsilon}\right)\delta_{Q_1} \quad \text{for } z, z' \in S_1 \quad \text{distinct}.$$

Note also that

$$(46) \quad S_1 \subset Q_1,\ S_2 \subset Q_2, \text{ and } \delta_{Q_2} \leq 1.$$

We prepare to apply Algorithm AUB1. Let $0 < \epsilon' < \epsilon$, with $\epsilon'$ to be picked below, and let $\eta' = c(\epsilon', \eta)$ be picked so that $0 < \eta' < (\epsilon')^2 \exp\left(-\frac{1}{\epsilon'}\right)$ and $c\eta \exp\left(\frac{-1}{\epsilon}\right) > 2\eta' \exp\left(\frac{1}{\epsilon'}\right)$ with $c$ as in (45). Then we have $S_1 \subset Q_1, \delta_{Q_1} < 1$, $0 < \eta' < (\epsilon')^2 \exp\left(\frac{-1}{\epsilon'}\right)$, and $|z - z'| > 2\eta' \exp\left(\frac{1}{\epsilon'}\right)\delta_{Q_1}$ for $z, z' \in S_1$ distinct. Hence, Algorithm AUB1 applies to $\epsilon', \eta', S_1, Q_1$. Thus, with work at most $C(\epsilon', \eta')$, we compute

$$(47) \quad K_1 \in AUB(C\epsilon', S_1, Q_1, L_1), \text{ with } L_1 \leq C(\epsilon', \eta').$$

We now pick $\epsilon' = c\epsilon$ so that $C\epsilon' < \epsilon$, with $C$ as in (47). Since we took $\eta' = c(\epsilon', \eta)$ above, we now have

$$(48) \quad K_1 \in AUB(\epsilon, S_1, Q_1, L_1), \text{ with } L_1 \leq C(\epsilon, \eta).$$

Moreover, $S_2$ is a singleton. Hence, applying Algorithm AUB1, with $\epsilon$ replaced by $\epsilon/C$ for large enough $C$, we obtain

$$(49) \quad K_2 \in AUB(\epsilon, S_2, Q_2, L_2), \text{ with } L_2 \leq C(\epsilon).$$

The work and storage used to compute $K_1$ are at most $C(\epsilon, \eta)$; and the work and storage used to compute $K_2$ are at most $C(\epsilon)$.

We now check that $K_1, K_2, S_1, S_2, Q_1, Q_2, \epsilon$ satisfy the assumptions of Algorithm AUB2. Aside from (46), (48), (49), these assumptions are as follows:

- $z_0 \in S_1 \cap S_2$; this holds by definition of $S_1, S_2$, since $z_0 \in S$.
- $\delta_{Q_2} \leq 1$; this holds by the assumptions of Algorithm AUB3, since $Q_2 = Q$.

- $2Q_1 \subset 2Q_2$; this holds since $z_0 \in Q_1 \cap Q_2$, and
$$\delta_{Q_1} \leq 10 \exp\left(\frac{1}{\epsilon}\right) r = 20 \exp\left(\frac{1}{\epsilon}\right) \mathrm{diam}\,(S) \leq 20 \exp\left(\frac{-9}{\epsilon}\right) \delta_{Q_2}\,.$$
- $S_1 \subset Q_2 \cap B(z_0, r)$; this holds since $S_1 = S \subset Q = Q_2$, $z_0 \in S$, and $r = 2\mathrm{diam}(S)$.
- $S_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon}\right) r) = \{z_0\}$; this holds, since $S_2 = \{z_0\}$.
- $Q_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon}\right) r) \subset Q_1$; this holds by definition of $Q_1, Q_2$.

Thus, the assumptions of Algorithm AUB2 are satisfied. Applying that algorithm, we obtain

(50) $K \in AUB(C\epsilon, S, Q, L_1 + L_2)$.

The work and storage used to apply Algorithm AUB2 are at most $C(\epsilon, \#(S_1), \#(S_2), L_1, L_2)$.

We have $\#(S_1) = \#(S) \leq C(\eta)$, since $|z - z'| \geq \eta\,\mathrm{diam}\,(S)$ for $z, z' \in S$ distinct. Also, $\#(S_2) = 1$; and we have seen that $L_1 \leq C(\epsilon, \eta)$ and $L_2 \leq C(\epsilon)$. Consequently, the work and storage used in applying Algorithm AUB2 are at most $C(\epsilon, \eta)$, and $L_1 + L_2 \leq C(\epsilon, \eta)$.

Thus, (50) shows that $K$ is an approximate unit ball, as promised. This completes our explanation of Algorithm AUB3.

At last, we pass to the general case.

**Algorithm AUB4.** Given: $\epsilon > 0$ satisfying the "small $\epsilon$ assumption"; an open square $Q \subset \mathbb{R}^2$, with $\delta_Q \leq 1$; and a non-empty finite set $S \subset Q$; we compute a $K \in AUB(\epsilon, S, Q, L)$, with $L \leq C(\epsilon, \#(S))$. The work and storage used to do so are at most $C(\epsilon, \#(S))$.

*Explanation*: We proceed recursively, using induction on $\#(S)$.

We first check whether $\#(S) = 1$. If so, then we may apply Algorithm AUB1 (with $\epsilon$ replaced by $\epsilon/C$ for large enough $C$), and we are done. If $\#(S) > 1$, then we proceed as follows:

Let $0 < \epsilon' < \epsilon$, with $\epsilon'$ to be picked below. (Later on, we will pick $\epsilon' = c\epsilon$ for small enough $c$, but we do not yet make that choice.)

We check whether

(51) $|z - z'| > \exp\left(\frac{-100 \cdot \#(S)}{\epsilon'}\right) \cdot \mathrm{diam}(S)$ for $z$, $z' \in S$ distinct.

Case 1: Suppose (51) holds. Then, taking $\eta' = \exp\left(\frac{-100\#(S)}{\epsilon'}\right)$, we can apply Algorithm AUB3, with inputs $\epsilon', \eta', S, Q$. That algorithm produces

(52) $K \in AUB(C\epsilon', S, Q, L)$, with $L \leq C(\epsilon', \eta') = C'(\epsilon', \#(S))$.

The work and storage used to produce $K$ are at most $C(\epsilon', \eta') = C'(\epsilon', \#(S))$. Moreover, we have $K \in AUB(\epsilon, S, Q, L)$, provided

(53) $C\epsilon' < \epsilon$, where $C$ is as in (52).

Case 2: Suppose (51) does not hold. We can trivially compute

(54) $z_0, z_1 \in S$ distinct, such that

(55) $|z_0 - z_1| \le \exp\left(\frac{-100\cdot\#(S)}{\epsilon'}\right) \cdot \mathrm{diam}(S)$.

Since there are at most $\#(S)$ distinct distances $|z - z_0| (z \in S)$, there exists an even integer $j$ $(2 \le j \le 98 \cdot \#(S))$, such that the distances $|z - z_0| (z \in S)$ do not lie in the interval $[\exp\left(\frac{-j}{\epsilon'}\right) \mathrm{diam}\,(S), \exp\left(\frac{2-j}{\epsilon'}\right) \mathrm{diam}(S))$. It is trivial to compute such a $j$; we fix that $j$. Now define:

$$r = 2\exp\left(\tfrac{-j}{\epsilon'}\right) \cdot \mathrm{diam}(S);$$

$Q_1 =$ open square centered at $z_0$, with sidelength

$$\delta_{Q_1} = 100 \exp\left(\tfrac{-(j-1)}{\epsilon'}\right) \mathrm{diam}(S) = 50 \exp\left(\tfrac{1}{\epsilon'}\right) r;$$

$$Q_2 = Q;$$

$$S_1 = S \cap B(z_0, r) \subset Q_1; \quad \text{and}$$

$$S_2 = \{z_0\} \cup (S \setminus B(z_0, r)) \subset Q_2.$$

We prepare to verify the assumptions of Algorithm AUB2, for the inputs $\epsilon', r, S_1, Q_1, S_2, Q_2$. In fact, (29) holds, since $\delta_{Q_2} = \delta_Q \le 1$ by the assumptions of Algorithm AUB4. To check (30), we note that $z_0 \in Q_1 \cap Q_2$, and

$$\delta_{Q_1} = 100\exp\left(\frac{-(j-1)}{\epsilon'}\right)\mathrm{diam}(S) \le 100\exp\left(\frac{-1}{\epsilon'}\right)\mathrm{diam}(S)$$
$$\le C \exp\left(\frac{-1}{\epsilon'}\right)\delta_Q = C \exp\left(\frac{-1}{\epsilon'}\right)\delta_{Q_2};$$

hence $2Q_1 \subset 2Q_2$.

Next, (31) holds, since $S_1 \subset S \subset Q_2$ and $S_1 \subset B(z_0, r)$ by definition.

To check (32), let $z \in (S_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon'}\right) r))$ be given. Then $z \in S$, and

$$|z - z_0| \le \exp\left(\frac{1}{\epsilon'}\right) r = 2\exp\left(\frac{1-j}{\epsilon'}\right)\mathrm{diam}(S).$$

The defining property of $j$ tells us that $|z - z_0|$ cannot lie in the interval

$$\left[\exp\left(\frac{-j}{\epsilon'}\right) \cdot \mathrm{diam}(S), \exp\left(\frac{2-j}{\epsilon'}\right) \cdot \mathrm{diam}(S)\right).$$

Consequently, $|z - z_0| < \exp\left(-\frac{j}{\epsilon'}\right) \cdot \mathrm{diam}(S) = \frac{1}{2} r$. Thus, $z \in S_2 \cap B(z_0, r)$. By definition of $S_2$, we have $S_2 \cap B(z_0, r) = \{z_0\}$. Hence, $z = z_0$. We have shown that $S_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon'}\right) r)$ contains no points other than $z_0$. On the other hand, by definition of $S_2$, we have $z_0 \in S_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon'}\right) r)$. This completes the proof of (32).

To check (33), we note that $Q_2 \cap B(z_0, \exp\left(\frac{1}{\epsilon'}\right) r) \subset B(z_0, \exp\left(\frac{1}{\epsilon'}\right) r) \subset Q_1$, by definition of $Q_1$. Thus,

(56) Conditions (29)–(33) hold for the data $\epsilon', r, S_1, Q_1, S_2, Q_2$.

Next, we check that

(57) $\#(S_1), \#(S_2) < \#(S)$.

Indeed, $S_1, S_2$ are subsets of $S$. We cannot have $S_1 = S$, since that would imply $\mathsf{diam}(S) = \mathsf{diam}(S_1) \le 2r = 4\exp\left(\frac{-j}{\epsilon'}\right) \cdot \mathsf{diam}(S) < \mathsf{diam}(S)$. (Recall that $j \ge 2$ and $\#(S) \ge 2$.) To check that $S_2 \ne S$, we show that $z_1 \notin S_2$. (See (54), (55).) Indeed, (55) gives

$$|z_1 - z_0| \le \exp\left(\frac{-100 \cdot \#(S)}{\epsilon'}\right) \cdot \mathsf{diam}(S) < \exp\left(\frac{-j}{\epsilon'}\right) \cdot \mathsf{diam}(S)$$
$$= \frac{1}{2}r \quad (\text{since } j \le 98 \cdot \#(S) \text{ by definition}).$$

Thus, $z_1$ cannot belong to $S \setminus B(z_0, r)$. Since also $z_1 \ne z_0$, we have $z_1 \notin S_2$, completing the proof of (57).

We have $S_1 \subset Q_1, S_2 \subset Q_2$, and $\delta_{Q_1}, \delta_{Q_2} \le 1$. Thanks to (57), we may recursively apply Algorithm AUB4 to the inputs $(\epsilon', S_1, Q_1)$ and $(\epsilon', S_2, Q_2)$. Thus, using work and storage at most $C(\epsilon', \#(S_1)) + C(\epsilon', \#(S_2))$, we compute

(58) $K_1 \in \mathsf{AUB}(\epsilon', S_1, Q_1, L_1)$ and $K_2 \in \mathsf{AUB}(\epsilon', S_2, Q_2, L_2)$, with

(59) $L_1 \le C(\epsilon', \#(S_1))$ and $L_2 \le C(\epsilon', \#(S_2))$.

Recall that $z_0 \in S_1 \cap S_2$. Hence, by (56) and (58), all the assumptions of Algorithm AUB2 hold, for the data $\epsilon', r, S_1, Q_1, S_2, Q_2, K_1, K_2$. Applying that algorithm, we compute

(60) $K \in \mathsf{AUB}(C\epsilon', S, Q, L_1 + L_2)$; the work and storage used to do so are at most $C(\epsilon', \#(S_1), \#(S_2), L_1, L_2)$.

Recalling (59), we see that the work and storage used to apply Algorithm AUB2 as above are at most $C(\epsilon', \#(S_1), \#(S_2))$, and furthermore, $L_1 + L_2 \le C(\epsilon', \#(S_1), \#(S_2))$. Moreover, we have $K \in \mathsf{AUB}(\epsilon, S, Q, L_1 + L_2)$, provided

(61) $C\epsilon' < \epsilon$, where $C$ is as in (60).

This concludes our analysis of Case 2.

We now pick $\epsilon' = c\epsilon$, with $c$ taken small enough, so that $\epsilon'$ will satisfy (53) and (61). Thus, in both Case 1 and Case 2, we can compute $K \in \mathsf{AUB}(\epsilon, S, Q, L)$ with $L \le C(\epsilon, \#(S))$; the work and storage used to do so (apart from the recursive calls to Algorithm AUB4) are at most $C(\epsilon, \#(S))$.

Since we make two recursive calls to Algorithm AUB4 in Case 2, and since $\#(S_1), \#(S_2) < \#(S)$, it follows that the total work and storage used by Algorithm AUB4 are at most $C(\epsilon, \#(S))$.

This completes our explanation of Algorithm AUB4.

**Remark.** By definition of $\mathrm{AUB}(\epsilon, S, Q, L)$, the $K$ computed in Algorithm AUB4 has the following properties:

- $K \subset Wh(S)$ is a polyhedron defined by at most $C(\epsilon, \#(S))$ constraints.
- Let $F \in C^2(2Q)$ with norm $\le 1$. Then $J_S(F) \in K$.
- Let $\vec{P} \in K$. Then there exists $F \in C^2(Q)$ with norm $\le 1 + \epsilon$, such that $J_S(F) = \vec{P}$.

## 7. The basic tree

(1) Let $\bar{E} \subset \mathbb{R}$ be a finite set, with $N \geq 2$ elements.

For $x \in \mathbb{R}$ we define a lengthscale

(2) $\delta_{LS}(x) = \inf\{r > 0 : [x - r, x + r]$ contains at least two points of $\bar{E}\}$.

We fix a dyadic grid $\mathcal{G}_\tau$; in this section, an interval I is called "dyadic" if $I \in \mathcal{G}_\tau$.

We use no "boiler-plate constants" in this section; thus, $c, C, C'$ here denote absolute constants.

Our goal here is to define a binary tree $T^{global}(\check{I})$, whose nodes are dyadic subintervals of a given dyadic interval $\check{I}$. The root of $T^{global}(\check{I})$ is the interval $\check{I}$. Each internal node $I \in T^{global}(\check{I})$ has two children in the tree $T^{global}(\check{I})$, namely its two dyadic children.

In trivial cases, $T^{global}(\check{I})$ consists merely of the single node $\check{I}$. Except for those trivial cases, each node $I \in T^{global}(\check{I})$ satisfies $\#(25I \cap \bar{E}) \geq 2$; and we will define two "representatives" $x^{rep}_{left}(I)$ and $x^{rep}_{rt}(I)$ in $25I \cap \bar{E}$. Moreover, the leaves of $T^{global}(\check{I})$ form a partition of $\check{I}$ into dyadic intervals I, such that

(3) $c\delta_{LS}(x) \leq |I| \leq \delta_{LS}(x)$ for any $x \in 3I$.

In view of (3), the number of nodes in $T^{global}(\check{I})$ depends on the spacing of the points of $\bar{E}$. We cannot bound the number of nodes solely in terms of $N = \#(\bar{E})$.

To make possible efficient computations using $T^{global}(\check{I})$, we therefore introduce a subset $T^{dist}(\check{I})$, consisting of at most CN "distinguished nodes". We will think about all the nodes of $T^{global}(\check{I})$, but we will make computations only for the nodes of $T^{dist}(\check{I})$. To illustrate, we discuss the representatives $x^{rep}_{left}(I), x^{rep}_{rt}(I)$.

A fundamental property of $T^{global}(\check{I})$ and $T^{dist}(\check{I})$ is that $T^{global}(\check{I}) \setminus \{\check{I}\}$ may be written as a disjoint union, over suitable $\tilde{I} \in T^{dist}(\check{I})$, of $T^{loc}(\tilde{I}) \setminus \{\tilde{I}\}$, where $T^{loc}(\tilde{I})$ is a "local tree" with root $\tilde{I}$. It turns out that we can define our representatives in such a way that $I \mapsto x^{rep}_{left}(I)$ and $I \mapsto x^{rep}_{rt}(I)$ are constant on $T^{loc}(\tilde{I}) \setminus \{\tilde{I}\}$, for each distinguished node $\tilde{I}$. Accordingly, we will compute representatives $x^{s}_{left}(\tilde{I}), x^{s}_{rt}(\tilde{I})$ for suitable $\tilde{I} \in T^{dist}(\check{I})$, and then define $x^{rep}_{left}(I) = x^{s}_{left}(\tilde{I})$, $x^{rep}_{rt}(I) = x^{s}_{rt}(\tilde{I})$, for each $I \in T^{loc}(\tilde{I}) \setminus \{\tilde{I}\}$, $\tilde{I} \in T^{dist}(\check{I})$. Thus, we will only have to compute at most CN representatives $x^{s}_{left}(\tilde{I})$, $x^{s}_{rt}(\tilde{I})$; these yield all the $x^{rep}_{left}(I)$, $x^{rep}_{rt}(I)$ for $I \in T^{global}(\check{I}) \setminus \{\check{I}\}$. Even though $T^{global}(\check{I})$ may contain far more than CN nodes, we can nevertheless compute all the representatives we need without excessive work.

We prepare to give the definitions of $T^{global}(\check{I})$ and $T^{dist}(\check{I})$. We begin with a few preliminary definitions. Let I be a dyadic interval. We define

(4) $J(I) =$ convex hull of $5I^+ \cap \bar{E}$.

Thus, $J(I)$ is the empty set, a single point, or a non-degenerate closed interval. We say that I is of:

**"type A"** if $|J(I)| \geq \frac{1}{32}|I|$;

**"type B"** if $\frac{1}{32}|I| > |J(I)| > 0$; and

**"type C"** if $|J(I)| = 0$.

If I is of type C, then $J(I)$ is the empty set or a single point. We say that I is of type C0 if $J(I)$ is empty, while I is of type C1 if $J(I)$ is a single point. For I of type C1, we write $x_!(I)$ to denote the one and only element of $J(I)$. Note that $\bar{E} \cap 5I^+ = \{x_!(I)\}$ when I is of type C1.

Since $\bar{E}$ is finite, we have the following:

(5) If I is of type A or B, then $J(I)$ is a non-degenerate closed interval, and the endpoints of $J(I)$ belong to $\bar{E}$.

(6) Every sufficiently small dyadic interval is of type C.

Next, for any dyadic interval $\tilde{I}$, we define the "local tree" $T^{loc}(\tilde{I})$, which consists of dyadic subintervals of $\tilde{I}$. Some of the leaves of this tree will be called "red offspring" of $\tilde{I}$; the set of all red offspring of $\tilde{I}$ will be denoted by $RO(\tilde{I})$. The definition of $T^{loc}(\tilde{I})$ and $RO(\tilde{I})$ proceeds by cases.

(7) **Suppose $\tilde{I}$ is of type A.** Then $T^{loc}(\tilde{I})$ consists of $\tilde{I}$ and its two dyadic children $\tilde{I}_1$ and $\tilde{I}_2$; and $RO(\tilde{I})$ consists of $\tilde{I}_1$ and $\tilde{I}_2$.

(8) **Suppose $\tilde{I}$ is of type B.** Then $T^{loc}(\tilde{I})$ consists of all dyadic $I \subseteq \tilde{I}$ such that $5I^+ \cap J(\tilde{I}) \neq \emptyset$ and $|I| \geq |J(\tilde{I})|$.

In this case, the leaves of $T^{loc}(\tilde{I})$ are precisely those dyadic intervals $I \subseteq \tilde{I}$ such that either

(8a) $5I \cap J(\tilde{I}) = \emptyset$, $5I^+ \cap J(\tilde{I}) \neq \emptyset$, $|I| \geq |J(\tilde{I})|$ or

(8b) $5I \cap J(\tilde{I}) \neq \emptyset, |J(\tilde{I})| \leq |I| < 2|J(\tilde{I})|$.

The set $RO(\tilde{I})$ consists of all $I \subseteq \tilde{I}$ that satisfy (8b).

(9) **Suppose $\tilde{I}$ is of type C.** Then $T^{loc}(\tilde{I})$ consists of the single node $\tilde{I}$, and $RO(\tilde{I})$ is the empty set.

Thus, we have defined $T^{loc}(\tilde{I})$ and $RO(\tilde{I})$. Note that

(10) Every $\tilde{I}' \in RO(\check{I})$ is a proper dyadic subinterval of $\check{I}$.

We are now ready to define $T^{global}(\check{I})$ and $T^{dist}(\check{I})$ for any given dyadic interval $\check{I}$. Our definitions of these objects will be recursive: Given $\check{I}$, we assume that we have already defined $T^{global}(\check{I}')$ and $T^{dist}(\check{I}')$ for all $\check{I}' \in RO(\check{I})$, and then we proceed to define $T^{global}(\check{I})$ and $T^{dist}(\check{I})$. Such a recursive definition makes sense, thanks to (6), (9) and (10).

Our recursive definitions of $T^{global}(\check{I})$ and $T^{dist}(\check{I})$ are as follows:

(11) $$T^{global}(\check{I}) = T^{loc}(\check{I}) \cup \bigcup_{\check{I}' \in RO(\check{I})} T^{global}(\check{I}').$$

(12) $$T^{dist}(\check{I}) = \{\check{I}\} \cup \bigcup_{\check{I}' \in RO(\check{I})} T^{dist}(\check{I}').$$

Note that

(13) If $\check{I}$ is of type C, then $T^{dist}(\check{I}) = T^{global}(\check{I}) = \{\check{I}\}$.

The basic properties of $\mathsf{T}^{\mathsf{global}}(\check{I})$ and $\mathsf{T}^{\mathsf{dist}}(\check{I})$ are given by the next three lemmas.

**Lemma BT1.** *Let $\check{I}$ be a dyadic interval. Then*

(I) $\mathsf{T}^{\mathsf{global}}(\check{I})$ *is a finite collection of dyadic subintervals of $\check{I}$, including $\check{I}$ itself.*

(II) *For each $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$, either*

  (A) ($I$ *is a "leaf"*): *No proper dyadic subinterval of $I$ belongs to $\mathsf{T}^{\mathsf{global}}(\check{I})$ or*

  (B) ($I$ *is an "internal node"*): *Both of the dyadic children of $I$ belong to $\mathsf{T}^{\mathsf{global}}(\check{I})$.*

(III) *The leaves of $\mathsf{T}^{\mathsf{global}}(\check{I})$ form a partition of $\check{I}$ into finitely many dyadic subintervals $I$. Each leaf $I$ satisfies $c\delta_{\mathsf{LS}}(x) \leq |I| \leq \delta_{\mathsf{LS}}(x)$ for all $x \in 3I$, provided $\check{I}$ is of type A or B, and $(5\check{I} \cap \bar{E}) \neq \emptyset$.* (*See* (2).)

(IV) *Any given point $x \in \mathbb{R}$ lies in $3I$ for at most $C$ distinct leaves $I$ in $\mathsf{T}^{\mathsf{global}}(\check{I})$.*

**Lemma BT2.** *Let $\check{I}$ be a dyadic interval. Then*

(I) $\mathsf{T}^{\mathsf{dist}}(\check{I}) \subset \mathsf{T}^{\mathsf{global}}(\check{I})$, *and* $\check{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$.

(II) *Let $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. Then the children of $\tilde{I}$ in the tree $\mathsf{T}^{\mathsf{dist}}(\check{I})$ are precisely the dyadic intervals in $\mathsf{RO}(\tilde{I})$. There are at most $C$ such intervals.*

(III) *Let $I$ be a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$. If $5I \cap \bar{E} \neq \emptyset$, then $I \in \mathsf{T}^{\mathsf{dist}}(\check{I})$, $I$ is of type $\mathsf{C1}$, and $\#(5I^{+} \cap \bar{E}) = 1$.*

(IV) *The tree $\mathsf{T}^{\mathsf{dist}}(\check{I})$ has at most $CN$ nodes.*

**Lemma BT3.** *Let $\check{I}$ be a dyadic interval. Then*

(I) $\mathsf{T}^{\mathsf{global}}(\check{I}) \setminus \{\check{I}\}$ *is the disjoint union of $\mathsf{T}^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$ over all $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$.* (*Here, we may restrict to $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$ of type $\mathsf{A}$ or $\mathsf{B}$, since $\mathsf{T}^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\} = \emptyset$ for $\tilde{I}$ of type $\mathsf{C}$.*)

(II) *Let $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I}) \cap \mathsf{T}^{\mathsf{loc}}(\tilde{I}')$, where $\tilde{I}, \tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. Then either*

  (A) $\tilde{I} = \tilde{I}'$,

  (B) $I = \tilde{I} \in \mathsf{RO}(\tilde{I}')$, *or*

  (C) $I = \tilde{I}' \in \mathsf{RO}(\tilde{I})$.

(III) *Let $I$ be an internal node of $\mathsf{T}^{\mathsf{global}}(\check{I})$. Then there exists $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$, such that $I$ and its two dyadic children all belong to $\mathsf{T}^{\mathsf{loc}}(\tilde{I})$.*

The above lemmas justify our assertions* regarding $\mathsf{T}^{\mathsf{global}}(\check{I})$ and $\mathsf{T}^{\mathsf{dist}}(\check{I})$ at the start of this section. To justify our assertions regarding "representatives", we make the following definitions:

Let $I$ be any dyadic interval. We write $x_{\mathsf{left}}(I), x_{\mathsf{rt}}(I)$ to denote the left and right endpoints of $I$, respectively.

*The assertion $\#(25I \cap E) \geq 2$ for $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$ (in non-trivial cases) will be justified by Lemma BT4 below.

If I is of type A or B, then we recall that $J(I)$ is a non-degenerate closed interval; we write $x^s_{\mathrm{left}}(I)$ and $x^s_{\mathrm{rt}}(I)$ to denote the left and right endpoints of $J(I)$, respectively. Note that $x^s_{\mathrm{left}}(I)$ and $x^s_{\mathrm{rt}}(I)$ are undefined for I of type C.
Recall that we defined $x_!(I)$ to be the one and only element of $J(I)$, in case I is of type C1. For intervals I of type A, B or C0, $x_!(I)$ is undefined.

Now let $\check{I}$ be a dyadic interval, and let $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. According to Lemma BT3 (conclusion I), we have $I \in T^{\mathrm{loc}}(\tilde{I}) \backslash \{\tilde{I}\}$ for precisely one $\tilde{I} \in T^{\mathrm{dist}}(\check{I})$; and this $\tilde{I}$ is of type A or B.

Using this $\tilde{I}$, we define

(14) $x^{\mathrm{rep}}_{\mathrm{left}}(I) = x^s_{\mathrm{left}}(\tilde{I})$ and $x^{\mathrm{rep}}_{\mathrm{rt}}(I) = x^s_{\mathrm{right}}(\tilde{I})$.

Thus, $x^{\mathrm{rep}}_{\mathrm{left}}(I)$ and $x^{\mathrm{rep}}_{\mathrm{rt}}(I)$ are defined for all $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. If $\check{I}$ is of type A or B, then we define

(15) $x^{\mathrm{rep}}_{\mathrm{left}}(\check{I}) = x^s_{\mathrm{left}}(\check{I})$ and $x^{\mathrm{rep}}_{\mathrm{rt}}(\check{I}) = x^s_{\mathrm{rt}}(\check{I})$, so that $x^{\mathrm{rep}}_{\mathrm{left}}(I)$, $x^{\mathrm{rep}}_{\mathrm{rt}}(I)$ are defined for all $I \in T^{\mathrm{global}}(\check{I})$.

If $\check{I}$ is of type C, then $x^{\mathrm{rep}}_{\mathrm{left}}(\check{I})$, $x^{\mathrm{rep}}_{\mathrm{rt}}(\check{I})$ are undefined. (In this case, $T^{\mathrm{global}}(\check{I})$ is trivial; see (13).)

**Lemma BT 4.** *Let $\check{I}$ be a dyadic interval, and let $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. Then $x^{\mathrm{rep}}_{\mathrm{left}}(I) < x^{\mathrm{rep}}_{\mathrm{rt}}(I)$ (strict inequality), and $x^{\mathrm{rep}}_{\mathrm{left}}(I)$, $x^{\mathrm{rep}}_{\mathrm{rt}}(I) \in 25I \cap \bar{E}$.*

*If $\check{I}$ is of type A or B, then the above conclusions hold also for $I = \check{I}$.*

This lemma justifies our earlier assertions regarding representatives.

Most of the conclusions of Lemmas BT1–BT4 are trivial, but, to help the careful reader, we provide some details of their proofs. At the end of this section, we give algorithms to compute the tree $T^{\mathrm{dist}}(\check{I})$ and the points $x_{\mathrm{left}}(\tilde{I})$, $x_{\mathrm{rt}}(\tilde{I})$, $x^s_{\mathrm{left}}(\tilde{I})$, $x^s_{\mathrm{rt}}(\tilde{I})$, $x_!(\tilde{I})$ for $\tilde{I} \in T^{\mathrm{dist}}(\check{I})$ (whenever those points are defined).

We prepare the way to the proofs of Lemmas BT1–BT4 by establishing a series of propositions.

The basic properties of $T^{\mathrm{loc}}(\tilde{I})$ and $RO(\tilde{I})$ are given by the following result:

**Proposition BT1.** *Let $\tilde{I}$ be a dyadic interval. Then*

(16) $T^{\mathrm{loc}}(\tilde{I})$ *is a finite tree whose nodes are dyadic subintervals of* $\tilde{I}$.

(17) $\tilde{I} \in T^{\mathrm{loc}}(\tilde{I})$.

(18) *Each* $I \in T^{\mathrm{loc}}(\tilde{I})$ *satisfies either*

- (a) (I *is a "leaf" of* $T^{\mathrm{loc}}(\tilde{I})$): *No proper subinterval of* I *belongs to* $T^{\mathrm{loc}}(\tilde{I})$, *or*
- (b) (I *is an "internal node" of* $T^{\mathrm{loc}}(\tilde{I})$): *Both of the dyadic children of* I *belong to* $T^{\mathrm{loc}}(\tilde{I})$.

(19) *Let* $I \neq \tilde{I}$ *be a leaf of* $T^{\mathrm{loc}}(\tilde{I})$. *If* $I \notin RO(\tilde{I})$, *then* $5I \cap \bar{E} = \emptyset$.

(20) *Each* $I \in RO(\tilde{I})$ *is a proper dyadic subinterval of* $\tilde{I}$.

(21) *The intervals of* $\mathsf{RO}(\tilde{I})$ *are pairwise disjoint.*

(22) *There are at most* $\mathsf{C}$ *intervals in* $\mathsf{RO}(\tilde{I})$.

(23) *For each* $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$, *we have* $J(\tilde{I}) \subset 25I$.

(24) *For each* $I \in \mathsf{RO}(\tilde{I})$, *we have* $\mathsf{diam}(\bar{E} \cap 25I) \geq \frac{1}{16}|I|$.

(25) *If* $\tilde{I}$ *is of type* $\mathsf{A}$ *or* $\mathsf{B}$, *and* $5\tilde{I} \cap \bar{E} \neq \emptyset$, *then the two dyadic children of* $\tilde{I}$ *belong to* $\mathsf{T}^{\mathsf{loc}}(\tilde{I})$.

*Proof.* Assertions (16), (17), (18), (20) are trivial from the definitions. Assertion (19) holds vacuously for $\tilde{I}$ of type $\mathsf{A}$ or $\mathsf{C}$. For $\tilde{I}$ of type $\mathsf{B}$, (19) follows from (8a), since $\bar{E} \cap 5\tilde{I}^+ \subset J(\tilde{I})$ by definition. Assertion (21) holds because the intervals $I \in \mathsf{RO}(\tilde{I})$ are leaves in a tree consisting of dyadic intervals. Assertion (22) is obvious for $\tilde{I}$ of type $\mathsf{A}$ or $\mathsf{C}$. For $\tilde{I}$ of type $\mathsf{B}$, (22) holds because there are at most $\mathsf{C}$ dyadic intervals $I$ satisfying (8b).

Assertion (23) holds trivially for $\tilde{I}$ of type $\mathsf{A}$ or $\mathsf{C}$. Suppose $\tilde{I}$ is of type $\mathsf{B}$. Then any $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I})$ satisfies $5I^+ \cap J(\tilde{I}) \neq \emptyset$ and $|I| \geq |J(\tilde{I})|$, hence $J(\tilde{I}) \subset 25I$. Thus, (23) holds in all cases.

To prove (24), let $I \in \mathsf{RO}(\tilde{I})$. Then $\mathsf{RO}(\tilde{I}) \neq \emptyset$, so $\tilde{I}$ cannot be of type $\mathsf{C}$. Thus, $\tilde{I}$ is of type $\mathsf{A}$ or $\mathsf{B}$. Hence, (5) and (23) show that the endpoints of $J(\tilde{I})$ lie in $\bar{E} \cap 25I$. Consequently, $\mathsf{diam}(\bar{E} \cap 25I) \geq |J(\tilde{I})|$. On the other hand, for $\tilde{I}$ of type $\mathsf{A}$ or $\mathsf{B}$ and $I \in \mathsf{RO}(\tilde{I})$, we can check that $|J(\tilde{I})| \geq \frac{1}{16}|I|$. Indeed, if $\tilde{I}$ is of type $\mathsf{A}$, then $I \in \mathsf{RO}(\tilde{I})$ is a dyadic child of $\tilde{I}$, and $|I| = \frac{1}{2}|\tilde{I}| \leq 16|J(\tilde{I})|$ (since $\tilde{I}$ is of type $\mathsf{A}$ ).

If instead $\tilde{I}$ is of type $\mathsf{B}$, then any $I \in \mathsf{RO}(\tilde{I})$ satisfies (8b). In particular, $|I| < 2|J(\tilde{I})|$. Thus, in all cases, $|J(\tilde{I})| \geq \frac{1}{16}|I|$ as claimed. We now know that $\mathsf{diam}(25I \cap \bar{E}) \geq |J(\tilde{I})| \geq \frac{1}{16}|I|$, proving (24).

Assertion (25) is trivial for $\tilde{I}$ of type $\mathsf{A}$. For $\tilde{I}$ of type $\mathsf{B}$, (25) asserts that $5\tilde{I} \cap J(\tilde{I}) \neq \emptyset$ and $\frac{1}{2}|\tilde{I}| \geq |J(\tilde{I})|$.

We know that $\frac{1}{2}|\tilde{I}| \geq |J(\tilde{I})|$, since $\tilde{I}$ is of type $\mathsf{B}$. We have also $5\tilde{I} \cap J(\tilde{I}) \supset 5\tilde{I} \cap [5\tilde{I}^+ \cap \bar{E}] = 5\tilde{I} \cap \bar{E}$. Hence, if $5\tilde{I} \cap \bar{E} \neq \emptyset$, then $5\tilde{I} \cap J(\tilde{I}) \neq \emptyset$, proving (25).

The proof of Proposition BT1 is complete. □

The next several propositions pertain to $\mathsf{T}^{\mathsf{global}}(\check{I})$ and $\mathsf{T}^{\mathsf{dist}}(\check{I})$. To prove those propositions, we will make repeated use of "induction on $\check{I}$", which means the following:

Let $\mathsf{Prop}(\check{I})$ be some assertion involving a given dyadic interval $\check{I}$. We want to prove that $\mathsf{Prop}(\check{I})$ holds for all $\check{I}$. To do so, it is enough to fix $\check{I}$, assume $\mathsf{Prop}(\check{I}')$ for all $\check{I}' \in \mathsf{RO}(\check{I})$, and then prove $\mathsf{Prop}(\check{I})$. (This establishes $\mathsf{Prop}(\check{I})$ for all $\check{I}$, thanks to (6), (9), (10). We have already used this idea in our recursive definitions (11) and (12).) We refer to the assumption that $\mathsf{Prop}(\check{I}')$ holds for all $\check{I}' \in \mathsf{RO}(\check{I})$ as the "induction hypothesis".

**Proposition BT2.** *Let* $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. *Then*

(A) $\mathsf{T}^{\mathsf{loc}}(\tilde{I}) \subset \mathsf{T}^{\mathsf{global}}(\check{I})$.

(B) *The children of* $\tilde{I}$ *in the tree* $\mathsf{T}^{\mathsf{dist}}(\check{I})$ *are precisely the intervals in* $\mathsf{RO}(\tilde{I})$.

*Proof.* By induction on $\check{\mathrm{I}}$. □

**Proposition BT3.** *Let* $\mathrm{I}' \subset \mathrm{I}$, *with* $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$ *and* $\mathrm{I} \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$. *Then either*

(A) $\mathrm{I}' \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$ *or*

(B) *For some* $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$, *we have* $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$ *and* $\check{\mathrm{I}}' \subset \mathrm{I}$.

*Proof.* Suppose (A) fails. By (11), we have $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$ for some $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$. In particular, $\mathrm{I}' \subset \check{\mathrm{I}}'$; and we are assuming that $\mathrm{I}' \subset \mathrm{I}$. Thus, the dyadic intervals I and $\check{\mathrm{I}}'$ are not disjoint. Consequently, either $\check{\mathrm{I}}' \subset \mathrm{I}$ or $\mathrm{I} \subsetneq \check{\mathrm{I}}'$.

The latter inclusion is impossible, since $\mathrm{I} \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$ and $\check{\mathrm{I}}'$ is a leaf of $\mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$. Thus, $\check{\mathrm{I}}' \subset \mathrm{I}$, $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$, and $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$ , i.e., (B) holds. □

**Proposition BT4.** *Let* $\mathrm{I}' \subset \mathrm{I}$, *with* $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$ *and* $\mathrm{I} \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$, *where* $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$. *Then* $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$.

*Proof.* We may suppose $\mathrm{I}' \subsetneq \mathrm{I}$. Suppose $\mathrm{I}' \notin \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$. Then by (11), either $\mathrm{I}' \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$ or $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}'')$, with $\check{\mathrm{I}}'' \in \mathsf{RO}(\check{\mathrm{I}})$ distinct from $\check{\mathrm{I}}'$. The latter case would imply $\mathrm{I}' \subset \check{\mathrm{I}}''$ and $\mathrm{I} \subset \check{\mathrm{I}}'$; hence I and $\mathrm{I}'$ are disjoint, by (21). That's impossible, since $\mathrm{I}' \subsetneq \mathrm{I}$. Therefore, $\mathrm{I}' \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$. However, that's also impossible, since $\mathrm{I}' \subsetneq \mathrm{I} \subset \check{\mathrm{I}}'$, and $\check{\mathrm{I}}'$ is a leaf of $\mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$. This contradiction completes the proof of the proposition. □

**Proposition BT5.** *Suppose* $\mathrm{I} \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$. *Then either*

(A) (I *is a "leaf"*): *No proper subinterval of* I *belongs to* $\mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$ *or*

(B) (I *is an "internal node"*): *There exists* $\tilde{\mathrm{I}} \in \mathsf{T}^{\mathsf{dist}}(\check{\mathrm{I}})$ *such that* I *and both its dyadic children belong to* $\mathsf{T}^{\mathsf{loc}}(\tilde{\mathrm{I}})$. *In particular, the dyadic children of* I *belong to* $\mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$.

*Proof.* We use induction on $\check{\mathrm{I}}$. Suppose (A) fails; we will prove (B). Fix $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$ such that $\mathrm{I}' \subsetneq \mathrm{I}$. (Such an $\mathrm{I}'$ exists, since (A) fails.) Thanks to Proposition BT3, Proposition BT4 and (11), we fall into one of the following cases:

Case 1: $\mathrm{I}, \mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$ for some $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$.

Case 2: $\mathrm{I} \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$ and $\mathrm{I}' \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$.

Case 3: $\mathrm{I} \in \mathsf{T}^{\mathsf{loc}}(\check{\mathrm{I}})$, $\mathrm{I}' \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}')$, and $\check{\mathrm{I}}' \subset \mathrm{I}$, for some $\check{\mathrm{I}}' \in \mathsf{RO}(\check{\mathrm{I}})$.

In Case 1, conclusion (B) follows from the induction hypothesis (i.e., Proposition BT5 for $\check{\mathrm{I}}'$). In Case 2, (18) shows that (B) holds, with $\tilde{\mathrm{I}} = \check{\mathrm{I}}$. In Case 3, we have $\mathrm{I}' \subset \check{\mathrm{I}}' \subset \mathrm{I}$, with $\mathrm{I}' \neq \mathrm{I}$. If $\check{\mathrm{I}}' \neq \mathrm{I}$, then again (18) shows that (B) holds, with $\tilde{\mathrm{I}} = \check{\mathrm{I}}$. If instead $\check{\mathrm{I}}' = \mathrm{I}$, then (B) follows from the induction hypothesis, i.e., Proposition BT5 for $\check{\mathrm{I}}'$.

Thus, assuming (A) fails, we have proven (B) in all cases. □

**Corollary.** *The leaves of* $\mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}})$ *form a partition of* $\check{\mathrm{I}}$ *into dyadic subintervals.*

**Proposition BT6.** *Each* $\mathrm{I} \in \mathsf{T}^{\mathsf{global}}(\check{\mathrm{I}}) \setminus \{\check{\mathrm{I}}\}$ *belongs to* $\mathsf{T}^{\mathsf{loc}}(\tilde{\mathrm{I}}) \setminus \{\tilde{\mathrm{I}}\}$ *for some* $\tilde{\mathrm{I}} \in \mathsf{T}^{\mathsf{dist}}(\check{\mathrm{I}})$.

*Proof.* An easy induction on $\check{I}$. □

**Proposition BT7.** *Let* $\tilde{I}, \tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I})$*, and suppose* $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I}) \cap \mathsf{T}^{\mathsf{loc}}(\tilde{I}')$*. Then either* $\tilde{I} = \tilde{I}'$*; or* $I = \tilde{I} \in \mathsf{RO}(\tilde{I}')$*; or* $I = \tilde{I}' \in \mathsf{RO}(\tilde{I})$*.*

*Proof.* We use induction on $\check{I}$. We may assume $\tilde{I} \neq \tilde{I}'$. We proceed by cases.

