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On the time derivative in an obstacle problem

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Dedicated to O. Martio on his seventieth birthday

Abstract. We prove that the time derivative of the solution for the obstacle problem related to the evolutionary p-Laplace equation exists in Sobolev's sense, provided that the given obstacle is smooth enough. We keep $p \geq 2$.

1. Introduction

The time derivative is notoriously difficult in parabolic problems. The celebrated *Evolutionary p-Laplace Equation* is much studied and the regularity theory for the solutions is almost complete. We refer to the book [8] about this fascinating equation. In general, the corresponding subsolutions and supersolutions do not possess that much regularity, they are semicontinuous, cf. [17]. We are interested in a special kind of (weak) supersolutions of the evolutionary p-Laplace equation, namely the solutions of an obstacle problem. In the presence of a smooth obstacle the regularity improves a lot. Given a function $\psi = \psi(x,t)$ in a bounded domain $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$, we consider all functions v such that

$$\frac{\partial v}{\partial t} \geq \nabla\cdot(|\nabla v|^{p-2}\nabla v) \quad \text{and} \quad v \geq \psi \quad \text{in } \Omega_T.$$

The function ψ acts as an *obstacle*. The smallest admissible v is the solution of the obstacle problem. (This makes sense because a comparison principle is valid.) However, the above description was only formal. We will instead use Definition 1 below, which is more adequate since it comes with a *variational inequality.* We will restrict ourselves to the case $p > 2$, the so-called slow diffusion case.

It is an established fact that if the obstacle ψ is smooth enough, the solution to the obstacle problem inherits some regularity. Our objective is the time derivative u_t of the solution u, which *a priori* is only known to be a distribution. Our main result, Theorem 2, states that, if ψ has continuous second derivatives,

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then the time derivative u_t exists in Sobolev's sense and it belongs to the space $L^{p/(p-1)}_{\text{loc}}(\Omega_T)$. A formula is given for the derivative. The most laborious part of the proof is to show that $\Delta_p u = \nabla \cdot (|\nabla v|^{p-2}\nabla v)$ is a function so that the rule

$$\int_0^T\int_\Omega \varphi\Delta_p u\, dx\, dt = -\int_0^T\int_\Omega \langle\, |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\, dx\, dt$$

with test functions applies. The equation has first to be regularized, keeping the obstacle unaffected, and then difference quotients are used. The test functions in [19] can be adjusted to work here.

An important feature, typical for obstacle problems, is that in the open set $\Upsilon = \{u > \psi\}$ where the obstacle does not hinder, u is, actually, a solution to the differential equation. Thus in Υ the equation $u_t = \Delta_p u$ holds in the weak sense. This enables us to get an identity for the integral $\iint u\varphi_t\, dx\, dt$ from which one can deduce the existence of the time derivative sought for. The boundary of the *coincidence set* $\Xi = \{u = \psi\}$ is crucial. The special case with no obstacle present was treated in [20]. See also [1].

To this we may add a curious fact valid for $\psi \in C^2(\Omega_T)$. At all points in the coincidence set Ξ the obstacle satisfies the inequality

$$\frac{\partial\psi}{\partial t} \geq \Delta_p\psi.$$

Thus a point at which $\frac{\partial\psi}{\partial t} < \Delta_p\psi$ cannot belong to the coincidence set. This piece of information follows from the characterization of weak supersolutions as *viscosity* supersolutions, cf. [12]. Then ψ itself can do as a test function for the pointwise testing required in the theory of viscosity solutions. (The reader may consult [15] and [7] for some basic concepts.) – We will not need this observation.

It is likely that the time derivative belongs to the space $L^2_{\text{loc}}(\Omega_T)$, but an eventual proof of this improvement would require much stronger regularity considerations for ∇u. We have kept $p > 2$, but one can expect a counterpart to Theorem 2 valid in the extended range $p > 2n/(n+2)$. The difficulty about further generalizations with $\Delta_p u$ replaced by some operator $\operatorname{div} \mathbf{H}_p$ is the following. It is absolutely necessary that the solutions of the differential equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{H}_p(x, t, u, \nabla u)$$

enjoy the property of having a time derivative themselves, in order that the corresponding results could be extended to the related obstacle problem. This considerably restricts the possibilities. – See also [4] for some general comments valid for "irregular" obstacles and [3] for a linear problem.