Assume $\tilde{I} = \check{I}$. Then $\tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I}) \setminus \{\check{I}\}$. Hence, by (12), there exists $\check{I}' \in \mathsf{RO}(\check{I})$ such that $\tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I}')$. Consequently, $\tilde{I}' \subset \check{I}'$. Also, since $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I}')$, we have $I \subset \tilde{I}'$. Thus $I \subset \tilde{I}' \subset \check{I}'$. However, $I$ cannot be properly contained in $\check{I}'$, since $I \in \mathsf{T}^{\mathsf{loc}}(\check{I})$ and $\check{I}'$ is a leaf of $\mathsf{T}^{\mathsf{loc}}(\check{I})$. Therefore, $I = \tilde{I}' = \check{I}' \in \mathsf{RO}(\check{I}) = \mathsf{RO}(\tilde{I})$. In particular, $I = \tilde{I}' \in \mathsf{RO}(\tilde{I})$.

Assume $\tilde{I}' = \check{I}$. Proceeding as above, with the rôles of $\tilde{I}$ and $\tilde{I}'$ interchanged, we see that $I = \tilde{I} \in \mathsf{RO}(\tilde{I}')$.

Assume $\tilde{I}, \tilde{I}' \neq \check{I}$. Then by (12), we have $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I}')$ and $\tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I}'')$, with $\check{I}', \check{I}'' \in \mathsf{RO}(\check{I})$. Also, since $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I})$, we have $I \subset \tilde{I} \subset \check{I}'$; similarly, $I \subset \tilde{I}' \subset \check{I}''$. Therefore, $\check{I}' = \check{I}''$ by (21). Thus, $\tilde{I}, \tilde{I}' \in \mathsf{T}^{\mathsf{dist}}(\check{I}')$.

The conclusion of Proposition BT7 therefore follows by induction hypothesis, i.e., Proposition BT7 for $\check{I}'$.

Thus, Proposition BT7 holds in all cases. □

**Corollary.** *The sets* $\mathsf{T}^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$*, for* $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$*, are pairwise disjoint.*

**Proposition BT8.** *Let* $I$ *be a leaf of* $\mathsf{T}^{\mathsf{global}}(\check{I})$*. Then* $\#(\bar{\mathsf{E}} \cap 5\,I) \leq 1$*.*

*Proof.* We suppose $I$ is a leaf with $\#(\bar{\mathsf{E}} \cap 5\,I) \geq 2$, and derive a contradiction. We proceed by cases.

Case 1: Suppose $I \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. Then $\mathsf{T}^{\mathsf{loc}}(I) \subset \mathsf{T}^{\mathsf{global}}(\check{I})$; hence $I$ is a leaf of $\mathsf{T}^{\mathsf{loc}}(I)$. That is, $\mathsf{T}^{\mathsf{loc}}(I) = \{I\}$. According to (25), $I$ cannot be of type A or B. However, since $\#(\bar{\mathsf{E}} \cap 5\,I) \geq 2$, $I$ cannot be of type C. Thus, we have derived a contradiction in Case 1.

Case 2: Suppose $I \notin \mathsf{T}^{\mathsf{dist}}(\check{I})$. Then $I \in \mathsf{T}^{\mathsf{global}}(\check{I}) \setminus \{\check{I}\}$, hence $I \in \mathsf{T}^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$ for some $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. Since $I$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$, it is a leaf of $\mathsf{T}^{\mathsf{loc}}(\tilde{I})$. Moreover, we cannot have $I \in \mathsf{RO}(\tilde{I})$, since then $I \in \mathsf{T}^{\mathsf{dist}}(\check{I})$. Hence, (19) tells us that $5\,I \cap \bar{\mathsf{E}} = \emptyset$, whereas we have assumed that $\#(5\,I \cap \bar{\mathsf{E}}) \geq 2$. Thus, we have derived a contradiction in Case 2. □

**Proposition BT9.** *Let* $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I})$*. If* $\tilde{I}$ *is of type* C*, then it is a leaf of* $\mathsf{T}^{\mathsf{global}}(\check{I})$*.*

*Proof.* We use induction on $\check{I}$, and proceed by cases.

If $\tilde{I} = \check{I}$, then, by (13), $\tilde{I}$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$.

If $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I}) \setminus \{\check{I}\}$, then, by (12), we have $\tilde{I} \in \mathsf{T}^{\mathsf{dist}}(\check{I}')$ for some $\check{I}' \in \mathsf{RO}(\check{I})$. Since $\mathsf{T}^{\mathsf{global}}(\check{I}') \subset \mathsf{T}^{\mathsf{global}}(\check{I})$, it follows by the inductive assumption (Proposition BT9 for $\check{I}'$), that $\tilde{I}$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I}')$. Consequently, $\tilde{I}$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$, thanks to Proposition BT4.

Thus, Proposition BT9 holds in all cases. □

**Proposition BT10.** *Let* $I$ *be a leaf of* $T^{\mathrm{global}}(\check{I})$. *If* $5I \cap \bar{E} \neq \emptyset$, *then* $I$ *is of type* $C1$.

*Proof.* We use induction on $\check{I}$, and proceed by cases.

Case 1: Suppose $I = \check{I}$. Then $\check{I}$ is a leaf of $T^{\mathrm{loc}}(\check{I})$, i.e., $T^{\mathrm{loc}}(\check{I}) = \{\check{I}\}$. This cannot happen if $\check{I}$ is of type A or B, thanks to (25). Hence, $\check{I}$ is of type C. Since $5I \cap \bar{E} \neq \emptyset$, I is of type C1.

Case 2: Suppose $I \in T^{\mathrm{loc}}(\check{I}) \setminus \{\check{I}\}$. Then I is a leaf of $T^{\mathrm{loc}}(\check{I})$ such that $I \neq \check{I}$ and $5I \cap \bar{E} \neq \emptyset$. By (19), we have $I \in RO(\check{I})$. Hence, by (11), $T^{\mathrm{global}}(I) \subset T^{\mathrm{global}}(\check{I})$.

Consequently, I is a leaf of $T^{\mathrm{global}}(I)$. It follows that I is a leaf of $T^{\mathrm{loc}}(I)$, i.e., $T^{\mathrm{loc}}(I) = \{I\}$. Since $5I \cap \bar{E} \neq \emptyset$, (25) now shows that I is of type C. In particular, I is of type C1, since $5I \cap \bar{E} \neq \emptyset$.

Case 3: Suppose $I \notin T^{\mathrm{loc}}(\check{I})$. By (11), we have $I \in T^{\mathrm{global}}(\check{I}')$ for some $\check{I}' \in RO(\check{I})$. Since $T^{\mathrm{global}}(\check{I}') \subset T^{\mathrm{global}}(\check{I})$, I is a leaf of $T^{\mathrm{global}}(\check{I}')$ with $5I \cap \bar{E} \neq \emptyset$. Induction hypothesis (Proposition BT10 for $\check{I}'$) now tells us that I is of type C1.

Thus, Proposition BT10 holds in all cases. □

**Proposition BT11.** *Let* $I$ *be a leaf of* $T^{\mathrm{global}}(\check{I})$. *If* $5I \cap \bar{E} \neq \emptyset$, *then* $I \in T^{\mathrm{dist}}(\check{I})$.

*Proof.* The desired conclusion is obvious for $I = \check{I}$. Suppose $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. Proposition BT6 gives $I \in T^{\mathrm{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$ for some $\tilde{I} \in T^{\mathrm{dist}}(\check{I})$. By Proposition BT2, $T^{\mathrm{loc}}(\tilde{I}) \subset T^{\mathrm{global}}(\check{I})$; hence, I is a leaf of $T^{\mathrm{loc}}(\tilde{I})$. Since $I \neq \tilde{I}$ and $5I \cap \bar{E} \neq \emptyset$, (19) gives $I \in RO(\tilde{I})$. Consequently, $I \in T^{\mathrm{dist}}(\check{I})$, by Proposition BT2. □

**Proposition BT12.** *Let* $x \in 3I$, $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. *Then* $\frac{1}{20}\delta_{LS}(x) \leq |I|$. *Also, if* $I$ *is any leaf of* $T^{\mathrm{global}}(\check{I})$ *and* $x \in 3I$, *then* $|I| \leq \delta_{LS}(x)$. (*See* (2).)

*Proof.* Let $x \in 3I$, $I \in T^{\mathrm{global}}(\check{I}) \setminus \{\check{I}\}$. By Proposition BT6, $I \in T^{\mathrm{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$ for some $\tilde{I}$. The interval $\tilde{I}$ cannot be of type C, since $T^{\mathrm{loc}}(\tilde{I}) \setminus \{\tilde{I}\} \neq \emptyset$. Hence, by (5) and (23), $J(\tilde{I})$ is a non-degenerate closed interval, whose endpoints both lie in $\bar{E} \cap 25I$. In particular, $\#(\bar{E} \cap 25I) \geq 2$. However, since $x \in 3I$, we have $[x - 20|I|, x + 20|I|] \supset 25I$. Consequently, $\#(\bar{E} \cap [x - 20|I|, x + 20|I|]) \geq 2$, so that, by definition, $20|I| \geq \delta_{LS}(x)$.

On the other hand, suppose I is a leaf of $T^{\mathrm{global}}(\check{I})$. By Proposition BT8, $\#(\bar{E} \cap 5I) \leq 1$. Since $x \in 3I$, we have $[x - |I|, x + |I|] \subset 5I$. Consequently, $\#(\bar{E} \cap [x - |I|, x + |I|]) \leq 1$, so that, by definition, $|I| \leq \delta_{LS}(x)$. □

**Corollary.** *Let* $\check{I}$ *be a dyadic interval, and let* $x \in \mathbb{R}$ *be given. Then* $x \in 3I$ *for at most* $C$ *distinct leaves* $I$ *of* $T^{\mathrm{global}}(\check{I})$.

*Proof.* We may suppose $\check{I}$ is not a leaf of $T^{\mathrm{global}}(\check{I})$. By Proposition BT12, each I as above is a dyadic interval, such that $\frac{1}{20}\delta_{LS}(x) \leq |I| \leq \delta_{LS}(x)$ and $x \in 3I$. There are at most C such I. □

**Proposition BT13.** *For any* $\tilde{I} \in T^{\mathrm{dist}}(\check{I}) \setminus \{\check{I}\}$, *we have* $\mathrm{diam}(25\tilde{I} \cap \bar{E}) \geq \frac{1}{16}|\tilde{I}|$.

*Proof.* We use induction on $\check{I}$. By (12), we have $\tilde{I} \in T^{\text{dist}}(\check{I}')$ for some $\check{I}' \in RO(\check{I})$. If $\tilde{I} = \check{I}'$, then (24) yields $\text{diam}(25\,\tilde{I} \cap \bar{E}) \geq \frac{1}{16}|\tilde{I}|$. Otherwise, we have $\tilde{I} \in T^{\text{dist}}(\check{I}') \setminus \{\check{I}'\}$. Induction hypothesis (Proposition BT13 for $\check{I}'$) tells us that $\text{diam}(25\,\tilde{I} \cap \bar{E}) \geq \frac{1}{16}|\tilde{I}|$. Thus, the proposition holds in all cases. □

**Proposition BT14.** *The number of nodes of* $T^{\text{dist}}(\check{I})$ *is at most* $CN$.

*Proof.* We bring in the Well-Separated Pairs Decomposition for $\bar{E}$. Recall ([3]) that $\bar{E} \times \bar{E} \setminus$ Diagonal can be partitioned into at most $CN$ (non-empty) Cartesian products $E'_\nu \times E''_\nu$, such that, for each $\nu$,

$$\text{diam}(E'_\nu) + \text{diam}(E''_\nu) < 10^{-3}\text{dist}(E'_\nu, E''_\nu). \tag{26}$$

Here, as usual, $\text{diam}(E) = \max\{|x - x'| : x, x' \in E\}$ and $\text{dist}(E, E') = \min\{|x - x'| : x \in E, x' \in E'\}$ for finite sets $E, E' \subset \mathbb{R}$. For each $\nu$, we pick a "representative" $(x'_\nu, x''_\nu) \in E'_\nu \times E''_\nu$.

Now let $I \in T^{\text{dist}}(\check{I}) \setminus \{\check{I}\}$. By Proposition BT13, there exist $x', x'' \in \bar{E} \cap 25\,I$ such that $|x' - x''| \geq \frac{1}{16}|I|$.

We know that $(x', x'') \in E'_\nu \times E''_\nu$ for some $\nu$. Then $x', x'_\nu \in E'_\nu$ and $x'', x''_\nu \in E''_\nu$, so that (26) yields $|x' - x'_\nu| + |x'' - x''_\nu| < 10^{-3}|x' - x''| \leq 25 \cdot 10^{-3}|I|$ (since $x', x'' \in 25\,I$). Consequently, $x'_\nu, x''_\nu \in 50\,I$, and $|x'_\nu - x''_\nu| \geq |x' - x''| - 25 \cdot 10^{-3}|I| \geq \frac{1}{100}|I|$.

Thus, we have proven the following: Let $I \in T^{\text{dist}}(\check{I}) \setminus \{\check{I}\}$. Then, for some $\nu$, we have

$$x'_\nu, x''_\nu \in 50\,I \text{ and } |I| \leq 100|x'_\nu - x''_\nu|. \tag{27}$$

For fixed $\nu$, there are at most $C$ distinct dyadic intervals $I$ satisfying (27). Since there are at most $CN$ distinct $\nu$ here, we conclude that $T^{\text{dist}}(\check{I}) \setminus \{\check{I}\}$ consists of at most $CN$ nodes. □

**Proposition BT15.** (A) *Let* $I$ *be a leaf of* $T^{\text{global}}(\check{I})$, *and suppose* $x \in \bar{E} \cap 5\,I$. *Then* $I \in T^{\text{dist}}(\check{I})$, $I$ *is of type* C1, *and* $x = x_!(I) \in 5\,I$.

(B) *Conversely, suppose* $I \in T^{\text{dist}}(\check{I})$, $I$ *is of type* C1, *and* $x_!(I) \in 5\,I$. *Then* $I$ *is a leaf of* $T^{\text{global}}(\check{I})$, *and* $x_!(I) \in \bar{E} \cap 5\,I$.

*Proof.* Let $I$ and $x$ be as assumed in (A). By Propositions BT10 and BT11, $I \in T^{\text{dist}}(\check{I})$ and $I$ is of type C1. Since $I$ is of type C1, we have $\bar{E} \cap 5\,I^+ = \{x_!(I)\}$. On the other hand, $x \in \bar{E} \cap 5\,I \subset \bar{E} \cap 5\,I^+$. Hence, $x = x_!(I) \in 5\,I$.

Conversely, let $I$ be as assumed in (B). Then by Proposition BT9, $I$ is a leaf of $T^{\text{global}}(\check{I})$. Since $I$ is of type C1, we have $x_!(I) \in \bar{E}$. By assumption, $x_!(I) \in 5\,I$. □

It is now easy to establish Lemmas BT1–BT4.

*Proof of Lemma* BT1:

(I) follows from an easy induction on $\check{I}$.

(II) follows from Proposition BT5.

(III) follows from the Corollary to Proposition BT5, Proposition BT12, and (25).

(IV) is just the Corollary to Proposition BT12. □

*Proof of Lemma* BT2:

(I) follows from an easy induction on $\check{I}$.

(II) follows from (22) and Proposition BT2.

(III) follows from Propositions BT10 and BT11, and the definition of "type C1".

(IV) is just Proposition BT14. □

*Proof of Lemma* BT3:

(I) is immediate from Proposition BT6 and the Corollary to Proposition BT7.

(II) is precisely Proposition BT7.

(III) is immediate from Proposition BT5. □

*Proof of Lemma* BT4: Let $I \in T^{\mathsf{global}}(\check{I}) \setminus \{\check{I}\}$. By definition,

$$x^{\mathsf{rep}}_{\mathsf{left}}(I) = x^{\mathsf{s}}_{\mathsf{left}}(\tilde{I}) = \text{ left endpoint of } J(\tilde{I}),$$

and $x^{\mathsf{rep}}_{\mathsf{rt}}(I) = x^{\mathsf{s}}_{\mathsf{rt}}(\tilde{I}) = $ right endpoint of $J(\tilde{I})$, where $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ satisfies that $I \in T^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$. In particular, $\tilde{I}$ is of type A or B (since $T^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\} \neq \emptyset$), hence $J(\tilde{I})$ is a non-degenerate closed interval.

Hence, it is obvious that $x^{\mathsf{rep}}_{\mathsf{left}}(I) < x^{\mathsf{rep}}_{\mathsf{rt}}(I)$; moreover, (5) and (23) show that $x^{\mathsf{rep}}_{\mathsf{left}}(I), x^{\mathsf{rep}}_{\mathsf{rt}}(I) \in \bar{E} \cap 25\,I$. Thus, we have proven the desired results for $I \in T^{\mathsf{global}}(\check{I}) \setminus \{\check{I}\}$.

Now suppose $\check{I}$ is of type A or B. Then $J(\check{I})$ is a non-degenerate closed interval, whose endpoints belong to $\bar{E}$, by (5). Moreover, by definition, $x^{\mathsf{rep}}_{\mathsf{left}}(\check{I}) = x^{\mathsf{s}}_{\mathsf{left}}(\check{I}) =$ left endpoint of $J(\check{I})$, and $x^{\mathsf{rep}}_{\mathsf{rt}}(\check{I}) = x^{\mathsf{s}}_{\mathsf{rt}}(\check{I}) = $ right endpoint of $J(\check{I})$. Thus, $x^{\mathsf{rep|}}_{\mathsf{left}}(\check{I}) < x^{\mathsf{rep}}_{\mathsf{rt}}(\check{I})$, and $x^{\mathsf{rep}}_{\mathsf{left}}(\check{I}), x^{\mathsf{rep}}_{\mathsf{rt}}(\check{I}) \in \bar{E}$. Moreover, by definition, $x^{\mathsf{rep}}_{\mathsf{left}}(\check{I}), x^{\mathsf{rep}}_{\mathsf{rt}}(\check{I}) \in J(\check{I}) \subset 5\,\check{I}^+ \subset 25\,\check{I}$. The proof of Lemma BT4 is complete. □

So far, we have discussed the mathematical properties of $T^{\mathsf{loc}}(\tilde{I})$, $T^{\mathsf{global}}(\check{I})$, $x^{\mathsf{s}}_{\mathsf{left}}(\tilde{I})$, $x^{\mathsf{s}}_{\mathsf{rt}}(\tilde{I})$, etc. We now present algorithms. We suppose that the set $\bar{E} \subset \mathbb{R}$, with $\#(\bar{E}) = N$, is given to us as a sorted list, $\bar{E} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$, with $\bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_N$.

**Algorithm BT1.** Suppose we are given a dyadic grid $\mathcal{G}_\tau$, an $N$-element set $\bar{E}$ (sorted, with $N \geq 2$), and a dyadic interval $\tilde{I} \in \mathcal{G}_\tau$. We compute $J(\tilde{I})$, and determine whether $\tilde{I}$ is of type A,B, C0 or C1. If $\tilde{I}$ is of type A or B, then we compute $x^{\mathsf{s}}_{\mathsf{left}}(\tilde{I})$ and $x^{\mathsf{s}}_{\mathsf{rt}}(\tilde{I})$. If $\tilde{I}$ is of type C1, then we compute $x_{!}(\tilde{I})$. Regardless of the type of $\tilde{I}$, we compute $x_{\mathsf{left}}(\tilde{I})$, $x_{\mathsf{rt}}(\tilde{I})$, and $\mathsf{RO}(\tilde{I})$.

The work used to do the above is at most $C \log N$, and the storage used (apart from that used to hold $\bar{E}$) is at most $C$.

*Explanation*: By binary searches, we first determine whether $\bar{E} \cap 5\,\tilde{I}^+$ is empty; and if it is non-empty, we then compute $\max(\bar{E} \cap 5\,\tilde{I}^+)$ and $\min(\bar{E} \cap 5\,\tilde{I}^+)$. This allows us to compute $J(\tilde{I})$ and determine whether $\tilde{I}$ is of type A,B,C0 or C1. It also allows us to write down $x^{\mathsf{s}}_{\mathsf{left}}(\tilde{I})$, $x^{\mathsf{s}}_{\mathsf{rt}}(\tilde{I})$ if $\tilde{I}$ is of type A or B; and $x_{!}(\tilde{I})$ if $\tilde{I}$ is of type C1.

The points $x_{\text{left}}(\tilde{I})$, $x_{\text{rt}}(\tilde{I})$ are simply the endpoints of $\tilde{I}$.

To compute $\text{RO}(\tilde{I})$ is trivial if $\tilde{I}$ is of type A or C. For $\tilde{I}$ of type B, the set $\text{RO}(\tilde{I})$ consists of all dyadic intervals $I \subset \tilde{I}$ satisfying (8b). We can easily list all such intervals.

The binary searches above require work $C \log N$ and storage $C$ (aside from the storage used to hold $\bar{E}$). The rest of the computation requires work and storage at most $C$.

**Algorithm BT 2.** Given a dyadic grid $\mathcal{G}_\tau$, an $N$-element set $\bar{E}$ (sorted, with $N \geq 2$), and a dyadic interval $\check{I} \in \mathcal{G}_\tau$, we compute the tree $T^{\text{dist}}(\check{I})$. We mark each node $\tilde{I} \in T^{\text{dist}}(\check{I})$ to indicate whether it is of type A, B, C0, or C1. We mark each node $\tilde{I} \in T^{\text{dist}}(\check{I})$ of type A or B with the points $x^s_{\text{left}}(\tilde{I})$ and $x^s_{\text{rt}}(\tilde{I})$. We mark each node $\tilde{I} \in T^{\text{dist}}(\check{I})$ of type C1 with the point $x_!(\tilde{I})$. We mark each node $\tilde{I} \in T^{\text{dist}}(\check{I})$ with the points $x_{\text{left}}(\tilde{I})$, $x_{\text{rt}}(\tilde{I})$.

The work used to do so is at most $CN \log N$, and the storage used is at most $CN$.

*Explanation*: We start with the root $\check{I}$, and apply Algorithm BT1.

This provides all the markings required for $\check{I}$, and provides also $\text{RO}(\check{I})$, the set of all the children of $\check{I}$ in $T^{\text{dist}}(\check{I})$. If $\text{RO}(\check{I}) \neq \emptyset$, then, recursively, we apply Algorithm BT2 to each $\check{I}' \in \text{RO}(\check{I})$, to compute and mark the tree $T^{\text{dist}}(\check{I}')$. Thus, we compute and mark $T^{\text{dist}}(\check{I})$.

Since the tree $T^{\text{dist}}(\check{I})$ has at most $CN$ nodes, the work and storage used by our algorithm are as claimed.

**Algorithm BT 3.** Given a dyadic grid $\mathcal{G}_\tau$, an $N$-element set $\bar{E}$ (sorted, with $N \geq 2$), and a dyadic interval $\check{I} \in \mathcal{G}_\tau$, we compute the set $\mathcal{L}$ of all pairs $(I, x)$, for which $I$ is a leaf of $T^{\text{global}}(\check{I})$ and $x \in 5I \cap \bar{E}$.

The work used to do so is at most $CN \log N$, and the storage used is at most $CN$.

*Explanation*: We execute Algorithm BT2 to compute $T^{\text{dist}}(\check{I})$ and mark its nodes. According to Proposition BT15, $\mathcal{L}$ consists of all pairs $(I, x_!(I))$ for $I \in T^{\text{dist}}(\check{I})$ of type C1, such that $x_!(I) \in 5I$.

Thus, we can trivially list all the elements of $\mathcal{L}$.

Since the tree $T^{\text{dist}}(\check{I})$ has at most $CN$ nodes, the work and storage used, once we have executed Algorithm BT2, are at most $CN$.

## 8. The basic set-up

In this section, we provide the basic assumptions that we will be making in several sections below. The assumptions below involve positive constants $\bar{c}_1, \bar{C}_1, \bar{C}_2, \bar{C}_3$, which we regard as given.

(1) We are given a real number $\tau$, used to fix a dyadic grid $\mathcal{G}_\tau$.

(2) We are given a positive real number $\epsilon < \bar{c}_1$.

(3) We are given a dyadic interval $I_0 \in \mathcal{G}_\tau$, with $|I_0| \leq \bar{C}_1 \epsilon$.

(4) We are given a finite set $\bar{E} \subset I_0$.

(5) We are given a function $\varphi \in C^2(\bar{c}_1 \epsilon^{-1} I_0)$, which is assumed to satisfy the estimates:

(6) $|\varphi'| \leq \bar{C}_2$ and $|\varphi''| \leq \bar{C}_2 \epsilon |I_0|^{-1}$ on $\bar{c}_1 \epsilon^{-1} I_0$.

We suppose we have access to a "$\varphi$-Oracle":

(7) Given a point $x_1 \in \bar{c}_1 \epsilon^{-1} I_0$, the $\varphi$-Oracle computes $\varphi(x_1)$, $\varphi'(x_1)$, $\varphi''(x_1)$, and changes us "work"

(8) $W_{\varphi O} \geq 1$ to do so.

(9) We define $E = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}\} \subset \mathbb{R}^2$.

(10) Let $N = \#(\bar{E}) = \#(E)$. We assume $N \geq 2$.

(11) We are given a function $f : E \longrightarrow \mathbb{R}$.

(12) We are given a real number $\xi$.

(13) We are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ $(z \in \mathbb{R}^2)$, and an Oracle, satisfying conditions (1)–(4) in Section 5. We define the $C^2$ norm as in that section.

(14) We assume that there exists $F_{\mathsf{crude}} \in C^2(\mathbb{R}^2)$, such that:

(15) $F_{\mathsf{crude}} = f$ on $E$,

(16) $\| F_{\mathsf{crude}} \|_{C^2(\mathbb{R}^2)} \leq \bar{C}_3$, and

(17) $|\partial_2 F_{\mathsf{crude}} - \xi| \leq \bar{C}_3 \epsilon^{-1} |I_0|$ on $E$.

We fix integers $k_1(\epsilon)$ and $\nu_0(\epsilon)$, such that

(18) $\frac{1}{10}\epsilon^{100} < 2^{-k_1(\epsilon)} < \epsilon^{100}$ and

(19) $\frac{1}{8}\epsilon^{-2} < 2^{\nu_0(\epsilon)} < \epsilon^{-2}$.

(20) We take the boiler–plate constants in this section to be $\bar{c}_1$, $\bar{C}_1$, $\bar{C}_2$ and $\bar{C}_3$ in (1)–(19) above, together with the constants called $c_0, C_0, C_1, C_2$ in Section 5.

As explained in Section 2, the notion of a "controlled constant" is well-defined thanks to (20). We make the Small $\epsilon$ Assumption:

(21) $\epsilon$ is less than a small enough controlled constant.

In this section, and in the next several sections, we assume (1)–(21) above.

We now define a function $e_2 \in C^2_{\mathrm{loc}}(\bar{c}_1 \epsilon^{-1} I_0^{\mathrm{interior}} \times \mathbb{R})$, by setting

(22) $e_2(x_1, x_2) = x_2 - \varphi(x_1)$ for $x_1 \in \bar{c}_1 \epsilon^{-1} I_0^{\mathrm{interior}}$, $x_2 \in \mathbb{R}$.

Note that (9) yields

(23) $e_2 = 0$ on $E$,

while (6) implies the estimates

$$|\partial^\alpha e_2| \leq C \quad \text{for } |\alpha| = 1, \quad \text{and} \quad |\partial^\alpha e_2| \leq C\epsilon |I_0|^{-1} \quad \text{for } |\alpha| = 2.$$

Together with (3), the above estimates imply the following:

(24) $\left[ \begin{array}{l} \text{Let } Q \subset \bar{c}_1 \epsilon^{-1} I_0 \times \mathbb{R} \text{ be an open square. If } e_2 = 0 \text{ at some point of } Q, \\ \text{then } \| e_2 \|_{C^2(Q)} \leq C\epsilon |I_0|^{-1}. \end{array} \right.$

Thanks to (7), we have the following:

(25) Given a point $z \in \bar{c}_1 \epsilon^{-1} I_0^{\text{interior}} \times \mathbb{R}$, we can compute the jet $J_z(\dot{} e_2)$ using $C$ operations and one call to the $\varphi$-Oracle.

Next, for each dyadic interval

(26) $I \subseteq I_0$, we define a square

(27) $Q(I) = (\tilde{C}_Q I \times J)^{\text{interior}} \in \mathbb{R}^2$,

where $\tilde{C}_Q$ is a large enough controlled constant (to be picked in a moment), and

(28) center $(J) = \varphi$ ( center $(I)$).

If we pick $\tilde{C}_Q$ large enough, then (5) and (6) guarantee that

(29) For all $x \in 1024I$, $\varphi(x)$ is well-defined and $(x, \varphi(x))$ belongs to the middle half of $Q(I)$. Also, for $\tilde{C}_Q$ large enough, (6) yields

(30) $Q(I') \subset Q(I)$ for $I' \subset I$.

We now pick $\tilde{C}_Q$ to be a controlled constant, large enough to guarantee (29) and (30).

In the next several sections, the function $e_2$ and the square $Q(I)$ ($I \subset I_0$ dyadic) are as defined in this section. Note that, given $I \subset I_0$ dyadic, we can trivially compute the square $Q(I)$ using work at most $C$, together with a single call to the $\varphi$-Oracle.

Let $I \subset I_0$ be dyadic. Then $1024\tilde{C}_Q I \subset \bar{c}_1 \epsilon^{-1} I_0$, by (21). Hence, $e_2$ is well-defined on $1024Q(I)$. Moreover, $e_2 = 0$ at the center of $Q(I)$, thanks to (28). Therefore, (24) and (25) yield the following:

(31) For any dyadic $I \subseteq I_0$, we have $\| e_2 \|_{C^2(1024Q(I))} \leq C\epsilon |I_0|^{-1}$.

(32) For any dyadic $I \subseteq I_0$ and any given $z \in 1024Q(I)$, we can compute $J_z(e_2)$ using $C$ operations and one call to the $\varphi$-Oracle.

## Remarks

(33) As explained in the Introduction, one of the main ideas in our proof of Theorem 1 is to introduce a Calderón–Zygmund decomposition of $\mathbb{R}^2$ into squares $Q_\nu$. For each $Q_\nu$, either

$$E \cap Q_\nu \subset \{(x_1, \varphi(x_1)) : x_1 \in \mathbb{R}\} \quad \text{or} \quad E \cap Q_\nu \subset \{(\varphi(x_2), x_2) : x_2 \in \mathbb{R}\}$$

for a $C^2$-function $\varphi$. (See (29), (30) in Section 0.)

Eventually, we will cut up each $Q_\nu$ into a grid of subsquares $\{Q_{\nu,i}\}$, with sidelengths $\delta_{Q_{\nu,i}}$ comparable to $\epsilon \cdot \delta_{Q_\nu}$.

The assumptions made in this section will (eventually) be applied to compute interpolants on a given $Q_{\nu,i}$. Our plan is then to combine our results for all the $Q_{\nu,i}$, to compute an interpolant on a given $Q_\nu$. (See Sections 18 and 19 below.)

(34) One might be tempted to reduce matters to the case $\varphi \equiv 0$ in (5)–(9) above, by making a change of variables such as $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2 - \varphi(x_1)$. Unfortunately, when we transform the $C^2$-norm to take such a change of variables into account, we lose the Approximate Translation Invariance property, because $\varphi$ is merely $C^2$.

(35) Recall that we have now postulated an Oracle and a $\varphi$-Oracle (see Section 5 and assumptions (7), (8)). From now on, whenever we present an algorithm, we regard the work charged by the Oracle, but *not* that charged by the $\varphi$-Oracle, as part of the work of the algorithm in question. We will always provide an upper bound for the number of calls made to the $\varphi$-Oracle in each algorithm given below.

## 9. Marking the basic tree

In this section, we adopt the notation, assumptions, and boiler-plate constants of Section 8. In particular, we recall the Small $\epsilon$ Assumption, and the assumption

(1) $\frac{1}{10}\epsilon^{100} < 2^{-k_1(\epsilon)} < \epsilon^{100}$.

For any dyadic interval

(2) $I \subseteq I_0$,

we define

(3) $\bigwedge(I) = (2^{-k_1(\epsilon)}|I|\mathbb{Z}^2) \cap Q(I)$.

Note that

(4) $\bigwedge(I) \subset Q(I)$ and $\#(\Lambda(I)) \leq C\epsilon^{-200}$, thanks to (1).

In this section, we suppose we are given an interval

(5) $\check{I} \subseteq I_0$ dyadic, such that $\#(5\check{I} \cap \bar{E}) \geq 2$.

We recall from the Section 7 that we have defined a tree $T^{\mathsf{global}}(\check{I})$ and a subset $T^{\mathsf{dist}}(\check{I}) \subset T^{\mathsf{global}}(\check{I})$. Each node $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ is of "type" $A, B, C0$ or $C1$. Each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type A or B is marked with two points $x^s_{\mathsf{left}}(\tilde{I})$ and $x^s_{\mathsf{rt}}(\tilde{I})$, the endpoints of the non-degenerate interval $J(\tilde{I})$= convex hull of $5\tilde{I}^+ \cap \bar{E}$. (In particular, $x^s_{\mathsf{left}}(\tilde{I})$, $x^s_{\mathsf{rt}}(\tilde{I}) \in 25\,\tilde{I} \cap \bar{E}$.) Each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type $C1$ is marked with a point $x_!(\tilde{I})$, the one and only element of $5\tilde{I}^+ \cap \bar{E}$. (In particular, $x_!(\tilde{I}) \in 25\,\tilde{I} \cap \bar{E}$.) For any $I \in T^{\mathsf{global}}(\check{I})$, we have also defined two points $x^{\mathsf{rep}}_{\mathsf{left}}(I)$ and $x^{\mathsf{rep}}_{\mathsf{rt}}(I)$ as follows: If $I = \check{I}$, then $x^{\mathsf{rep}}_{\mathsf{left}}(I) = x^s_{\mathsf{left}}(\check{I})$ and $x^{\mathsf{rep}}_{\mathsf{rt}}(I) = x^s_{\mathsf{rt}}(\check{I})$. (This makes sense, since $\check{I}$ is of type A or B, thanks to (5)). If $I \in T^{\mathsf{global}}(\check{I}) \setminus \{\check{I}\}$, then $I \in T^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$ for one and only one $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$; and $\tilde{I}$ is of type A or B. (Here, $T^{\mathsf{loc}}(\tilde{I})$ is the "local tree" associated to $\tilde{I}$; see Section 7.) We then have $x^{\mathsf{rep}}_{\mathsf{left}}(I) = x^s_{\mathsf{left}}(\tilde{I})$, and $x^{\mathsf{rep}}_{\mathsf{rt}}(I) = x^s_{\mathsf{rt}}(\tilde{I})$. We have seen that $x^{\mathsf{rep}}_{\mathsf{left}}(I)$, $x^{\mathsf{rep}}_{\mathsf{rt}}(I) \in 25\,I \cap \bar{E}$.

For each $I \in T^{\mathsf{global}}(\check{I})$, we have also defined $x_{\mathsf{left}}(I)$, $x_{\mathsf{rt}}(I)$ to be the endpoints of I.

We recall from Section 8 that $E = \{(x, \varphi(x)) : x \in \bar{E}\}$, and that $(x, \varphi(x)) \in Q(I)$ whenever $x \in 25\, I$, $I \subseteq I_0$ dyadic. Thus, we may define

(6) $z^{\mathsf{rep}}_{\mathsf{left}}(I) = (x^{\mathsf{rep}}_{\mathsf{left}}(I),\ \varphi(x^{\mathsf{rep}}_{\mathsf{left}}(I))) \in E \cap Q(I)$ and

(7) $z^{\mathsf{rep}}_{\mathsf{rt}}(I) = (x^{\mathsf{rep}}_{\mathsf{rt}}(I),\ \varphi(x^{\mathsf{rep}}_{\mathsf{rt}}(I))) \in E \cap Q(I)$, for all $I \in T^{\mathsf{global}}(\check{I})$.

Also, we may define

(8) $z_{\mathsf{left}}(I) = (x_{\mathsf{left}}(I),\ \varphi(x_{\mathsf{left}}(I))) \in Q(I)$ and

(9) $z_{\mathsf{rt}}(I) = (x_{\mathsf{rt}}(I),\ \varphi(x_{\mathsf{rt}}(I))) \in Q(I)$, for all $I \in T^{\mathsf{global}}(\check{I})$.

Similarly, for all $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type A or B, we define

(10) $z^{s}_{\mathsf{left}}(\tilde{I}) = (x^{s}_{\mathsf{left}}(\tilde{I}),\ \varphi(x^{s}_{\mathsf{left}}(\tilde{I}))) \in E \cap Q(\tilde{I})$ and

(11) $z^{s}_{\mathsf{rt}}(\tilde{I}) = (x^{s}_{\mathsf{rt}}(\tilde{I}),\ \varphi(x^{s}_{\mathsf{rt}}(\tilde{I}))) \in E \cap Q(\tilde{I})$.

For all $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type C1, we define

(12) $z_{!}(\tilde{I}) = (x_{!}(\tilde{I}),\ \varphi(x_{!}(\tilde{I}))) \in E \cap Q(\tilde{I})$.

Next, for each $I \in T^{\mathsf{global}}(\check{I})$, we define a finite subset $S(I) \subset Q(I)$. We take $S(I)$ to consist of the following points:

(13) All the points of $\bigwedge(I)$ (see (3)).

(14) The points $z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$, $z_{\mathsf{left}}(I)$, $z_{\mathsf{rt}}(I)$.

(15) The points $z^{s}_{\mathsf{left}}(I)$, $z^{s}_{\mathsf{rt}}(I)$ if $I \in T^{\mathsf{dist}}(\check{I})$ and I is of type A or B.

(16) The point $z_{!}(I)$ if $I \in T^{\mathsf{dist}}(\check{I})$ and I is of type C1.

Thanks to (4) and (6)–(12), we have

(17) $S(I) \subset Q(I)$ and $\#(S(I)) \le C\epsilon^{-200}$, for all $I \in T^{\mathsf{global}}(\check{I})$.

The following algorithm computes the sets $S(\tilde{I})$ and the Whitney fields $J_{S(\tilde{I})}(e_2)$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$.

**Algorithm MMBT.** ("Make and Mark the Basic Tree"): Given a dyadic interval $\check{I} \subseteq I_0$ such that $\#(5\,\check{I} \cap \bar{E}) \ge 2$, we compute the tree $T^{\mathsf{dist}}(\check{I})$, and mark its nodes as follows:

- We mark each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ to indicate its type (A, B, C0 or C1).
- We mark each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type A or B to indicate the points $z^{s}_{\mathsf{left}}(\tilde{I})$, $z^{s}_{\mathsf{rt}}(\tilde{I})$ and the function values $f(z^{s}_{\mathsf{left}}(\tilde{I}))$, $f(z^{s}_{\mathsf{rt}}(\tilde{I}))$.
- We mark each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type C1 to indicate the point $z_{!}(\tilde{I})$ and the function value $f(z_{!}(\tilde{I}))$.
- We mark each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ to indicate the points $z_{\mathsf{left}}(\tilde{I})$, $z_{\mathsf{rt}}(\tilde{I})$, $z^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$, $z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$ and the function values $f(z^{\mathsf{rep}}_{\mathtt{left}}(\tilde{I}))$, $f(z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I}))$.

- We mark each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ to indicate the square $Q(\tilde{I})$, the sets $\Lambda(\tilde{I})$, $S(\tilde{I})$, and the Whitney field $J_{S(\tilde{I})}(e_2)$.

The total work used to perform the above computations is at most $C\epsilon^{-200}N + CN\log N$, together with at most $C\epsilon^{-200}N$ calls to the $\varphi$-Oracle. The storage used is at most $C\epsilon^{-200}N$.

*Explanation*: First, we apply Algorithm BT2 from Section 7. Thus, we obtain the tree $T^{\mathsf{dist}}(\check{I})$; mark each of its nodes as having type $A, B, C0$ or $C1$; compute the points $x_{\mathsf{left}}(\tilde{I})$, $x_{\mathsf{rt}}(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$; compute the points $x^s_{\mathsf{left}}(\tilde{I})$, $x^s_{\mathsf{rt}}(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type $A$ or $B$; and compute $x_!(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ of type $C1$.

Applying the $\varphi$-Oracle as needed, we can easily compute the points $z_{\mathsf{left}}(\tilde{I})$, $z_{\mathsf{rt}}(\tilde{I})$, $z^s_{\mathsf{left}}(\tilde{I})$, $z^s_{\mathsf{rt}}(\tilde{I})$, $z_!(\tilde{I})$ as in the statement of Algorithm MMBT. We can then look up the values of $f$ at the points $z^s_{\mathsf{left}}(\tilde{I})$ and $z^s_{\mathsf{rt}}(\tilde{I})$ for $\tilde{I}$ of type $A$ or $B$, and $z_1(\tilde{I})$ for $\tilde{I}$ of type $C1$.

We next explain how to compute the points $z^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$, $z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$. For $\tilde{I} = \check{I}$, we have $x^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I}) = x^s_{\mathsf{left}}(\tilde{I})$. and $x^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I}) = x^s_{\mathsf{rt}}(\tilde{I})$; the points $x^s_{\mathsf{left}}(\tilde{I})$, $x^s_{\mathsf{rt}}(\tilde{I})$ are already known. For $\tilde{I} \in T^{\mathsf{dist}}(\check{I}) \setminus \{\check{I}\}$, let $I^{\#}$ be the parent of $\tilde{I}$ in the tree $T^{\mathsf{dist}}(\check{I})$. Then $I^{\#} \in T^{\mathsf{dist}}(\check{I})$ is of type $A$ or $B$, and $\tilde{I} \in \mathsf{RO}(I^{\#}) \subset T^{\mathsf{loc}}(I^{\#}) \setminus \{I^{\#}\}$. (See Section 7.) Consequently, $x^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I}) = x^s_{\mathsf{left}}(I^{\#})$ and $x^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I}) = x^s_{\mathsf{rt}}(I^{\#})$; the points $x^s_{\mathsf{left}}(I^{\#})$ and $x^s_{\mathsf{rt}}(I^{\#})$ are already known. Thus, we can compute $x^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$, $x^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$. Invoking the $\varphi$-Oracle, we obtain $z^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$, $z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$, as promised.

We can then look up the values of $f$ at the points $z^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$, $z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$.

Next, for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we compute the square $Q(\tilde{I})$, as explained in the section on "The Basic Set-up". Each $Q(\tilde{I})$ requires a single application of the $\varphi$-Oracle. It is now trivial to compute the $\bigwedge(\tilde{I})$ for all $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, using (3). We can then read off the set $S(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, by recalling (13)–(16). Finally, once we know $S(\tilde{I})$, we can read off the Whitney field $J_{S(\tilde{I})}(e_2)$ by using (32) from Section 8.

Thus, we have computed everything promised in Algorithm MMBT. To estimate the work and storage used by Algorithm MMBT, we have only to recall the resources used by Algorithm BT2, as well as the estimates $\#(S(\tilde{I})) \le C\epsilon^{-200}$ (each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$), and $\#(T^{\mathsf{dist}}(\check{I})) \le CN$.

It is now trivial to check that the resources used by Algorithm MMBT are as promised.

## 10. A partition of unity

In this section, we keep the assumptions, conventions and boiler-plate constants of Section 8. As in Section 9, we suppose that we are given

(1) $\check{I} \subseteq I_0$ dyadic, such that $\#(5\check{I} \cap \bar{E}) \ge 2$.

Define an open set

(2) $\Omega(\check{I}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \check{I}^{\mathsf{interior}}, |x_2 - \varphi(x_1)| < |\check{I}|\} \subset \mathbb{R}^2$.

We will introduce functions $\tilde{\theta}$ defined on $\mathbb{R}^2$, and functions $\theta$ defined only on $\Omega(\check{I})$. We write $\mathsf{supp}\,\theta$ to denote the set of all points $z$ in $\Omega(\check{I})$ such that $\theta$ is not identically zero on any disc centered at $z$. As usual, we write $\mathsf{supp}\,\tilde{\theta}$ to denote the set of all points $z$ in $\mathbb{R}^2$ such that $\tilde{\theta}$ is not identically zero on any disc centered at $z$.

For each $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$, we will define a function

(3) $\theta_I \in C^2(\Omega(\check{I}))$, defined only on $\Omega(\check{I})$, such that the following hold:

(4) $\displaystyle\sum_{I \in \mathsf{T}^{\mathsf{global}}(\check{I})} \theta_I = 1$ on $\Omega(\check{I})$.

(5) $\mathsf{supp}\,\theta_I \subset Q(I)$ for each $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$.

(6) $|\partial^\alpha \theta_I| \leq C|I|^{-|\alpha|}$ on $\Omega(\check{I})$, for $|\alpha| \leq 2$, $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$.

(7) $\theta_I \geq 0$ on $\Omega(\check{I})$ for each $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$.

Moreover, we will compute the jets of the $\theta_I$ at each point of $E$.

To define the functions $\theta_I$, we start by fixing cutoff functions $\chi_0, \chi_1 \in C^2(\mathbb{R})$, with the following properties:

(8) $\chi_1(t) = 1$ for $\frac{1}{2} \leq |t| \leq 1$.

(9) $\chi_1(t) = 0$ for $|t| \notin \left[\frac{1}{2.02}, 1.01\right]$

(10) $\chi_1(t) \geq 0$ for all $t \in \mathbb{R}$.

(11) $\left|\left(\frac{d}{dt}\right)^\ell \chi_1(t)\right| \leq C$ for $0 \leq \ell \leq 2$, $t \in \mathbb{R}$.

(12) $\chi_0(t) = 1$ for $|t| \leq 1$.

(13) $\chi_0(t) = 0$ for $|t| \geq 1.01$.

(14) $\chi_0(t) \geq 0$ for all $t \in \mathbb{R}$.

(15) $\left|\left(\frac{d}{dt}\right)^\ell \chi_0(t)\right| \leq C$ for $0 \leq \ell \leq 2$, $t \in \mathbb{R}$.

We suppose that, given $t \in \mathbb{R}$ and $0 \leq \ell \leq 2$, we can compute $\left(\frac{d}{dt}\right)^\ell \chi_i(t)$ for $i = 0, 1$, with work at most $C$. It is trivial to construct such $\chi_0, \chi_1$.

Also, for each dyadic interval $I$, we fix a cutoff function $\chi_I \in C^2(\mathbb{R})$, with the following properties:

(16) $\chi_I = 1$ on $I$, $\mathsf{supp}\chi_I \subset (1.01)I$, $\chi_I \geq 0$ on $\mathbb{R}$, and

(17) $\left|\left(\frac{d}{dt}\right)^\ell \chi_I(t)\right| \leq C|I|^{-\ell}$ for $0 \leq \ell \leq 2$, $t \in \mathbb{R}$.

We suppose that, given $t \in \mathbb{R}$, $0 \leq \ell \leq 2$, and $I$ a dyadic interval, we can compute $\left(\frac{d}{dt}\right)^\ell \chi_I(t)$ with work at most $C$. It is trivial to construct such $\chi_I$.

Next, given $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$, we define a cutoff function $\tilde{\theta}_I \in C^2(\mathbb{R}^2)$. The definition of $\tilde{\theta}_I$ proceeds by cases.

Suppose $I$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$. Then we define

(18) $\tilde{\theta}_I(x_1, x_2) = \chi_I(x_1) \cdot \chi_0\left(\frac{x_2 - \varphi(x_1)}{|I|}\right)$ for $(x_1, x_2) \in \mathbb{R}^2$.

Suppose $I$ is an internal node of $\mathsf{T}^{\mathsf{global}}(\check{I})$. Then we define

(19) $\tilde{\theta}_I(x_1,x_2) = \chi_I(x_1)\cdot\chi_1\left(\frac{x_2-\varphi(x_1)}{|I|}\right)$ for $(x_1,x_2)\in\mathbb{R}^2$.

We recall that $|\varphi'|\le\bar{C}_2$ and $|\varphi''|\le\bar{C}_2\epsilon|I_0|^{-1}\le\bar{C}_2|I|^{-1}$ on $\bar{c}_1\epsilon^{-1}I_0\supset(1.01)I_0\supset(1.01)I$; see (6) and (21) in Section 8. Hence, one obtains easily the estimate

(20) $|\partial^\alpha\tilde{\theta}_I|\le C|I|^{-|\alpha|}$ on $\mathbb{R}^2$, for $|\alpha|\le 2$, $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$. Also

(21) $\tilde{\theta}_I\ge 0$ on $\mathbb{R}^2$, for all $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$.

Let us study $\mathsf{supp}\,\tilde{\theta}_I$. From (18), (19) and the defining properties of $\chi_0,\chi_1,\chi_I$, we have:

(22) $\mathsf{supp}\,\tilde{\theta}_I\subset\{(x_1,x_2)\in\mathbb{R}^2\colon x_1\in(1.01)I,\ |x_2-\varphi(x_1)|\le(1.01)|I|\}$ for any $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$.

Moreover,

(23) $\mathsf{supp}\,\tilde{\theta}_I\subset\{(x_1,x_2)\in\mathbb{R}^2 : x_1\in(1.01)I,\ \frac{1}{2.02}|I|\le|x_2-\varphi(x_1)|\le(1.01)|I|\}$ for any internal node $I$ of $\mathsf{T}^{\mathsf{global}}(\check{I})$.

Note that $\{(x_1,x_2)\in\mathbb{R}^2 : x_1\in(1.01)I,\ |x_2-\varphi(x_1)|\le(1.01)|I|\}\subset Q(I)$, for all $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$. (This follows from (29) in Section 8.)

Together with (22), this yields the inclusion

(24) $\mathsf{supp}\,\tilde{\theta}_I\subset Q(I)$ for all $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$.

We establish an additional property of $\mathsf{supp}\,\tilde{\theta}_I$. From (2) in Section 7, we recall the function $\delta_{LS}(x)$, defined for $x\in\mathbb{R}$.

Now suppose $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$, and let $(x_1,x_2)\in\mathsf{supp}\,\tilde{\theta}_I$. We will show that

(25) $c[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]\le|I|\le C[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]$.

Indeed, (22) gives $|x_2-\varphi(x_1)|\le(1.01)|I|$, $x_1\in(1.01)I$; and Proposition BT12 in Section 7 gives $\delta_{LS}(x_1)\le C|I|$, except for the case $I=\check{I}$.

In the case $I=\check{I}$, we still have $\delta_{LS}(x_1)\le C|I|$, by definition of $\delta_{LS}(x_1)$, and thanks to (1).

Thus, $\delta_{LS}(x_1)+|x_2-\varphi(x_1)|\le C|I|$, which is half of (25).

To prove the other half, we proceed by cases. Suppose first that $I$ is a leaf of $\mathsf{T}^{\mathsf{global}}(\check{I})$.

Then, by conclusion (III) of Lemma BT1 in Section 7, we have $|I|\le\delta_{LS}(x_1)\le[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]$. Note that (III) applies, thanks to our assumption (1). On the other hand, suppose that $I$ is an internal node of $\mathsf{T}^{\mathsf{global}}(\check{I})$. Then, since $(x_1,x_2)\in\mathsf{supp}\,\tilde{\theta}_I$, we learn from (23) that $|I|\le(2.02)\cdot|x_2-\varphi(x_1)|\le(2.02)[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]$. Thus, in either case, we have $|I|\le C\cdot[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]$, completing the proof of (25).

From (22) and (25), we obtain the following useful result:

(26) Let $I\in\mathsf{T}^{\mathsf{global}}(\check{I})$, and let $(x_1,x_2)\in\mathsf{supp}\,\tilde{\theta}_I$. Then $x_1\in(1.01)I$, and $c[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]\le|I|\le C[\delta_{LS}(x_1)+|x_2-\varphi(x_1)|]$.

From (26), we obtain at once the following consequences:

(27) Let $I_1, I_2 \in T^{\mathsf{global}}(\check{I})$, and suppose $\operatorname{supp}\tilde{\theta}_{I_1} \cap \operatorname{supp}\tilde{\theta}_{I_2} \neq \emptyset$. Then $c|I_1| \leq |I_2| \leq C|I_1|$.

(28) Any given point $z \in \mathbb{R}^2$ lies in $\operatorname{supp}\tilde{\theta}_I$ for at most C distinct $I \in T^{\mathsf{global}}(\check{I})$.

Next, we establish the following:

(29) Let $z = (x_1, x_2) \in \Omega(\check{I})$ be given. Then there exists $I \in T^{\mathsf{global}}(\check{I})$ such that $\tilde{\theta}_I(z) = 1$.

In fact, (2) gives $x_1 \in \check{I}$ and $|x_2 - \varphi(x_1)| < |\check{I}|$.

The leaves of $T^{\mathsf{global}}(\check{I})$ form a partition of $\check{I}$. Hence, there exists a leaf $I_1$ of $T^{\mathsf{global}}(\check{I})$ containing $x_1$. The nodes of $T^{\mathsf{global}}(\check{I})$ containing $x_1$ are $I_1 \subset I_2 \subset \cdots \subset I_L = \check{I}$, where $I_{\ell+1} = (I_\ell)^+$ (the dyadic parent of $I_\ell$) for each $\ell < L$. Since $|x_2 - \varphi(x_1)| < |I_L|$, we have either

(30) $|x_2 - \varphi(x_1)| \leq |I_1|$, or

(31) $\frac{1}{2}|I_\ell| = |I_{\ell-1}| < |x_2 - \varphi(x_1)| \leq |I_\ell|$ for some $\ell\,(2 \leq \ell \leq L)$.

If (30) holds, then, since $I_1$ is a leaf, and since $x_1 \in I_1$, we see from (12), (16), (18) that $\theta_{I_1}(x_1, x_2) = 1$.

If instead (31) holds, then, since $I_\ell$ is an internal node and $x_1 \in I_\ell$, we learn from (8), (16) and (19) that $\tilde{\theta}_{I_\ell}(x_1, x_2) = 1$.

Thus (29) holds in all cases.

From (21) and (29), we see that

(32) $$\sum_{I' \in T^{\mathsf{global}}(\check{I})} \tilde{\theta}_{I'} \geq 1 \text{ on } \Omega(\check{I}).$$

Now it is easy to define our partition of unity on $\Omega(\check{I})$. We set

(33) $\theta_I = \tilde{\theta}_I \Big/ \Big[\sum_{I' \in T^{\mathsf{global}}(\check{I})} \tilde{\theta}_{I'}\Big]$ on $\Omega(\check{I})$, for each $I \in T^{\mathsf{global}}(\check{I})$.

Note that $\theta_I$ is defined only on $\Omega(\check{I})$. The desired properties (3)–(7) of the $\theta_I$ now follow trivially from the properties of the $\tilde{\theta}_I$ established above. Moreover, from (26)–(28), we have:

(34) Let $I \in T^{\mathsf{global}}(\check{I})$, and let $(x_1, x_2) \in \operatorname{supp}\theta_I$. Then $x_1 \in (1.01)I$, and $c[\delta_{LS}(x_1) + |x_2 - \varphi(x_1)|] \leq |I| \leq C[\delta_{LS}(x_1) + |x_2 - \varphi(x_1)|]$.

(35) Let $I_1, I_2 \in T^{\mathsf{global}}(\check{I})$, and suppose $\operatorname{supp}\theta_{I_1} \cap \operatorname{supp}\theta_{I_2} \neq \emptyset$. Then $c|I_1| \leq |I_2| \leq C|I_1|$.

(36) Any given point of $\Omega(\check{I})$ lies in $\operatorname{supp}\theta_I$ for at most C distinct $I \in T^{\mathsf{global}}(\check{I})$.

To prepare to compute the jets $J_z(\theta_I)$ for all $z \in E$, $I \in T^{\mathsf{global}}(\check{I})$, we establish the following result:

(37) Let $z = (x_1, x_2) \in E$, and let $I \in T^{\mathsf{global}}(\check{I})$. If $z \in \operatorname{supp}\tilde{\theta}_I$, then $I \in T^{\mathsf{dist}}(\check{I})$, I is of type C1, $x_1 = x_!(I)$, and $x_1 \in (1.01)I$.

Indeed, suppose $z = (x_1, x_2) \in E \cap \mathrm{supp}\, \tilde{\theta}_I$, with $I \in T^{\mathrm{global}}(\check{I})$. Then, since $z \in E$, we have $x_1 \in \bar{E}$ and $x_2 = \varphi(x_1)$; see (9) in Section 8. Since $(x_1, x_2) \in \mathrm{supp}\, \tilde{\theta}_I$ and $x_2 - \varphi(x_1) = 0$, it follows from (23) that I cannot be an internal node of $T^{\mathrm{global}}(\check{I})$. Moreover, since $(x_1, x_2) \in \mathrm{supp}\, \tilde{\theta}_I$, (22) shows that $x_1 \in (1.01)I$.

Thus, I is a leaf of $T^{\mathrm{global}}(\check{I})$, and $x_1 \in (1.01)I \cap \bar{E}$. Hence, conclusion (III) of Lemma BT2 tells us that $I \in T^{\mathrm{dist}}(\check{I})$, I is of type C1, and $\#(5I^+ \cap \bar{E}) = 1$. By definition $x_!(I)$ is the one and only element of $5I^+ \cap \bar{E}$. On the other hand, we know that $x_1 \in (1.01)I \cap \bar{E} \subset 5I^+ \cap \bar{E}$. Consequently, $x_1 = x_!(I)$, completing the proof of (37).

Thanks to (37), we have

(38) $$J_{(x_1,x_2)}\left( \sum_{I' \in T^{\mathrm{global}}(\check{I})} \tilde{\theta}_{I'} \right) = \sum_{I' \in \bigwedge(x_1)} J_{(x_1,x_2)}(\tilde{\theta}_{I'}) \text{ for each } (x_1, x_2) \in E, \text{ where}$$

(39) $\bigwedge(x_1) = \{I' \in T^{\mathrm{dist}}(\check{I}) : I' \text{ is of type C1}, x_1 = x_!(I'), \text{ and } x_1 \in (1.01)I'\}$ for each $x_1 \in \bar{E}$.

Also from (37), we have

(40) $J_{(x_1,x_2)}(\tilde{\theta}_I) = 0$ for any $I \in T^{\mathrm{global}}(\check{I}) \setminus \bigwedge(x_1)$, whenever $(x_1, x_2) \in \bar{E}$.

Regarding the set $\bigwedge(x_1)$ in (39), we note that

(41) Each $I \in \bigwedge(x_1)$ is a leaf of $T^{\mathrm{global}}(\check{I})$, and

(42) There are at most C distinct $I \in \bigwedge(x_i)$.

Here (41) and (42) hold for any $x_1 \in \bar{E}$. In fact, (41) is immediate from Proposition BT9 in Section 7, and (42) follows from (41), together with Lemma BT1 (conclusion (IV)) in that same section.

We are now ready to compute the jets of the $\theta_I$ at the points of E.

**Algorithm JPU.** ("Jets for the Partition of Unity"). Assume we have already carried out Algorithm MMBT in the section on "Marking the Basic Tree".

For each $x_1 \in \bar{E}$, we compute the set $\bigwedge(x_1) \subset T^{\mathrm{dist}}(\check{I})$ as in (39).

For each $z = (x_1, x_2) \in E \cap \Omega(\check{I})$, and for each $I \in \bigwedge(x_1)$, we compute the jet $J_z(\theta_I)$.

We have $\#(\bigwedge(x_1)) \le C$ for each $x_1 \in \bar{E}$, and $J_z(\theta_I) = 0$ whenever $z = (x_1, x_2) \in E \cap \Omega(\check{I})$, $I \notin \bigwedge(x_1)$.

The work used to carry out the above is at most CN, together with at most CN calls to the φ-Oracle. The storage used is at most CN.

*Explanation*: The assertions regarding $\#(\bigwedge(x_1))$ and $J_z(\theta_I)$ for $I \notin \bigwedge(x_1)$ are immediate from (40) and (42). We recall that Algorithm MMBT marks each node I of $T^{\mathrm{dist}}(\check{I})$ to indicate whether it is of type C1; and in case I is of type C1, then Algorithm MMBT marks I with the point $x_!(I)$.

We compute the $\bigwedge(x_1)$ for all $x_1 \in \bar{E}$ by the following obvious procedure:

First, we set all the $\bigwedge(x_1) = \emptyset$. We then loop over all the nodes $I \in T^{\mathsf{dist}}(\check{I})$. For each such I, we check whether I is of type C1, and $(1.01)I \ni x_!(I)$.

If so, then we set $x_1 := x_!(I)$, and we add I to the set $\bigwedge(x_1)$.

Thus, we can compute all the $\bigwedge(x_1)$, $x_1 \in \bar{E}$. Since $\#(T^{\mathsf{dist}}(\check{I})) \le CN$, the work and storage used to compute the $\bigwedge(x_1)$ (all $x_1 \in \bar{E}$) are also at most CN.

Next, for each $z = (x_1, x_2) \in E$, and for each $I \in \bigwedge(x_1)$, we compute the jet $J_z(\tilde{\theta}_I)$. Thanks to (41), this computation is accomplished by (18). For each such $z, I$, the computation of $J_z(\tilde{\theta}_I)$ takes work and storage at most C, together with a single appeal to the φ-Oracle.

Finally, for each $z = (x_1, x_2) \in E$, we check to see whether $z \in \Omega(\check{I})$; this holds if and only if $x_1 \in \check{I}^{\mathsf{interior}}$, since $x_2 = \varphi(x_1)$. If $z \in \Omega(\check{I})$, then, thanks to (38), we have

$$(43)\quad J_z(\theta_I) = J_z(\tilde{\theta}_I) \Big/ \sum_{I' \in \bigwedge(x_1)} J_z(\tilde{\theta}_{I'}),$$

where the division is performed in the ring of jets at $z$. (Thanks to (32), we have that $\sum_{I' \in \bigwedge(x_1)} \tilde{\theta}_{I'}(z) \ge 1$, so that (43) makes sense.)

Since $\#(\bigwedge(x_1)) \le C$, and since we have already computed all the $J_z(\tilde{\theta}_{I'})$ in (43), we can compute a single $J_z(\theta_I)$ from (43) using work and storage at most C.

Thus, we have computed all the $\bigwedge(x_1)$ $(x_1 \in \bar{E})$, and all the $J_z(\theta_I)$ $(z = (x_1, x_2) \in E \cap \Omega(\check{I}), I \in \bigwedge(x_1))$.

The work and storage used are as promised. This completes our explanation of Algorithm JPU.

We close this section by observing two simple consequences of (33), (39), (40):

(44) Let $I \in T^{\mathsf{global}}(\check{I})$, and suppose $z \in \operatorname{supp} \theta_I \cap E$. Then $I \in T^{\mathsf{dist}}(\check{I})$, I is of type C1, and $z = z_!(I)$.

More precisely,

(45) Let $z = (x_1, x_2) \in E$, and suppose $I \in \bigwedge(x_1)$. Then $I \in T^{\mathsf{dist}}(\check{I})$, I is of type C1, and $z = z_!(I)$.

(Here, we use also the definition of $z_!(I)$ in Section 9, together with assumption (9) in Section 8.)

## 11. Simplifying a convex set

In this section, we use no boiler-plate constants. We work with the standard Euclidean norm in $\mathbb{R}^D$. Our goal here is to present the following elementary algorithm:

**Algorithm SCS.** Suppose we are given the following data:

(1) A convex polyhedron $K \subset \mathbb{R}^D$, given by I constraints.

(2) A real number $A > 0$ such that $|v| \le A$ for all $v \in K$.

(3) A real number $\epsilon > 0$.

(4) A linear functional $\lambda : \mathbb{R}^D \longrightarrow \mathbb{R}$.

Then we compute a convex polyhedron $\tilde{K} \subset \mathbb{R}^D$, with the following properties:

(5) $K \subseteq \tilde{K}$.

(6) Given $\tilde{v} \in \tilde{K}$, there exists $v \in K$ such that $|v - \tilde{v}| \leq \epsilon$ and $\lambda(\tilde{v} - v) = 0$.

(7) $\tilde{K}$ is defined by at most $C(A, \epsilon, D)$ constraints.

(In particular, the number of constraints defining $\tilde{K}$ is bounded independently of I.)

The work and storage used to compute $\tilde{K}$ are at most a $C(A, \epsilon, D, I)$.

*Explanation*: We may trivially reduce matters to the case

(8) $\lambda(v_1, \ldots, v_D) = v_D$ for $(v_1, \ldots, v_D) \in \mathbb{R}^D$.

Assuming (8), we proceed as follows: From (2), we have

$$K \subset Q := \{(v_1, \ldots, v_D) \in \mathbb{R}^D : |v_1|, |v_2|, \ldots, |v_D| \leq A\}.$$

We subdivide Q into a grid of (closed) cubes $\{Q_\nu\}$, each $Q_\nu$ having diameter between $10^{-3}\epsilon$ and $\epsilon$.

For each $Q_\nu$, we compute $I_\nu := \lambda(K \cap Q_\nu)$. Each $I_\nu$ is a (possibly empty) closed interval. Define

$$K^\# = \bigcup_\nu \{ v \in Q_\nu : \lambda(v) \in I_\nu \} \subset \mathbb{R}^D.$$

Thus, $K \subset K^\# \subset \mathbb{R}^D$, but $K^\#$ needn't be convex.

Let $v^\# \in K^\#$. Then, for some $\nu$, we have $v^\# \in Q_\nu$ and $\lambda(v^\#) \in I_\nu$. By definition of $I_\nu$, there exists $v \in K \cap Q_\nu$ such that $\lambda(v) = \lambda(v^\#)$. Since $v^\#, v \in Q_\nu$, we have $|v - v^\#| \leq \epsilon$. Thus, we have proven the following:

Given $v^\# \in K^\#$, there exists $v \in K$ such that $|v - v^\#| \leq \epsilon$ and $\lambda(v - v^\#) = 0$.

We define $\tilde{K} =$ convex hull $(K^\#)$. Thus, $K \subset \tilde{K} \subset \mathbb{R}^D$.

Note that $K^\#$ is a union of at most $I^\#$ closed rectangular boxes, where $I^\#$ may be computed from $A, \epsilon, D$. Consequently, $\tilde{K}$ is a closed, convex polyhedron, defined by at most $\tilde{I}$ constraints, where $\tilde{I}$ may be computed from $A, \epsilon, D$.

Moreover, suppose $\tilde{v} \in \tilde{K}$. Then we can write

$$\tilde{v} = \sum_{j=1}^{J} t_j v_j^\#,$$

with $t_1 + \cdots + t_J = 1$, $t_j \geq 0$ (each j), and $v_j^\# \in K^\#$ (each j).

For each $v_j^\#$, there exists $v_j \in K$ such that $|v_j - v_j^\#| \leq \epsilon$ and $\lambda(v_j - v_j^\#) = 0$. Setting

$$v = \sum_{j=1}^{J} t_j v_j \in K,$$

we have $|v - \tilde{v}| \leq \epsilon$ and $\lambda(v - \tilde{v}) = 0$. Thus, $\tilde{K}$ satisfies (5), (6), (7). Moreover, one checks easily that the work and storage needed to compute $\tilde{K}$ are less than a constant computed from $A, \epsilon, D, I$. This completes our explanation of Algorithm SCS.

**Remark.** Let $\lambda, K, A, \epsilon, \tilde{K}$ be as in Algorithm SCS. Suppose we are given a point $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_D) \in \tilde{K}$. Then by (6), there exists $v = (v_1, \dots, v_D) \in K$, such that $|v_1 - \tilde{v}_1| \le \epsilon, \dots, |v_D - \tilde{v}_D| \le \epsilon$, and $\lambda(v - \tilde{v}) = 0$. In particular, $|v - \tilde{v}| \le D^{1/2}\epsilon$. Note that we can compute such a $v$ by routine linear programming, once $\tilde{v}, K, \lambda, \epsilon$ are given.

The work and storage used to compute $v$ are less than a constant computed from D and I.

## 12. Simplifying a convex set of Whitney fields

In this section, we retain the notation, assumptions, and boiler-plate constants of Section 8. We suppose we are given

(1) $\check{I} \subseteq I_0$ (dyadic), such that $\#(5\check{I} \cap \bar{E}) \ge 2$.

We suppose that we have already carried out Algorithm MMBT with input $\check{I}$; see Section 9. Thus, for each node $I \in T^{\mathsf{dist}}(\check{I})$, we have computed the set $S(I)$ and the points $z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$, $z_{\mathsf{left}}(I)$, $z_{\mathsf{rt}}(I)$.

Recall that f is defined at $z^{\mathsf{rep}}_{\mathsf{left}}(I)$ and at $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$. Also, recall the real numbers $\xi$ and $\epsilon$ from Section 8.

Under the above assumptions, we can carry out the following algorithm:

**Algorithm SCSWF.** ("Simplifying Convex Set of Whitney Fields"). Suppose we are given an interval

(2) $I \in T^{\mathsf{dist}}(\check{I})$

and a convex polyhedron

(3) $K \subset Wh(S(I))$, defined by at most NC constraints.

Assume that for every $\vec{P} = (P^z)_{z \in S(I)} \in K$, we have

(4) $\mathsf{val}(\vec{P}, z) = f(z)$ for $z = z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$;

(5) $\mathsf{val}(\partial_2 \vec{P}, z_{\mathsf{left}}(I)) = \xi$; and

(6) There exists $F \in C^2(Q(I))$ such that $\| F \|_{C^2(Q(I))} \le C$ and $J_{S(I)}(F) = \vec{P}$.

Then we compute a convex polyhedron $\tilde{K} \subset Wh(S(I))$, after which we can respond to queries. (See (10)–(15) below.) The polyhedron $\tilde{K}$ satisfies the following conditions:

(7) $K \subseteq \tilde{K}$

(8) $\tilde{K}$ is defined by at most $\widetilde{NC}$ constraints, where $\widetilde{NC}$ may be computed from $\epsilon$ and the boiler-plate constants. (In particular, $\widetilde{NC}$ is independent of NC.)

(9) Every $\vec{P} \in \tilde{K}$ satisfies (4) and (5).

(10) A "query" consists of a Whitney field $\vec{\tilde{P}} = (\tilde{P}^z)_{z \in S(I)} \in \tilde{K}$.

The response to a query (10) consists of a Whitney field

(11) $\vec{P} = (P^z)_{z\in S(I)} \in K$,

such that there exists a function $F_I^{err} \in C^2(Q(I))$, satisfying the following conditions:

(12) $J_{S(I)}(F_I^{err}) = \vec{\tilde{P}} - \vec{P}$.

(13) $|\partial^\alpha F_I^{err}| \le \epsilon^{100}|I|^{2-|\alpha|}$ on $Q(I)$, for $|\alpha| \le 2$.

(14) $F_I^{err} = 0$ at $z^{rep}_{left}(I)$, $z^{rep}_{rt}(I)$.

(15) $\partial_2 F_I^{err} = 0$ at $z_{left}(I)$ and at $z_{rt}(I)$.

In particular, given any $\vec{\tilde{P}} \in \tilde{K}$, there exist $\vec{P} \in K$ and $F_I^{err} \in C^2(Q(I))$ satisfying (11)–(15).

The work and storage used to compute $\tilde{K}$, and the work and storage used to answer a query, are less than $C(\epsilon, NC)$.

*Explanation*: Let $V$ be the vector space of all Whitney fields $\vec{P} = (P^z)_{z\in S(I)} \in Wh(S(I))$, such that: $\mathsf{val}(\vec{P}, z) = 0$ for $z = z^{rep}_{left}(I)$, $z^{rep}_{rt}(I)$; and $\mathsf{val}(\partial_2\vec{P}, z_{left}(I)) = 0$. Let $L : (x_1, x_2) \mapsto A_0 + A_1x_1 + A_2x_2$ be the one and only linear function on $\mathbb{R}^2$ such that: $L(z) = f(z)$ for $z = z^{rep}_{left}(I)$, $z^{rep}_{rt}(I)$; and $\partial_2 L = \xi$.

Next, let $\mu_1, \dots, \mu_m$ be an enumeration of the following linear functionals on $Wh(S(I))$:

$(P^z)_{z\in S(I)} \mapsto \partial^\alpha P^z(z) \big/ |I|^{2-|\alpha|}$, for $z \in S(I), |\alpha| \le 2$; and

$(P^z)_{z\in S(I)} \mapsto [\partial^\alpha(P^z - P^{z'})(z)] \big/ |z - z'|^{2-|\alpha|}$, for $z, z' \in S(I)$ distinct, $|\alpha| \le 2$.

Since the intersection of the nullspaces of $\mu_1, \dots, \mu_m$ is just $\{0\}$, we can define a Hilbert space norm on $V \subset Wh(S(I))$ by setting

$$|||\vec{P}|||^2 = \sum_{j=1}^{m} (\mu_j(\vec{P}))^2 \quad \text{for } \vec{P} \in V.$$

We may trivially identify $V$ with $\mathbb{R}^D (D = \mathsf{dim} V)$ so that the above norm agrees with the usual Euclidean norm on $\mathbb{R}^D$.

Recall that $\#(S(I)) \le C(\epsilon)$. Hence, Taylor's theorem and the classical Whitney extension theorem for finite sets tell us the following:

(16) Let $F \in C^2(Q(I))$ satisfy:

(a) $|\partial^\alpha F| \le |I|^{2-|\alpha|}$ on $Q(I)$, for $|\alpha| \le 2$;

(b) $F = 0$ at $z^{rep}_{left}(I)$, $z^{rep}_{rt}(I)$.

(c) $\partial_2 F = 0$ at $z_{left}(I)$.

Then

(d) $J_{S(I)}(F) \in V$, and $|||J_{S(I)}(F)||| \le C(\epsilon)$.

Conversely,

(17) Let $\vec{P} \in V$ satisfy $|||\vec{P}||| \leq 1$. Then there exists $F \in C^2(Q(I))$ such that:

(a) $|\partial^\alpha F| \leq C|I|^{2-|\alpha|}$ on $Q(I)$, for $|\alpha| \leq 2$; and

(b) $J_{S(I)}(F) = \vec{P}$.

Now let $\vec{P} = (P^z)_{z \in S(I)} \in K$ be given. By (4), (5) and definitions of $L$ and $V$, we have $\vec{P} - J_{S(I)}(L) = (P^z - L)_{z \in S(I)} \in V$. Moreover, let $F \in C^2(Q(I))$ be as in (6). Then (4), (5), (6) yield $F = f$ at $z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$; and $\partial_2 F(z_{\mathsf{left}}(I)) = \xi$. Since also $|\partial^\alpha F| \leq C$ on $Q(I)$ for $|\alpha| \leq 2$ (by (6)), it follows that $|\partial^\alpha(F - L)| \leq C|I|^{2-|\alpha|}$ on $Q(I)$, for $|\alpha| \leq 2$. Hence, applying (16) to the function $F - L$, we learn that $J_{S(I)}(F - L) \in V$ and $|||J_{S(I)}(F - L)||| \leq C(\epsilon)$. Recalling that $J_{S(I)}(F) = \vec{P}$ by (6), we conclude that

(18) $\vec{P} - J_{S(I)}(L) \in V$, and $|||\vec{P} - J_{S(I)}(L)||| \leq C(\epsilon)$, for all $\vec{P} \in K$.

Let $\hat{\epsilon} > 0$ be a small enough number, to be picked below. Let $\lambda : V \longrightarrow \mathbb{R}$ be the linear functional

(19) $\lambda(\vec{P}) = \mathsf{val}(\partial_2 \vec{P}, z_{\mathsf{rt}}(I))$. Define

(20) $K^{\mathsf{red}} := K - J_{S(I)}(L)$.

Thus, by (3) and (18), we have the following:

(21) $K^{\mathsf{red}} \subset V$ is a convex polyhedron defined by at most $NC$ constraints. Moreover,

(22) $|||\vec{P}||| \leq C(\epsilon)$ for each $\vec{P} \in K^{\mathsf{red}}$.

We now apply Algorithm SCS to the convex polyhedron $K^{\mathsf{red}}$, the small number $\hat{\epsilon}$, the large constant $C(\epsilon)$, and the linear functional $\lambda$. Here, we identify $V$ with $\mathbb{R}^D$ as noted above. The assumptions of Algorithm SCS hold here, thanks to (21) and (22). (See the Section 11, on "Simplifying a Convex Set".)

Thus, Algorithm SCS produces a convex polyhedron

(23) $\tilde{K}^{\mathsf{red}} \subset V$, defined by at most $\tilde{I}$ constraints, where

(24) $\tilde{I}$ may be computed from $\hat{\epsilon}, \epsilon$ and the boiler-plate constants. Moreover,

(25) $K^{\mathsf{red}} \subseteq \tilde{K}^{\mathsf{red}}$,

and the Remark at the end of Section 11 yields the following:

(26) Given $\vec{\tilde{P}}_{\mathsf{red}} \in \tilde{K}^{\mathsf{red}}$, we can compute $\vec{P}_{\mathsf{red}} \in K^{\mathsf{red}}$, such that $|||\vec{\tilde{P}}_{\mathsf{red}} - \vec{P}_{\mathsf{red}}||| \leq C(\epsilon) \cdot \hat{\epsilon}$ and $\lambda(\vec{\tilde{P}}_{\mathsf{red}} - \vec{P}_{\mathsf{red}}) = 0$.

Furthermore, the work and storage used to compute $\tilde{K}^{red}$, and to compute $\vec{P}_{red}$ in (26), are less than a constant that may be computed from $\hat{\epsilon}, \epsilon, NC$, and the boiler-plate constants.

Let $\tilde{\vec{P}}_{red}$ and $\vec{P}_{red}$ be as in (26). Then, since $\tilde{\vec{P}}_{red} \in \tilde{K}^{red} \subset V$ and $\vec{P}_{red} \in K^{red} \subset V$, we have

(27) $\mathsf{val}(\tilde{\vec{P}}_{red} - \vec{P}_{red}, z) = 0$ for $z = z^{rep}_{left}(I),\ z^{rep}_{rt}(I)$; and

(28) $\mathsf{val}(\partial_2[\tilde{\vec{P}}_{red} - \vec{P}_{red}],\ z_{left}(I)) = 0$.

Also, since $\lambda(\tilde{\vec{P}}_{red} - \vec{P}_{red}) = 0$, we have

(29) $\mathsf{val}(\partial_2[\tilde{\vec{P}}_{red} - \vec{P}_{red}],\ z_{rt}(I)) = 0$.

Putting (26)–(29) into (17), we conclude that there exists $F^{err}_I \in C^2(Q(I))$, such that

(30) $|\partial^\alpha F^{err}_I| \leq C(\epsilon) \cdot \hat{\epsilon} \cdot |I|^{2-|\alpha|}$ on $Q(I)$ for $|\alpha| \leq 2$; and

(31) $J_{S(I)}(F^{err}_I) = \tilde{\vec{P}}_{red} - \vec{P}_{red}$.

In particular, (27), (28), (29) and (31) yield

(32) $F^{err}_I(z) = 0$ for $z = z^{rep}_{left}(I),\ z^{rep}_{rt}(I)$; and

(33) $\partial_2 F^{err}_I(z) = 0$ for $z = z_{left}(I),\ z_{rt}(I)$.

Now define

(34) $\tilde{K} := \tilde{K}^{red} + J_{S(I)}(L) \subset V + J_{S(I)}(L) \subset Wh(S(I))$.

Thus, (23) and (24) show that:

(35) $\tilde{K} \subset Wh(S(I))$ is a convex polyhedron defined by at most $\widetilde{NC}$ constraints, where $\widetilde{NC}$ may be computed from $\hat{\epsilon}, \epsilon$ and the boiler-plate constants.

Moreover, in view of (20) and (34), the inclusion (25) yields

(36) $K \subseteq \tilde{K}$.

Also, (34) and the definitions of $V$ and $L$ imply the following:

(37) Every $\tilde{\vec{P}} \in \tilde{K}$ satisfies: $\mathsf{val}(\tilde{\vec{P}}, z) = f(z)$ for $z = z^{rep}_{left}(I),\ z^{rep}_{rt}(I)$; and

$\mathsf{val}(\partial_2 \tilde{\vec{P}},\ z_{left}(I)) = \xi$.

Next, suppose we are given $\tilde{\vec{P}} \in \tilde{K}$. Then we set $\tilde{\vec{P}}_{red} := \tilde{\vec{P}} - J_{S(I)}(L) \in \tilde{K}^{red}$ (see (34)).

From $\tilde{\vec{P}}_{red}$, we compute $\vec{P}_{red} \in K^{red}$ as in (26). We have seen that there exists $F^{err}_I \in C^2(Q(I))$ satisfying (30)–(33). We now set $\vec{P} := \vec{P}_{red} + J_{S(I)}(L) \in K$ (see (20)). Since $\tilde{\vec{P}} - \vec{P} = \tilde{\vec{P}}_{red} - \vec{P}_{red}$, (31) is equivalent to

(38) $J_{S(I)}(F^{err}_I) = \tilde{\vec{P}} - \vec{P}$.

Thus, given $\vec{\tilde{P}} \in \tilde{K}$, we have computed $\vec{P} \in K$ such that there exists $F_I^{err} \in C^2(Q(I))$ with the following properties:

$$\text{(39)} \quad \begin{cases} J_{S(I)}(F_I^{err}) = \vec{\tilde{P}} - \vec{P}. \\ |\partial^\alpha F_I^{err}| \le C(\epsilon) \cdot \hat{\epsilon} |I|^{2-|\alpha|} \text{ on } Q(I) \text{ for } |\alpha| \le 2. \\ F_I^{err} = 0 \text{ at } z_{left}^{rep}(I),\ z_{rt}^{rep}(I); \text{ and } \partial_2 F_I^{err} = 0 \text{ at } z_{left}(I),\ z_{rt}(I). \end{cases}$$

We now take $\hat{\epsilon}$ to be a constant of the form $c(\epsilon)$, picked small enough to guarantee that

(40) $C(\epsilon) \cdot \hat{\epsilon} < \epsilon^{100}$, with $C(\epsilon)$ as in (39).

With this choice of $\hat{\epsilon}$, the above computations produce a polyhedron $\tilde{K}$, and answer queries, as promised in Algorithm SCSWF. Indeed, (7) holds, as we have seen in (36). Also, (8) follows at once from (35), since we have taken $\hat{\epsilon} = c(\epsilon)$. Property (9) is precisely our result (37).

Regarding queries of the form (10), we see from (39), (40) that our algorithm produces $\vec{P} \in K$ as in (11), such that (12)–(15) are satisfied for some $F_I^{err} \in C^2(Q(I))$.

Thus, our algorithm computes $\tilde{K}$ and answers queries, as promised in Algorithm SCSWF.

Finally, the reader may easily check that the work and storage used to compute $\tilde{K}$ or answer a query are as promised in Algorithm SCSWF.

Our explanation of that algorithm is complete.

## 13. Computing the basic polyhedra

In this section, we adopt the notation, assumptions and boiler-plate constants of Section 8. We suppose that we are given an interval

(0) $\check{I} \subseteq I_0$ (dyadic), with $\#(5\check{I} \cap \bar{E}) \ge 2$.

We suppose that we have already carried out Algorithm MMBT from Section 9. Thus, we have computed the tree $T^{dist}(\check{I})$ and its markings. In particular, for each node $\tilde{I} \in T^{dist}(\check{I})$, we have computed a subset $S(\tilde{I}) \subset Q(\tilde{I})$, and the Whitney field $J_{S(\tilde{I})}(e_2)$. We recall that $Q(\tilde{I}') \subset Q(\tilde{I})$ for $\tilde{I}' \in RO(\tilde{I})$, and that $RO(\tilde{I})$ is the set of children of $\tilde{I}$ in the tree $T^{dist}(\check{I})$. Recall also that $\#(S(\tilde{I})) \le C(\epsilon)$ and $\#(RO(\tilde{I})) \le C$ for each $\tilde{I} \in T^{dist}(\check{I})$; and that $T^{dist}(\check{I})$ has at most $CN$ nodes.

Let $\tilde{I} \in T^{dist}(\check{I})$. We will say that a Whitney field $\vec{P} \in Wh(S(\tilde{I}))$ is "adapted" to $\tilde{I}$ if the following hold:

- $val(\vec{P}, z) = f(z)$ for $z = z_{left}^{rep}(\tilde{I})$, and for $z = z_{rt}^{rep}(\tilde{I})$.
- $val(\partial_2 \vec{P}, z_{left}(\tilde{I})) = \xi$ (see Section 8 for $\xi$).

- If $\tilde{I}$ is of type $A$ or $B$, then $\mathsf{val}(\vec{P}, z) = f(z)$ for $z = z^s_{\text{left}}(\tilde{I})$, and for $z = z^s_{\text{rt}}(\tilde{I})$.
- If $\tilde{I}$ is of type $C1$, then $\mathsf{val}(\vec{P}, z) = f(z)$ for $z = z_!(\tilde{I})$.