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2. Preliminaries

Let Ω be a bounded domain in the n-dimensional space $\mathbb{R}^n$ having a Lipschitz regular boundary. Suppose that a function $\psi = \psi(x,t)$ is given in the closure of the space-time cylinder $\Omega_T = \Omega \times (0,T)$. The function ψ acts as an *obstacle* so that the admissible functions are forced to lie above ψ in Ω_T. We make the

$$\text{Assumption:} \quad \psi \in C(\overline{\Omega_T}) \cap W^{2,p}(\Omega_T).$$

For simplicity, the obstacle ψ also determines the values of the admissible functions on the *parabolic boundary*

$$\Gamma_T = \Omega \times \{0\} \,\cup\, \partial\Omega \times [0,T].$$

The *class of admissible functions* is

$$\mathcal{F}_\psi = \big\{ v \in L^p(0,T;W^{1,p}(\Omega)) \,|\, v \in C(\overline{\Omega_T}),\ v \geq \psi \text{ in } \Omega_T,\ v = \psi \text{ on } \Gamma_T \big\}.$$

– We keep $p \geq 2$.

Definition 1. We say that the function $u \in \mathcal{F}_\psi$ is the solution to the obstacle problem if the inequality

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\langle |\nabla u|^{p-2} \nabla u, \nabla(\phi - u) \rangle + (\phi - u)\,\frac{\partial \phi}{\partial t} \Big) dx\,dt \\ &\qquad\qquad \geq \frac{1}{2} \int_\Omega (\phi(x,T) - u(x,T))^2\,dx \end{aligned} \tag{2.1}$$

holds for all *smooth* functions $\phi \in \mathcal{F}_\psi$.

The solution exists and is unique, cf. [2] and [6]. See also [16]. It is also a supersolution of the equation $u_t \geq \Delta_p u$, i.e.,

$$\int_0^T \int_\Omega \Big(\langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle - u\,\frac{\partial \varphi}{\partial t} \Big) dx\,dt \geq 0 \tag{2.2}$$

for all non-negative $\varphi \in C_0^\infty(\Omega_T)$.

Notice that nothing is assumed about the time derivative u_t. Our main result is the theorem below.

Theorem 2. *The time derivative u_t of the solution u to the obstacle problem exists in the Sobolev sense and $u_t \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega_T)$. It is the function*

$$u_t = \begin{cases} \psi_t & \text{in } \Xi, \\ \Delta_p u & \text{in } \Omega_T \setminus \Xi, \end{cases}$$

where $\Xi = \{u = \psi\}$ denotes the coincidence set. In other words, the formula

$$u_t = \Delta_p u + (\psi_t - \Delta_p \psi)\chi_\Xi$$

holds.

In order to avoid the difficulty with the "forbidden" time derivative u_t in the proof, we have to regularize the equation, keeping the obstacle unchanged. We replace $|\nabla u|^{p-2}\nabla u$ by

$$\Big(|\nabla u|^2+\varepsilon^2\Big)^{\frac{p-2}{2}}\nabla u$$

to obtain an equation which does not degenerate as $\nabla u = 0$.

Lemma 3. *For $\varepsilon \geq 0$ there is a unique $u^\varepsilon \in \mathcal{F}_\psi$ such that*

$$\begin{aligned}\int_0^T\!\!\int_\Omega\Big(\langle\big(|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon,\nabla(\phi-u^\varepsilon)\rangle+(\phi-u^\varepsilon)\,\frac{\partial\phi}{\partial t}\Big)dx\,dt&\\ \geq\frac{1}{2}\int_\Omega(\phi(x,T)-u^\varepsilon(x,T))^2\,dx&\end{aligned}\tag{2.3}$$

for all smooth functions ϕ in the class $\mathcal{F}_\psi$. In the open set $\{u^\varepsilon > \psi\}$ the function u^ε is a solution of the equation

$$\frac{\partial u^\varepsilon}{\partial t}=\nabla\cdot\Big(\big(|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon\Big).$$

In the case $\varepsilon \neq 0$ we have $u^\varepsilon \in C^\infty(\Omega_T)$ and $\frac{\partial u^\varepsilon}{\partial t} \in L^2(\Omega_T)$.