To understand this definition, we recall that Algorithm MMBT has marked each node $\tilde{I} \in T^{\text{dist}}(\check{I})$ as having "type" $A, B, C0$ or $C1$. If $\tilde{I}$ is of type $A$ or $B$, then we have marked $\tilde{I}$ with the points $z^s_{\text{left}}(\tilde{I})$ and $z^s_{\text{rt}}(\tilde{I})$. If $\tilde{I}$ is of type $C1$, then we have marked it with the point $z_!(\tilde{I})$. Each $\tilde{I} \in T^{\text{dist}}(\check{I})$ has been marked with the two points $z^{\text{rep}}_{\text{left}}(\tilde{I})$, $z^{\text{rep}}_{\text{rt}}(\tilde{I})$. All these points lie in the set $E$. We have also marked each $\tilde{I} \in T^{\text{dist}}(\check{I})$ with the points $z_{\text{left}}(\tilde{I})$, $z_{\text{rt}}(\tilde{I})$, which need not belong to $E$.

We are ready to present our algorithms.

**Algorithm MOK.** ("Make One $K$"): Suppose we are given the following data:

(1) A positive integer $NC$.

(2) A node $\tilde{I} \in T^{\text{dist}}(\check{I})$.

(3) For each $\tilde{I}' \in RO(\tilde{I})$, a convex polyhedron $K(\tilde{I}') \subset Wh(S(\tilde{I}'))$ defined by at most $NC$ constraints.

Assume that

(4) $\vec{P}$ is adapted to $\tilde{I}'$ for each $\vec{P} \in K(\tilde{I}')$, $\tilde{I}' \in RO(\tilde{I})$.

Then we compute a convex polyhedron

$$K(\tilde{I}) \subset Wh(S(\tilde{I}))$$

satisfying conditions (5), (6), (7) below; after which, we can respond to queries as in (8). The polyhedron $K(\tilde{I})$ satisfies the following conditions:

(5) $K(\tilde{I})$ is defined at most $C_1(\epsilon)$ constraints.

(In particular, the number of constraints in (5) is bounded independently of the number of constraints in (3).)

(6) Each $\vec{P} \in K(\tilde{I})$ is adapted to $\tilde{I}$.

(7) Let $F \in C^2(2Q(\tilde{I}))$ with norm $\leq 1$. Suppose that $J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$, and that for each $\tilde{I}' \in RO(\tilde{I})$ there exists $\lambda_{\tilde{I}'} \in \mathbb{R}$ such that $J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) \in K(\tilde{I}')$. Then $J_{S(\tilde{I})}(F) \in K(\tilde{I})$.

Conversely, we can answer queries, as follows:

(8) Given $\vec{\vec{P}} \in K(\tilde{I})$, we can compute $\vec{P}^{\tilde{I}'} \in K(\tilde{I}')$ and $\lambda_{\tilde{I}'} \in \mathbb{R}$ for each $\tilde{I}' \in RO(\tilde{I})$, such that there exist functions $F, F^{\text{err}} \in C^2(Q(\tilde{I}))$, satisfying the following conditions:

(a) $\| F \|_{C^2(Q(\tilde{I}))} \leq 1 + \epsilon$.

(b) $|\partial^\alpha F^{\text{err}}| \leq \epsilon^{100} |\tilde{I}|^{2-|\alpha|}$ on $Q(\tilde{I})$, for $|\alpha| \leq 2$.

(c) $F^{\text{err}} = 0$ at $z^{\text{rep}}_{\text{left}}(\tilde{I})$ and at $z^{\text{rep}}_{\text{rt}}(\tilde{I})$.

(d) $\partial_2 F^{\text{err}} = 0$ at $z_{\text{left}}(\tilde{I})$ and at $z_{\text{rt}}(\tilde{I})$.

(e) $J_{S(\tilde{I})}(F + F^{\text{err}}) = \vec{\vec{P}}$.

(f) $J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) = \vec{P}^{\tilde{I}'}$ for each $\tilde{I}' \in RO(\tilde{I})$.

(g) If $\tilde{I}$ is of type $A$ or $B$, then $F = f$ and $F^{err} = 0$ at $z^s_{left}(\tilde{I})$ and at $z^s_{rt}(\tilde{I})$.

(h) If $\tilde{I}$ is of type $C1$, then $F = f$ and $F^{err} = 0$ at $z_!(\tilde{I})$.

The work and storage used to compute $K(\tilde{I})$, and the work and storage used to answer a query as in (8), are less than $C(\epsilon, NC)$. We make no calls to the $\varphi$-Oracle here.

*Explanation*: Set

$$S^+ = S(\tilde{I}) \cup \bigcup_{\tilde{I}' \in RO(\tilde{I})} S(\tilde{I}').$$

Then $S^+ \subset Q(\tilde{I})$, and $\#(S^+) \le C(\epsilon)$.

Let $V$ be the vector space of all families of real numbers $(\lambda_{\tilde{I}'})_{\tilde{I}' \in RO(\tilde{I})}$ indexed by the nodes $\tilde{I}' \in RO(\tilde{I})$. (If $RO(\tilde{I})$ is empty, then $V = \{0\}$.)

Applying Algorithm AUB4 from Section 6, we obtain a convex polyhedron $K^+_{AUB} \subset Wh(S^+)$, with the following properties (see the Remark after the explanation of that algorithm):

(9) $K^+_{AUB}$ is defined by at most $C(\epsilon)$ constraints.

(10) Let $F \in C^2(2Q(\tilde{I}))$, with norm $\le 1$. Then $J_{S^+}(F) \in K^+_{AUB}$.

(11) Let $\vec{P}^+ \in K^+_{AUB}$. Then there exists $F \in C^2(Q(\tilde{I}))$ with norm $\le 1 + \epsilon$, such that $J_{S^+}(F) = \vec{P}^+$.

The work and storage used to compute $K^+_{AUB}$ are at most $C(\epsilon)$. We define $K^{++}$ to be the set of all $(\vec{P}^+, (\lambda_{\tilde{I}'})_{\tilde{I} \in RO(\tilde{I})}) \in Wh(S^+) \oplus V$, satisfying the following conditions:

- $\vec{P}^+ \in K^+_{AUB}$.
- $\vec{P}^+|_{S(\tilde{I})}$ is adapted to $\tilde{I}$.
- $\vec{P}^+|_{S(\tilde{I}')} - \lambda_{\tilde{I}'} J_{S(\tilde{I}')}(e_2) \in K(\tilde{I}')$, for each $\tilde{I}' \in RO(\tilde{I})$.

Also, we define

(12) $K = \{\vec{P}^+|_{S(\tilde{I})} : (\vec{P}^+, (\lambda_{\tilde{I}'})_{\tilde{I}' \in RO(\tilde{I})}) \in K^{++}\}$.

Then $K^{++} \subset Wh(S^+) \oplus V$ and $K \subset Wh(S(\tilde{I}))$ are convex polyhedra defined by at most $C(\epsilon, NC)$ constraints. (See (3).)

We can compute $K^{++}$ and $K$ using work and storage less than $C(\epsilon, NC)$.

Note that, by definition of $K^{++}$ and $K$, we have

(13) Each $\vec{P} \in K$ is adapted to $\tilde{I}$.

Moreover, let $\vec{P} \in K$ be given. By definition of $K$, $K^{++}$, we have $\vec{P} = \vec{P}^+|_{S(\tilde{I})}$ for some $\vec{P}^+ \in K^+_{AUB}$. From (11), we obtain a function $F \in C^2(Q(\tilde{I}))$ with norm $\le 1 + \epsilon$, such that $J_{S^+}(F) = \vec{P}^+$, and consequently, $J_{S(\tilde{I})}(F) = \vec{P}^+|_{S(\tilde{I})} = \vec{P}$.

Thus, we have proven the following:

(14) Given $\vec{P} \in K$, there exists $F \in C^2(Q(\tilde{I}))$ with norm $\leq 1+\epsilon$, such that $J_{S(\tilde{I})}(F) = \vec{P}$.

We can now apply Algorithm SCSWF, with inputs $\tilde{I}, K, C(\epsilon, NC)$ in place of $I, K, NC$. (See Section 12.) Note that assumptions (4), (5), (6) of that algorithm are satisfied here, thanks to (13) and (14).

Algorithm SCSWF computes a convex polyhedron $\tilde{K} \subset Wh(S(\tilde{I}))$; after the computation of $\tilde{K}$, we can answer queries, as explained below. The polyhedron $\tilde{K}$ satisfies the following conditions:

(15) $K \subset \tilde{K}$.

(16) $\tilde{K}$ is defined by at most $C(\epsilon)$ constraints.

Moreover, we can answer queries, as follows:

(17) A query consists of a Whitney field $\vec{\tilde{P}} \in \tilde{K}$.

(18) The response to a query $\vec{\tilde{P}}$ consists of a Whitney field $\vec{P} \in K$, such that there exists a function $F^{err} \in C^2(Q(\tilde{I}))$, for which the following hold:

(a) $J_{S(\tilde{I})}(F^{err}) = \vec{\tilde{P}} - \vec{P}$.

(b) $|\partial^\alpha F^{err}| \leq \epsilon^{100}|\tilde{I}|^{2-|\alpha|}$ on $Q(\tilde{I})$, for $|\alpha| \leq 2$.

(c) $F^{err} = 0$ at $z^{rep}_{left}(\tilde{I})$ and at $z^{rep}_{rt}(\tilde{I})$.

(d) $\partial_2 F^{err} = 0$ at $z_{left}(\tilde{I})$ and at $z_{rt}(\tilde{I})$.

In particular, given $\vec{\tilde{P}} \in \tilde{K}$, there exist $\vec{P} \in K$ and $F^{err} \in C^2(Q(\tilde{I}))$ satisfying (18)(a)–(d).

The work and storage used to compute $\tilde{K}$ from $K$, and the work and storage used to answer a query as in (17), (18), are less than $C(\epsilon, NC)$.

Finally, we set

(19) $K(\tilde{I}) = \{\vec{\tilde{P}} \in \tilde{K} : \vec{\tilde{P}} \text{ is adapted to } \tilde{I}\}$.

Thus, we have computed $K(\tilde{I})$, as promised in Algorithm MOK.

Let us check that $K(\tilde{I})$ satisfies (5), (6), (7), and then pass to the query algorithm (8). First of all, (5) is immediate from (16) and (19); and (6) is immediate from the definition (19).

We check (7). Thus, let $F \in C^2(2Q(\tilde{I}))$ with norm $\leq 1$. Suppose that $J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$, and that $J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) \in K(\tilde{I}')$ for each $\tilde{I}' \in RO(\tilde{I})$. We must show that $J_{S(\tilde{I})}(F) \in K(\tilde{I})$.

From (10), we see that the Whitney field $\vec{P}^+ = J_{S^+}(F)$ belongs to $K^+_{AUB}$.

Also, $\vec{P}^+|_{S(\tilde{I})} = J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$, by assumption. Furthermore, for each $\tilde{I}' \in \mathrm{RO}(\tilde{I})$, we have

$$\vec{P}^+|_{S(\tilde{I}')} - \lambda_{\tilde{I}'} J_{S(\tilde{I}')}(e_2) = J_{S(\tilde{I}')}(F) - \lambda_{\tilde{I}'} J_{S(\tilde{I}')}(e_2) = J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) \in K(\tilde{I}'),$$

again by assumption. Comparing the above remarks with the definition of $K^{++}$, we see that $(J_{S^+}(F), (\lambda_{\tilde{I}'})_{\tilde{I}' \in \mathrm{RO}(\tilde{I})}) \in K^{++}$, and consequently, (12) and (15) imply that $J_{S(\tilde{I})}(F) \in \tilde{K}$. Since also $J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$ (by assumption), we see from (19) that $J_{S(\tilde{I})}(F) \in K(\tilde{I})$, completing the proof of (7).

Thus, we have proven (5), (6), (7) for our polyhedron $K(\tilde{I})$. One checks easily that the work and storage used to compute $K(\tilde{I})$ as above are at most $C(\epsilon, NC)$. Moreover, we have made no use of the $\varphi$-Oracle here. We now provide the query algorithm (8).

Thus, let $\vec{\tilde{P}} \in K(\tilde{I})$ be given. By definition (19), we have

(20) $\vec{\tilde{P}} \in \tilde{K}$, and

(21) $\vec{\tilde{P}}$ is adapted to $\tilde{I}$.

Applying the query algorithm (17), (18), we compute a Whitney field $\vec{P} \in K$ such that there exists $F^{\mathrm{err}} \in C^2(Q(\tilde{I}))$ satisfying (18)(a)–(d). Let us fix such an $F^{\mathrm{err}}$. The work and storage used to compute $\vec{P}$ are at most $C(\epsilon, NC)$. We now recall the definition (12). By routine linear programming, we can compute a point

(22) $(\vec{P}^+, (\lambda_{\tilde{I}'})_{\tilde{I}' \in \mathrm{RO}(\tilde{I})}) \in K^{++}$, such that

(23) $\vec{P}^+|_{S(\tilde{I})} = \vec{P}$.

Since $K^{++}$ is defined by at most $C(\epsilon, NC)$ constraints, the work and storage used to compute the point (22) are at most $C(\epsilon, NC)$.

Comparing (22) with the definition of $K^{++}$, we see that the following hold:

(24) $\vec{P}^+ \in K^+_{\mathrm{AUB}}$.

(25) $\vec{P}^+|_{S(\tilde{I})}$ is adapted to $\tilde{I}$.

(26) $\vec{P}^+|_{S(\tilde{I}')} - \lambda_{\tilde{I}'} J_{S(\tilde{I}')}(e_2) \in K(\tilde{I}')$ for each $\tilde{I}' \in \mathrm{RO}(\tilde{I})$.

Let us define

(27) $\vec{P}^{\tilde{I}'} = \vec{P}^+|_{S(\tilde{I}')} - \lambda_{I'} J_{S(\tilde{I}')}(e_2)$ for $\tilde{I}' \in \mathrm{RO}(\tilde{I})$.

Thus, $\vec{P}^{\tilde{I}'} \in K(\tilde{I}')$ for $\tilde{I}' \in \mathrm{RO}(\tilde{I})$, as asserted in (8). Since the $J_{S(\tilde{I}')}(e_2)$ have been precomputed by Algorithm MMBT, the work and storage used to compute the $\vec{P}^{\tilde{I}'}$ from (27) are at most $C(\epsilon)$.

Next, note that (11) and (24) show that there exists

(28) $F \in C^2(Q(\tilde{I}))$ with norm $\leq 1 + \epsilon$, such that

(29) $J_{S^+}(F) = \vec{P}^+$.

Thus, starting from our given $\vec{\tilde{P}} \in K(\tilde{I})$, we have computed $\vec{P}^{\tilde{I}'} \in K(\tilde{I}')$ and $\lambda_{\tilde{I}'} \in \mathbb{R}$, for each $\tilde{I}' \in RO(\tilde{I})$; and we have defined the functions $F, F^{err} \in C^2(Q(\tilde{I}))$.

One checks easily that the work and storage used to compute the $\vec{P}^{\tilde{I}'}$ and $\lambda_{\tilde{I}'}$ from $\vec{\tilde{P}}$ are at most $C(\epsilon, NC)$. Moreover, we have made no calls to the $\varphi$-Oracle here.

It remains to show that $\vec{\tilde{P}}, \vec{P}^{\tilde{I}'}$, $\lambda_{\tilde{I}'}$ satisfy (8)(a)–(h). Let us check that these assertions are correct. In fact, (8)(a) is just our result (28); and (8)(b), (c), (d) hold, because our $F^{err}$ satisfies (18)(b), (c), (d). To check (8)(e), we note that

$$J_{S(\tilde{I})}(F + F^{err}) = \vec{P}^{+}\big|_{S(\tilde{I})} + J_{S(\tilde{I})}(F^{err}) \overset{\text{(by (29))}}{=} \vec{P} + (\vec{\tilde{P}} - \vec{P}) \overset{\text{(by (18)(a) and (23))}}{=} \vec{\tilde{P}}.$$

Thus, (8)(e) holds.

Next, (27) and (29) show that

$$J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) = \vec{P}^{+}\big|_{S(\tilde{I}')} - \lambda_{\tilde{I}'} J_{S(\tilde{I}')}(e_2) = \vec{P}^{\tilde{I}'}$$

for each $\tilde{I}' \in RO(\tilde{I})$, proving (8)(f).

Finally, to check (8)(g) and (8)(h), we argue as follows: From (25) and (29), we see that $J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$. From (21) and (8)(e) (which we have already proven), we see that $J_{S(\tilde{I})}(F + F^{err})$ is adapted to $\tilde{I}$. Assertions (8)(g) and (8)(h) follow trivially from the above remarks and the definition of "adapted".

This completes our explanation of Algorithm MOK.

**Algorithm MAK.** ("Make All K's"): For each $\tilde{I} \in T^{dist}(\check{I})$, we compute a convex polyhedron $K(\tilde{I}) \subset Wh(S(\tilde{I}))$ satisfying the following conditions:

(30) $K(\tilde{I})$ is defined by at most $C(\epsilon)$ constraints.

(31) Each $\vec{P} \in K(\tilde{I})$ is adapted to $\tilde{I}$.

(32) Let $F \in C^2(2Q(\tilde{I}))$. Suppose that the following hold:

  (a) $\| F \|_{C^2(2Q(\tilde{I}))} \leq 1 - C_1\epsilon$ (for large enough $C_1$).

  (b) $F = f$ on $E \cap Q(\tilde{I})$.

  (c) $\partial_2 F(z_{left}(\tilde{I})) = \xi$.

  Then $J_{S(\tilde{I})}(F) \in K(\tilde{I})$.

Moreover, after we have computed all the $K(\tilde{I})$, we can answer queries as follows:

(33) A query consists of a Whitney field $\vec{P}^{\check{I}} \in K(\check{I})$.

(34) The response to a query (33) consists of a family of Whitney fields $\vec{P}^{\tilde{I}} \in K(\tilde{I})$ and real numbers $\lambda(\tilde{I})$ (all $\tilde{I} \in T^{dist}(\check{I})$), such that there exist functions $F_{\tilde{I}}, F^{err}_{\tilde{I}} \in C^2(Q(\tilde{I}))$ (all $\tilde{I} \in T^{dist}(\check{I})$) for which the following hold:

  (a) $\lambda(\check{I}) = 0$ and $\vec{P}^{\check{I}}$ is the given Whitney field from (33).

Moreover, for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we have:

(b) $J_{S(\tilde{I})}(F_{\tilde{I}} + F^{\mathsf{err}}_{\tilde{I}}) = \vec{P}^{\tilde{I}}$

(c) $J_{S(\tilde{I}')}(F_{\tilde{I}} + [\lambda(\tilde{I}) - \lambda(\tilde{I}')]e_2) = \vec{P}^{\tilde{I}'}$ for each $\tilde{I}' \in RO(\tilde{I})$.

(d) $\| F_{\tilde{I}} \|_{C^2(Q(\tilde{I}))} \leq 1 + \epsilon$.

(e) $|\partial^\alpha F^{\mathsf{err}}_{\tilde{I}}| \leq \epsilon^{100} |\tilde{I}|^{2-|\alpha|}$ on $Q(\tilde{I})$, for $|\alpha| \leq 2$.

(f) $F^{\mathsf{err}}_{\tilde{I}} = 0$ at $z^{\mathsf{rep}}_{\mathsf{left}}(\tilde{I})$ and at $z^{\mathsf{rep}}_{\mathsf{rt}}(\tilde{I})$.

(g) $\partial_2 F^{\mathsf{err}}_{\tilde{I}} = 0$ at $z_{\mathsf{left}}(\tilde{I})$ and at $z_{\mathsf{rt}}(\tilde{I})$.

(h) If $\tilde{I}$ is of type $A$ or B, then $F_{\tilde{I}} = f$ and $F^{\mathsf{err}}_{\tilde{I}} = 0$ at $z^{s}_{\mathsf{left}}(\tilde{I})$ and at $z^{s}_{\mathsf{rt}}(\tilde{I})$.

(i) If $\tilde{I}$ is of type C1, then $F_{\tilde{I}} = f$ and $F^{\mathsf{err}}_{\tilde{I}} = 0$ at $z_!(\tilde{I})$.

In particular, for any $\vec{P}^{\check{I}} \in K(\check{I})$, there exist $\vec{P}^{\tilde{I}} \in K(\tilde{I})$, $\lambda(\tilde{I}) \in \mathbb{R}$ and functions $F_{\tilde{I}}, F^{\mathsf{err}}_{\tilde{I}}$, satisfying (34)(a)–(i).

The work and storage used to compute all the $K(\tilde{I})$, and the work and storage used to answer a query as in (33), (34), are at most $C(\epsilon)N$. We make no calls to the $\varphi$-Oracle.

*Explanation*: In our explanation of Algorithm MAK, the expression $C_1(\epsilon)$ will always denote the constant $C_1(\epsilon)$ in (5).

By bottom-up recursion in the tree $T^{\mathsf{dist}}(\check{I})$, we define a convex polyhedron $K(\tilde{I})$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, such that

(35) $K(\tilde{I}) \subset Wh(S(\tilde{I}))$ is defined by at most $C_1(\epsilon)$ constraints, and each $\vec{P} \in K(\tilde{I})$ is adapted to $\tilde{I}$.

Given $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we make the inductive assumption that such polyhedra $K(\tilde{I}')$ have already been computed for all $\tilde{I}' \in RO(\tilde{I})$. Taking $NC := C_1(\epsilon)$, we see that assumptions (1)–(4) hold.

Accordingly, we perform Algorithm MOK, to produce a polyhedron $K(\tilde{I}) \subset Wh(S(\tilde{I}))$, again satisfying (35), as well as (7) and (8). Thus, we compute all the $K(\tilde{I})$ ($\tilde{I} \in T^{\mathsf{dist}}(\check{I})$), and we know that (30), (31) are satisfied. Note that the constant $C(\epsilon)$ in (30) does not grow as we proceed recursively up the tree $T^{\mathsf{dist}}(\check{I})$.

The work and storage used to compute a single $K(\tilde{I})$ from Algorithm MOK are at most $C(\epsilon)$. Since $T^{\mathsf{dist}}(\check{I})$ has at most $CN$ nodes, the total work and storage used to compute all the $K(\tilde{I})$ are at most $C(\epsilon)N$. Moreover, once we have computed all the $K(\tilde{I})$, we can answer queries as follows, for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$. (See (8).)

(36) Given $\vec{P}^{\tilde{I}} \in K(\tilde{I})$, we can compute $\vec{P}^{\tilde{I}'} \in K(\tilde{I}')$ and $\lambda(\tilde{I}, \tilde{I}') \in \mathbb{R}$ for each $\tilde{I}' \in RO(\tilde{I})$, such that there exist functions

(a) $F_{\tilde{I}}, F^{\mathsf{err}}_{\tilde{I}} \in C^2(Q(\tilde{I}))$, satisfying the following:

(b) $J_{S(\tilde{I})}(F_{\tilde{I}} + F^{\mathsf{err}}_{\tilde{I}}) = \vec{P}^{\tilde{I}}$.

(c) $J_{S(\tilde{I}')}(F_{\tilde{I}} - \lambda(\tilde{I}, \tilde{I}')e_2) = \vec{P}^{\tilde{I}'}$ for each $\tilde{I}' \in RO(\tilde{I})$.

(d) $\| F_{\tilde{I}} \|_{C^2(Q(\tilde{I}))} \leq 1 + \epsilon$.

(e) $|\partial^\alpha F^{err}_{\tilde{I}}| \leq \epsilon^{100} |\tilde{I}|^{2-|\alpha|}$ on $Q(\tilde{I})$, for $|\alpha| \leq 2$.

(f) $F^{err}_{\tilde{I}} = 0$ at $z^{rep}_{left}(\tilde{I})$, and at $z^{rep}_{rt}(\tilde{I})$.

(g) $\partial_2 F^{err}_{\tilde{I}} = 0$ at $z_{left}(\tilde{I})$ and at $z_{rt}(\tilde{I})$.

(h) If $\tilde{I}$ is of type $A$ or $B$, then $F_{\tilde{I}} = f$ and $F^{err}_{\tilde{I}} = 0$ at $z^s_{left}(\tilde{I})$ and at $z^s_{rt}(\tilde{I})$.

(i) If $\tilde{I}$ is of type $C1$, then $F_{\tilde{I}} = f$ and $F^{err}_{\tilde{I}} = 0$ at $z_!(\tilde{I})$.

The work and storage used to produce the $\vec{P}^{\tilde{I}'}$ and $\lambda(\tilde{I}, \tilde{I}')$ from $\vec{P}^{\tilde{I}}$ are at most $C(\epsilon)$. Furthermore, our $K(\tilde{I})$ satisfy (7).

Our next task is to prove (32). To do so, let $A$ be a large enough constant, to be fixed below. (Later, we will take $A$ to be a large enough controlled constant $C'$, but not yet.) We recall that each $\tilde{I} \in T^{dist}(\check{I})$ satisfies $\tilde{I} \subseteq \check{I} \subseteq I_0$. (See (0).)

By induction on $\tilde{I} \in T^{dist}(\check{I})$, we will prove the following:

(37) Let $F \in C^2(2Q(\tilde{I}))$. Suppose that the following hold:

(a) $\| F \|_{C^2(2Q(\tilde{I}))} \leq 1 - A\epsilon \frac{|\tilde{I}|}{|I_0|}$.

(b) $F = f$ on $E \cap Q(\tilde{I})$.

(c) $\partial_2 F(z_{left}(\tilde{I})) = \xi$.

Then $J_{S(\tilde{I})}(F) \in K(\tilde{I})$.

Indeed, let us fix $\tilde{I} \in T^{dist}(\check{I})$ and assume that (37) holds for each $\tilde{I}' \in RO(\tilde{I})$. We will then prove (37) for the given $\tilde{I}$.

Thus, let $F \in C^2(2Q(\tilde{I}))$, and assume (37)(a), (b), (c). From (37)(a) and the Bounded Distortion Property, we have $|\partial^\alpha F| \leq C$ on $2Q(\tilde{I})$ for $|\alpha| \leq 2$.

Since also $Q(\tilde{I})$ has diameter at most $C|\tilde{I}|$, and since $z_{left}(\tilde{I}), z_{left}(\tilde{I}') \in Q(\tilde{I})$ for $\tilde{I}' \in RO(\tilde{I})$, it follows that $|\partial_2 F(z_{left}(\tilde{I})) - \partial_2 F(z_{left}(\tilde{I}'))| \leq C|\tilde{I}|$. Hence, by (37)(c), the number

(38) $\lambda_{\tilde{I}'} := \partial_2 F(z_{left}(\tilde{I}')) - \xi$

satisfies

(39) $|\lambda_{\tilde{I}'}| \leq C|\tilde{I}|$ for each $\tilde{I}' \in RO(\tilde{I})$.

Now fix $\tilde{I}' \in RO(\tilde{I})$, and define

(40) $\tilde{F} := F - \lambda_{\tilde{I}'} e_2 \in C^2(2Q(\tilde{I}'))$.

We recall that $e_2 = 0$ on $E$, $\partial_2 e_2 = 1$ on $2Q(\tilde{I}')$, and

(41) $\| e_2 \|_{C^2(2Q(\tilde{I}'))} \leq C\epsilon |I_0|^{-1}$.

(See (22)–(24) and (31) in Section 8.)

Consequently,

(42) $\tilde{F} = f$ on $E \cap Q(\tilde{I}')$, thanks to (37)(b); and

(43) $\partial_2 \tilde{F}(z_{\text{left}}(\tilde{I}')) = \xi$, by (38) and (40).

Moreover, from (37)(a), (39), (40) and (41), we obtain the estimate

(44) $\| \tilde{F} \|_{C^2(2Q(\tilde{I}'))} \leq 1 - \frac{A\epsilon|\tilde{I}|}{|I_0|} + \frac{C\epsilon|\tilde{I}|}{|I_0|}$.

If we take $A$ to satisfy

(45) $A \geq 2C$, with $C$ as in (44),

then we obtain from (44) the estimate

(46) $\| \tilde{F} \|_{C^2(2Q(\tilde{I}'))} \leq 1 - \frac{1}{2} A\epsilon \frac{|\tilde{I}|}{|I_0|}$.

However, since $\tilde{I}' \in RO(\tilde{I})$, we know that $\tilde{I}'$ is a proper dyadic subinterval of $\tilde{I}$, and therefore $|\tilde{I}'| \leq \frac{1}{2}|\tilde{I}|$. Hence (46) yields

(47) $\| \tilde{F} \|_{C^2(2Q(\tilde{I}'))} \leq 1 - \frac{A\epsilon|\tilde{I}'|}{|I_0|}$.

We now pick $A$ to be a controlled constant $C'$, large enough to satisfy (45). Thus, (47) holds.

Our inductive assumption tells us that (37) holds for $\tilde{I}'$. Moreover, the function $\tilde{F}$ satisfies (37)(a), (b), (c), with $\tilde{I}'$ in place of $\tilde{I}$, as we see from (42), (43) and (47). Consequently, $J_{S(\tilde{I}')}(\tilde{F}) \in K(\tilde{I}')$, i.e.,

(48) $J_{S(\tilde{I}')}(F - \lambda_{\tilde{I}'} e_2) \in K(\tilde{I}')$.

(See (40).) We have proven (48) for each $\tilde{I}' \in RO(\tilde{I})$.

Recall that we have assumed that $F \in C^2(2Q(\tilde{I}))$ satisfies (37)(a), (b), (c). Comparing the definition of "adapted to $\tilde{I}$" with (37)(b), (c), and see that

(49) $J_{S(\tilde{I})}(F)$ is adapted to $\tilde{I}$.

Also from (37)(a), we have

(50) $\| F \|_{C^2(2Q(\tilde{I}))} \leq 1$.

Recall that our polyhedron $K(\tilde{I})$ satisfies (7). Therefore, from (48), (49), (50), we conclude that

(51) $J_{S(\tilde{I})}(F) \in K(\tilde{I})$.

We have thus shown that every $F \in C^2(2Q(\tilde{I}))$ satisfying (37)(a), (b), (c) must also satisfy (51).

This completes our inductive proof of (37).

Since we have picked $A$ in (37) to be a controlled constant $C'$, and since $|\tilde{I}| \le |I_0|$ for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, assertion (32) now follows from (37). We take up the query algorithm (33), (34).

Let $\vec{P}^{\check{I}} \in K(\check{I})$ be given, as in (33). By top-down recursion on $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we compute for each such $\tilde{I}$ a Whitney field $\vec{P}^{\tilde{I}} \in K(\tilde{I})$ and real numbers $\lambda(\tilde{I}, \tilde{I}')$ indexed by $\tilde{I}' \in RO(\tilde{I})$. Also, for each such $\tilde{I}$, we define functions $F_{\tilde{I}}$, $F^{\mathsf{err}}_{\tilde{I}} \in C^2(Q(\tilde{I}))$.

The recursion proceeds as follows: Suppose we have already computed $\vec{P}^{\tilde{I}}$ for a given $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$. Applying the query algorithm (36), we compute $\vec{P}^{\tilde{I}'} \in K(\tilde{I}')$ and $\lambda(\tilde{I}, \tilde{I}') \in \mathbb{R}$ for each $\tilde{I}' \in RO(\tilde{I})$. These are such that there exist functions $F_{\tilde{I}}, F^{\mathsf{err}}_{\tilde{I}} \in C^2(Q(\tilde{I}))$, satisfying (36)(a)–(i). We fix such functions $F_{\tilde{I}}, F^{\mathsf{err}}_{\tilde{I}}$.

Since we are given $\vec{P}^{\check{I}}$ to start with, the above recursion computes all the $\vec{P}^{\tilde{I}}$ (for $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$) and $\lambda(\tilde{I}, \tilde{I}')$ (for $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ and $\tilde{I}' \in RO(\tilde{I})$), and defines all the $F_{\tilde{I}}$, $F^{\mathsf{err}}_{\tilde{I}}$ (for $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$).

Each application of (36) uses work and storage at most $C(\epsilon)$. Since the tree $T^{\mathsf{dist}}(\check{I})$ has at most $CN$ nodes, the total work and storage used to compute all the $\vec{P}^{\tilde{I}}$ and $\lambda(\tilde{I}, \tilde{I}')$ are at most $C(\epsilon)N$.

Our Whitney fields $\vec{P}^{\tilde{I}}$, numbers $\lambda(\tilde{I}, \tilde{I}')$, and functions $F_{\tilde{I}}$, $F^{\mathsf{err}}_{\tilde{I}}$ satisfy (36)(a)–(i).

Comparing (36)(a)–(i) with (34)(a)–(i) we see that (34)(a)–(i) hold, provided the real numbers $\lambda(\tilde{I})$ ($\tilde{I} \in T^{\mathsf{dist}}(\check{I})$) satisfy the following conditions:

(52) $\lambda(\check{I}) = 0$.

(53) $\lambda(\tilde{I}') - \lambda(\tilde{I}) = \lambda(\tilde{I}, \tilde{I}')$ for $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, $\tilde{I}' \in RO(\tilde{I})$.

However, since the $\lambda(\tilde{I}, \tilde{I}')$ have already been computed, an obvious top-down recursion in the tree $T^{\mathsf{dist}}(\check{I})$ computes numbers $\lambda(\tilde{I})$ (all $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$) satisfying (52) and (53).

The work and storage used to compute all the $\lambda(\tilde{I})$ are at most $CN$.

Thus, we have computed $\vec{P}^{\tilde{I}} \in K(\tilde{I})$ and $\lambda(\tilde{I}) \in \mathbb{R}$, for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$; and we have defined functions $F_{\tilde{I}}, F^{\mathsf{err}}_{\tilde{I}} \in C^2(Q(\tilde{I}))$, such that (34)(a)–(i) are satisfied. Moreover, the work and storage used to perform the above computations are at most $C(\epsilon)N$.

This completes our explanation of the query algorithm (33), (34). Our explanation of Algorithm MAK is also complete.

## 14. Local interpolants

In this section, we adopt the notation, assumptions and boiler-plate constants of Section 8. We suppose we are given an interval

(0) $\check{I} \subseteq I_0$ (dyadic), with $\#(5\check{I} \cap \bar{E}) \ge 2$.

We suppose that we have carried out Algorithm MMBT from Section 9, and the one-time work of Algorithm MAK from Section 13. Thus, for each node $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we have computed the convex polyhedron $K(\tilde{I})$.

Finally, we suppose that we are given a Whitney field

(1) $\vec{P}^{\check{I}} \in K(\check{I})$.

Using the query algorithm within Algorithm MAK, we obtain from $\vec{P}^{\check{I}}$ a family of Whitney fields $\vec{P}^{\tilde{I}}$ and real numbers $\lambda(\tilde{I})$ (each $\tilde{I} \in T^{dist}(\check{I})$), for which there exist functions $F_{\tilde{I}}, F^{err}_{\tilde{I}} \in C^2(Q(\tilde{I}))$ (each $\tilde{I} \in T^{dist}(\check{I})$), such that the following hold:

(2) For $\tilde{I} = \check{I}$, we have $\lambda(\tilde{I}) = 0$ and $\vec{P}^{\tilde{I}} = \vec{P}^{\check{I}}$ as in (1).

For each $\tilde{I} \in T^{dist}(\check{I})$, the following hold:

(3) $\vec{P}^{\tilde{I}} \in K(\tilde{I})$; in particular, $\vec{P}^{\tilde{I}}$ is adapted to $\tilde{I}$.

(4) $J_{S(\tilde{I})}(F_{\tilde{I}} + F^{err}_{\tilde{I}}) = \vec{P}^{\tilde{I}}$.

(5) $J_{S(\tilde{I}')}(F_{\tilde{I}} + [\lambda(\tilde{I}) - \lambda(\tilde{I}')]e_2) = \vec{P}^{\tilde{I}'}$ for each $\tilde{I}' \in RO(\tilde{I})$.

(6) $\| F_{\tilde{I}} \|_{C^2(Q(\tilde{I}))} \leq 1 + \epsilon$.

(7) $|\partial^\alpha F^{err}_{\tilde{I}}| \leq \epsilon^{100} |\tilde{I}|^{2-|\alpha|}$ on $Q(\tilde{I})$, for $|\alpha| \leq 2$.

(8) $F^{err}_{\tilde{I}} = 0$ at $z^{rep}_{left}(\tilde{I})$ and at $z^{rep}_{rt}(\tilde{I})$.

(9) $\partial_2 F^{err}_{\tilde{I}} = 0$ at $z_{left}(\tilde{I})$ and at $z_{rt}(\tilde{I})$.

(10) If $\tilde{I}$ is of type $A$ or B, then $F_{\tilde{I}} = f$ and $F^{err}_{\tilde{I}} = 0$ at $z^s_{left}(\tilde{I})$ and at $z^s_{rt}(\tilde{I})$.

(11) If $\tilde{I}$ is of type C1, then $F_{\tilde{I}} = f$ and $F^{err}_{\tilde{I}} = 0$ at $z_!(\tilde{I})$.

Fix functions $F_{\tilde{I}}$, $F^{err}_{\tilde{I}}$ as above. For each $I \in T^{global}(\check{I})$, we will define a function $F^\#_I \in C^2(Q(I))$. To do so, we recall from Section 7 the following facts about the trees $T^{global}(\check{I})$, $T^{dist}(\check{I})$, $T^{loc}(\tilde{I})$:

(12) The children of a given node $\tilde{I}$ in the tree $T^{dist}(\check{I})$ are precisely the intervals $\tilde{I}' \in RO(\tilde{I})$.

(13) $T^{global}(\check{I}) \setminus \{\check{I}\}$ is the disjoint union of $T^{loc}(\tilde{I}) \setminus \{\tilde{I}\}$ over all $\tilde{I} \in T^{dist}(\check{I})$.

(14) Let $I \in T^{loc}(\tilde{I}) \cap T^{loc}(\tilde{I}')$, with $\tilde{I}, \tilde{I}' \in T^{dist}(\check{I})$ distinct.
Then either $I = \tilde{I}' \in RO(\tilde{I})$, or $I = \tilde{I} \in RO(\tilde{I}')$.

(15) Let $I$ be an internal node in the tree $T^{global}(\check{I})$, and let $I_1, I_2$ be its two dyadic children. Then there exists $\tilde{I} \in T^{dist}(\check{I})$ such that $I, I_1, I_2 \in T^{loc}(\tilde{I})$.

In particular, from (13), (14), we obtain the following:

(16) For any $\tilde{I} \in T^{dist}(\check{I})$, we have $T^{dist}(\check{I}) \cap T^{loc}(\tilde{I}) = \{\tilde{I}\} \cup RO(\tilde{I})$.

(17) Any $I \in T^{global}(\check{I}) \setminus T^{dist}(\check{I})$ belongs to $T^{loc}(\tilde{I})$ for one and only one $\tilde{I} \in T^{dist}(\check{I})$.

We are now ready to define the functions $F^\#_I$. Let $I \in T^{global}(\check{I})$.

Case 1: If $I \in T^{dist}(\check{I})$, then we define $F^\#_I = F_I + F^{err}_I + \lambda(I)e_2$.

Case 2: Suppose $I \notin T^{dist}(\check{I})$. Then $I \in T^{loc}(\tilde{I})$ for one and only one $\tilde{I} \in T^{dist}(\check{I})$. We define $F_I^{\#} = [F_{\tilde{I}} + \lambda(\tilde{I})e_2]\big|_{Q(I)}$.

In either case, we have

(18) $F_I^{\#} \in C^2(Q(I))$ for $I \in T^{global}(\check{I})$.

Observe that we have defined $F_I^{\#}$ for each $I \in T^{global}(\check{I})$, but we haven't computed it. There are an uncontrolled number of nodes $I \in T^{global}(\check{I})$, so we cannot afford to make computations for each such I.

The goal of this section is to establish the basic properties of the functions $F_I^{\#}$. To do so, we first prepare to estimate the numbers $\lambda(I)$ in (2)–(11). Recall from (3) and the definition of "adapted" that the following hold, for each $\tilde{I} \in T^{dist}(\check{I})$:

(19) $\mathrm{val}(\vec{P}^{\tilde{I}}, z) = f(z)$ for $z = z^{rep}_{left}(\tilde{I})$ and for $z = z^{rep}_{rt}(\tilde{I})$.

(20) $\mathrm{val}(\partial_2 \vec{P}^{\tilde{I}}, z_{left}(\tilde{I})) = \xi$.

(21) $\mathrm{val}(\vec{P}^{\tilde{I}}, z) = f(z)$ for $z = z^{s}_{left}(\tilde{I})$ and for $z = z^{s}_{rt}(\tilde{I})$, if $\tilde{I}$ is of type A or B.

(22) $\mathrm{val}(\vec{P}^{\tilde{I}}, z) = f(z)$ for $z = z_!(\tilde{I})$, if $\tilde{I}$ is of type C1.

We are ready to estimate the $\lambda(\tilde{I})$.

**Lemma 1.** *We have*

(23) $|\lambda(\tilde{I}) - \lambda(\tilde{I}')| \le C|\tilde{I}|$ *for each* $\tilde{I} \in T^{dist}(\check{I})$ *and* $\tilde{I}' \in RO(\tilde{I})$.

*Moreover,*

(24) $|\lambda(\tilde{I})| \le C|\check{I}|$ *for each* $\tilde{I} \in T^{dist}(\check{I})$.

*Proof.* Let $\tilde{I} \in T^{dist}(\check{I})$, $\tilde{I}' \in RO(\tilde{I})$. Since $\partial_2 e_2 \equiv 1$ (see Section 8), we learn from (5) and (20) that

$$\xi = \partial_2 F_{\check{I}}(z_{left}(\tilde{I}')) + [\lambda(\tilde{I}) - \lambda(\tilde{I}')].$$

Similarly, (4), (9) and (20) yield $\xi = \partial_2 F_{\check{I}}(z_{left}(\tilde{I}))$. Consequently,

(25) $\lambda(\tilde{I}) - \lambda(\tilde{I}') = \partial_2 F_{\check{I}}(z_{left}(\tilde{I})) - \partial_2 F_{\check{I}}(z_{left}(\tilde{I}'))$.

Recall that $z_{left}(\tilde{I}) \in Q(\tilde{I})$, $z_{left}(\tilde{I}') \in Q(\tilde{I}') \subset Q(\tilde{I})$, and that $Q(\tilde{I})$ has diameter less than $C|\tilde{I}|$. Hence, from (6), we have

(26) $|\partial_2 F_{\check{I}}(z_{left}(\tilde{I})) - \partial_2 F_{\check{I}}(z_{left}(\tilde{I}'))| \le C|\tilde{I}|$.

Assertion (23) now follows from (25) and (26).

We turn to assertion (24). Let $\tilde{I} \in T^{dist}(\check{I})$ be given. Passing from the root $\check{I}$ down to $\tilde{I}$ in the tree $T^{dist}(\check{I})$, we obtain a finite sequence $\tilde{I}_0, \tilde{I}_1, \dots, \tilde{I}_L$, with $\tilde{I}_0 = \check{I}$, $\tilde{I}_L = \tilde{I}$, and $\tilde{I}_{\ell+1} \in RO(\tilde{I}_\ell)$ for $0 \le \ell < L$.

In particular, $\tilde{I}_{\ell+1}$ is a proper dyadic subinterval of $\tilde{I}_\ell$, hence

(27) $|\tilde{I}_{\ell+1}| \le \frac{1}{2}|\tilde{I}_\ell|$ for $0 \le \ell < L$.

Moreover (23) tells us $|\lambda(\tilde{I}_{\ell+1}) - \lambda(\tilde{I}_\ell)| \leq C|\tilde{I}_\ell|$ for $0 \leq \ell < L$. Consequently, since $\lambda(\tilde{I}_0) = \lambda(\check{I}) = 0$ by (2), we have

$$|\lambda(\tilde{I})| = |\lambda(\tilde{I}_L)| \leq \sum_{0\leq\ell<L} |\lambda(\tilde{I}_{\ell+1}) - \lambda(\tilde{I}_\ell)| \leq C \sum_{0\leq\ell<L} |\tilde{I}_\ell| \overset{(\text{by } (27))}{\leq} C'|\tilde{I}_0| = C'|\check{I}|,$$

proving (24). □

**Lemma 2.** *For each* $I \in T^{\mathsf{global}}(\check{I})$, *we have*

(28) $\| F_I^\# \|_{C^2(Q(I))} \leq 1 + C\epsilon.$

*Proof.* Let $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$. Then $\tilde{I} \subseteq \check{I} \subseteq I_0$ (see (0)). From Section 8, we recall that $\| e_2 \|_{C^2(Q(\tilde{I}))} \leq C\epsilon|I_0|^{-1}$. Hence, by (24),

(29) $\| \lambda(\tilde{I})e_2 \|_{C^2(Q(\tilde{I}))} \leq C\epsilon|\check{I}| \cdot |I_0|^{-1} \leq C\epsilon.$

Turning to (28), we proceed by cases.

Case 1: Suppose $I \in T^{\mathsf{dist}}(\check{I})$. Then since $|I| \leq |I_0| \leq C\epsilon$ (see (3) in Section 8), we learn from (7) that

(30) $\| F_I^{\mathsf{err}} \|_{C^2(Q(I))} \leq C\epsilon,$

while (6) yields

(31) $\| F_I \|_{C^2(Q(I))} \leq 1 + \epsilon.$

Since $F_I^\# = F_I + F_I^{\mathsf{err}} + \lambda(I)e_2$ in this case, (28) follows from (29), (30), (31).

Case 2: Suppose $I \notin T^{\mathsf{dist}}(\check{I})$. Let $\tilde{I}$ be the unique node of $T^{\mathsf{dist}}(\check{I})$ such that $I \in T^{\mathsf{loc}}(\tilde{I})$. Then $Q(I) \subset Q(\tilde{I})$, and, as in Case 1, we have $\| F_{\tilde{I}} \|_{C^2(Q(\tilde{I}))} \leq 1 + \epsilon$, and $\| \lambda(\tilde{I})e_2 \|_{C^2(Q(\tilde{I}))} \leq C\epsilon$. Since

$$F_I^\# = [F_{\tilde{I}} + \lambda(\tilde{I})e_2]\big|_{Q(I)}$$

in this case, we again have (28). □

**Lemma 3.** *For each* $I \in T^{\mathsf{global}}(\check{I})$, *we have*

(32) $F_I^\# = f$ *at* $z_{\mathsf{left}}^{\mathsf{rep}}(I)$ *and at* $z_{\mathsf{rt}}^{\mathsf{rep}}(I)$.

*Proof.* Recall that $e_2 = 0$ on $E$, and that $z_{\mathsf{left}}^{\mathsf{rep}}(I), z_{\mathsf{rt}}^{\mathsf{rep}}(I) \in E$. To check (32), we proceed by cases.

Case 1: Suppose $I \in T^{\mathsf{dist}}(\check{I})$. Then, for $z = z_{\mathsf{left}}^{\mathsf{rep}}(I)$ or $z = z_{\mathsf{rt}}^{\mathsf{rep}}(I)$, we have

$$\begin{aligned} F_I^\#(z) &= F_I(z) + F_I^{\mathsf{err}}(z) + \lambda(I)e_2(z) \\ &= F_I(z) + F_I^{\mathsf{err}}(z) \overset{(\text{by } (4))}{=} \mathsf{val}(\vec{P}^I, z) \overset{(\text{by } (19))}{=} f(z). \end{aligned}$$

Thus, (32) holds in Case 1.

Case 2: Suppose $I \notin T^{\mathsf{dist}}(\check{I})$. Let $\tilde{I}$ be the one and only node in $T^{\mathsf{dist}}(\check{I})$ for which $I \in T^{\mathsf{loc}}(\tilde{I})$. Then $I \in T^{\mathsf{loc}}(\tilde{I}) \setminus \{\tilde{I}\}$, and consequently, $\tilde{I}$ is of type $\mathcal{A}$ or $\mathcal{B}$, and $z^{\mathsf{rep}}_{\mathsf{left}}(I) = z^{s}_{\mathsf{left}}(\tilde{I})$, and $z^{\mathsf{rep}}_{\mathsf{rt}}(I) = z^{s}_{\mathsf{rt}}(\tilde{I})$. (See (14) in Section 7 and (6), (7), (10), (11) in Section 9.) We know that $z^{s}_{\mathsf{left}}(\tilde{I}), z^{s}_{\mathsf{rt}}(\tilde{I}) \in E$, hence $e_2 = 0$ at those points. By definition, at $z = z^{\mathsf{rep}}_{\mathsf{left}}(I) = z^{s}_{\mathsf{left}}(\tilde{I})$ and at $z = z^{\mathsf{rep}}_{\mathsf{rt}}(I) = z^{s}_{\mathsf{rt}}(\tilde{I})$, we have

$$F^{\#}_{I}(z) = F_{\tilde{I}}(z) + \lambda(\tilde{I})e_2(z) = F_{\tilde{I}}(z) = f(z),$$

thanks to (10). Thus (32) holds also in Case 2. □

**Lemma 4.** *Let* $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$*, and let* $I \in T^{\mathsf{loc}}(\tilde{I})$*. Then, for* $|\alpha| \le 1$*, we have*

$$(33)\quad |\partial^{\alpha}(F^{\#}_{I} - [F_{\tilde{I}} + \lambda(\tilde{I})e_2])| \le C\epsilon^{100}|I|^{2-|\alpha|} \text{ on } Q(I).$$

*Proof.* We proceed by cases.

Case 1: Suppose $I \in T^{\mathsf{dist}}(\check{I})$. Then by (16), either $I = \tilde{I}$ or $I \in RO(\tilde{I})$. By definition, we have $F^{\#}_{I} = F_I + F^{\mathsf{err}}_{I} + \lambda(I)e_2$ in that case. Therefore, if $I = \tilde{I}$, then (33) follows at once from (7). On the other hand, suppose $I \in RO(\tilde{I})$. Then

$$(34)\quad \begin{aligned} J_{S(I)}(F^{\#}_{I} - [F_{\tilde{I}} + \lambda(\tilde{I})e_2]) &= J_{S(I)}(F_I + F^{\mathsf{err}}_{I} + \lambda(I)e_2) - J_{S(I)}(F_{\tilde{I}} + \lambda(\tilde{I})e_2) \\ &= J_{S(I)}(F_I + F^{\mathsf{err}}_{I}) - J_{S(I)}(F_{\tilde{I}} + [\lambda(\tilde{I}) - \lambda(I)]e_2) = 0, \end{aligned}$$

since $J_{S(I)}(F_I + F^{\mathsf{err}}_{I}) = \vec{P}^{I}$ by (4), and $J_{S(I)}(F_{\tilde{I}} + [\lambda(\tilde{I}) - \lambda(I)]e_2) = \vec{P}^{I}$ by (5).

Also, from (6), (28) and (29), we see that

$$(35)\quad \| F^{\#}_{I} - [F_{\tilde{I}} + \lambda(\tilde{I})e_2] \|_{C^2(Q(I))} \le C.$$

Recall from Section 9 that $\bigwedge(I) \subseteq S(I)$, and therefore any given point of $Q(I)$ lies within distance $C\epsilon^{100}|I|$ of $S(I)$. Therefore, (33) follows from (34), (35) and Taylor's theorem. Thus, (33) holds in Case 1.

Case 2: Suppose $I \notin T^{\mathsf{dist}}(\check{I})$. Then, by definition, $F^{\#}_{I} = [F_{\tilde{I}} + \lambda(\tilde{I})e_2]|_{Q(I)}$. Hence, the left-hand side of (33) is zero, and thus (33) holds trivially in Case 2. □

**Lemma 5.** *Let* $I \in T^{\mathsf{global}}(\check{I})$ *be an internal node, and let* $I'$ *be one of the two dyadic children of* $I$*. Then, for* $|\alpha| \le 1$*, we have*

$$(36)\quad |\partial^{\alpha}(F^{\#}_{I} - F^{\#}_{I'})| \le C\epsilon^{100}|I|^{2-|\alpha|} \text{ on } Q(I').$$

*Proof.* By (15), there exists $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ such that $I, I' \in T^{\mathsf{loc}}(\tilde{I})$. By Lemma 4, we have $|\partial^{\alpha}(F^{\#}_{I} - [F_{\tilde{I}} + \lambda(\tilde{I})e_2])| \le C\epsilon^{100}|I|^{2-|\alpha|}$ on $Q(I)$, and $|\partial^{\alpha}(F^{\#}_{I'} - [F_{\tilde{I}} + \lambda(\tilde{I})e_2])| \le C\epsilon^{100}|I'|^{2-|\alpha|}$ on $Q(I')$, for $|\alpha| \le 1$. Estimate (36) now follows trivially, since $|I'| = \frac{1}{2}|I|$ and $Q(I') \subset Q(I)$. □

Next, we discuss $\partial_2 F^{\#}_{I}$ at the points $z_{\mathsf{left}}(I)$, $z_{\mathsf{rt}}(I)$. Recall that $\partial_2 e_2 \equiv 1$. (See Section 8.)

**Lemma 6.** *Let* $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$*, and let* $I \in T^{\mathsf{loc}}(\tilde{I})$*. Then*

(37) $\partial_2 F^{\#}_I = \partial_2 F_{\tilde{I}} + \lambda(\tilde{I})$ *at* $z_{\mathsf{left}}(I)$ *and at* $z_{\mathsf{rt}}(I)$.

*Proof.* We proceed by cases.

Case 1: Suppose $I \in T^{\mathsf{dist}}(\check{I})$. Then by (16), either $I = \tilde{I}$ or $I \in \mathsf{RO}(\tilde{I})$. If $I = \tilde{I}$, then by definition, $F^{\#}_{\tilde{I}} = F_{\tilde{I}} + F^{\mathsf{err}}_{\tilde{I}} + \lambda(\tilde{I})e_2$. Hence, at $z_{\mathsf{left}}(\tilde{I})$ and at $z_{\mathsf{rt}}(\tilde{I})$, we have $\partial_2 F^{\#}_{\tilde{I}} = \partial_2 F_{\tilde{I}} + \partial_2 F^{\mathsf{err}}_{\tilde{I}} + \lambda(\tilde{I}) = \partial_2 F_{\tilde{I}} + \lambda(\tilde{I})$ by (9).

Thus, (37) holds for $I = \tilde{I}$. On the other hand, suppose $I \in \mathsf{RO}(\tilde{I})$. Then by definition, $F^{\#}_I = F_I + F^{\mathsf{err}}_I + \lambda(I)e_2$. For $z = z_{\mathsf{left}}(I)$ or $z = z_{\mathsf{rt}}(I)$, we have

(38) $\partial_2 F^{\#}_I(z) = \partial_2(F_I + F^{\mathsf{err}}_I)(z) + \lambda(I) = \mathsf{val}\,(\partial_2 \vec{P}^I, z) + \lambda(I)$, thanks to (4).

Moreover, (5) yields

(39) $\partial_2 F_{\tilde{I}}(z) + [\lambda(\tilde{I}) - \lambda(I)] = \mathsf{val}\,(\partial_2 \vec{P}^I, z)$.

Combining (38) and (39), we obtain (37). Thus (37) holds in Case 1.

Case 2: Suppose $I \notin T^{\mathsf{dist}}(\tilde{I})$. Then by definition, $F^{\#}_I = [F_{\tilde{I}} + \lambda(\tilde{I})e_2]\big|_{Q(I)}$, from which (37) follows trivially.

Thus, (37) holds in all cases. □

**Lemma 7.** *Let* $I \in T^{\mathsf{global}}(\check{I})$ *be an internal node, and let* $I_1, I_2$ *be its two dyadic children, with* $I_1$ *to the left of* $I_2$*. Then*

(40) $\partial_2 F^{\#}_I = \partial_2 F^{\#}_{I_1}$ *at* $z_{\mathsf{left}}(I) = z_{\mathsf{left}}(I_1)$,

(41) $\partial_2 F^{\#}_I = \partial_2 F^{\#}_{I_2}$ *at* $z_{\mathsf{rt}}(I) = z_{\mathsf{rt}}(I_2)$, *and*

(42) $\partial_2 F^{\#}_{I_1} = \partial_2 F^{\#}_{I_2}$ *at* $z_{\mathsf{rt}}(I_1) = z_{\mathsf{left}}(I_2)$.

*Proof.* By (15), there exists $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ such that $I, I_1, I_2 \in T^{\mathsf{loc}}(\tilde{I})$. Applying Lemma 6, we learn that $\partial_2 F^{\#}_I = \partial_2 F_{\tilde{I}} + \lambda(\tilde{I})$ at $z_{\mathsf{left}}(I)$ and at $z_{\mathsf{rt}}(I)$; $\partial_2 F^{\#}_{I_1} = \partial_2 F_{\tilde{I}} + \lambda(\tilde{I})$ at $z_{\mathsf{left}}(I_1)$ and at $z_{\mathsf{rt}}(I_1)$; and $\partial_2 F^{\#}_{I_2} = \partial_2 F_{\tilde{I}} + \lambda(\tilde{I})$ at $z_{\mathsf{left}}(I_2)$ and at $z_{\mathsf{rt}}(I_2)$. Since $z_{\mathsf{left}}(I) = z_{\mathsf{left}}(I_1)$, $z_{\mathsf{rt}}(I) = z_{\mathsf{rt}}(I_2)$, and $z_{\mathsf{rt}}(I_1) = z_{\mathsf{left}}(I_2)$, we obtain (40), (41) and (42). □

Next, we compare each $F^{\#}_I$ to the function $F_{\mathsf{crude}}$ from Section 8. Recall from that section that $F_{\mathsf{crude}} \in C^2(\mathbb{R}^2)$, and that:

(43) $F_{\mathsf{crude}} = f$ on $E$;

(44) $\| F_{\mathsf{crude}} \|_{C^2(\mathbb{R}^2)} \le C$; and

(45) $|\partial_2 F_{\mathsf{crude}} - \xi| \le C\epsilon^{-1}|I_0|$ on $E$.

**Lemma 8.** *Let* $I \in T^{\mathsf{global}}(\check{I})$*, and let* $z_0 \in Q(I)$*. Define*

(46) $G = F^{\#}_I - \{F_{\mathsf{crude}} + [\partial_2 F^{\#}_I(z_0) - \partial_2 F_{\mathsf{crude}}(z_0)]e_2\}$ *on* $Q(I)$*. Then*

(47) $|\partial^\alpha G| \le C|I|^{2-|\alpha|}$ *on* $Q(I)$*, for* $|\alpha| \le 2$.

*Proof.* We start by estimating $[\partial_2 F^{\#}_{I}(z_0) - \partial_2 F_{\mathsf{crude}}(z_0)]$. Since $z^{\mathsf{rep}}_{\mathsf{left}}(I) \in E$, (45) gives

(48) $|\partial_2 F_{\mathsf{crude}}(z^{\mathsf{rep}}_{\mathsf{left}}(I)) - \xi| \leq C\epsilon^{-1}|I_0|$.

Also, $z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z_0 \in Q(I)$, and

(49) $\mathrm{diam}\, Q(I) \leq C|I|$.

Hence, (44) yields

(50) $|\partial_2 F_{\mathsf{crude}}(z_0) - \partial_2 F_{\mathsf{crude}}(z^{\mathsf{rep}}_{\mathsf{left}}(I))| \leq C|I|$.

From (48) and (50), we obtain

(51) $|\partial_2 F_{\mathsf{crude}}(z_0) - \xi| \leq C\epsilon^{-1}|I_0|$, thanks to the inclusions

(52) $I \subseteq \check{I} \subseteq I_0$ (see (0)).

On the other hand, there exists $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$ such that $I \in T^{\mathsf{loc}}(\tilde{I})$. Lemma 4 then tells us that

(53) $|\partial_2 F^{\#}_{I} - \partial_2 F_{\tilde{I}} - \lambda(\tilde{I})| \leq C\epsilon^{100}|I|$ on $Q(I)$.

(Recall that $\partial_2 e_2 \equiv 1$.)

Moreover, (3) and (4) imply $\partial_2(F_{\tilde{I}} + F^{\mathsf{err}}_{\tilde{I}}) = \xi$ at $z_{\mathsf{left}}(\tilde{I})$; hence, by (9), we have

(54) $\partial_2 F_{\tilde{I}}(z_{\mathsf{left}}(\tilde{I})) = \xi$.

Since $z_{\mathsf{left}}(\tilde{I}) \in Q(\tilde{I})$ and $z_0 \in Q(I) \subset Q(\tilde{I})$, with $\mathrm{diam}\, Q(\tilde{I}) \leq C|\tilde{I}|$, we learn from (6) that $|\partial_2 F_{\tilde{I}}(z_0) - \partial_2 F_{\tilde{I}}(z_{\mathsf{left}}(\tilde{I}))| \leq C|\tilde{I}| \leq C|\check{I}|$. Together with (54), this gives

(55) $|\partial_2 F_{\tilde{I}}(z_0) - \xi| \leq C|\check{I}|$.

From (53), (55), Lemma 1, and (52), we obtain the estimate

(56) $|\partial_2 F^{\#}_{I}(z_0) - \xi| \leq C|\check{I}| \leq C|I_0|$.

Combining (51) and (56), we obtain our basic estimate for $[\partial_2 F^{\#}_{I}(z_0) - \partial_2 F_{\mathsf{crude}}(z_0)]$, namely

(57) $|\partial_2 F^{\#}_{I}(z_0) - \partial_2 F_{\mathsf{crude}}(z_0)| \leq C\epsilon^{-1}|I_0|$.

On the other hand, we recall from Section 8 that $\| e_2 \|_{C^2(Q(I))} \leq C\epsilon|I_0|^{-1}$. Therefore, by (57), we have $\| [\partial_2 F^{\#}_{I}(z_0) - \partial_2 F_{\mathsf{crude}}(z_0)] e_2 \|_{C^2(Q(I))} \leq C$.

Together with Lemma 2, estimate (44), and definition (46), this tells us that

(58) $\| G \|_{C^2(Q(I))} \leq C$.

Next, we recall that $z^{\mathsf{rep}}_{\mathsf{left}}(I)$, $z^{\mathsf{rep}}_{\mathsf{rt}}(I) \in E$; hence, $e_2 = 0$ and $F_{\mathsf{crude}} = f$ at those points. Recalling also Lemma 3 and definition (46), we conclude that

(59) $G = 0$ at $z^{\mathsf{rep}}_{\mathsf{left}}(I)$ and at $z^{\mathsf{rep}}_{\mathsf{rt}}(I)$.

Also, since $\partial_2 e_2 \equiv 1$, definition (46) gives $\partial_2 G(z_0) = 0$; hence, by (49) and (58), we have:

(60) $|\partial_2 G(z_{\mathrm{left}}^{\mathrm{rep}}(I))| \leq C|I|$.

Let us write

(61) $z_{\mathrm{left}}^{\mathrm{rep}}(I) = (\bar{x}_1, \bar{x}_2)$ and $z_{\mathrm{rt}}^{\mathrm{rep}}(I) = (\bar{\bar{x}}_1, \bar{\bar{x}}_2)$.

From Section 9, we recall that $\bar{x}_2 = \varphi(\bar{x}_1)$ and $\bar{\bar{x}}_2 = \varphi(\bar{\bar{x}}_1)$. Moreover, from Section 8, we recall that $|\varphi'| \leq C$; hence,

(62) $|\bar{\bar{x}}_2 - \bar{x}_2| \leq |z_{\mathrm{left}}^{\mathrm{rep}}(I) - z_{\mathrm{rt}}^{\mathrm{rep}}(I)| \leq C|\bar{\bar{x}}_1 - \bar{x}_1| \leq C|I|$.

(The last inequality in (62) follows from (49).)

Also, we recall from Section 9 that the points $z_{\mathrm{left}}^{\mathrm{rep}}(I)$ and $z_{\mathrm{rt}}^{\mathrm{rep}}(I)$ are distinct. Hence, (62) yields

(63) $\bar{\bar{x}}_1 \neq \bar{x}_1$.

Our plan is to estimate $\partial_1 G(z_{\mathrm{left}}^{\mathrm{rep}}(I))$ by comparing $G(z_{\mathrm{rt}}^{\mathrm{rep}}(I))$ with its Taylor expansion about $z_{\mathrm{left}}^{\mathrm{rep}}(I)$. From Taylor's theorem and (58), we see that

$$|G(z_{\mathrm{rt}}^{\mathrm{rep}}(I)) - [G(z_{\mathrm{left}}^{\mathrm{rep}}(I)) + \partial_1 G(z_{\mathrm{left}}^{\mathrm{rep}}(I)) \cdot (\bar{\bar{x}}_1 - \bar{x}_1) + \partial_2 G(z_{\mathrm{left}}^{\mathrm{rep}}(I)) \cdot (\bar{\bar{x}}_2 - \bar{x}_2)]| \leq$$
$$\leq C|\bar{\bar{x}}_1 - \bar{x}_1|^2 + C|\bar{\bar{x}}_2 - \bar{x}_2|^2.$$

Hence, by (59), (60), (62), it follows that

$$|\partial_1 G(z_{\mathrm{left}}^{\mathrm{rep}}(I)) \cdot (\bar{\bar{x}}_1 - \bar{x}_1)| \leq |\partial_2 G(z_{\mathrm{left}}^{\mathrm{rep}}(I)) \cdot (\bar{\bar{x}}_2 - \bar{x}_2)| + C|\bar{\bar{x}}_1 - \bar{x}_1|^2 + C|\bar{\bar{x}}_2 - \bar{x}_2|^2$$
$$\leq C|I| \cdot C|\bar{\bar{x}}_1 - \bar{x}_1| + C|\bar{\bar{x}}_1 - \bar{x}_1|^2 + C|\bar{\bar{x}}_2 - \bar{x}_2|^2 \leq C'|I| \cdot |\bar{\bar{x}}_1 - \bar{x}_1|.$$

This in turn tells us that

(64) $|\partial_1 G(z_{\mathrm{left}}^{\mathrm{rep}}(I))| \leq C'|I|$,

thanks to (63).

From (58), (59), (60), (64) (together with Taylor's theorem and (49)), we conclude that $|G| \leq C|I|^2$, $|\nabla G| \leq C|I|$, and $|\nabla^2 G| \leq C$ on $Q(I)$.

Thus, (47) holds. □

We can use Lemmas 7 and 8 to give a crude estimate for $\partial^\alpha(F_{I'}^{\#} - F_{I''}^{\#})$ on $Q(I') \cap Q(I'')$, for certain $I', I'' \in T^{\mathrm{global}}(\check{I})$.

**Lemma 9.** *Let $I', I'' \in T^{\mathrm{global}}(\check{I})$. Suppose that the right endpoint of $I'$ coincides with the left endpoint of $I''$. Then*

(65) $|\partial^\alpha(F_{I'}^{\#} - F_{I''}^{\#})| \leq C(|I'| + |I''|)^{2-|\alpha|}$ *on* $Q(I') \cap Q(I'')$, *for* $|\alpha| \leq 2$.

*Proof.* Let $x_0$ = right endpoint $(I')$ = left endpoint $(I'')$. Note that $I'$ lies to the left of $I''$. Let $I$ be the least common ancestor of $I'$ and $I''$ in the tree $T^{\mathrm{global}}(\check{I})$.

Descending from I to I′ in that tree, we obtain a finite sequence $I'_0, I'_1, \dots, I'_{L'} \in \mathsf{T}^{\mathsf{global}}(\check{I})$, such that

(66) $I'_0 = I, I'_{L'} = I'$, and $I'_{\ell+1}$ is a dyadic child of $I'_\ell$ for $0 \le \ell < L'$.

Similarly, we obtain $I''_0, I''_1, \dots, I''_{L''} \in \mathsf{T}^{\mathsf{global}}(\check{I})$, such that

(67) $I''_0 = I$, $I''_{L''} = I''$, and $I''_{\ell+1}$ is a dyadic child of $I''_\ell$ for $0 \le \ell < L''$.

We prove a few elementary properties of the $I'_\ell$, $I''_\ell$. First, we show that

(68) $I'_1$ is the left dyadic child of I, and $I''_1$ is the right dyadic child of I.

Indeed, (66) and (67) tell us that $I'_1$, $I''_1$ are dyadic children of I. We have $I'_1 \ne I''_1$, since I is the least common ancestor of I′ and I″. Therefore, (68) holds, unless we have

(69) $I''_1$ is the left dyadic child of I, and $I'_1$ is the right dyadic child of I.

However, (69) cannot hold, since $I' \subset I'_1$, $I'' \subset I''_1$, and I′ lies to the left of I″. This completes the proof of (68). Next we check that

(70) $x_0$ is the right endpoint of $I'_{\ell'}$ for $1 \le \ell' \le L'$ and

(71) $x_0$ is the left endpoint of $I''_{\ell''}$ for $1 \le \ell'' \le L''$.

To see (70), (71), we note that $I'_{\ell'} \subset I'_1$ and $I''_{\ell''} \subset I''_1$ for $1 \le \ell' \le L'$, $1 \le \ell'' \le L''$. Hence (69) shows that

(72) $I'_{\ell'}$ lies to the left of $I''_{\ell''}$.

On the other hand,

(73) $x_0 \in (I')^{\mathsf{closure}} \cap (I'')^{\mathsf{closure}} \subset (I'_{\ell'})^{\mathsf{closure}} \cap (I''_{\ell''})^{\mathsf{closure}}$.

Assertions (70) and (71) follow from (72) and (73).

From (66) and (70), we see that

(74) $I'_{\ell+1}$ is the right dyadic child of $I'_{\ell'}$ for $1 \le \ell < L'$.

Similarly, (67) and (71) yield

(75) $I''_{\ell+1}$ is the left dyadic child of $I''_\ell$, for $1 \le \ell < L''$.

This concludes our discussion of the elementary properties of the $I'_\ell, I''_\ell$.

Next, we bring in Lemma 7. From (70), (71) and the definition of $z_{\mathsf{left}}(I)$, $z_{\mathsf{rt}}(I)$ in Section 9, we see that the points $z_{\mathsf{rt}}(I'_\ell)(1 \le \ell \le L')$ and $z_{\mathsf{left}}(I''_\ell)$ $(1 \le \ell \le L'')$ are all equal. Let $z_0$ denote this common point.

From (74) and conclusion (41) of Lemma 7, we learn that

(76) $\partial_2 F^{\#}_{I'_1}(z_0) = \partial_2 F^{\#}_{I'_2}(z_0) = \cdots = \partial_2 F^{\#}_{I'_{L'}}(z_0)$.

Similarly, (75) and (40) yield the equalities

(77) $\partial_2 F^{\#}_{I''_1}(z_0) = \partial_2 F^{\#}_{I''_2}(z_0) = \cdots = \partial_2 F^{\#}_{I''_{L''}}(z_0)$.

Moreover, (68) and (42) tell us that

(78) $\partial_2 F^{\#}_{I'_1}(z_0) = \partial_2 F^{\#}_{I''_1}(z_0)$.

In view of (76), (77), (78), we have $\partial_2 F^{\#}_{I'_{L'}}(z_0) = \partial_2 F^{\#}_{I''_{L''}}(z_0)$. That is,

(79) $\partial_2 F^{\#}_{I'}(z_0) = \partial_2 F^{\#}_{I''}(z_0)$, where $z_0 = z_{\mathrm{rt}}(I') = z_{\mathrm{left}}(I'')$.

(See (66) and (67).)

We now bring in Lemma 8. Note that

(80) $z_0 = z_{\mathrm{rt}}(I') = z_{\mathrm{left}}(I'') \in Q(I') \cap Q(I'')$.

Hence, Lemma 8 tells us that

(81) $|\partial^\alpha(F^{\#}_{I'} - \{F_{\mathrm{crude}} + [\partial_2 F^{\#}_{I'}(z_0) - \partial_2 F_{\mathrm{crude}}(z_0)]e_2\})| \leq C|I'|^{2-|\alpha|}$ on $Q(I')$, for $|\alpha| \leq 2$.

Another application of Lemma 8 (and (80)) yields the estimate

(82) $|\partial^\alpha(F^{\#}_{I''} - \{F_{\mathrm{crude}} + [\partial_2 F^{\#}_{I''}(z_0) - \partial_2 F_{\mathrm{crude}}(z_0)]e_2\})| \leq C|I''|^{2-|\alpha|}$ on $Q(I'')$, for $|\alpha| \leq 2$.

The conclusion (65) of Lemma 9 follows at once from (79), (81) and (82). □

From Lemma 5, we can sometimes obtain a sharper estimate for $|\partial^\alpha(F^{\#}_{I'} - F^{\#}_{I''})|$.

**Lemma 10.** *Let* $I', I'' \in T^{\mathrm{global}}(\check{I})$. *Assume* $I', I'' \subset \hat{I}$ *for some dyadic interval* $\hat{I}$ *of length at most* $\epsilon^{-2} \cdot (|I'| + |I''|)$. *Then*

(83) $|\partial^\alpha(F^{\#}_{I'} - F^{\#}_{I''})| \leq C\epsilon^{96} \cdot (|I'| + |I''|)^{2-|\alpha|}$ *on* $Q(I') \cap Q(I'')$ *for* $|\alpha| \leq 1$.

*Proof.* Let $I$ be the least common ancestor of $I'$ and $I''$ in $T^{\mathrm{global}}(\check{I})$. Then

(84) $|I| \leq \epsilon^{-2} \cdot (|I'| + |I''|)$.

Descending from $I$ to $I'$ in the tree $T^{\mathrm{global}}(\check{I})$, we obtain a finite sequence $I_0, I_1, \dots, I_L \in T^{\mathrm{global}}(\check{I})$, such that $I_0 = I$, $I_L = I'$, and $I_{\ell+1}$ is a dyadic child of $I_\ell$ for $0 \leq \ell < L$. Note that $|I_\ell| = 2^{-\ell}|I|$ for each $\ell$, and that $Q(I_0) \supset Q(I_1) \supset \cdots \supset Q(I_L) = Q(I')$.

Lemma 5 gives the estimate

$$|\partial^\alpha(F^{\#}_{I_\ell} - F^{\#}_{I_{\ell+1}})| \leq C\epsilon^{100}|I_\ell|^{2-|\alpha|} \quad \text{on } Q(I_{\ell+1}), \text{ for } |\alpha| \leq 1,\ 0 \leq \ell < L.$$

Summing on $\ell$, we find that

(85) $|\partial^\alpha(F^{\#}_{I} - F^{\#}_{I'})| \leq C\epsilon^{100}|I|^{2-|\alpha|}$ on $Q(I')$, for $|\alpha| \leq 1$.

Similarly,

(86) $|\partial^\alpha(F^{\#}_{I} - F^{\#}_{I''})| \leq C\epsilon^{100}|I|^{2-|\alpha|}$ on $Q(I'')$, for $|\alpha| \leq 1$.

Our desired conclusion (83) follows from (84), (85), (86). □

**Remark.** In Lemma 10, we do not assume that the dyadic interval $\hat{I}$ is a node of the tree $T^{\mathsf{global}}(\check{I})$.

We close this section with two simple observations.

**Lemma 11.** *For the root* $\check{I}$*, we have*

(87) $J_{S(\check{I})}(F^{\#}_{\check{I}}) = \vec{P}^{\check{I}}$. (*See* (1).)

*Proof.* Since $\check{I} \in T^{\mathsf{dist}}(\check{I})$, we have by definition: $F^{\#}_{\check{I}} = F_{\check{I}} + F^{\mathsf{err}}_{\check{I}} + \lambda(\check{I})e_2$. Recalling (2) and (4), we obtain (87). □

**Lemma 12.** *Let* $I \in T^{\mathsf{global}}(\check{I})$*. If* $I \in T^{\mathsf{dist}}(\check{I})$ *and* $I$ *is of type* $C1$*, then* $F^{\#}_I = f$ *at* $z_!(I)$*.*

*Proof.* By definition $F^{\#}_I = F_I + F^{\mathsf{err}}_I + \lambda(I)e_2$. At $z = z_!(I) \in E$, we have $e_2 = 0$. Hence, $F^{\#}_I(z_!(I)) = (F_I + F^{\mathsf{err}}_I)(z_!(I)) = f(z_!(I))$, thanks to (11). □

## 15. Global interpolants

In this section, we adopt the notation, assumptions and boiler-plate constants from Section 8. We suppose we are given an interval

(0) $\check{I} \subset I_0$ (dyadic), with $\#(5\check{I} \cap \bar{E}) \geq 2$.

We suppose that we have carried out Algorithm MMBT from Section 9, Algorithm JPU from Section 10, and the one-time work of Algorithm MAK from Section 13. Thus, for each $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$, we have computed the convex polyhedron $K(\tilde{I})$.

Finally, we suppose that we are given a Whitney field

(1) $\vec{P}^{\check{I}} \in K(\check{I})$.

Using the query algorithm within Algorithm MAK, we obtain from $\vec{P}^{\check{I}}$ a family of Whitney fields $\vec{P}^{\tilde{I}}$ and real numbers $\lambda(\tilde{I})$, indexed by the nodes $\tilde{I} \in T^{\mathsf{dist}}(\check{I})$.

In Section 14, we defined a function

(2) $F^{\#}_I \in C^2(Q(I))$ for each $I \in T^{\mathsf{global}}(\check{I})$.

From Section 10, we recall the open set

(3) $\Omega(\check{I}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \check{I}^{\mathsf{interior}}, |x_2 - \varphi(x_1)| < |\check{I}|\}$,

and the partition of unity

(4) $$\sum_{I \in T^{\mathsf{global}}(\check{I})} \theta_I = 1 \text{ on } \Omega(\check{I}), \text{ with}$$

(5) $\mathsf{supp}\, \theta_I \subset Q(I)$ for each $I \in T^{\mathsf{global}}(\check{I})$.

Recall that the $\theta_I$ are defined only on $\Omega(\check{I})$, and that $\operatorname{supp}\theta_I$ is the set of all points $z$ in $\Omega(\check{I})$ such that $\theta_I$ does not vanish identically in any neighborhood of $z$.

In this section, we establish the basic properties of the function

(6) $$F^\# := \sum_{I \in T^{\text{global}}(\check{I})} \theta_I F_I^\# \in C^2(\Omega(\check{I})).$$

Recall that, in Section 8, we supposed that we are given a real number $\tau$, used to fix a dyadic grid $\mathcal{G}_\tau$. Whenever we speak of a "dyadic" interval I in this section, or in Sections 10 or 14, our interval I is dyadic with respect to $\mathcal{G}_\tau$.

Recall also, from Section 4, the following definition:

(7) Let $\tau, x_1 \in \mathbb{R}$, and let $k_0, \ell \in \mathbb{Z}$, with $k_0 > 0$. Let $\hat{I}$ be the interval of length $2^{k_0+\ell}$ in $\mathcal{G}_\tau$ containing $x_1$. Then we say that $(x_1, \ell)$ is "$k_0$-regular" for $\mathcal{G}_\tau$, if every $I \in \mathcal{G}_\tau$ such that $|I| \le 2^\ell$ and $3I \ni x_1$ satisfies $I \subset \hat{I}$.

Definition (7) will enter into the basic properties of the function $F^\#$. To see this, we make the following further definitions:

We fix an integer $k_0$, such that

(8) $$2^{k_0-10} < \epsilon^{-1} < 2^{k_0}.$$

For $z = (x_1, x_2) \in \Omega(\check{I})$, we define

(9) $$\delta_{LS}(z) = \delta_{LS}(x_1) + |x_2 - \varphi(x_1)| = \delta_{LS}(x_1) + |e_2(z)|.$$

(See equation (2) in Section 7, for the definition of $\delta_{LS}(x_1)$.)

Let $I \in T^{\text{global}}(\check{I})$, and let $z = (x_1, x_2) \in \Omega(\check{I})$. Recall (see (34) in Section 10) that if $z \in \operatorname{supp}\theta_I$, then

(10) $$x_1 \in (1.01)I \text{ and } c_1\delta_{LS}(z) < |I| < C_1\delta_{LS}(z).$$

For the rest of this section, we fix $c_1, C_1$ as in (10).

For $z \in \Omega(\check{I})$, we define an integer $\ell(z)$ by

(11) $$2^{\ell(z)-1} < C_1\delta_{LS}(z) \le 2^{\ell(z)}.$$

Thus, whenever $z = (x_1, x_2) \in \Omega(\check{I})$ and $I \in T^{\text{global}}(\check{I})$ satisfy $z \in \operatorname{supp}\theta_I$, we then have

(12) $$x_1 \in (1.01)I \text{ and } c \cdot 2^{\ell(z)} < |I| < 2^{\ell}(z).$$

**Lemma 1.** *Let $z = (x_1, x_2) \in \Omega(\check{I})$. Assume that either*

(13) $(x_1, \ell(z))$ *is $k_0$-regular for the grid $\mathcal{G}_\tau$ or*

(14) $$2^{\ell(z)+k_0} > \epsilon^{1/2}|\check{I}|.$$

*Then there exists a dyadic interval $\hat{I}$, such that for every $I \in T^{\text{global}}(\check{I})$ satisfying $z \in \operatorname{supp}\theta_I$, we have*

(15) $I \subset \hat{I}$ *and* $|\hat{I}| < \epsilon^{-2}|I|$.

*Proof.* First, assume (13). Let $\hat{I}$ be the dyadic interval of length $2^{k_0+\ell(z)}$ containing $x_1$. By assumption (13) and definition (7), we know that:

Any dyadic interval $I$ such that $x_1 \in 3I$ and $|I| \le 2^{\ell(z)}$ satisfies $I \subset \hat{I}$.

Hence, (12) shows that any $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$ such that $z \in \operatorname{supp}\theta_I$ satisfies $I \subset \hat{I}$. Any such $I$ also satisfies $|I| > c \cdot 2^{\ell(z)} = c \cdot 2^{-k_0}|\hat{I}| > c'\epsilon|\hat{I}|$, thanks to (12) and (8). Thus, (15) holds for every $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$ such that $z \in \operatorname{supp}\theta_I$. This proves our lemma under hypothesis (13).

On the other hand, suppose (14) holds. We take $\hat{I} = \check{I}$, and check that (15) holds for every $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$ such that $z \in \operatorname{supp}\theta_I$. Indeed, any such $I$ satisfies $I \subset \check{I}$, simply because $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$. Moreover, for such $I$, (8), (12) and (14) yield

$$|I| > c \cdot 2^{\ell(z)} > c \cdot 2^{-k_0}\epsilon^{1/2}|\check{I}| > c'' \cdot \epsilon^{3/2}|\check{I}| > \epsilon^2|\check{I}|.$$

Thus, (15) holds for all $I \in \mathsf{T}^{\mathsf{global}}(\check{I})$ such that $z \in \operatorname{supp}\theta_I$. This proves Lemma 1 under assumption (14). □

**Lemma 2.** *Let $z = (x_1, x_2) \in \Omega(\check{I})$. Assume that either*

(16) $(x_1, \ell(z))$ *is* $k_0$*-regular for the grid* $\mathcal{G}_\tau$, *or*

(17) $2^{\ell(z)+k_0} > \epsilon^{1/2}|\check{I}|$.

*Let* $I', I'' \in \mathsf{T}^{\mathsf{global}}(\check{I})$, *and suppose* $z \in \operatorname{supp}\theta_{I'} \cap \operatorname{supp}\theta_{I''}$. *Then*

(18) $|\partial^\alpha(F^{\#}_{I'} - F^{\#}_{I''})(z)| \le C\epsilon^{96}|I'|^{2-|\alpha|}$ *for* $|\alpha| \le 1$.

*Proof.* By Lemma 1, together with Lemma 10 from Section 14, we have

(19) $|\partial^\alpha(F^{\#}_{I'} - F^{\#}_{I''})| \le C\epsilon^{96} \cdot (|I'| + |I''|)^{2-|\alpha|}$ on $Q(I') \cap Q(I'')$ for $|\alpha| \le 1$.

Recalling (5) and (12), we see that

(20) $z \in \operatorname{supp}\theta_{I'} \cap \operatorname{supp}\theta_{I''} \subset Q(I') \cap Q(I'')$, and

(21) $|I''| < 2^{\ell(z)} < C|I'|$.

Conclusion (18) now follows at once from (19), (20), (21). □

Dropping the assumptions (16), (17), we can still prove a crude version of (18).

**Lemma 3.** *Let $z = (x_1, x_2) \in \Omega(\check{I})$, and let $I', I'' \in \mathsf{T}^{\mathsf{global}}(\check{I})$. If $z \in \operatorname{supp}\theta_{I'} \cap \operatorname{supp}\theta_{I''}$, then*

(22) $|\partial^\alpha(F^{\#}_{I'} - F^{\#}_{I''})(z)| \le C|I'|^{2-|\alpha|}$ *for* $|\alpha| \le 1$.

*Proof.* From (12), we have $c|I'| \le |I''| \le C|I'|$. Hence, without loss of generality, we may suppose $|I''| \le |I'|$. Thus

(23) $|I''| \le |I'| \le C|I''|$.

Let $\tilde{I}''$ be the dyadic interval containing $I''$, of length

(24) $|\tilde{I}''| = |I'|$.

Since $I', I'' \in T^{\mathsf{global}}(\check{I})$, we know that $|I'| \le |\check{I}|$ and $I'' \subset \check{I}$. Thus, $\check{I}$ and $\tilde{I}''$ are both dyadic intervals containing $I''$, and moreover $|\tilde{I}''| \le |\check{I}|$ by (24). Consequently,

(25) $I'' \subset \tilde{I}'' \subset \check{I}$.

Next, we observe a useful corollary of conclusion (II) of Lemma BT1 in Section 7, namely:

(26) Let $I_1, I_2, I_3$ be dyadic intervals. If $I_1, I_3 \in T^{\mathsf{global}}(\check{I})$ and $I_1 \subset I_2 \subset I_3$, then also $I_2 \in T^{\mathsf{global}}(\check{I})$.