Proof. The existence can be extracted from the proof of Theorem 3.2 in [2]. The regularity for the nondegenerate case $\varepsilon \neq 0$ is according to the standard parabolic theory described in the celebrated book [18]. The proof of the Hölder continuity for the degenerate case $\varepsilon = 0$ is in [6]. □

When $\varepsilon \neq 0$, we can rewrite equation (2.3) in the more convenient form

$$\int_0^T\!\!\int_\Omega\Big(\langle\underbrace{\big(|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon}_{\mathbf{A}_\varepsilon(x,t)},\nabla\eta\rangle+\eta\,\frac{\partial u^\varepsilon}{\partial t}\Big)dx\,dt\geq 0\tag{2.4}$$

valid for all test functions η such that $\eta \geq \psi - u^\varepsilon$ in Ω_T and $\eta = 0$ on Γ_T. We may even use any continuous $\eta \in L^p(0,T;W_0^{1,p}(\Omega))$ with $\eta(x,0) = 0$.

In order to proceed to the limit under the integral sign in the forthcoming equations we need the convergence result below, where u denotes the solution to the original obstacle problem, the one with $\varepsilon = 0$.

Lemma 4.

$$\lim_{k\to 0}\int_0^T\!\!\int_\Omega\Big(|u^\varepsilon-u|^p+|\nabla u^\varepsilon-\nabla u|^p\Big)\,dx\,dt\;=\;0.\tag{2.5}$$

Proof. It was established in Lemma 3.2 of [13] that

$$\lim_{k\to 0}\int_0^T\!\!\int_\Omega|\nabla u^\varepsilon-\nabla u|^p\,dx\,dt\;=\;0,\tag{2.6}$$

but the strong convergence of the functions themselves requires, as it were, an extra compactness argument. Since u^ε is a weak supersolution, there exists a Radon measure μ_ε such that

$$\int_0^T\int_\Omega\Big(\langle\big|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon,\nabla\varphi\rangle-u^\varepsilon\,\frac{\partial\varphi}{\partial t}\Big)dx\,dt=\int_{\Omega_T}\varphi\,d\mu_\varepsilon$$

for all functions $\varphi\in C_0^\infty(\Omega_T)$, whether positive or not. This is a consequence of Riesz's Representation Theorem, cf. §1.8 of [9]. See [14] for details.

Given a regular open set (for example a polyhedron) $U\subset\subset\Omega_T$, we have to verify that

$$\mu_\varepsilon(U)\le M_U$$

with a bound independent of ε, $0<\varepsilon<1$. Then the lemma follows as at pages 720–721 of [14]. (There [21] was used.) To this end, choose a test function $\varphi\in C_0^\infty(\Omega_T)$ such that $0\le\varphi\le 1$ and $\varphi=1$ in U. A rough estimation yields

$$\begin{aligned}\mu_\varepsilon(U)=\int_U d\mu_\varepsilon\le\int_{\Omega_T}\varphi\,d\mu_\varepsilon&=\int_0^T\int_\Omega\Big(\langle\big|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon,\nabla\varphi\rangle-u^\varepsilon\,\frac{\partial\varphi}{\partial t}\Big)dx\,dt\\&\le C_1\left(\|\nabla u^\varepsilon\|^p_{L^p(\Omega_T)}+\varepsilon^{\frac{p(p-2)}{p-1}}\right)+C_2\|u^\varepsilon\|_\infty.\end{aligned}$$

By the maximum principle $\|u^\varepsilon\|_\infty\le\|\psi\|_\infty$ and $\|\nabla u^\varepsilon\|^p_{L^p(\Omega_T)}$ is uniformly bounded, since the gradients converge strongly. This yields the desired bound M_U. □