From (25), (26) we conclude that

(27) $\tilde{I}'' \in T^{\mathsf{global}}(\check{I})$.

Moreover, since $z = (x_1, x_2) \in \mathsf{supp}\, \theta_{I'} \cap \mathsf{supp}\, \theta_{I''}$, we know from (12) that

(28) $x_1 \in (1.01)I' \cap (1.01)I'' \subset (1.01)I' \cap (1.01)\tilde{I}''$.

Together with (24) and the fact that $I'$ and $\tilde{I}''$ are dyadic, this implies that either

(a) $I' = \tilde{I}''$, or

(b) The right endpoint of $I'$ coincides with the left endpoint of $\tilde{I}''$, or

(c) The right endpoint of $\tilde{I}''$ coincides with the left endpoint of $I'$.

Consequently, Lemma 9 in Section 14 tells us that $|\partial^\alpha(F^\#_{I'} - F^\#_{\tilde{I}''})| \le C \cdot (|I'| + |\tilde{I}''|)^{2-|\alpha|}$ on $Q(I') \cap Q(\tilde{I}')$, for $|\alpha| \le 2$. Thanks to (24), this is equivalent to

(29) $|\partial^\alpha(F^\#_{I'} - F^\#_{\tilde{I}''})| \le C|I'|^{2-|\alpha|}$ on $Q(I') \cap Q(\tilde{I}'')$, for $|\alpha| \le 2$.

On the other hand, by (25), there exists a finite sequence of dyadic intervals,

(30) $\tilde{I}_0 \supset \tilde{I}_1 \supset \cdots \supset \tilde{I}_L$, such that

(31) $\tilde{I}_0 = \tilde{I}''$, $\tilde{I}_L = I''$, and $\tilde{I}_{\ell+1}$ is a dyadic child of $\tilde{I}_\ell$ for $0 \le \ell < L$.

In particular,

(32) $Q(\tilde{I}'') = Q(\tilde{I}_0) \supset Q(\tilde{I}_1) \supset \cdots \supset Q(\tilde{I}_L) = Q(I'')$ and

(33) $|\tilde{I}_\ell| = 2^{-\ell}|\tilde{I}_0| = 2^{-\ell}|\tilde{I}''| = 2^{-\ell}|I'|$ for $0 \le \ell \le L$.

By (31), together with Lemma 5 from Section 14, we have

$$\left|\partial^\alpha(F^\#_{\tilde{I}_\ell} - F^\#_{\tilde{I}_{\ell+1}})\right| \le C\epsilon^{100}|\tilde{I}_\ell|^{2-|\alpha|}$$

on $Q(\tilde{I}_{\ell+1})$, for $|\alpha| \le 1$, $\ell < L$. Hence, by (32) and (33), we have

$$\left|\partial^\alpha(F^\#_{\tilde{I}_\ell} - F^\#_{\tilde{I}_{\ell+1}})\right| \le C\epsilon^{100} \cdot 2^{-\ell}|I'|^{2-|\alpha|}$$

on $Q(I'')$, for $0 \le \ell < L$, $|\alpha| \le 1$.

Summing over $\ell$, and recalling (31), we conclude that

(34) $|\partial^\alpha(F^\#_{\tilde{I}''} - F^\#_{I''})| \le C\epsilon^{100}|I'|^{2-|\alpha|}$ on $Q(I'')$, for $|\alpha| \le 1$.

Now from (29), (32) and (34), we obtain

(35) $|\partial^\alpha(F^\#_{I'} - F^\#_{I''})| \le C|I'|^{2-|\alpha|}$ on $Q(I') \cap Q(I'')$, for $|\alpha| \le 1$.

From (5), we have $z \in \operatorname{supp}\theta_{I'} \cap \operatorname{supp}\theta_{I''} \subset Q(I') \cap Q(I'')$. Consequently, the conclusion (22) of Lemma 3 follows at once from (35). □

Recall that $F^\#$ is defined by (6), with the $\theta_I$ and $F^\#_I$ satisfying the following:

(36) $\sum_{I \in T^{\mathsf{global}}(\check{I})} \theta_I = 1$ on $\Omega(\check{I})$.

(37) $\operatorname{supp}\theta_I \subset Q(I)$ for each $I \in T^{\mathsf{global}}(\check{I})$.

(38) $|\partial^\alpha \theta_I| \le C|I|^{-|\alpha|}$ for $|\alpha| \le 2$, $I \in T^{\mathsf{global}}(\check{I})$.

(39) $\theta_I \ge 0$ for each $I \in T^{\mathsf{global}}(\check{I})$.

(40) Any given $z \in \Omega(\check{I})$ belongs to $\operatorname{supp}\theta_I$ for at most $C$ distinct $I \in T^{\mathsf{global}}(\check{I})$.

(41) Let $I', I'' \in T^{\mathsf{global}}(\check{I})$. If $\operatorname{supp}\theta_{I'} \cap \operatorname{supp}\theta''_I \ne \emptyset$, then $c|I'| \le |I''| \le C|I'|$

(42) $\| F^\#_I \|_{C^2(Q(I))} \le 1 + C\epsilon$, for $I \in T^{\mathsf{global}}(\check{I})$.

Indeed, (36)–(41) may be found in Section 10, and (42) is Lemma 2 in Section 14. (See (3)–(7), (35), (36) in Section 10.)

Recall that $|I| \le 1$ for $I \in T^{\mathsf{global}}(\check{I})$, thanks to (0), together with assumption (3) in Section 8.

The above remarks, together with Lemmas 2 and 3, allow us to apply Lemma GPU from Section 5. Thus, we obtain Lemmas 4 and 5 below.

**Lemma 4.** *Let $z = (x_1, x_2) \in \Omega(\check{I})$. Assume that either*

(43) $(x_1, \ell(z))$ *is* $k_0$*-regular for the grid* $\mathcal{G}_\tau$, *or*

(44) $2^{\ell(z)+k_0} > \epsilon^{1/2}|\check{I}|$.

*Then*

(45) $|J_z(F^\#)|_z \le 1 + C\epsilon$.

**Lemma 5.** *For any $z \in \Omega(\check{I})$, we have*

(46) $|J_z(F^\#)|_z \le C$.

We next investigate how well $J_{S(I)}(F^\#)$ agrees with the given Whitney field $\vec{P}^{\check{I}}$ in (1).

**Lemma 6.** *Let $\vec{P}^{\check{I}}$ in (1) be given by*

(47) $\vec{P}^{\check{I}} = (P^z)_{z \in S(\check{I})}$.

*Then for $z \in S(\check{I}) \cap \Omega(\check{I})$ such that*

(48) $|e_2(z)| \ge \epsilon^4|\check{I}|$, *we have*

(49) $|\partial^\alpha(F^\# - P^z)(z)| \le C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ *for* $|\alpha| \le 1$.

*Proof.* Let $I \in T^{\mathsf{global}}(\check{I})$. Descending from $\check{I}$ to $I$ in the tree $T^{\mathsf{global}}(\check{I})$, we obtain a finite sequence of dyadic intervals $I'_0 \supset I'_1 \supset \cdots \supset I'_L$, such that each $I'_\ell$ belongs to $T^{\mathsf{global}}(\check{I})$, $I'_0 = \check{I}$, $I'_L = I$, and $I'_{\ell+1}$ is a dyadic child of $I'_\ell$ for $0 \le \ell < L$. In particular, $|I'_\ell| = 2^{-\ell}|\check{I}|$ for each $\ell$, and $Q(I'_0) \supset Q(I'_1) \supset \cdots \supset Q(I'_L) = Q(I)$. By Lemma 5 in Section 14, we have

$$|\partial^\alpha(F^\#_{I'_\ell} - F^\#_{I'_{\ell+1}})| \le C\epsilon^{100}|I'_\ell|^{2-|\alpha|}$$

on $Q(I'_{\ell+1})$, for $|\alpha| \le 1$, $0 \le \ell < L$. Consequently,

$$|\partial^\alpha(F^\#_{I'_\ell} - F^\#_{I'_{\ell+1}})| \le C\epsilon^{100} \cdot 2^{-\ell}|\check{I}|^{2-|\alpha|}$$

on $Q(I)$, for $|\alpha| \le 1$, $0 \le \ell < L$. Summing over $\ell$, we see that

(50) $|\partial^\alpha(F^\#_{\check{I}} - F^\#_I)| \le C\epsilon^{100}|\check{I}|^{2-|\alpha|}$ on $Q(I)$, for $|\alpha| \le 1$.

Estimate (50) holds for all $I \in T^{\mathsf{global}}(\check{I})$.

Now suppose $z = (x_1, x_2) \in \Omega(\check{I})$, and suppose that $|e_2(z)| = |x_2 - \varphi(x_1)| \ge \epsilon^4|\check{I}|$. Then, for $|\alpha| \le 1$, we have

(51) $$|\partial^\alpha(F^\#_{\check{I}} - F^\#)(z)| = |\partial^\alpha\{\sum_{I \in T^{\mathsf{global}}(\check{I})} \theta_I \cdot (F^\#_{\check{I}} - F^\#_I)\}(z)| \le C \sum_{\substack{I \in T^{\mathsf{global}}(\check{I}) \\ \mathrm{supp}\,\theta_I \ni z}} \sum_{\alpha' + \alpha'' = \alpha} |\partial^{\alpha'}\theta_I(z)| \cdot |\partial^{\alpha''}(F^\#_{\check{I}} - F^\#_I)(z)|.$$

(52) There are at most $C$ terms on the right-hand side of (51). For each of those terms, we have $c \cdot [\delta_{LS}(x_1) + |x_2 - \varphi(x_1)|] \le |I| \le C \cdot [\delta_{LS}(x_1) + |x_2 - \varphi(x_1)|]$, and thus $|I| \ge c|x_2 - \varphi(x_1)| \ge c\epsilon^4|\check{I}|$.

Consequently, for each term on the right in (51), we have

(53) $|\partial^{\alpha'}\theta_I(z)| \le C|I|^{-|\alpha'|} \le C\epsilon^{-4}|\check{I}|^{-|\alpha'|}$, while (50) gives

(54) $|\partial^{\alpha''}(F^\#_{\check{I}} - F^\#_I)(z)| \le C\epsilon^{100}|\check{I}|^{2-|\alpha''|}$, since $z \in \mathrm{supp}\,\theta_I \subset Q(I)$.

Using (52), (53), (54) to estimate the right-hand side of (51), we learn that

(55) $|\partial^\alpha(F^\#_{\check{I}} - F^\#)(z)| \le C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ for $|\alpha| \le 1$, whenever $z \in \Omega(\check{I})$ and $|e_2(z)| \ge \epsilon^4|\check{I}|$.

Finally, suppose $z \in \Omega(\check{I}) \cap S(\check{I})$, and suppose that $|e_2(z)| \ge \epsilon^4|\check{I}|$. Recalling (47) and Lemma 11 from Section 14, we see that conclusion (49) follows from estimate (55). □

Next, we study how $F^\#$ behaves at the points of $E \cap \Omega(\check{I})$.

**Lemma 7.** $F^\# = f$ *on* $E \cap \Omega(\check{I})$.

*Proof.* Fix $z \in E \cap \Omega(\check{I})$. Let $I \in T^{\text{global}}(\check{I})$, and suppose $\text{supp}\,\theta_I \ni z$. Then by (44) in Section 10, $I \in T^{\text{dist}}(\check{I})$, $I$ is of type C1, and $z = z_!(I)$. Hence, by Lemma 12 in Section 14, $F_I^{\#}(z) = f(z)$. Thus, we have shown that

(56) $F_I^{\#}(z) = f(z)$ whenever $z \in E \cap \Omega(\check{I})$, $I \in T^{\text{global}}(\check{I})$, $\text{supp}\,\theta_I \ni z$.

The conclusion of Lemma 7 follows at once from (4), (6) and (56). □

We close this section with the following algorithm:

**Algorithm CJF$^{\#}$** ("Compute the Jet of $F^{\#}$"). Given the Whitney field $\vec{P}^{\check{I}} \in K(\check{I})$ as in (1), we compute the Whitney field $J_{E\cap\Omega(\check{I})}(F^{\#})$. The work and storage used to do so are at most CN. (Here, we do not count the work or storage of Algorithm MMBT, Algorithm JPU, or Algorithm MAK.) We make no calls to the φ-Oracle.

*Explanation*: Fix a point

(57) $z = (x_1, x_2) \in E \cap \Omega(\check{I})$.

From Section 10, we recall the following:

(58) Any $I \in T^{\text{global}}(\check{I})$ such that $\text{supp}\,\theta_I \ni z$ belongs to $\bigwedge(x_1)$. Each $I \in \bigwedge(x_1)$ belongs to $T^{\text{dist}}(\check{I})$, is of type C1, and satisfies $z = z_!(I)$. We have $\#(\bigwedge(x_1)) \le C$.

(59) Moreover, we have precomputed $\bigwedge(x_1)$ as well as $J_z(\theta_I)$ for each $I \in \bigwedge(x_1)$; see Algorithm JPU.

In view of (58) and the definition of $F^{\#}$, we have

(60) $$J_z(F^{\#}) = \sum_{I\in\bigwedge(x_1)} J_z(F_I^{\#}) \odot_z J_z(\theta_I),$$

and there are at most C summands in (60).

Let $I \in \bigwedge(x_1)$. Then (58) gives

(61) $z = z_!(I) \in S(I)$.

Moreover, since $I \in T^{\text{dist}}(\check{I})$, equation (4) in Section 14, and the definition of $F_I^{\#}$ in that section, together yield

(62) $J_{S(I)}(F_I^{\#}) = \vec{P}^I + \lambda(I) J_{S(I)}(e_2)$.

We have precomputed $J_{S(I)}(e_2)$ in Algorithm MMBT, and we have precomputed $\vec{P}^I$ and $\lambda(I)$ in the query algorithm within Algorithm MAK. Consequently, $J_z(F_I^{\#})$ may be computed from (61), (62) using work and storage at most C. Therefore, $J_z(F^{\#})$ may be computed from (59), (60) using work and storage at most C. (Here, we do not count the work or storage of the Algorithm MMBT, Algorithm JPU, or Algorithm MAK.)

Looping over all $z \in E \cap \Omega(\check{I})$, we thus compute $J_{E\cap\Omega(\check{I})}(F^{\#})$ using work and storage at most CN. Note that we have made no calls to the φ-Oracle here. This concludes our explanation of Algorithm CJF$^{\#}$.

## 16. An almost OK interpolant

The set-up of this section is as follows: We adopt the notation, assumptions and boiler-plate constants of Section 8.

We suppose we are given an interval

(0) $\check{I} \subset I_0$ (dyadic), such that $\bar{E} \subset \check{I}^{\text{interior}}$ and $N = \#(\bar{E}) \geq 2$.

Observe that (0) strengthens the assumption made on $\check{I}$ in several previous sections.

We work with the open rectangle

(1) $R(\check{I}) = Q(\check{I}) \cap [\check{I}^{\text{interior}} \times \mathbb{R}] \subset \mathbb{R}^2$.

Note that $E \subset R(\check{I})$. We suppose we are given a finite subset

(2) $S_{00} \subset R(\check{I})$, such that

(3) $\#(S_{00}) \leq \epsilon^{-200}$, and

(4) For every $z = (x_1, x_2) \in S_{00}$, we have $|e_2(z)| = |x_2 - \varphi(x_1)| > 4\epsilon^4|\check{I}|$.

Finally, we suppose we are given a base point

(5) $z_{00} \in S_{00}$.

This completes the list of the assumptions made in this section. We recall a few relevant definitions. For any $x_1 \in \mathbb{R}$, we define

(6) $\delta_{LS}(x_1) = \inf\{r > 0 : [x_1 - r, x_1 + r]$ contains at least two points of $\bar{E}\}$, as in Section 7.

For any point $z = (x_1, x_2) \in \check{I} \times \mathbb{R}$, we define

(7) $\delta_{LS}(z) = \delta_{LS}(x_1) + |x_2 - \varphi(x_1)|$, as in Section 15.

Moreover, for all such $z$, we define an integer $\ell(z)$ by

(8) $2^{\ell(z)-1} < C_1\delta_{LS}(z) \leq 2^{\ell(z)}$, with $C_1$ as in equations (10), (11) in Section 15.

(In Section 15, we defined $\delta_{LS}(z)$ and $\ell(z)$ only for $z \in \Omega(\check{I})$; here, we define these quantities for all $z = (x_1, x_2) \in \check{I} \times \mathbb{R}$.)

As in equation (8) of Section 15, we fix an integer $k_0$, such that

(9) $2^{k_0-10} < \epsilon^{-1} < 2^{k_0}$.

The notion "$(x_1, \ell)$ is $k_0$-regular for the grid $\mathcal{G}_\tau$" has been defined in Section 4, and used, for example, in Lemma 1 in Section 15. Recall that we have picked a dyadic grid $\mathcal{G}_\tau$ in Section 8; see (1) in that section. Our goal here is to present the following algorithm:

**Algorithm AOK** ("Almost OK Interpolant"): We compute a convex polyhedron $K_{00} \subset Wh(S_{00})$, defined by at most $C(\epsilon)$ constraints, such that the following hold:

(A) Let $F \in C^2(2Q(\check{I}))$. Suppose $F = f$ on $E$, $\partial_2 F(z_{00}) = \xi$, and $\| F \|_{C^2(2Q(\check{I}))} \leq 1 - C\epsilon$ for a large enough controlled constant $C$. Then $J_{S_{00}}(F) \in K_{00}$.

(B) (Query Algorithm) After computing $K_{00}$, we can answer queries as follows:

- A query consists of a Whitney field $\vec{P} \in K_{00}$.
- The response to a query $\vec{P} \in K_{00}$ is a Whitney field $\vec{P}^E \in Wh(E)$, such that there exists a function $F \in C^2(R(\check{I}))$, having the following properties:

$$J_{S_{00}}(F) = \vec{P}; \quad J_E(F) = \vec{P}^E; \quad F = f \text{ on } E; \quad \partial_2 F(z_{00}) = \xi;$$

moreover, if a given point $z = (x_1, x_2) \in R(\check{I})$ satisfies either

(i) $(x_1, \ell(z))$ is $k_0$-regular for the grid $\mathcal{G}_\tau$ or

(ii) $2^{k_0+\ell(z)} > \epsilon^{1/2}|\check{I}|$,

then

$$|J_z(F)|_z \leq 1 + C\epsilon.$$

For any $z \in R(\check{I})$, we have $|J_z(F)|_z \leq C$.

The computation of $K_{00}$ uses work at most $C(\epsilon)N \log N$, storage at most $C(\epsilon)N$, and at most $C(\epsilon)N$ calls to the $\varphi$-Oracle.

The work and storage used to answer a query are at most $C(\epsilon)N$. The query algorithm makes no calls to the $\varphi$-Oracle.

Explanation: We start by making some simple observations on the geometry of the sets $S(\check{I})$, $\Omega(\check{I})$, $R(\check{I})$, $Q(\check{I})$. Next, we present the construction of the polyhedron $K_{00}$. Then we prove that $K_{00}$ has property (A) above. After that, we present the query algorithm in (B). Finally, we estimate the computer resources used to compute $K_{00}$ and answer queries. The geometrical observations are as follows: Recall that

(10) $\Omega(\check{I}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \check{I}^{\text{interior}}, |x_2 - \varphi(x_1)| < |\check{I}|\} \subset \mathbb{R}^2.$

See equation (2) in Section 10. We check the inclusions

(11) $\Omega(\check{I}) \subset R(\check{I}) \subset Q(\check{I}).$

Indeed, let $z = (x_1, x_2) \in \Omega(\check{I})$. Since $\sum_{I \in T^{\text{global}}(\check{I})} \theta_I = 1$ on $\Omega(\check{I})$, we have $z \in \text{supp}\, \theta_I$ for some $I \in T^{\text{global}}(\check{I})$. Recalling that $\text{supp}\, \theta_I \subset Q(I) \subset Q(\check{I})$, we conclude that $z \in Q(\check{I})$, and thus $\Omega(\check{I}) \subset Q(\check{I})$. Since also every $(x_1, x_2) \in \Omega(\check{I})$ satisfies $x_1 \in \check{I}^{\text{interior}}$, (11) is now obvious from (1).

Next, we prove the following:

(12) $$\begin{cases} \text{Let } z = (x_1, x_2) \in \mathbb{R}^2, \text{ with } x_1 \in \check{I}^{\text{interior}} \text{ and } 2\epsilon^4|\check{I}| \leq |x_2 - \varphi(x_1)| \leq \frac{1}{10}|\check{I}|. \\ \text{Then there exists } z' = (x_1', x_2') \in S(\check{I}) \cap \Omega(\check{I}), \text{ with the following properties:} \\ \bullet\ |z' - z| < C\epsilon^{100}|\check{I}|. \\ \bullet\ \epsilon^4|\check{I}| < |x_2' - \varphi(x_1')| < \frac{1}{4}|\check{I}|. \\ \bullet\ \text{The closed line segment joining } z' \text{ to } z \text{ is contained in } \Omega(\check{I}). \end{cases}$$

To see this, we recall from Section 9 that

(13) $S(\check{I}) \supset \bigwedge(\check{I}) = (2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}^2) \cap Q(\check{I})$, where

(14) $\frac{1}{10}\epsilon^{100} < 2^{-k_1(\epsilon)} < \epsilon^{100}$.

Now let $z = (x_1, x_2) \in \mathbb{R}^2$, with

(15) $x_1 \in \check{I}^{\text{interior}}$ and $2\epsilon^4|\check{I}| \leq |x_2 - \varphi(x_1)| \leq \frac{1}{10}|\check{I}|$.

Then there exists

(16) $x_1' \in \check{I}^{\text{interior}} \cap 2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}$, such that

(17) $|x_1' - x_1| < C\epsilon^{100}|\check{I}|$.

(This follows from the fact that any interval of length greater than 1 contains an integer.)

Fix such an $x_1'$, and fix

(18) $x_2' \in 2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}$ such that

(19) $|x_2' - x_2| < C\epsilon^{100}|\check{I}|$. Then

(20) $z' := (x_1', x_2') \in 2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}^2$ satisfies

(21) $|z' - z| < C\epsilon^{100}|\check{I}|$.

Let $z'' = (x_1'', x_2'')$ lie on the closed line segment joining $z'$ to $z$. Then

(22) $x_1'' \in \check{I}^{\text{interior}}$, since $x_1, x_1' \in \check{I}^{\text{interior}}$.

Since $|\nabla e_2| \leq C$ on $\check{I}^{\text{interior}} \times \mathbb{R}$, it follows that $|e_2(z'') - e_2(z)| \leq C\epsilon^{100}|\check{I}|$, i.e.,

$$|[x_2'' - \varphi(x_1'')] - [x_2' - \varphi(x_1')]| \leq C\epsilon^{100}|\check{I}|.$$

Hence, by (15), we have

(23) $\epsilon^4|\check{I}| < |x_2'' - \varphi(x_1'')| < \frac{1}{4}|\check{I}|$.

Comparing (22) and (23) with definition (10), we see that $z'' \in \Omega(\check{I})$. Thus,

(24) The closed line segment joining $z'$ to $z$ is contained in $\Omega(\check{I})$.

Also, note that

(25) $\epsilon^4|\check{I}| < |x_2' - \varphi(x_1')| < \frac{1}{4}|\check{I}|$,

since we may take $z'' = z'$ in (23). Moreover, (20) and (24) tell us that $z' \in (2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}^2) \cap \Omega(\check{I})$, and therefore

$$z' \in (2^{-k_1(\epsilon)}|\check{I}|\mathbb{Z}^2) \cap Q(\check{I}) = \bigwedge(\check{I}) \subset S(\check{I}),$$

thanks to (11) and (13). Thus,

(26) $z' \in S(\check{I}) \cap \Omega(\check{I})$.

The proof of (12) is complete, thanks to (21), (24), (25), (26).

Next, we present the algorithm to construct the polyhedron $K_{00}$. The algorithm proceeds in several steps.

Step 0: Using the $\varphi$-Oracle, we compute the Whitney field $J_E(e_2)$.

Step 1: We execute Algorithm MMBT from Section 9.

Step 2: We execute Algorithm JPU from Section 10.

Step 3: We perform the one-time work of Algorithm MAK from Section 13.

Thus, we compute the convex polyhedron $K(\check{I}) \subset Wh(S(\check{I}))$, which satisfies conditions (30)–(32) in Section 13. In particular, $K(\check{I})$ is defined by at most $C(\epsilon)$ constraints.

After Step 3, we will be able to respond to queries, as in (33), (34) of Section 13.

Step 4: We compute the set

(27) $S^+ := S_{00} \cup S(\check{I})$.

Note that

(28) $S^+ \subset Q(\check{I})$ and $\#(S^+) \le C\epsilon^{-200}$.

This follows from (1), (2), (3), together with the definition of the set $S(\check{I})$ in Section 9.

Step 5: By applying algorithm AUB4 from Section 6, we compute a convex polyhedron

(29) $K^+_{AUB} \subset Wh(S^+)$, defined by at most $C(\epsilon)$ constraints, such that the following hold:

(30) Let $F \in C^2(2Q(\check{I}))$ with norm $\le 1$. Then $J_{S^+}(F) \in K^+_{AUB}$.

(31) Let $\vec{P} \in K^+_{AUB}$. Then there exists $F \in C^2(Q(\check{I}))$ with norm $\le 1+\epsilon$ such that $J_{S^+}(F) = \vec{P}^+$.

Step 6: Let $A$ be a constant to be specified later. (We will later take $A$ to be a large enough controlled constant.) We compute the convex polyhedron $K^{++} \subset Wh(S^+) \oplus \mathbb{R}$, defined as follows:

(32) $K^{++} = \{(\vec{P}^+, \lambda) \in Wh(S^+) \oplus \mathbb{R} : \vec{P}^+ \in K^+_{AUB}, |\lambda| \le A|\check{I}|,$
$\vec{P}^+|_{S(\check{I})} + \lambda J_{S(\check{I})}(e_2) \in K(\check{I}), \mathrm{val}\,(\partial_2 \vec{P}^+, z_{00}) = \xi\}.$

Recall that $J_{S(\check{I})}(e_2)$ was already computed (along with many other things) in Step 1 above. Also, $K(\check{I})$ and $K^+_{AUB}$ were computed in Steps 3 and 5. Hence, we can compute $K^{++}$ from (32), once we know the constant $A$.

Step 7: We compute the convex polyhedron

(33) $K_{00} := \{(\vec{P}^+|_{S_{00}}) : (\vec{P}^+, \lambda) \in K^{++}\} \subset Wh(S_{00})$.

Since $K(\check{I})$ and $K^+_{AUB}$, are defined by at most $C(\epsilon)$ constraints, we see by examining Steps 6 and 7 that

(34) $K_{00} \subset Wh(S_{00})$ and $K^{++} \subset Wh(S^+) \oplus \mathbb{R}$ are defined by at most $C(\epsilon)$ constraints.

This completes the computation of $K_{00}$, except that we have not yet picked the constant $A$ in Step 6.

We now prove that $K_{00}$ has property (A) for a suitable choice of the constant $A$. To do so, let

(35) $F \in C^2(2Q(\check{I}))$ with norm $\leq 1 - A\epsilon$, and assume that

(36) $F = f$ on $E$, and $\partial_2 F(z_{00}) = \xi$.

We will prove that

(37) $J_{S_{00}}(F) \in K_{00}$,

under certain assumptions on the constant $A$. To prove (37), we define

(38) $\vec{P}^+ = J_{S^+}(F)$ and

(39) $\lambda = \partial_2 F(z_{00}) - \partial_2 F(z_{\text{left}}(\check{I})) = \xi - \partial_2 F(z_{\text{left}}(\check{I}))$.

By (30) and (35), we have

(40) $\vec{P}^+ \in K^+_{AUB}$.

From (35), we obtain the estimate

(41) $|\lambda| = |\partial_2 F(z_{00}) - \partial_2 F(z_{\text{left}}(\check{I}))| \leq C \operatorname{diam} Q(\check{I}) \leq C'|\check{I}|$. We suppose that

(42) $A > C'$, with $C'$ as in (41).

Then (41) yields at once

(43) $|\lambda| \leq A|\check{I}|$.

Recall that $\| e_2 \|_{C^2(2Q(\check{I}))} \leq C\epsilon |I_0|^{-1}$; see estimate (31) in Section 8. Hence, (41) yields also

(44) $\| \lambda e_2 \|_{C^2(2Q(\check{I}))} \leq C\epsilon |\check{I}| \cdot |I_0|^{-1} \leq C\epsilon$, thanks to (0).

From (35) and (44), we see that the function

(45) $\tilde{F} := F + \lambda e_2 \in C^2(2Q(\check{I}))$ has norm

(46) $\| \tilde{F} \|_{C^2(2Q(\check{I}))} \leq 1 - A\epsilon + C\epsilon$.

We recall that $e_2 = 0$ on $E$ and $\partial_2 e_2 = 1$ on $\check{I} \times \mathbb{R}$. Hence, (36), (39) and (45) imply the equalities

(47) $\tilde{F} = f$ on $E$, and

(48) $\partial_2 \tilde{F}(z_{\text{left}}(\check{I})) = \partial_2 F(z_{\text{left}}(\check{I})) + \lambda = \xi$.

Comparing (46), (47), (48) with property (32) in the statement of Algorithm MAK in Section 13, we see that

(49) $J_{S(\check{I})}(\tilde{F}) \in K(\check{I})$, provided

(50) $A \geq C''$ for a large enough controlled constant $C''$.

Recalling (27), (38) and (45), we see that (49) is equivalent to the inclusion

(51) $\vec{P}^+|_{S(\check{I})} + \lambda J_{S(\check{I})}(e_2) \in K(\check{I})$.

Note also that

(52) $\mathsf{val}\,(\partial_2\vec{P}^+, z_{00}) = \xi$, since (38) holds and $\partial_2 F(z_{00}) = \xi$.

Thus, assuming that $A$ satisfies (42) and (50), we find that $(\vec{P}^+,\lambda) \in Wh(S^+) \oplus \mathbb{R}$ satisfies (40), (43), (51) and (52). Hence, recalling the definitions (32), (33), we conclude that

(53) $(\vec{P}^+,\lambda) \in K^{++}$ and $\vec{P}^+|_{S_{00}} \in K_{00}$, if $A$ satisfies (42), (50).

By (27) and (38), we have $\vec{P}^+|_{S_{00}} = J_{S_{00}}(F)$. Thus, (53) tells us that (37) holds, provided $A$ satisfies (42), (50).

We now pick $A$ to be a controlled constant, large enough to satisfy (42) and (50). Then we have proven that (35), (36) together imply (37).

Since $A$ is a controlled constant, the fact that (35) and (36) together imply (37) completes the proof of property (A) for our polyhedron $K_{00}$.

We pass to the query algorithm (B) of Algorithm AOK. Suppose we are given a query

(54) $\vec{P} \in K_{00}$.

We first explain how to compute the response $\vec{P}^E \in Wh(E)$ to the query (54); then we prove that there exists a function $F \in C^2(R(\check{I}))$ having the properties asserted in (B).

To compute the response $\vec{P}^E$, we proceed as follows: Recall that $K_{00}$ and $K^{++}$ are convex polyhedra, defined by at most $C(\epsilon)$ constraints. Hence, thanks to (33) and (54), routine linear programming allows us to compute a point

(55) $(\vec{P}^+,\lambda) \in K^{++}$, such that

(56) $\vec{P}^+|_{S_{00}} = \vec{P}$.

Recall that we have picked $A$ to be a controlled constant. Hence, from (55) and definition (32), we see that:

(57) $\vec{P}^+ \in K^+_{AUB}$;

(58) $|\lambda| \leq C|\check{I}|$;

(59) $\vec{P}^+|_{S(\check{I})} + \lambda J_{S(\check{I})}(e_2) \in K(\check{I})$; and

(60) $\mathrm{val}\,(\partial_2 \vec{P}^+, z_{00}) = \xi$.

Recall that we have computed $J_{S(\check{I})}(e_2)$ in Step 1 of the one-time work.

Hence, we can now compute the Whitney field

(61) $\vec{P}^{\check{I}} = (P^{\check{I},z})_{z\in S(\check{I})} := \vec{P}^+|_{S(\check{I})} + \lambda J_{S(\check{I})}(e_2)$.

Inclusion (59) tells us that

(62) $\vec{P}^{\check{I}} \in K(\check{I})$.

Accordingly, we can carry out the query algorithm within Algorithm MAK, for the query $\vec{P}^{\check{I}}$ in (62). (See (33), (34) in the statement of that algorithm in Section 13, and recall that we have carried out the one-time work of Algorithm MAK in Step 3 above.)

We are now in position to apply Algorithm CJF$^\#$ in Section 15, taking as data the Whitney field $\vec{P}^{\check{I}}$ in (61), (62). (Note that by this point, we have already executed Algorithms MMBT, JPU and Algorithm MAK, as assumed in Section 15.)

Applying Algorithm CJF$^\#$, we compute a Whitney field

(63) $\vec{P}^{CJ} = (P^{CJ,z})_{z\in E} \in Wh(E)$

such that there exists a function $F^\# \in C^2(\Omega(\check{I}))$, with the following properties:

(64) Let $z = (x_1, x_2) \in \Omega(\check{I})$. Then $|J_z(F^\#)|_z \le C$.

(65) Moreover, let $z = (x_1, x_2) \in \Omega(\check{I})$. If either $(x_1, \ell(z))$ is $k_0$-regular for the grid $\mathcal{G}_\tau$ or $2^{k_0+\ell(z)} > \epsilon^{1/2}|\check{I}|$ then $|J_z(F^\#)|_z \le 1 + C\epsilon$.

(66) $F^\# = f$ on $E \cap \Omega(\check{I}) = E$; see (0).

(67) $J_E(F^\#) = \vec{P}^{CJ}$.

(68) For $z \in S(\check{I}) \cap \Omega(\check{I})$ such that $|e_2(z)| > \epsilon^4|\check{I}|$, we have $|\partial^\alpha(F^\# - P^{\check{I},z})(z)| \le C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ for $|\alpha| \le 1$.

Indeed, (64), (65), (66) and (68) hold, thanks to Lemmas 5, 4, 7 and 6 in Section 15 (respectively); (67) holds because $\vec{P}^{CJ}$ is the Whitney field computed from $\vec{P}^{\check{I}}$ by Algorithm CJF$^\#$, and because $E \cap \Omega(\check{I}) = E$ thanks to (0).

We now compute the Whitney field

(69) $\vec{P}^{E} := \vec{P}^{CJ} - \lambda J_E(e_2) \in Wh(E)$.

(Recall that the Whitney field $J_E(e_2)$ has been computed in Step 0.)

The Whitney field $\vec{P}^E$ is the answer to our query (54).

Thus, we have shown how to compute the response $\vec{P}^E$ to a query $\vec{P} \in K_{00}$. We prepare to show that there exists a function $F \in C^2(R(\check{I}))$ having the properties asserted in (B) of Algorithm AOK.

To do so, we first return to (57). By (31) and (57), there exists a function

(70) $F^+ \in C^2(Q(\check{I}))$ with norm $\le 1 + \epsilon$, such that

(71) $J_{S^+}(F^+) = \vec{P}^+$.

Let us fix such an $F^+$, as well as an $F^\#$ as in (64)–(68). By (61) and (71), we have

(72) $\vec{P}^{\check{I}} = J_{S(\check{I})}(F^+ + \lambda e_2)$, hence

(73) $\vec{P}^{\check{I},z} = J_z(F^+ + \lambda e_2)$ for all $z \in S(\check{I})$.

Therefore (68) tells us that

(74) $|\partial^\alpha(F^\# - \lambda e_2 - F^+)(z)| \leq C\epsilon^{96}|\check{I}|^{2-|\alpha|}$, for $|\alpha| \leq 1$ and $z \in S(\check{I}) \cap \Omega(\check{I})$ such that $|e_2(z)| > \epsilon^4|\check{I}|$.

Note that $F^\# - \lambda e_2 - F^+ \in C^2(\Omega(\check{I}))$ thanks to (11), since $F^\# \in C^2(\Omega(\check{I}))$ and $e_2, F^+ \in C^2(Q(\check{I}))$.

We estimate the norm of $F^\# - \lambda e_2 - F^+$ in $C^2(\Omega(\check{I}))$.

We recall that $\| e_2 \|_{C^2(Q(\check{I}))} \leq C\epsilon|I_0|^{-1}$ by (31) in Section 8. Hence, (11), (58) and (0) together imply that

(75) $\| \lambda e_2 \|_{C^2(\Omega(\check{I}))} \leq C\epsilon|\check{I}| \cdot |I_0|^{-1} \leq C\epsilon$.

Also, (11) and (70) yield

(76) $\| F^+ \|_{C^2(\Omega(\check{I}))} \leq 1 + \epsilon$.

From (64), we have also

(77) $\| F^\# \|_{C^2(\Omega(\check{I}))} \leq C$.

From (75), (76), (77) and the Bounded Distortion Property, we see that

(78) $|\partial^\alpha(F^\# - \lambda e_2 - F^+)| \leq C$ on $\Omega(\check{I})$ for $|\alpha| \leq 2$.

Now let $z = (x_1, x_2) \in \Omega(\check{I})$, and suppose that

(79) $2\epsilon^4|\check{I}| \leq |x_2 - \varphi(x_1)| \leq \frac{1}{10}|\check{I}|$.

Applying observation (12), we obtain a point

(80) $z' = (x_1', x_2') \in S(\check{I}) \cap \Omega(\check{I})$ such that

(81) $|z' - z| < C\epsilon^{100}|\check{I}|$,

(82) $\epsilon^4|\check{I}| < |x_2' - \varphi(x_1')| < \frac{1}{4}|\check{I}|$, and

(83) The closed line segment joining $z'$ to $z$ is contained in $\Omega(\check{I})$.

By (74), (80) and (82), we have

(84) $|\partial^\alpha(F^\# - \lambda e_2 - F^+)(z')| \leq C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ for $|\alpha| \leq 1$.

From (78) and (83), we see that

(85) $|\partial^\alpha(F^\# - \lambda e_2 - F^+)| \leq C$ for $|\alpha| \leq 2$, everywhere on the closed line segment joining $z'$ to $z$.

From (81), (84), (85) and Taylor's theorem, we conclude that

$$|\partial^\alpha(F^\# - \lambda e_2 - F^+)(z)| \le C\epsilon^{96}|\check{I}|^{2-|\alpha|} \quad \text{for } |\alpha| \le 1.$$

Thus, we have proven the following:

(86) Let $z = (x_1, x_2) \in \Omega(\check{I})$, and suppose $2\epsilon^4|\check{I}| \le |x_2 - \varphi(x_1)| \le \frac{1}{10}|\check{I}|$. Then $|\partial^\alpha(F^\# - \lambda e_2 - F^+)(z)| \le C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ for $|\alpha| \le 1$.

We note also that

(87) $\partial_2 F^+(z_{00}) = \xi$,

thanks to (60) and (71).

We prepare to patch together the functions $F^+$ and $F^\# - \lambda e_2$, using a partition of unity.

Fix functions $\theta_{in}, \theta_{out} \in C^2(\mathbb{R})$, with the following properties:

(88) $\theta_{in} + \theta_{out} = 1$ on $\mathbb{R}$.

(89) $0 \le \theta_{in} \le 1$ and $0 \le \theta_{out} \le 1$ on $\mathbb{R}$.

(90) $\theta_{in}(t) = 1$ and $\theta_{out}(t) = 0$ for $|t| \le 2$.

(91) $\theta_{in}(t) = 0$ and $\theta_{out}(t) = 1$ for $|t| \ge 4$.

(92) $\left|\left(\frac{d}{dt}\right)^k \theta_{in}(t)\right|, \left|\left(\frac{d}{dt}\right)^k \theta_{out}(t)\right| \le C$ for $k = 0, 1, 2$ and $t \in \mathbb{R}$.

For $(x_1, x_2) \in R(\check{I}) = Q(\check{I}) \cap [\check{I}^{\text{interior}} \times \mathbb{R}]$, we set

(93) $\chi_{in}(x_1, x_2) = \theta_{in}\left(\frac{x_2 - \varphi(x_1)}{\epsilon^4|\check{I}|}\right)$ and $\chi_{out}(x_1, x_2) = \theta_{out}\left(\frac{x_2 - \varphi(x_1)}{\epsilon^4|\check{I}|}\right)$.

Note that $\chi_{in}$, $\chi_{out}$ are defined only on $R(\check{I})$. Let us establish the basic properties of these cutoff functions. Immediately from (88)–(93), we have the following:

(94) $\chi_{in} + \chi_{out} = 1$ on $R(\check{I})$.

(95) $0 \le \chi_{in} \le 1$ and $0 \le \chi_{out} \le 1$ on $R(\check{I})$.

(96) $\chi_{in}(x_1, x_2) = 1$ and $\chi_{out}(x_1, x_2) = 0$ for $|x_2 - \varphi(x_1)| \le 2\epsilon^4|\check{I}|$.

(97) $\chi_{in}(x_1, x_2) = 0$ and $\chi_{out}(x_1, x_2) = 1$ for $|x_2 - \varphi(x_1)| \ge 4\epsilon^4|\check{I}|$.

We check that also

(98) $|\partial^\alpha \chi_{in}|, |\partial^\alpha \chi_{out}| \le C \cdot (\epsilon^4|\check{I}|)^{-|\alpha|}$ on $R(\check{I})$, for $|\alpha| \le 2$.

To verify (98), recall that $|\nabla e_2| \le C$ and $|\nabla^2 e_2| \le C \cdot (\epsilon^{-1}|I_0|)^{-1}$ on $\check{I} \times \mathbb{R}$; see Section 8. Thus, on $R(\check{I})$,

(99) $|\partial^\alpha e_2| \le C \cdot (\epsilon^{-1}|I_0|)^{1-|\alpha|} \le C \cdot (\epsilon^4|\check{I}|)^{1-|\alpha|}$ for $1 \le |\alpha| \le 2$. (See (0).)

On the other hand, for $|\alpha| \le 2$, $\partial^\alpha \chi_{in}(z)$ is a sum of terms of the form

(100) $\left\{\left(\frac{d}{dt}\right)^k \theta_{in}(t)\Big|_{t = \frac{e_2(z)}{\epsilon^4|\check{I}|}}\right\} \cdot \prod_{\mu=1}^{k} \left(\frac{\partial^{\alpha_\mu} e_2(z)}{\epsilon^4|\check{I}|}\right)$,

with $\alpha_1 + \cdots + \alpha_k = \alpha$, and with each $|\alpha_\mu| \geq 1$. By (92) and (99), each term (100) has absolute value

$$\leq C \cdot \prod_{\mu=1}^{k} \left[ \frac{(\epsilon^4 |\check{I}|)^{1-|\alpha_\mu|}}{\epsilon^4 |\check{I}|} \right] = C \cdot (\epsilon^4 |\check{I}|)^{-|\alpha|}.$$

Hence, $|\partial^\alpha \chi_{in}(z)| \leq C \cdot (\epsilon^4 |\check{I}|)^{-|\alpha|}$ on $R(\check{I})$, for $|\alpha| \leq 2$. The same argument applies to $|\partial^\alpha \chi_{out}(z)|$, completing the proof of (98).

As a consequence of (1), (10) and (97), we have the following:

(101) Let $(x_1, x_2) \in R(\check{I}) \setminus \Omega(\check{I})$. Then $\chi_{in} = 0$ and $\chi_{out} = 1$ in a neighborhood of $(x_1, x_2)$.

The basic properties of $\chi_{in}$, $\chi_{out}$ are (94)–(98) and (101). We now define

(102) $F = \chi_{in} \cdot (F^\# - \lambda e_2) + \chi_{out} \cdot F^+ \in C^2(R(\check{I}))$.

(This makes sense thanks to (101). Recall that $F^\# \in C^2(\Omega(\check{I}))$ and $e_2, F^+ \in C^2(Q(\check{I}))$.) We establish the basic properties of $F$. Let $z \in S_{00}$. Then by (4), (97) and (102), we have $F = F^+$ in a neighborhood of $z$. Consequently, (56), (60) and (71) tell us that

(103) $J_{S_{00}}(F) = \vec{P}$ and

(104) $\partial_2 F(z_{00}) = \xi$.

On the other hand, let $z = (x_1, x_2) \in E$. Then $z \in R(\check{I})$ (as we observed just after (1)); and $e_2(z) = x_2 - \varphi(x_1) = 0$. Hence, (96) applies, and therefore $F = F^\# - \lambda e_2$ in a neighborhood of $z$. In view of the above remarks and (66), (67), (69) we have

(105) $F = f$ on $E$ and

(106) $J_E(F) = J_E(F^\#) - \lambda J_E(e_2) = \vec{P}^{CJ} - \lambda J_E(e_2) = \vec{P}^E$.

Next, we establish the following: Let $z = (x_1, x_2) \in R(\check{I})$. Then

(107) $|J_z(F)|_z \leq C$.

(108) Moreover, if either $(x_1, \ell(z))$ is $k_0$-regular for the grid $\mathcal{G}_\tau$ or $2^{k_0 + \ell(z)} > \epsilon^{1/2} |\check{I}|$, then $|J_z(F)|_z \leq 1 + C\epsilon$.

To check (107), (108), we proceed by cases.

Case 1: Suppose $|x_2 - \varphi(x_1)| < 2\epsilon^4 |\check{I}|$.

Then $(x_1, x_2) \in \Omega(\check{I})$ (see (1) and (10)), and (96) applies. Hence, $F = F^\# - \lambda e_2$ in a neighborhood of $z$. Consequently, (107) and (108) follow from (64), (65) and (75).

Case 2: Suppose $|x_2 - \varphi(x_1)| > 4\epsilon^4 |\check{I}|$.

Then (97) applies. Hence, $F = F^+$ in a neighborhood of $z$. Consequently, (107) and (108) follow from (70).

Case 3: Suppose $2\epsilon^4|\check{I}| \leq |x_2 - \varphi(x_1)| \leq 4\epsilon^4|\check{I}|$.

Then, as in Case 1, $(x_1, x_2) \in \Omega(\check{I})$, and (75) applies. Hence, (64), (65) imply the following:

(109) $|J_z(F^\# - \lambda e_2)|_z \leq C$.

(110) Moreover, if either $(x_1, \ell(z))$ is $k_0$-regular for the grid $\mathcal{G}_\tau$ or $2^{k_0+\ell(z)} > \epsilon^{1/2}|\check{I}|$, then $|J_z(F^\# - \lambda e_2)|_z \leq 1 + C\epsilon$.

Also, (70) tells us that

(111) $|J_z(F^+)|_z \leq 1 + \epsilon$.

Since we are in Case 3, (86) applies. Thus,

(112) $|\partial^\alpha(F^\# - \lambda e_2 - F^+)(z)| \leq C\epsilon^{96}|\check{I}|^{2-|\alpha|}$ for $|\alpha| \leq 1$.

Recall that $F$ is given by (102), with $\chi_{\text{in}}$ and $\chi_{\text{out}}$ satisfying (94), (95), (98).

We define $\delta_z := \epsilon^4|\check{I}|$, and note that

(113) $0 < \delta_z < 1$,

thanks to (0) and assumption (3) in Section 8. The above remarks, together with Lemma GPU in Section 5, imply (107), (108). Thus, (107) and (108) hold in all cases.

Given a query $\vec{P}$ as in (54), we have thus computed a response $\vec{P}^E$ in (69), and proven that the function $F \in C^2(R(\check{I}))$ satisfies (103)–(108). This completes our explanation of part (B) of Algorithm AOK.

It remains to estimate the computer resources used by Algorithm AOK. We begin with the one-time work.

Step 0 requires work and storage $CN$ and $N$ calls to the $\varphi$-Oracle.

Step 1 uses work less than $C(\epsilon)N \log N$, storage at most $C(\epsilon)N$, and at most $C(\epsilon)N$ calls to the $\varphi$-Oracle.

Step 2 uses work and storage at most $CN$, and at most $CN$ calls to the $\varphi$-Oracle.

Step 3 uses work and storage at most $C(\epsilon)N$, and makes no calls to the $\varphi$-Oracle.

Step 4 uses work and storage at most $C(\epsilon)$, and makes no calls to the $\varphi$-Oracle.

Step 5 uses work and storage at most $C(\epsilon)$ (thanks to (28)), and makes no calls to the $\varphi$-Oracle.

Step 6 uses work and storage at most $C(\epsilon)$, and makes no calls to the $\varphi$-Oracle. (Here, we use the fact that $K^+_{\text{AUB}}$ and $K(\check{I})$ are defined by at most $C(\epsilon)$ constraints, and that $\#(S^+) \leq C(\epsilon)$.)

Step 7 uses work and storage at most $C(\epsilon)$ (thanks to (34)), and makes no calls to the $\varphi$-Oracle.

The above remarks may be easily verified by looking up the computer resources used by Algorithms MMBT, JPU, and MAK.

Consequently, the computer resources used to compute $K_{00}$ are as promised.

We turn our attention to the query algorithm (B).

Starting from a query $\vec{P}$ as in (54), we first compute $(\vec{P}, \lambda)$ as in (55), (56). This linear programming requires work and storage at most $C(\epsilon)$, and makes no calls to the $\varphi$-Oracle.

Next, we compute the Whitney field $\vec{P}^{\check{I}}$ in (61). Again, this step requires work and storage at most $C(\epsilon)$, and makes no calls to the $\varphi$-Oracle (because we already computed $J_{S(\check{I})}(e_2)$ in Step 1).

We then carry out the query algorithm from Algorithm MAK, for the query $\vec{P}^{\check{I}}$. This requires work and storage at most $C(\epsilon)N$, and makes no calls to the $\varphi$-Oracle.

The next step is to execute Algorithm CJF$^{\#}$, taking as data the Whitney field $\vec{P}^{\check{I}}$. This step produces the Whitney field $\vec{P}^{CJ}$, and uses work and storage at most $CN$, without calling on the $\varphi$-Oracle.

Finally, we compute $\vec{P}^{E}$ from $\vec{P}^{CJ}$ using (69).

Since the Whitney field $J_E(e_2)$ was computed already in Step 0, this last step uses work and storage at most $CN$, and makes no calls on the $\varphi$-Oracle.

In view of the above remarks, the computer resources used to answer a query as in (B) are as promised in Algorithm AOK.

## 17. Almost optimal interpolants

The interpolants $F$ produced by Algorithm AOK fall short of satisfying the desired estimate $\| F \|_{C^2} \leq 1 + C\epsilon$. In this section, we remedy this defect by averaging over an ensemble of dyadic grids. We recall our convention that $(i.j)$ denotes equation $(j)$ in Section $i$.

The setting for this section is slightly different from that given in Section 8. In this section, we make the following assumptions:

(1) We are given a positive real number $\epsilon$.

(2) We are given an open square $Q_{00} = I_{00} \times J_{00} \subset \mathbb{R}^2$ with sidelength $\delta_{Q_{00}} = |I_{00}| \leq \bar{\bar{C}}_1 \epsilon$.

(3) We are given a finite set $\bar{E} \subset I_{00}$.

(4) Let $\varphi \in C^2(\bar{\bar{c}}_1 \epsilon^{-1} I_{00})$, where $\bar{\bar{c}}_1 \epsilon^{-1} > 1$.

(5) On the interval $\bar{\bar{c}}_1 \epsilon^{-1} I_{00}$, we have $|\varphi'| \leq \bar{\bar{C}}_2$ and $|\varphi''| \leq \bar{\bar{C}}_3 \epsilon |I_{00}|^{-1}$.

(6) Given a point $x_1 \in \bar{\bar{c}}_1 \epsilon^{-1} I_{00}$, a "$\varphi$-Oracle" computes $\varphi(x_1)$, $\varphi'(x_1)$, $\varphi''(x_1)$, charging us "work"

(7) $W_{\varphi O} \geq 1$ for the service.

(8) Let $E = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}\} \subset \mathbb{R}^2$.

(9) We assume $E \subset Q_{00}$.

(10) We assume that $N = \#(E) = \#(\bar{E}) \geq 2$.

(11) We are given a function $f : E \longrightarrow \mathbb{R}$.

(12) We are given a real number $\xi$.

(13) We are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ $(z \in \mathbb{R}^2)$, and an Oracle, satisfying conditions (1)–(4) in Section 5. We define $C^2$-norms as in that section.

(14) We assume there exists $F_{\mathsf{crude}} \in C^2(\mathbb{R}^2)$, such that $\| F_{\mathsf{crude}} \|_{C^2(\mathbb{R}^2)} \leq \bar{\bar{C}}_4$, $F_{\mathsf{crude}} = f$ on $E$, and $|\partial_2 F_{\mathsf{crude}} - \xi| \leq \bar{\bar{C}}_5 \epsilon^{-1} |I_{00}|$ on $E$.

(15) In this section, we take the boiler–plate constants to be $\bar{\bar{C}}_1, \bar{\bar{c}}_1, \bar{\bar{C}}_2, \bar{\bar{C}}_3, \bar{\bar{C}}_4, \bar{\bar{C}}_5$, and the constants called $c_0, C_0, C_1, C_2$ in Section 5.

Our choice (15) gives meaning to the notion of a "controlled constant" in this section.

We make the following

Small $\epsilon$ Assumption

(16) $\epsilon$ is less than a small enough controlled constant.

This concludes our enumeration of the assumptions made in this section.

Given the above objects and assumptions, we introduce the following auxiliary objects. We fix a half-open interval $I_0 = [a, b)$, such that

(17) $I_{00}$ is contained in the middle half of $I_0$,

(18) $50|I_{00}| \leq |I_0| \leq 200|I_{00}|$, and

(19) $|I_0|$ is an integer power of 2.

Thanks to (19), we can fix a real number $t_0$, such that

(20) $I_0 \in \mathcal{G}_{t_0}$ (see Section 4.)

We introduce integers $k_0$, $k_{\mathsf{max}}$, $k_1(\epsilon)$, $\nu_0(\epsilon)$ such that

(21) $10^{-5}|I_{00}| \leq 2^{k_{\mathsf{max}}} \leq 10^{-4}|I_{00}|$.

(22) $2^{k_0 - 10} < \epsilon^{-1} < 2^{k_0}$,

(23) $\frac{1}{10}\epsilon^{100} < 2^{-k_1(\epsilon)} < \epsilon^{100}$, and

(24) $\frac{1}{8}\epsilon^{-2} < 2^{\nu_0(\epsilon)} < \epsilon^{-2}$.

From Section 4, we recall the set $T(k_0, k_{\mathsf{max}})$, having the following properties:

(25) $T(k_0, k_{\mathsf{max}}) \subset [0, 2^{k_{\mathsf{max}}}]$.

(26) $\#(T(k_0, k_{\mathsf{max}})) = 2^{k_0}$.

(27) Let $x_1 \in \mathbb{R}, \ell \in \mathbb{Z}$. If $\ell \leq k_{\mathsf{max}} - k_0$, then there are at most 100 distinct $t \in T(k_0, k_{\mathsf{max}})$ such that $(x_1, \ell)$ is not $k_0$-regular for the grid $\mathcal{G}_{t_0+t}$.

For $t \in T(k_0, k_{max})$, we define

(28) $\tau(t) = t_0 + t$ and

(29) $I_{0,t} = I_0 + t$.

From (20), (28), (29), we have

(30) $I_{0,t} \in \mathcal{G}_{\tau(t)}$ for $t \in T(k_0, k_{max})$.

Also, (21) and (25) give $T(k_0, k_{max}) \subset [0, 10^{-4}|I_{00}|]$. Hence, by (17) and (29), we have

(31) $I_{00} \subset (I_{0,t})^{\text{interior}}$.

In addition, (18) and (29) give

(32) $50|I_{00}| \le |I_{0,t}| \le 200|I_{00}|$.

From (31) and (32), we have

(33) $10^{-4}\bar{\bar{c}}_1\epsilon^{-1}I_{0,t} \subset \bar{\bar{c}}_1\epsilon^{-1}I_{00}$,

with $\bar{\bar{c}}_1$ as in (4), (5), (6). Observations (30)–(33) hold for all $t \in T(k_0, k_{max})$.

**Lemma 1.** *Let* $t \in T(k_0, k_{max})$. *Then assumptions* (8.1)–(8.21) *hold here, with our present* $\tau(t)$ *and* $I_{0,t}$ *in place of* $\tau$ *and* $I_0$ *in Section* 8, *respectively. Moreover, we can take the constants in* (8.20) *to be controlled constants independent of* $t$.

*Proof.* (8.1) simply asserts that $\tau(t)$ is a real number.

(8.2) asserts that $0 < \epsilon < \bar{c}_1$. We take $\bar{c}_1 = 10^{-4}\bar{\bar{c}}_1$; (8.2) holds thanks to our Small $\epsilon$ Assumption (16).

(8.3) asserts that $I_{0,t} \in \mathcal{G}_{\tau(t)}$ and $|I_{0,t}| \le \bar{C}_1\epsilon$. This follows from (30), (32) and (2), with $\bar{C}_1 = 200\bar{\bar{C}}_1$.

(8.4) asserts that $\bar{E} \subset I_{0,t}$ is finite. This follows from (3) and (31).

(8.5) asserts that $\varphi \in C^2(\bar{c}_1\epsilon^{-1}I_{0,t})$. This follows from (4) and (33), since we have taken $\bar{c}_1 = 10^{-4}\bar{\bar{c}}_1$.

(8.6) asserts that on $\bar{c}_1\epsilon^{-1}I_{0,t}$ we have $|\varphi'| \le \bar{C}_2$ and $|\varphi''| \le \bar{C}_2\epsilon|I_{0,t}|^{-1}$. Thanks to (33), and thanks to our choice $\bar{c}_1 = 10^{-4}\bar{\bar{c}}_1$, this follows from (5) and (32), with $\bar{C}_2 = \bar{\bar{C}}_2 + \bar{\bar{C}}_3 \cdot 200$.

(8.7), (8.8) assert that, given any $x_1 \in \bar{c}_1\epsilon^{-1}I_{0,t}$, the $\varphi$-Oracle computes $\varphi(x_1)$, $\varphi'(x_1), \varphi''(x_1)$, and charges us "work" $W_{\varphi O} \ge 1$. Since we have taken $\bar{c}_1 = 10^{-4}\bar{\bar{c}}_1$, this follows from (6), (7), (33).

(8.9) asserts that $E = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}\} \subset \mathbb{R}^2$, which is just (8).

(8.10) asserts that $N = \#(E) = \#(\bar{E}) \ge 2$, which is just (10).

(8.11) asserts that we are given $f : E \to \mathbb{R}$, which is just (11).

(8.12) asserts that we are given a real number $\xi$; that's just (12).

(8.13) asserts that we are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ $(z \in \mathbb{R}^2)$, and an $\mathsf{Oracle}$, as in Section 5. That's just (13). We use the same constants $c_0, C_0, C_1, C_2$ in (8.13) as in (13).

(8.14)–(8.17) assert the existence of a function $F_{\mathsf{crude}} \in C^2(\mathbb{R}^2)$ such that $F_{\mathsf{crude}} = f$ on $E$, $|\partial_2 F_{\mathsf{crude}} - \xi| \leq \bar{C}_3 \epsilon^{-1} |I_{0,t}|$ on $E$, and $\| F_{\mathsf{crude}} \|_{C^2(\mathbb{R}^2)} \leq \bar{C}_3$.
This follows from (14), with $\bar{C}_3 = \bar{\bar{C}}_4 + \bar{\bar{C}}_5$, thanks to (32).

(8.18), (8.19) are just (23) and (24).

(8.20) declares that the boiler-plate constants for Section 8 are $\bar{c}_1$, $\bar{C}_1$, $\bar{C}_2, \bar{C}_3$, together with $c_0, C_0, C_1, C_2$. Recall from our discussion of (8.1)–(8.19) that $\bar{c}_1 = 10^{-4}\bar{\bar{c}}_1$, $\bar{C}_1 = 200\bar{\bar{C}}_1$, $\bar{C}_2 = \bar{\bar{C}}_2 + 200\bar{\bar{C}}_3$, $\bar{C}_3 = \bar{\bar{C}}_4 + \bar{\bar{C}}_5$; and $c_0, C_0, C_1, C_2$ are as in (13), (i.e., as in (1)–(4) in Section 5).
Hence, by (15), all the boiler-plate constants of Section 8 are controlled constants in the sense of the present section.

It follows that any controlled constant in the sense of Section 8 is also a controlled constant (in the sense of the present section), that does not depend on $t$. Consequently, (8.21) follows from our $\mathsf{Small}\ \epsilon\ \mathsf{Assumption}$ (16). The proof of Lemma 1 is complete □

Thanks to Lemma 1, the definitions made in Section 8 make sense here, and the observations made there are valid here (with $\tau(t)$ and $I_{0,t}$ in place of $\tau$ and $I_0$, respectively). Also, as observed in the proof of Lemma 1, any controlled constant in the sense of Section 8 is also a controlled constant (in the sense of the present section), that does not depend on $t$.

From Section 8, we recall that we have defined the open square

(34) $Q(I_{0,t}) \subset \mathbb{R}^2$, with sidelength

(35) $\delta_{Q(I_{0,t})} = \tilde{C}_Q |I_{0,t}|$,

for a controlled constant $\tilde{C}_Q$ independent of $t$. We recall (8.29), which tells us in particular that

(36) $(x_1, \varphi(x_1))$ belongs to the middle half of $Q(I_{0,t})$ for all $x_1 \in I_{0,t}$.

Note that (36) implies

(37) $\delta_{Q(I_{0,t})} \geq |I_{0,t}|$, i.e., $\tilde{C}_Q \geq 1$ in (35).

As in Section 16, we introduce the open rectangle

(38) $R(I_{0,t}) = Q(I_{0,t}) \cap [I_{0,t}^{\mathsf{interior}} \times \mathbb{R}]$.

We have (34)–(38) for all $t \in T(k_0, k_{\mathsf{max}})$. The following result relates $Q(I_{0,t})$ and $R(I_{0,t})$ to $Q_{00}$ (see (2), (9)).

**Lemma 2.** *For any* $t \in T(k_0, k_{\mathsf{max}})$, *we have*

(39) $Q_{00} \subset R(I_{0,t})$ *and*

(40) $2Q(I_{0,t}) \subset CQ_{00}$.

*Proof.* Let $\tilde{z} = (\tilde{x}_1, \tilde{x}_2) \in E$. Then $\tilde{x}_1 \in I_{00} \subset I_{0,t}^{\text{interior}}$, and $\tilde{x}_2 = \varphi(\tilde{x}_1)$; see (3), (8) and (31). Consequently, $\tilde{z}$ lies in the middle half of $Q(I_{0,t})$; see (36). Also, $\tilde{z} \in Q_{00}$ by (9). Hence,

(41) $Q_{00}$ intersects the middle half of $Q(I_{0,t})$.

In addition, we know from (2), (32), (35) and (37) that

$$\delta_{Q_{00}} = |I_{00}| \leq \frac{1}{50}|I_{0,t}| \leq \frac{1}{50}\delta_{Q(I_{0,t})} = \frac{\tilde{C}_Q}{50}|I_{0,t}| \leq C|I_{00}|.$$

Thus,

(42) $50\delta_{Q_{00}} \leq \delta_{Q(I_{0,t})} \leq C\delta_{Q_{00}}$.

From (41) and (42), we see that (40) holds, and

(43) $Q_{00} \subset Q(I_{0,t})$.

Moreover, $Q_{00} = I_{00} \times J_{00} \subset I_{00} \times \mathbb{R} \subset (I_{0,t})^{\text{interior}} \times \mathbb{R}$; see (2) and (31). Together with (43), this gives $Q_{00} \subset Q(I_{0,t}) \cap [(I_{0,t})^{\text{interior}} \times \mathbb{R}] = R(I_{0,t})$; see (38). This proves (39), completing the proof of Lemma 2. □

We are now ready to present the main result of this section.

**Algorithm AOI.** ("Almost Optimal Interpolant"): We suppose we are given the objects and assumptions (1)–(16). Suppose also that we are given a finite subset

(44) $S_{00} \subset Q_{00}$, such that

(45) $\#(S_{00}) \leq \epsilon^{-200}$, and

(46) $|x_2 - \varphi(x_1)| > \epsilon^3 |I_{00}|$ for all $(x_1, x_2) \in S_{00}$.

Finally, suppose we are given a base point

(47) $z_{00} \in S_{00}$.

Then we compute a convex polyhedron $K(S_{00}, Q_{00}, z_{00}) \subset Wh(S_{00})$, defined by at most $C(\epsilon)$ constraints, such that the following hold for a large enough controlled constant $C_A$:

(48) Let $F \in C^2(C_A Q_{00})$ with norm $\leq 1 - C_A\epsilon$, and suppose that $F = f$ on $E$ and $\partial_2 F(z_{00}) = \xi$. Then $J_{S_{00}}(F) \in K(S_{00}, Q_{00}, z_{00})$.

(49) After we have computed $K(S_{00}, Q_{00}, z_{00})$, we can answer queries as follows: A query consists of a Whitney field $\vec{P} \in K(S_{00}, Q_{00}, z_{00})$. The response to a query $\vec{P}$ consists of a Whitney field $\vec{P}^E \in Wh(E)$, such that there exists $F \in C^2(Q_{00})$ with norm $\leq 1 + C\epsilon$, such that $F = f$ on $E$, $\partial_2 F(z_{00}) = \xi$, $J_{S_{00}}(F) = \vec{P}$, $J_E(F) = \vec{P}^E$.

The computation of $K(S_{00}, Q_{00}, z_{00})$ uses work at most $C(\epsilon) N \log N$, and storage at most $C(\epsilon)N$; and makes at most $C(\epsilon)N$ calls to the $\varphi$-Oracle. To answer a given query as in (49), we use work and storage at most $C(\epsilon)N$, and we make no calls to the $\varphi$-Oracle.

*Explanation*: We start by computing $I_0$, $t_0$, $k_0$, $k_{max}$, $k_1(\epsilon)$, $\nu_0(\epsilon)$ and $T(k_0, k_{max})$ as in (17)–(27). For each $t \in T(k_0, k_{max})$, we compute $\tau(t)$ and $I_{0,t}$ from (28), (29), as well as $Q(I_{0,t})$ as in Section 8 and (34)–(38). These trivial computations use work and storage at most $C(\epsilon)$, and make at most $C(\epsilon)$ calls to the $\varphi$-Oracle.

We prepare to apply Algorithm AOK (from Section 16), for each $t \in T(k_0, k_{max})$. To do so, we first check that the assumptions of Section 16 hold here, provided we set $\check{I} = I_{0,t}$, and take $I_{0,t}$ and $\tau(t)$ in place of $I_0, \tau$ respectively.

Indeed, Lemma 1 tells us that the assumptions of Section 8 hold. The boiler-plate constants of Section 16 are those of Section 8. We have seen in Lemma 1 that those constants are controlled (in the sense of the present section) and independent of $t$.

The remaining assumptions of Section 16 are (16.0)–(16.5). Let us check that those assumptions hold here.

(16.0) asserts that $I_{0,t} \subset I_{0,t}, I_{0,t} \in \mathcal{G}_{\tau(t)}, \bar{E} \subset I_{0,t}^{\text{interior}}$, and $N = \#(\bar{E}) \geq 2$. These assertions follow from (30), (3), (31), and (10).

(16.1) defines $R(I_{0,t})$, precisely as in (38).

(16.2) asserts that $S_{00} \subset R(I_{0,t})$, which follows from (44) and (39).

(16.3) asserts that $\#(S_{00}) \leq \epsilon^{-200}$, which is just (45).

(16.4) asserts that $|x_2 - \varphi(x_1)| > 4\epsilon^4 |I_{0,t}|$ for all $(x_1, x_2) \in S_{00}$.

This assertion follows from (46), since $\epsilon^3 |I_{00}| > 4\epsilon^4 |I_{0,t}|$, by (32) and (16).

(16.5) asserts that $z_{00} \in S_{00}$, which is just (47).

Thus, as claimed, the assumptions of Section 16 hold here, with $I_{0,t}$ in place of $\check{I}$, $I_{0,t}$ in place of $I_0$, and $\tau(t)$ in place of $\tau$. The boiler-plate constants of Section 16 may be taken here to be controlled constants independent of $t$.

Note that our present $k_0$ is the same as the $k_0$ in Section 16. (See (22) and (16.9).) We now make the following definitions:

For $x_1 \in \mathbb{R}$, we define

(50) $\delta_{LS}(x_1) = \inf\{r > 0 : [x_1 - r, x_1 + r]$ contains at least two points of $\bar{E}\}$.

For $z = (x_1, x_2) \in Q_{00}$, we define

(51) $\delta_{LS}(z) = \delta_{LS}(x_1) + |x_2 - \varphi(x_1)|$;

and we define $\ell(z)$ for such $z$ by

(52) $2^{\ell(z)-1} < C_1 \delta_{LS}(z) \leq 2^{\ell(z)}$.

Here, $C_1$ is the controlled constant from equation (16.8). Note that $C_1$ is independent of $t$, since it is computed from the boiler-plate constants of Section 16.

Our definitions (50), (51), (52) agree with definitions (16.6), (16.7), (16.8). More precisely, (50) is the same as (16.6), while definitions (16.7), (16.8) are more general than our present definitions (51), (52). (Here, we assume $z \in Q_{00}$, whereas for (16.7), (16.8) we assume merely that $z \in I_{0,t} \times \mathbb{R}$. As usual, we are taking $I_{0,t}$ in place of $\check{I}$. We know that $Q_{00} \subset I_{0,t} \times \mathbb{R}$; see (38) and (39).) Note that $\ell(z)$ is independent of $t$, for fixed $z \in Q_{00}$.

We can now pass to the next steps in Algorithm AOI.

For each $t \in T(k_0, k_{\mathsf{max}})$, we perform the one-time work of Algorithm AOK, taking $I_{0,t}$ in place of $\check{I}$, $I_{0,t}$ in place of $I_0$, and $\tau(t)$ in place of $\tau$.

Thus, for each $t \in T(k_0, k_{\mathsf{max}})$, we obtain a convex polyhedron

(53) $K_{00}(t) \subset Wh(S_{00})$, defined by at most $C(\epsilon)$ constraints, such that the following hold:

(54) Let $F \in C^2(2Q(I_{0,t}))$ with norm $\leq 1 - C\epsilon$ for large enough $C$, and suppose $F = f$ on $E$ and $\partial_2 F(z_{00}) = \xi$. Then $J_{S_{00}}(F) \in K_{00}(t)$.

(55) After computing $K_{00}(t)$, we can answer queries as follows: A query consists of a Whitney field $\vec{P} \in K_{00}(t)$. The response to a query $\vec{P}$ is a Whitney field $\vec{P}^E_t \in Wh(E)$, such that there exists $F_t \in C^2(R(I_{0,t}))$ with the following properties:

(a) $F_t = f$ on $E$, $\partial_2 F_t(z_{00}) = \xi$, $J_{S_{00}}(F_t) = \vec{P}$, $J_E(F_t) = \vec{P}^E_t$.

(b) Let $z = (x_1, x_2) \in Q_{00}$. Then $|J_z(F_t)|_z \leq C$. Moreover, if either

(i) $(x_1, \ell(z))$ is $k_0$-regular for the grid $\mathcal{G}_{\tau(t)}$ or

(ii) $2^{k_0 + \ell(z)} > \epsilon^{1/2} |I_{0,t}|$

then $|J_z(F_t)|_z \leq 1 + C\epsilon$.

(In (55)(b), we have taken $z \in Q_{00}$ rather than $z \in R(I_{0,t})$. That's allowed, thanks to (39).)

We recall the following from Algorithm AOK:

(56) The computation of a single $K_{00}(t)$ uses work at most $C(\epsilon) N \log N$, and storage at most $C(\epsilon)N$; and it makes at most $C(\epsilon)N$ calls to the $\varphi$-Oracle. To answer a query as in (55), we use work and storage at most $C(\epsilon)N$, and make no calls to the $\varphi$-Oracle.

The polyhedron $K(S_{00}, Q_{00}, z_{00})$ is defined as

(57) $$K(S_{00}, Q_{00}, z_{00}) = \bigcap_{t \in T(k_0, k_{\mathsf{max}})} K_{00}(t).$$

Since $\#(T(k_0, k_{\mathsf{max}})) = 2^{k_0} \leq C\epsilon^{-1}$, we have

(58) $K(S_{00}, Q_{00}, z_{00}) \subset Wh(S_{00})$ is a convex polyhedron, defined by at most $C(\epsilon)$ constraints.

Once we have computed all the $K_{00}(t)(t \in T(k_0, k_{max}))$, we can compute $K(S_{00}, Q_{00}, z_{00})$ from (57), using work and storage at most $C(\epsilon)$, and making no calls to the $\varphi$-Oracle.

We now check that our polyhedron $K(S_{00}, Q_{00}, z_{00})$ satisfies (48).

Indeed, let $F \in C^2(C_A Q_{00})$ with norm $\leq 1 - C_A \epsilon$, and suppose $F = f$ on $E$ and $\partial_2 F(z_{00}) = \xi$. If $C_A$ is a large enough controlled constant, then (40) shows that $F \in C^2(2Q(I_{0,t}))$ with norm $\leq 1 - C_A \epsilon$, for each $t \in T(k_0, k_{max})$. Consequently, (54) yields $J_{S_{00}}(F) \in K_{00}(t)$ for each $t \in T(k_0, k_{max})$. By definition (57), we therefore have $J_{S_{00}}(F) \in K(S_{00}, Q_{00}, z_{00})$, completing the proof of (48).

Next, we provide the query algorithm (49). Suppose we have finished the computation of $K(S_{00}, Q_{00}, z_{00})$. Then, for each $t \in T(k_0, k_{max})$, we can answer queries as in (55). Now let $\vec{P} \in K(S_{00}, Q_{00}, z_{00})$ be a query. We must compute a response $\vec{P}^E \in Wh(E)$ such that there exists $F \in C^2(Q_{00})$ with the properties asserted in (49). By (57), we have $\vec{P} \in K_{00}(t)$ for each $t \in T(k_0, k_{max})$. Hence, applying (55) for each such $t$, we compute Whitney fields $\vec{P}^E_t \in Wh(E)$ for which there exist $F_t \in C^2(R(I_{0,t}))$ satisfying (55)(a) and (55)(b). Recalling (39), we see that each $F_t$ belongs to $C^2(Q_{00})$. We now define

(59) $F = [\#(T(k_0, k_{max}))]^{-1} \sum_{t \in T(k_0, k_{max})} F_t \in C^2(Q_{00})$ and

(60) $\vec{P}^E = [\#(T(k_0, k_{max}))]^{-1} \sum_{t \in T(k_0, k_{max})} \vec{P}^E_t \in Wh(E).$

This is the long-promised averaging over the ensemble of dyadic grids. Note that, once we know the $\vec{P}^E_t$ for each $t \in T(k_0, k_{max})$, we can compute $\vec{P}^E$ from (60) using work and storage at most $C(\epsilon)N$, without calling on the $\varphi$-Oracle.

Let us check that $\vec{P}^E$ and $F$ have the desired properties given in (49). First of all, since (55)(a) holds for each $t \in T(k_0, k_{max})$, (59) and (60) yield at once that

(61) $F = f$ on $E$, $\partial_2 F(z_{00}) = \xi$, $J_{S_{00}}(F) = \vec{P}$, and $J_E(F) = \vec{P}^E$.

Using (55)(b), we will prove that

(62) $\| F \|_{C^2(Q_{00})} \leq 1 + C\epsilon$.

From (61), (62), we will be assured that our $\vec{P}^E$ from (60) correctly answers the query $\vec{P}$, as in (49). To prove (62), we fix $z = (x_1, x_2) \in Q_{00}$, and note that

(63) $|J_z(F)|_z \leq [\#(T(k_0, k_{max}))]^{-1} \cdot \sum_{t \in T(k_0, k_{max})} |J_z(F_t)|_z$, thanks to (59).

We distinguish two cases.

Case 1: Suppose $2^{k_0 + \ell(z)} > \epsilon^{1/2} |I_0|$. Then, for each $t \in T(k_0, k_{max})$, we have $2^{k_0 + \ell(z)} > \epsilon^{1/2} |I_{0,t}|$; see (29). Consequently, (55)(b)(ii) applies, and thus $|J_z(F_t)|_z \leq 1 + C\epsilon$ for each $t \in T(k_0, k_{max})$. Therefore, (63) yields the estimate

(64) $|J_z(F)|_z \leq 1 + C\epsilon$ in Case 1.

Case 2: Suppose $2^{k_0+\ell(z)} \leq \epsilon^{1/2}|I_0|$. Then (18), (21) and (16) tell us that

$$2^{k_0+\ell(z)} \leq 200\epsilon^{1/2}|I_{00}| \leq 200\epsilon^{1/2} \cdot (10^5 \cdot 2^{k_{max}}) \leq 2^{k_{max}}.$$

Thus, $k_0 + \ell(z) \leq k_{max}$. Accordingly, (27) applies.

From (27), (28), we learn that:

(65) There are at most 100 distinct $t \in T(k_0, k_{max})$ such that $(x_1, \ell(z))$ is not $k_0$-regular for the grid $\mathcal{G}_{\tau(t)}$.

Together with (55)(b), this tells us the following:

(66) We have $|J_z(F_t)|_z \leq 1 + C\epsilon$ for all but at most 100 distinct $t \in T(k_0, k_{max})$. Moreover, $|J_z(F_t)|_z \leq C$ for all $t \in T(k_0, k_{max})$.

From (63), (66), we obtain the estimate

$$|J_z(F)|_z \leq 1 + C\epsilon + \frac{100}{\#[T(k_0, k_{max})]} \cdot C.$$

Recalling that $\#[T(k_0, k_{max})] = 2^{k_0} > c\epsilon^{-1}$ (see (26) and (22)), we conclude that

(67) $|J_z(F)|_z \leq 1 + C\epsilon$ in Case 2.

From (67) and (64), we see that $|J_z(F)|_z \leq 1 + C\epsilon$ for all $z \in Q_{00}$, completing the proof of (62). Thus, our query algorithm answers queries correctly.

In view of the comments provided above (regarding work, storage and calls to the $\varphi$-Oracle), it is now trivial to check that our use of computer resources is as promised in the statement of Algorithm AOI.

## 18. Almost optimal interpolants, version 2

In this section, we adopt the following assumptions:

(1) We are given a positive real number $\epsilon$.

(2) We are given an open square $Q_{00} = I_{00} \times J_{00} \subset \mathbb{R}^2$, with $\delta_{Q_{00}} = |I_{00}| \leq \bar{C}_1\epsilon$.

(3) We are given a finite set $\bar{E} \subset I_{00}$.

(4) A function $\varphi$ is given in $C^2(\bar{c}_1\epsilon^{-1}I_{00})$, where $\bar{c}_1\epsilon^{-1} > 1$.

(5) On $\bar{c}_1\epsilon^{-1}I_{00}$, we have $|\varphi'| \leq \bar{C}_2$ and $|\varphi''| \leq \bar{C}_3\epsilon|I_{00}|^{-1}$.

(6) Given a point $x_1 \in \bar{c}_1\epsilon^{-1}I_{00}$, a "$\varphi$-Oracle" computes $\varphi(x_1)$, $\varphi'(x_1)$, $\varphi''(x_1)$, charging us "work"

(7) $W_{\varphi O} \geq 1$ for the service.

(8) $E = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}\} \subset \mathbb{R}^2$.

(9) We assume $E \subset Q_{00}$.

(10) $N = \#(E) = \#(\bar{E}) \geq 2$.

(11) We are given a function $f : E \to \mathbb{R}$.

(12) We are given an interval $I_\Gamma \subset \mathbb{R}$, with $|I_\Gamma| \leq \bar{C}_4 \epsilon^{-1} |I_{00}|$.

(13) We are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ $(z \in \mathbb{R}^2)$, and an Oracle, satisfying conditions (1)–(4) in Section 5. We define $C^2$ norms as in that section.

(14) We assume that there exists $F_{\text{crude}} \in C^2(\mathbb{R}^2)$, such that $\| F_{\text{crude}} \|_{C^2(\mathbb{R}^2)} \leq \bar{C}_5$, $F_{\text{crude}} = f$ on $E$, and $\partial_2 F_{\text{crude}}(z) \in I_\Gamma$ for all $z \in E$.

(15) In this section, we take the boiler-plate constants to be $\bar{c}_1, \bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4, \bar{C}_5$ above, together with the constants called $c_0, C_0, C_1, C_2$ in Section 5.

We make the following Small $\epsilon$ Assumption:

(16) $\epsilon$ is less than a small enough controlled constant.

The main result of this section is as follows:

**Algorithm AOI, Version 2.** Suppose we are given the above objects and assumptions, as well as the following data:

(17) A finite subset $S_{00} \subset Q_{00}$, such that

(18) $\#(S_{00}) < \epsilon^{200}$ and

(19) $|x_2 - \varphi(x_1)| > \epsilon^3 |I_{00}|$ for all $(x_1, x_2) \in S_{00}$.

(20) A base point $z_{00} \in S_{00}$.

Then we compute a convex polyhedron $K \subset Wh(S_{00})$, defined by at most $C(\epsilon)$ constraints, such that the following hold for a large enough controlled constant $C_A$:

(21) Let $F \in C^2(C_A Q_{00})$ with norm $\leq 1 - C_A \epsilon$. Suppose $F = f$ on $E$ and $\partial_2 F(z_{00}) \in I_\Gamma$. Then $J_{S_{00}}(F) \in K$.

(22) After we have computed $K$, we can answer queries as follows: A query consists of a Whitney field $\vec{P} \in K$. The response to a query $\vec{P} \in K$ is a Whitney field $\vec{P}^E \in Wh(E)$ such that there exists $F \in C^2(Q_{00})$ with the following properties: $\| F \|_{C^2(Q_{00})} \leq 1 + C\epsilon$; $F = f$ on $E$; $\partial_2 F(z_{00}) \in I_\Gamma$; $J_{S_{00}}(F) = \vec{P}$; $J_E(F) = \vec{P}^E$.

The computation of $K$ uses work at most $C(\epsilon) N \log N$, storage at most $C(\epsilon) N$, and at most $C(\epsilon) N$ calls to the $\varphi$-Oracle. To answer a query as in (22), we use work and storage at most $C(\epsilon) N$, and make no calls to the $\varphi$-Oracle.

*Explanation*: We first compute a finite list of points

(23) $\xi_1, \ldots, \xi_{\nu_{\max}} \in I_\Gamma$, such that

(24) Any given $\xi \in I_\Gamma$ satisfies $|\xi - \xi_\nu| \leq \epsilon |I_\Gamma|$ for some $\nu$, and

(25) $\nu_{\max} \leq C\epsilon^{-1}$.

For each $\nu$, we note that assumptions (1)–(14) of Section 17, as well as (16) in Section 17, hold here, with $\xi_\nu$ in place of $\xi$. Moreover, the boiler-plate constants given in (15) of Section 17 are controlled constraints in the sense of this section. (See (15).)

In addition, assumptions (44)–(47) of Section 17 hold here. (We leave to the reader the trivial verification of the above remarks.) Hence, for each $\nu$, we may perform Algorithm AOI, with $\xi_\nu$ in place of $\xi$. Thus, for each $\nu$, we obtain a convex polyhedron

(26) $K_\nu \subset Wh(S_{00})$, defined by at most $C(\epsilon)$ constraints, such that the following hold, for a large enough controlled constant $\hat{C} > 1$:

(27) Let $F \in C^2(\hat{C}Q_{00})$ with norm $\leq 1 - \hat{C}\epsilon$, and suppose that $F = f$ on $E$ and $\partial_2 F(z_{00}) = \xi_\nu$. Then $J_{S_{00}}(F) \in K_\nu$.

(28) After we have computed $K_\nu$, we can answer queries, as follows: A query consists of a Whitney field $\vec{P} \in K_\nu$. The response to a query $\vec{P} \in K_\nu$ is a Whitney field $\vec{P}^E_\nu \in Wh(E)$ such that there exists $F_\nu \in C^2(Q_{00})$ with the following properties: $\| F_\nu \|_{C^2(Q_{00})} \leq 1 + C\epsilon$; $F_\nu = f$ on $E$; $\partial_2 F_\nu(z_{00}) = \xi_\nu$; $J_{S_{00}}(F_\nu) = \vec{P}$, $J_E(F_\nu) = \vec{P}^E_\nu$.

The computation of a single $K_\nu$ requires work $\leq C(\epsilon) N \log N$, storage $\leq C(\epsilon)N$, and at most $C(\epsilon)N$ calls to the $\varphi$-Oracle. Thanks to (25), the same holds for the computation of all the $K_\nu$.

Once we have computed all the $K_\nu$, the work and storage used to answer a query in (28) are at most $C(\epsilon)N$; and we make no calls to the $\varphi$-Oracle in (28). For the rest of this section, we fix $\hat{C}$ as in (27). As in many previous sections, we work with the function

(29) $e_2(x_1, x_2) = x_2 - \varphi(x_1)$, defined for $(x_1, x_2) \in \hat{C}Q_{00}$.

From (5), we see that on $\hat{C}Q_{00}$, we have: $|\partial_1 e_2| \leq C$, $\partial_2 e_2 \equiv 1$, $\partial^2_{12} e_2 = \partial^2_{22} e_2 = 0$, and $|\partial^2_1 e_2| \leq C\epsilon |I_{00}|^{-1}$. Note also that $e_2 = 0$ at any point of $E$, and $\emptyset \neq E \subset Q_{00} \subset \hat{C}Q_{00}$. Moreover, $\epsilon |I_{00}|^{-1} \geq c$; see (2). By the above remarks and Taylor's theorem, we have

(30) $\| e_2 \|_{C^2(\hat{C}Q_{00})} \leq C\epsilon |I_{00}|^{-1}$.

We have

(31) $e_2 = 0$ on $E$, and $\partial_2 e_2 \equiv 1$ on $\hat{C}Q_{00}$, as noted many times before.

Next, let $F \in C^2(\hat{C}Q_{00})$, and suppose $F = f$ on $E$ and $\partial_2 F(z_{00}) \in I_\Gamma$. Then there exists $\nu$ such that $|\xi_\nu - \partial_2 F(z_{00})| \leq \epsilon |I_\Gamma| \leq C|I_{00}|$ (see (12) and (24)). Fix such a $\nu$, and define

$$\tilde{F} = F + [\xi_\nu - \partial_2 F(z_{00})] e_2 \in C^2(\hat{C}Q_{00}), \quad \mu = [\partial_2 F(z_{00}) - \xi_\nu].$$

Then $\tilde{F} = f$ on $E$, $\partial_2 \tilde{F}(z_{00}) = \xi_\nu$, and

$$\| \tilde{F} \|_{C^2(\hat{C}Q_{00})} \leq \| F \|_{C^2(\hat{C}Q_{00})} + C|I_{00}| \cdot C\epsilon |I_{00}|^{-1} \quad \text{(by (30))}.$$

If $\| F \|_{C^2(\hat{C}Q_{00})} \leq 1 - C_A \epsilon$ for a large enough controlled constant $C_A > \hat{C}$ then we have shown that $\| \tilde{F} \|_{C^2(\hat{C}Q_{00})} \leq 1 - \hat{C}\epsilon$, $\tilde{F} = f$ on $E$, and $\partial_2 \tilde{F}(z_{00}) = \xi_\nu$; hence $J_{S_{00}}(\tilde{F}) \in K_\nu$, by (27). For the rest of this section, we fix $C_A$ as above.

Thus, we have proven the following:

(32) Let $F \in C^2(C_A Q_{00})$ with norm $\leq 1 - C_A \epsilon$. Suppose $F = f$ on $E$ and $\partial_2 F(z_{00}) \in I_\Gamma$. Then for some $\nu$ $(1 \leq \nu \leq \nu_{\mathsf{max}})$, and for some real number $\mu$, we have $J_{S_{00}}(F) \in K_\nu + \mu J_{S_{00}}(e_2)$ and $|\mu| \leq C_B |I_{00}|$.

For the rest of this section, we fix $C_B$ as in (32).

We now continue with our description of Algorithm AOI-Version 2.

(33) We compute the Whitney fields $J_{S_{00}}(e_2)$ and $J_E(e_2)$.

Thanks to (10) and (18), this requires work and storage at most $C(\epsilon)N$, and at most $C(\epsilon)N$ calls to the $\varphi$-Oracle.

We define a convex polyhedron $K \subset Wh(S_{00})$ as follows:

(34) A given $\vec{P} \in Wh(S_{00})$ belongs to $K$ if and only if

(35) $\mathsf{val}(\partial_2 \vec{P}, z_{00}) \in I_\Gamma$,

and moreover $\vec{P}$ can be represented in the form

(36) $\vec{P} = \sum_{\nu=1}^{\nu_{\mathsf{max}}} \lambda_\nu \vec{P}_\nu + \mu J_{S_{00}}(e_2)$, where $\mu, \lambda_1, \dots, \lambda_{\nu_{\max}}$ are real numbers,

(37) $|\mu| \leq C_B |I_{00}|$; $\lambda_\nu \geq 0$ for each $\nu$, $\lambda_1 + \cdots + \lambda_{\nu_{\mathsf{max}}} = 1$, and

(38) $\vec{P}_\nu \in K_\nu$ for each $\nu$.

Then $K \subset Wh(S_{00})$ is a convex polyhedron defined by at most $C(\epsilon)$ constraints. We can compute $K$ from $J_{S_{00}}(e_2)$ and the $K_\nu$, using work and storage at most $C(\epsilon)$.

Moreover, given $\vec{P} \in K$, we can compute $\mu, \lambda_1, \dots, \lambda_{\nu_{\mathsf{max}}}, \vec{P}_1, \dots, \vec{P}_{\nu_{\mathsf{max}}}$ satisfying (36), (37), (38). The work and storage used to do so are at most $C(\epsilon)$, and no calls to the $\varphi$-Oracle are involved.

We will prove that (21), (22) hold for the above $K$.

Let us start with (21). Thus, suppose $F \in C^2(C_A Q_{00})$ with norm $\leq 1 - C_A \epsilon$, and suppose that $F = f$ on $E$ and $\partial_2 F(z_{00}) \in I_\Gamma$. We will show that $J_{S_{00}}(F) \in K$. Indeed, $\mathsf{val}(\partial_2 J_{S_{00}}(F), z_{00}) = \partial_2 F(z_{00}) \in I_\Gamma$, hence (35) holds for $\vec{P} = J_{S_{00}}(F)$. Moreover, (32) shows that we can represent $\vec{P}$ in the form (36), (37), (38), by taking a single $\lambda_\nu$ to equal $1$, and all the rest of the $\lambda_\nu$ equal to $0$. Thus, $\vec{P} = J_{S_{00}}(F)$ belongs to $K$, as claimed. This proves (21).

We turn to the query algorithm (22).

Suppose we have finished the computation of $K$. Then we have computed $J_{S_{00}}(e_2)$, $J_E(e_2)$, and all the $K_\nu$. Let $\vec{P} \in K$ be given. We compute $\mu, \lambda_1, \dots, \lambda_{\nu_{\mathsf{max}}}$, $\vec{P}_1, \dots, \vec{P}_{\nu_{\mathsf{max}}}$ as in (36), (37), (38).

For each $\nu = 1, \dots, \nu_{\max}$, we then apply the query algorithm (28) for the query $\vec{P}_\nu \in K_\nu$; see (38). Thus, for each $\nu$, we compute a Whitney field $\vec{P}^E_\nu \in Wh(E)$ for which there exists a function $F_\nu \in C^2(Q_{00})$ of norm $\leq 1 + C\epsilon$, such that

(39) $F_\nu = f$ on $E$, $\partial_2 F_\nu(z_{00}) = \xi_\nu$, $J_{S_{00}}(F_\nu) = \vec{P}_\nu, J_E(F_\nu) = \vec{P}^E_\nu$.

Let us fix functions $F_\nu$ as above. We compute

(40) $\vec{P}^E = \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu \vec{P}^E_\nu + \mu J_E(e_2) \in Wh(E)$, and define

(41) $F = \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu F_\nu + \mu e_2 \in C^2(Q_{00})$.

From (31), (36), (37), (39), (40), (41), we conclude that

(42) $F = \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu f = f$ on $E$;

(43) $J_{S_{00}}(F) = \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu \vec{P}_\nu + \mu J_{S_{00}}(e_2) = \vec{P}$ and

(44) $J_E(F) = \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu \vec{P}^E_\nu + \mu J_E(e_2) = \vec{P}^E$.

Moreover, since $\vec{P} \in K$, we know that $\vec{P}$ satisfies (35). Hence, (43) implies that

(45) $\partial_2 F(z_{00}) \in I_\Gamma$.

Also, since each $F_\nu$ has norm at most $1 + C\epsilon$ in $C^2(Q_{00})$, estimates (30) and (37) tell us that

(46) $\| F \|_{C^2(Q_{00})} \leq \sum_{\nu=1}^{\nu_{\max}} \lambda_\nu \cdot (1 + C\epsilon) + C_B |I_{00}| \cdot C\epsilon |I_{00}|^{-1} \leq 1 + C'\epsilon$.

Our results (42)–(46) show that $\vec{P}^E$ and $F$ are as asserted in the query algorithm (22). Thus, we have successfully responded to the query $\vec{P}$, proving (22).

The reader may easily check that the computer resources used to carry out Algorithm AOI-Version 2 are as promised. This completes our explanation of Algorithm AOI-Version 2.

## 19. Almost optimal interpolants, version 3

In this section, we make the following assumptions:

(1) We are given a number $\epsilon > 0$.

(2) We are given an open square $Q_0 = I_0 \times J_0$ such that $|I_0| \leq \bar{C}_1$.

(3) A given function $\varphi \in C^2(2I_0)$ satisfies $|\varphi'| \leq \bar{C}_2$ and $|\varphi''| \leq \bar{C}_3 |I_0|^{-1}$ on $2I_0$.

(4) Given $x_1 \in 2I_0$, a $\varphi$-Oracle returns $\varphi(x_1)$, $\varphi'(x_1)$, $\varphi''(x_1)$, charging us "work" $W_{\varphi O} \geq 1$ to do so.

(5) We are given a finite set $\bar{E} \subset I_0$; let $N = \#(\bar{E})$.

(6) We define $E = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}\}$.

(7) We assume that $E \subset Q_0$.

(8) We are given a function $f : E \longrightarrow \mathbb{R}$.

(9) We are given a family of norms $|\cdot|_z$ on $\mathcal{P}$ ($z \in \mathbb{R}^2$), and an Oracle, satisfying conditions (1)–(4) in Section 5. We define $C^2$ norms as in that section.

(10) We are given a finite set $S_0 \subset Q_0$.

(11) We assume that $\#(S_0) < \epsilon^{-100}$.

(12) We assume that $|x_2 - \varphi(x_1)| > \epsilon^3 \delta_{Q_0}$ for all $(x_1, x_2) \in S_0$.

(13) We are given a base point $z_0 \in S_0$.

(14) We are given a convex polyhedron $\Gamma(z_0) \subset \mathcal{P}$, defined by at most $\bar{C}_4$ constraints.

(15) We assume that $|\partial^\alpha (P_1 - P_2)(z_0)| \leq \bar{C}_5 \delta_{Q_0}^{2-|\alpha|}$ for $|\alpha| \leq 2$, $P_1, P_2 \in \Gamma(z_0)$.

(16) We assume that there exists $F_{\text{crude}} \in C^2(\mathbb{R}^2)$ such that $\| F_{\text{crude}} \|_{C^2(\mathbb{R}^2)} \leq \bar{C}_6$, $F_{\text{crude}} = f$ on $E$, and $J_{z_0}(F_{\text{crude}}) \in \Gamma(z_0)$.

(17) We take the boiler-plate constants for this section to be $\bar{C}_1, \ldots, \bar{C}_6$ in assumptions (1)–(16) above, together with $c_0, C_0, C_1, C_2$ in (1)–(4) of Section 5. We assume that $\bar{C}_6 \geq 1$ in (16).

(18) We assume that $\epsilon$ is less than a small enough controlled constant.

This concludes the list of assumptions made in this section.

Our goal here is to present the following algorithm:

**Algorithm AOI, Version 3.** Given the above assumptions we compute a convex polyhedron $K \subset Wh(S_0)$, defined by at most $C(\epsilon)$ constraints, such that the following hold for a large enough controlled constant $C_A$:

(19) Let $F \in C^2(2Q_0)$ with norm $\leq 1 - C_A \epsilon$. Suppose that $F = f$ on $E$ and $J_{z_0}(F) \in \Gamma(z_0)$. Then $J_{S_0}(F) \in K$.

(20) After computing $K$, we can answer queries as follows: A query consists of a Whitney field $\vec{P} \in K$. The response to a query $\vec{P} \in K$ is a Whitney field $\vec{P}^E \in Wh(E)$ for which there exists $F \in C^2(Q_0)$ with norm $\leq 1 + C\epsilon$, such that $F = f$ on $E$, $J_{z_0}(F) \in \Gamma(z_0)$, $J_{S_0}(F) = \vec{P}$ and $J_E(F) = \vec{P}^E$.

The computation of $K$ uses work $\leq C(\epsilon)(N+2)\log(N+2)$ and storage $\leq C(\epsilon)(N+2)$, and makes $\leq C(\epsilon)(N+2)$ calls to the $\varphi$-Oracle.

To answer a query as in (20), we use work and storage at most $C(\epsilon) \cdot (N+2)$, and we make no calls to the $\varphi$-Oracle.

*Explanation:* First of all, note that our present assumptions, unlike those made in many previous sections, allow for the possibilities $E = \emptyset$, $\#(E) = 1$. Hence, we write $N + 2$ above, so that we get a sensible result for $\log(N + 2)$ in case $N = 0$. When $E = \emptyset$, we have $Wh(E) = \{0\}$, and the equation $J_E(F) = \vec{P}^E$ in (20) holds vacuously for $\vec{P}^E = 0$.

We begin the work of achieving (19), (20). We first compute a set $S^+$ such that

(21) $S_0 \subset S^+ \subset Q_0$,

(22) $\#(S^+) \le 2 \cdot \epsilon^{-100}$,

(23) $|x_2 - \varphi(x_1)| > \epsilon^3 \delta_{Q_0}$ for all $(x_1, x_2) \in S^+$, and

(24) For any $z \in Q_0$ there exists $z^+ \in S^+$ such that $|z - z^+| < \epsilon^2 \delta_{Q_0}$.

We can trivially compute such an $S^+$, using work and storage at most $C(\epsilon)$, and making at most $C(\epsilon)$ calls to the $\varphi$-Oracle.

We will compute a convex polyhedron $K^+ \subset Wh(S^+)$, defined by at most $C(\epsilon)$ constraints, such that the following hold for a large enough controlled constant $C_A$:

(25) Let $F \in C^2(2Q_0)$ with norm $\le 1 - C_A \epsilon$. Suppose that $F = f$ on $E$, and that $J_{z_0}(F) \in \Gamma(z_0)$. Then $J_{S^+}(F) \in K^+$.

(26) After computing $K^+$, we can respond to queries as follows: A query consists of a Whitney field $\vec{P}^+ \in K^+$. The response to a query $\vec{P}^+ \in K^+$ is a Whitney field $\vec{P}^E$ for which there exists $F \in C^2(Q_0)$ with norm $\le 1 + C\epsilon$, such that $F = f$ on $E$, $J_{z_0}(F) \in \Gamma(z_0)$, $J_{S^+}(F) = \vec{P}^+$, and $J_E(F) = \vec{P}^E$.

(27) Moreover, the computation of $K^+$ uses work $\le C(\epsilon)(N + 2)\log(N + 2)$, storage at most $C(\epsilon) \cdot (N + 2)$, and at most $C(\epsilon) \cdot (N + 2)$ calls to the $\varphi$-Oracle. To respond to a query as in (26), we use work and storage at most $C(\epsilon) \cdot (N + 2)$, and we make no calls to the $\varphi$-Oracle.

Once we compute $K^+$ as above, we can simply set $K = \{\vec{P}|_{S_0} : \vec{P} \in K^+\}$. It is then trivial to check all the assertions made in the statement of Algorithm AOI-Version 3. Thus, our task is to compute $K^+$, as above. We explain how to do so.

We partition $Q_0$ into a grid of congruent squares $\{\tilde{Q}_\nu\}$, with

(28) $\delta_{\tilde{Q}_\nu} = \bar{\delta}$ for each $\nu$, where $c\epsilon\delta_{Q_0} \le \bar{\delta} \le C\epsilon\delta_{Q_0}$. Thus,

(29) The number of squares $\tilde{Q}_\nu$ is at most $C\epsilon^{-2}$.

For each $\tilde{Q}_\nu$, we can trivially compute an open square $Q_\nu$, such that

(30) $Q_0 \cap 3\tilde{Q}_\nu \subset Q_\nu \subset Q_0$, and

(31) $\delta_{Q_\nu} = 5\bar{\delta} = 5\delta_{\tilde{Q}_\nu}$, for each $\nu$.

The computation of all the $\tilde{Q}_\nu$ and $Q_\nu$ uses work and storage $C(\epsilon)$, and involves no calls to the $\varphi$-Oracle.

Next, we introduce cutoff functions $\tilde{\theta}_\nu \in C^2(\mathbb{R}^2)$, with the following properties:

(32) $\tilde{\theta}_\nu \geq 0$ on $\mathbb{R}^2$, $\tilde{\theta}_\nu \geq 1$ on $\tilde{Q}_\nu$, $\tilde{\theta}_\nu = 0$ outside $2\tilde{Q}_\nu$;

(33) $|\partial^\alpha \tilde{\theta}_\nu| \leq C\bar{\delta}^{-|\alpha|} = C\delta_{\tilde{Q}_\nu}^{-|\alpha|}$ on $\mathbb{R}^2$, for $|\alpha| \leq 2$.

We may take $\tilde{\theta}_\nu$ such that, given $z$ and $\tilde{Q}_\nu$, we can compute $J_z(\tilde{\theta}_\nu)$ using work and storage at most $C$. We then define a partition of unity on $Q_0$, by setting

(34) $\theta_\nu = \tilde{\theta}_\nu / \sum_{\nu'} \tilde{\theta}_{\nu'}$ on $Q_0$, for each $\nu$.

Thus, the $\theta_\nu$ are defined only on $Q_0$. For each $\nu$, we have

(35) $\theta_\nu \in C^2(Q_0)$, $\theta_\nu \geq 0$ on $Q_0$, $\operatorname{supp} \theta_\nu \subset Q_0 \cap 2\tilde{Q}_\nu \subset Q_\nu$.

Here and below, $\operatorname{supp} \theta_\nu$ denotes the set of points $z \in Q_0$ such that $\theta_\nu$ does not vanish identically in any neighborhood of $z$. Moreover,

(36) $|\partial^\alpha \theta_\nu| \leq C\bar{\delta}^{-|\alpha|} \leq C'\delta_{Q_\nu}^{-|\alpha|}$ on $Q_0$, for $|\alpha| \leq 2$; and

(37) $\sum_\nu \theta_\nu = 1$ on $Q_0$.

In addition,

(38) Given $z \in \mathbb{R}^2$, and given $\tilde{Q}_\nu$, we can compute $J_z(\theta_\nu)$ using work and storage at most $C$.

For each $\nu$, we compute $J_E(\theta_\nu)$; this takes work and storage at most $C(\epsilon) \cdot (N+2)$, and requires no calls to the $\varphi$-Oracle.

Let $z \in \operatorname{supp} \theta_\nu \cap \sup \theta_{\nu'}$. Then by (24), there exists $z^+ \in S^+$ such that $|z - z^+| < \epsilon^2 \delta_{Q_0} < C\epsilon\bar{\delta}$. (See (28).) Since $z \in 2\tilde{Q}_\nu \cap 2\tilde{Q}_{\nu'}$ by (35), and since $\delta_{\tilde{Q}_\nu} = \delta_{\tilde{Q}_{\nu'}} = \bar{\delta}$, it follows that $z^+ \in 3\tilde{Q}_\nu \cap 3\tilde{Q}_{\nu'}$.

Also, $z^+ \in S^+ \subset Q_0$. Thus, $z^+ \in Q_0 \cap 3\tilde{Q}_\nu$ and $z^+ \in Q_0 \cap 3\tilde{Q}_{\nu'}$. Consequently, $z^+ \in Q_\nu \cap Q_{\nu'}$, thanks to (30). Thus, we have proven the following:

(39) Let $z \in \operatorname{supp} \theta_\nu \cap \operatorname{supp} \theta_{\nu'}$. Then there exists $z^+ \in S^+$ such that $z^+ \in Q_\nu \cap Q_{\nu'}$, and $|z - z^+| < \epsilon^2 \delta_{Q_0}$.

For each $\nu$, let us write

(40) $Q_\nu = I_\nu \times J_\nu$, and define

(41) $E_\nu = E \cap Q_\nu$,

(42) $\bar{E}_\nu = \{x_1 : (x_1, x_2) \in E_\nu\}$,

(43) $S_\nu = S^+ \cap Q_\nu$.

Note that $S_\nu$ is non-empty, thanks to (30), (31), and (24) applied to the center of $Q_\nu$.

For each $\nu$, we pick

(44) $z_{0,\nu} \in S_\nu$.

The computation of all the $E_\nu$, $\bar{E}_\nu$, $S_\nu$, $z_{0,\nu}$ uses work at storage at most $C(\epsilon) \cdot (N+2)$, and requires no calls to the $\varphi$-Oracle. Next, let

(45) $I_\Gamma^0 = \{\partial_2 P(z_0) : P \in \Gamma(z_0)\}$.

Since $\Gamma(z_0) \subset \mathcal{P}$ is a convex polyhedron defined by at most $C$ constraints, it follows that $I_\Gamma^0$ is an interval. Moreover, we can compute $I_\Gamma^0$ using work and storage at most $C$, without making calls to the $\varphi$-Oracle.

Thanks to our assumption (15), we have

(46) $|I_\Gamma^0| \leq C\delta_{Q_0}$.

Suppose $F \in C^2(Q_0)$, with norm at most $\bar{C}_6$. (See (16).) Then $|\partial_2 F(z) - \partial_2 F(z_0)| \leq C\delta_{Q_0}$ for all $z \in Q_0$. Hence, for a large enough controlled constant $C^\#$, the following holds:

(47) Let $I_\Gamma = \{\xi \in \mathbb{R} : \text{distance } (\xi, I_\Gamma^0) \leq C^\# \delta_{Q_0}\}$.

(48) Let $F \in C^2(Q_0)$ with norm at most $\bar{C}_6$ (as in (16)), and suppose that $J_{z_0}(F) \in \Gamma(z_0)$. Then $\partial_2 F(z) \in I_\Gamma$ for all $z \in Q_0$.

Moreover $I_\Gamma$ is an interval of length

(49) $|I_\Gamma| \leq C\delta_{Q_0}$.

The main step in our computation of $K^+$ is to do the following, for each of the $Q_\nu$:

(50) We compute a convex polyhedron $K_\nu \subset Wh(S_\nu)$, defined by at most $C(\epsilon)$ constraints, such that the following hold for a large enough controlled constant $C_A$:

(51) Let $F \in C^2(C_A Q_\nu)$ with norm $\leq 1 - C_A\epsilon$. Suppose $F = f$ on $E_\nu$ and $\partial_2 F(z_{0,\nu}) \in I_\Gamma$. Then $J_{S_\nu}(F) \in K_\nu$.

(52) After computing $K_\nu$, we can answer queries as follows: A query consists of a Whitney field $\vec{P} \in K_\nu$. The response to a query $\vec{P} \in K_\nu$ is a Whitney field $\vec{P}_\nu^E \in Wh(E_\nu)$ such that there exists $F \in C^2(Q_\nu)$ with norm $\leq 1 + C\epsilon$, satisfying $F = f$ on $E_\nu$, $\partial_2 F(z_{0,\nu}) \in I_\Gamma, J_{S_\nu}(F) = \vec{P}$, $J_{E_\nu}(F) = \vec{P}_\nu^E$.

(53) The computation of $K_\nu$ uses work $\leq C(\epsilon)(N+2)\log(N+2)$, storage $\leq C(\epsilon)(N+2)$, and at most $C(\epsilon)(N+2)$ calls to the $\varphi$-Oracle.

(54) To respond to a query as in (52), we use work and storage at most $C(\epsilon) \cdot (N+2)$, and make no calls to the $\varphi$-Oracle.

We first explain how to achieve (50)–(54) for each $\nu$; then we explain how to use (50)–(54) to compute $K^+$ and satisfy (25), (26), (27).

To achieve (50)–(54), we distinguish two cases.

The Easy Case: $\#(E_\nu) < 2$.

The Hard Case: $\#(E_\nu) \geq 2$.

We first tackle the Hard Case. Recall that (18.x) denotes expression (x) in Section 18.

We check the following:

(55) Assumptions (18.1)–(18.14) hold here, with our present $Q_\nu = I_\nu \times J_\nu$, $E_\nu, \bar{E}_\nu$ and $N_\nu := \#(E_\nu)$, respectively, in place of $Q_{00} = I_{00} \times J_{00}$, $E, \bar{E}$ and $N$ in Section 18. Moreover, the constants listed in (18.15) may be taken here to be controlled constants (in the sense of the present section; see (17)). Consequently, the "small $\epsilon$ assumption" (18.16) follows from our present small $\epsilon$ assumption (18).

Indeed,

(18.1) just says that $\epsilon$ is a positive real number; this is our assumption (1).

(18.2) says that $Q_\nu = I_\nu \times J_\nu$ is an open square, with $\delta_{Q_\nu} \leq C\epsilon$. We have defined $Q_\nu$ to be an open square with sidelength $5\bar{\delta}$. Hence, $\delta_{Q_\nu} \leq C\epsilon$, by (2) and (28).

(18.3) says that $\bar{E}_\nu \subset I_\nu$ is a finite set. This follows from (40), (41), (42), since $E$ is a finite set.

(18.4) says that $\varphi \in C^2(c\epsilon^{-1}I_\nu)$, with $c\epsilon^{-1} > 1$. We know that $I_\nu \subset I_0$, by (2), (30) and (40). Also, $|I_\nu| = 5\bar{\delta} < C\epsilon|I_0|$. Hence,

(56) $c\epsilon^{-1}I_\nu \subset 2I_0$ and $|I_0|^{-1} \leq C\epsilon|I_\nu|^{-1}$.

Consequently, (18.4) follows from (3) and (18).

(18.5) asserts that $|\varphi'| \leq C$ and $|\varphi''| \leq C\epsilon|I_\nu|^{-1}$ on $c\epsilon^{-1}I_\nu$. These estimates follow from (3) and (56).

(18.6) and (18.7) assert that, given $x_1 \in c\epsilon^{-1}I_\nu$, the $\varphi$-Oracle returns $\varphi(x_1)$, $\varphi'(x_1)$, $\varphi''(x_1)$, and charges us work $W_{\varphi O} \geq 1$. This follows from (4) and (56).

(18.8) asserts that $E_\nu = \{(x_1, \varphi(x_1)) : x_1 \in \bar{E}_\nu\}$. This follows from (6), (41), (42).

(18.9) says that $E_\nu \subset Q_\nu$. This follows from (41).

(18.10) says that $N_\nu = \#(E_\nu) = \#(\bar{E}_\nu) \geq 2$.

We have $N_\nu = \#(E_\nu)$ by definition, and $\#(E_\nu) = \#(\bar{E}_\nu)$ thanks to (18.8), which we just proven. We have $N_\nu \geq 2$ because we are in the Hard Case.

(18.11) says that $f : E_\nu \to \mathbb{R}$, which follows from (8), (41).

(18.12) says that $I_\Gamma$ is an interval, and that $|I_\Gamma| \leq C\epsilon^{-1}|I_\nu|$. Since $|I_\nu| = 5\bar{\delta}$, this follows from (28) and (49).

(18.13) is the same as our present (9).

(18.14) asserts here that $F_{\mathsf{crude}} \in C^2(\mathbb{R}^2)$, $\| F_{\mathsf{crude}} \|_{C^2(\mathbb{R}^2)} \leq C$, $F_{\mathsf{crude}} = f$ on $E_\nu$, and $\partial_2 F_{\mathsf{crude}}(z) \in I_\Gamma$ for all $z \in E_\nu$.

Except for the assertion $\partial_2 F_{\mathsf{crude}}(z) \in I_\Gamma$, the above properties of $F_{\mathsf{crude}}$ are assumed here in (16), since $E_\nu \subset E$ by (41).

Moreover, $J_{z_0}(F_{\mathsf{crude}}) \in \Gamma(z_0)$ and $\| F_{\mathsf{crude}} \|_{C^2(Q_0)} \leq \bar{C}_6$, by (16). Hence, $\partial_2 F_{\mathsf{crude}}(z) \in I_\Gamma$ for all $z \in Q_0$, thanks to (48). In particular, $\partial_2 F_{\mathsf{crude}}(z) \in I_\Gamma$ for all $z \in E_\nu \subset E$; see (7) and (41).

This completes the proof of our Claim (55).

Next, we check that

(57) Assumptions (18.17)–(18.20) hold here, with $S_\nu, Q_\nu, z_{0,\nu}$ in place of $S_{00}$, $Q_{00}$, $z_{00}$ in Section 18.

Indeed,

(18.17) says that $S_\nu \subset Q_\nu$ (and $S_\nu$ finite), which follows from (43).

(18.18) says that $\#(S_\nu) < \epsilon^{-200}$, which follows from (22) and (43).

(18.19) asserts that $|x_2 - \varphi(x_1)| > \epsilon^3 \delta_{Q_\nu}$ for all $(x_1, x_2) \in S_\nu$. This estimate follows from (23) and (43), since $Q_\nu \subset Q_0$ by (30).

Finally, (18.20) says that $z_{0,\nu} \in S_\nu$, which we know from (44).

This completes our verification of (57).

Thanks to (55) and (57), we may apply Algorithm AOI-Version 2. Thus, we compute a convex polyhedron $K_\nu$, satisfying (50)–(54). This concludes our discussion of (50)–(54) in the Hard Case $\#(E_\nu) \geq 2$.

We pass to the Easy Case $\#(E_\nu) < 2$.

We refer to Section 6, and to the Remark in that section, following the discussion of Algorithm AUB4.

Thus, using work and storage at most $C(\epsilon)$ (and no calls to the $\varphi$-Oracle), we compute a convex polyhedron $K_\nu^{AUB} \subset Wh(S_\nu \cup E_\nu)$, defined by at most $C(\epsilon)$ constraints, such that the following hold:

Let $F \in C^2(2Q_\nu)$ with norm $\leq 1$. Then $J_{S_\nu \cup E_\nu}(F) \in K_\nu^{AUB}$.

Let $\vec{P} \in K_\nu^{AUB}$. Then there exists $F \in C^2(Q_\nu)$ with norm $\leq 1 + C\epsilon$, such that $J_{S_\nu \cup E_\nu}(F) = \vec{P}$. We now set

$$K_\nu = \{\vec{P}|_{S_\nu} : \vec{P} \in K_\nu^{AUB}, \mathsf{val}\,(\vec{P}, z) = f(z) \text{ for all } z \in E_\nu, \mathsf{val}\,(\partial_2 \vec{P}, z_{0,\nu}) \in I_\Gamma\}.$$

It is trivial to check that (50)–(54) hold.

This completes our discussion of (50)–(54) in the Easy Case. Thus, we have achieved (50)–(54) in all cases.

We now use our $K_\nu$ from (50)–(54) to define and compute $K^+$, and establish (25), (26), (27). We take

(58) $K^+ = \{\vec{P} \in Wh(S^+) : \vec{P}|_{S_\nu} \in K_\nu \text{ for each } \nu, \text{ and } \vec{P}|_{z_0} \in \Gamma(z_0)\}$.

Thus, $K^+ \subset Wh(S^+)$ is a convex polyhedron defined by at most $C(\epsilon)$ constraints. Moreover, once we have computed the $K_\nu$, it takes work and storage at most $C(\epsilon)$ to compute $K^+$ from (58).

Let us check that (25) holds for our $K^+$. Suppose $F \in C^2(2Q_0)$ with norm $\leq 1 - C_A\epsilon$, and suppose also that $F = f$ on $E$ and $J_{z_0}(F) \in \Gamma(z_0)$. For each $\nu$, we have $C_A Q_\nu \subset 2Q_0$, since $Q_\nu \subset Q_0$ and $\delta_{Q_\nu} = 5\bar{\delta} \leq C\epsilon\delta_{Q_0}$. Therefore, $F \in C^2(C_A Q_\nu)$ with norm $\leq 1 - C_A\epsilon$, and $F = f$ on $E_\nu$ by (41). Also, $\partial_2 F(z_{0,\nu}) \in I_\Gamma$, by (48). (Recall that $z_{0,\nu} \in S_\nu \subset S^+ \subset Q_0$.) Consequently, (51) tells us that $J_{S_\nu}(F) \in K_\nu$ for each $\nu$. Since also $J_{z_0}(F) \in \Gamma(z_0)$ by assumption, a glance at (58) shows that $J_{S^+}(F) \in K^+$, completing the proof of (25).

We pass to the query algorithm (26). Thus, suppose we are given a query $\vec{P}^+ \in K^+$. By definition (58), we have $\vec{P}^+|_{S_\nu} \in K_\nu$ for each $\nu$, and

(59) $\vec{P}^+ = (P^z)_{z \in S^+}$, with $P^{z_0} \in \Gamma(z_0)$.

Applying the query algorithm (52) to the query $\vec{P}^+|_{S_\nu}$, we obtain for each $\nu$ a Whitney field

(60) $\vec{P}^E_\nu = (P^{E,z}_\nu)_{z \in E_\nu} \in Wh(E_\nu)$

for which there exists

(61) $F_\nu \in C^2(Q_\nu)$ with norm $\leq 1 + C\epsilon$ such that

(62) $F_\nu = f$ on $E_\nu$, $\partial_2 F(z_{0,\nu}) \in I_\Gamma$,

(63) $J_{S_\nu}(F_\nu) = \vec{P}^+|_{S_\nu}$, and

(64) $J_{E_\nu}(F_\nu) = \vec{P}^E_\nu$.

The $\vec{P}^E_\nu$ may be computed from $\vec{P}^+$ using work and storage at most $C(\epsilon)\cdot(N+2)$, and without calls to the $\varphi$-Oracle; see (43) and (54).

From (63) and (59), we obtain in particular that $J_{z_0}(F_\nu) = \vec{P}^{z_0} \in \Gamma(z_0)$ whenever $z_0 \in S_\nu$. Also, since $z_0 \in S_0 \subset S^+$ (see (13) and (21)), we learn from (43) that $z_0 \in Q_\nu$ implies $z_0 \in S_\nu$. Consequently,

(65) $J_{z_0}(F_\nu) = P^{z_0}$ for all $\nu$ such that $Q_\nu \ni z_0$.

We bring in the partition of unity (35)–(38). Let us define $F \in C^2(Q_0)$ and

(66) $\vec{P}^E = (P^{E,z})_{z \in E} \in Wh(E)$, by setting

(67) $F = \sum_\nu \theta_\nu F_\nu$ on $Q_0$, and

(68) $P^{E,z} = \sum_{Q_\nu \ni z} J_z(\theta_\nu) \odot_z P^{E,z}_\nu$ for all $z \in E$.

Since $\operatorname{supp}\theta_\nu \subset Q_\nu$ and $F_\nu \in C^2(Q_\nu)$ for each $\nu$, (67) makes sense, and $F \in C^2(Q_0)$. Also, (41), (60) and (64) yield $J_z(F_\nu) = P^{E,z}_\nu$ for $Q_\nu \ni z$, $z \in E$. Hence, comparing (67) and (68), we find that $J_z(F) = P^{E,z}$ for each $z \in E$, i.e.,

(69) $J_E(F) = \vec{P}^E$.

Let us check that $F$ and $\vec{P}^E$ have all the properties promised in (26).

We start by estimating the $C^2$ norm of $F$.

Fix $z \in Q_0$, and suppose $z \in \operatorname{supp}\theta_\nu \cap \operatorname{supp}\theta_{\nu'}$. Then $z \in Q_\nu \cap Q_{\nu'}$. By (39), there exists $z^+ \in S^+ \cap Q_\nu \cap Q_{\nu'}$ such that

(70) $|z - z^+| < \epsilon^2 \delta_{Q_0} < C\epsilon\bar{\delta}$ (see (28)).

Thus, $z^+ \in S_\nu \cap S_{\nu'}$, by (43). Therefore, (59) and (63) tell us that $J_{z^+}(F_\nu) = J_{z^+}(F_{\nu'}) = P^{z^+}$. In particular,

(71) $J_{z^+}(F_\nu - F_{\nu'}) = 0$.

Also, (61) and the Bounded Distortion Property imply that

$$F_\nu - F_{\nu'} \in C^2(Q_\nu \cap Q_{\nu'}),$$

and

(72) $|\partial^\alpha(F_\nu - F_{\nu'})| \le C$ on $Q_\nu \cap Q_{\nu'}$, for $|\alpha| \le 2$.

From (70), (71), (72) and Taylor's theorem, we learn that

(73) $|\partial^\alpha(F_\nu - F_{\nu'})(z)| \le C \cdot (\epsilon\bar{\delta})^{2-|\alpha|} \le C\epsilon\bar{\delta}^{2-|\alpha|}$ for $|\alpha| \le 1$,

whenever $\operatorname{supp}\theta_\nu \cap \operatorname{supp}\theta_{\nu'} \ni z$. We recall that

(74) $|\partial^\alpha\theta_\nu(z)| \le C\bar{\delta}^{-|\alpha|}$ for $|\alpha| \le 2$, and that

(75) $|J_z(F_\nu)|_z \le 1 + C\epsilon$ for $\operatorname{supp}\theta_\nu \ni z$;

see (36) and (61). Note also that $z \in \operatorname{supp}\theta_\nu$ for at most $C$ distinct $\theta_\nu$; see (30), (31), (35), and recall that the $\tilde{Q}_\nu$ form a grid of squares of sidelength $\bar{\delta}$. Also, recall that $\bar{\delta} \le C\epsilon\delta_{Q_0} \le C\epsilon < 1$ (see (2) and (28)), and that $\theta_\nu \ge 0$, $\sum_\nu \theta_\nu = 1$ on $Q_0$.

The above remarks and Lemma GPU from Section 5 tell us that

(76) $|J_z(F)|_z \le 1 + C\epsilon$.

Since $z \in Q_0$ is arbitrary in (76), we conclude that

(77) $\| F \|_{C^2(Q_0)} \le 1 + C\epsilon$.

Next, let $z \in E$. For any $\nu$ such that $\operatorname{supp}\theta_\nu \ni z$, we have $z \in Q_\nu \cap E = E_\nu$; hence $F_\nu(z) = f(z)$ by (62). Consequently, (67) and (37) give $F(z) = \sum_\nu \theta_\nu(z) f(z) = f(z)$. Thus,

(78) $F = f$ on $E$.

Next, let $z \in S^+$. For any $\nu$ such that $\operatorname{supp}\theta_\nu \ni z$, we have $z \in Q_\nu \cap S^+ = S_\nu$; hence, (59) and (63) tell us that $J_z(F_\nu) = P^z$. Consequently, (67) and (37) give $J_z(F) = \sum_\nu J_z(\theta_\nu) \odot_z P^z = P^z$, for all $z \in S^+$. That is,

(79) $J_{S^+}(F) = \vec{P}^+$.

In particular,

$$(80)\quad J_{z_0}(F) = P^{z_0} \in \Gamma(z_0),$$

thanks to (13) and (59).

Our results (69), (77), (78), (79), (80) show that $F$ and $\vec{P}^E$ are as promised in (26). Thus, we have succeeded in answering the query $\vec{P}^+ \in K^+$. This completes our discussion of the query algorithm (26).

The reader may easily check that the computer resources used to compute $K^+$ and answer queries (26) are as asserted in (27).

Our explanation of Algorithm AOI-Version 3 is complete. With AOI-Version 3, we have carried out Step I of the strategy presented in the Introduction. In [4], we will carry out Step II and complete our interpolation algorithm for $C^2(\mathbb{R}^2)$.

## References

[1] Bierstone, E. and Milman, P.: $C^m$-norms on finite sets and $C^m$ extension criteria. *Duke Math. J.* **137** (2007), no. 1, 1–18.

[2] Brudnyi, Y. and Shvarstman, P.: Generalization of Whitney's extension theorem. *Int. Math. Res. Notices* **1994**, no. 3, 129–139.

[3] Callahan, P. B. and Kosaraju, S. R.: A decomposition of multidimensional point sets with applications to $k$-nearest-neighbors and $n$-body potential fields. *J. Assoc. Comput. Mach.* **42** (1995), no. 1, 67–90.

[4] Fefferman, C.: Nearly optimal interpolation of data in $C^2(\mathbb{R}^2)$. Part II. In preparation.

[5] Fefferman, C.: The $C^m$ norm of a function with prescribed jets I. *Rev. Mat. Iberoam.* **26** (2010), no. 3, 1075–1098.

[6] Fefferman, C.: The $C^m$ norm of a function with prescribed jets. II. *Rev. Mat. Iberoam.* **25** (2009), no. 1, 275–421.

[7] Fefferman, C. and Klartag, B.: Fitting a $C^m$-smooth function to data. I. *Ann. of Math. (2)* **169** (2009), 315–346.

[8] Fefferman, C. and Klartag, B.: Fitting a $C^m$-smooth function to data. II. *Rev. Mat. Iberoam.* **25** (2009), no. 1, 49–273.

[9] Fefferman, C. and Klartag, B.: An example related to Whitney extension with almost minimal $C^m$ norm. *Rev. Mat. Iberoam.* **25** (2009), no. 2, 423–446.

[10] Fefferman, C.: A sharp form of Whitney's extension theorem. *Ann. of Math. (2)* **161** (2005), no. 1, 509–577.

[11] LeGruyer, E.: Minimal Lipschitz extensions to differentiable functions defined on a Hilbert space. *Geom. Funct. Anal.* **19** (2009), no. 4, 1101–1118.

[12] Hartmanis, J. and Simon, J.: On the power of multiplication in random access machines. In *15th Annual Symposium on Switching and Automata Theory (1974),* 13–23. IEEE Comput. Soc., Long Beach, Calif., 1974.

[13] Malgrange, B.: *Ideals of differentiable functions.* Tata Institute of Fundamental Research Studies in Mathematics 3, Tata Institute of Fundamental Research, Bombay; Oxford University Press, 1966.

[14] PREPARATA, F. AND SHAMOS, M.: *Computational geometry: An introduction,* (second edition). Texts and monographs in Computer Science, Springer-Verlag, New York, 1985.

[15] VON NEUMANN, J.: First draft of a report on the EDVAC. Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, Pa, 1945. Reprinted in *IEEE Ann. Hist. Comput.* **15** (1993), no. 4, 27–75.

[16] SHVARTSMAN, P.: The Whitney extension problem and Lipschitz selections of set-valued mappings in jet spaces. *Trans. Amer. Math. Soc.* **360** (2008), no. 10, 5529–5550.

[17] SHVARTSMAN, P.: Lipschitz selections of multivalued mappings and traces of the Zygmund class of functions to an arbitrary compact. *Dokl. Acad. Nauk SSSR* **276** (1984), 559–562. English translation in *Soviet Math. Dokl.* **29** (1984), 565–568.

[18] SCHÖNHAGE, A.: On the power of random-acess machines. In *Automata, languages and programming,* 520–529. Lecture Notes in Computer Science 71, Springer, Berlin–Heidelberg, 1979.

[19] STEIN, E. M.: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series 30, Princeton University Press, Princeton, NJ, 1970.

[20] WHITNEY, H.: Analytic extensions of differentiable functions defined on closed sets. *Trans. Amer. Math. Soc.* **36** (1934), 63–89.

[21] WHITNEY, H.: Functions differentiable on the boundaries of regions. *Ann. of Math. (2)* **35** (1934), 482–485.

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