3. The gradient estimate

In order to prove that $\Delta_p u$ is a *function*, u denoting the solution to the obstacle problem, we show that the function $\mathbf{F}=|\nabla u|^{(p-2)/2}\nabla u$, where the usual power $p-2$ has been replaced by $(p-2)/2$, is in a suitable first order Sobolev x-space. This will immediately imply the desired result. At a first reading one had better to assume that the obstacle ψ is as smooth as one pleases, say of class $C^2(\overline{\Omega_T})$. Actually, only the Sobolev derivatives $\psi_{x_ix_j}$ and ψ_{x_it} are needed, while ψ_{tt} does not appear at all. We recall our assumption $\psi\in C(\overline{\Omega_T})\cap W^{2,p}(\Omega_T)$ and use the abbreviation

$$|D^2\psi|^2=\sum\psi^2_{x_ix_j}.$$

Under these assumptions about the obstacle $\psi=\psi(x,t)$ we have the following result. (The elliptic case is in [5].)

Theorem 5. *For the solution u to the obstacle problem, the derivative $D\boldsymbol{F}$ of*

$$\boldsymbol{F}=|\nabla u|^{\frac{p-2}{2}}\nabla u$$

exists in Sobolev's sense and belongs to $L^{p/(p-1)}_{\mathrm{loc}}(\Omega_T)$.

The estimate

$$\int_0^T\int_\Omega \zeta^p|D\boldsymbol{F}|^2\,dx\,dt \le C\int_0^T\int_\Omega (\zeta^p+|\nabla\zeta|^p)|\nabla u|^p\,dx\,dt + C\int_0^T\int_\Omega \zeta^p|\nabla u|^2\,dx\,dt + C\int_0^T\int_\Omega |\nabla\zeta|^p|\nabla\psi|^p\,dx\,dt + C\int_0^T\int_\Omega \zeta^p(|D^2\psi|^p+|\nabla\psi_t|^2)\,dx\,dt + C\int_\Omega \zeta^p|\nabla\psi(x,T)|^2\,dx$$

holds for each non-negative test function $\zeta=\zeta(x)$ in $C_0^\infty(\Omega)$; and $C=C(p)$.

Proof. The proof is based on the regularized obstacle problem and equation (2.4), where we abbreviate

$$\mathbf{A}_\varepsilon(x,t)=\big(|\nabla u^\varepsilon|^2+\varepsilon^2\big)^{\frac{p-2}{2}}\nabla u^\varepsilon.$$

We denote its solution by u, suppressing the index ε. Thus u means u^ε, to begin with. Given ζ, the variable x is given a small increment h so that the test function

$$\begin{aligned}\eta &= \psi(x,t)-u(x,t)+\zeta(x)^p[u(x+h,t)-\psi(x+h,t)]\\ &= \zeta(x)^p\overbrace{[u(x+h,t)-u(x,t)]}^{\mathbf{\Delta_h}u}-\zeta(x)^p\overbrace{[\psi(x+h,t)-\psi(x,t)]}^{\mathbf{\Delta_h}\psi}\\ &\quad-(1-\zeta(x)^p)[u(x,t)-\psi(x,t)]\end{aligned}$$

is admissible in the regularized equation

$$\int_0^T\int_\Omega\Big(\langle\mathbf{A}_\varepsilon(x,t),\nabla\eta\rangle+\eta\,\frac{\partial u}{\partial t}\Big)dx\,dt\ge 0. \tag{3.1}$$

Inserting the test function, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega\Big(\langle\mathbf{A}_\varepsilon(x,t),\nabla(\zeta^p\mathbf{\Delta_h}u)\rangle+\zeta^p\mathbf{\Delta_h}u\frac{\partial u}{\partial t}\Big)\,dx\,dt\\ &\qquad-\int_0^T\int_\Omega\Big(\langle\mathbf{A}_\varepsilon(x,t),\nabla(\zeta^p\mathbf{\Delta_h}\psi)\rangle+\zeta^p\mathbf{\Delta_h}\psi\frac{\partial u}{\partial t}\Big)\,dx\,dt\\ &\ge\int_0^T\int_\Omega\Big(\langle\mathbf{A}_\varepsilon(x,t),\nabla\big((1-\zeta(x)^p)[u(x,t)-\psi(x,t)]\big)\rangle\\ &\qquad+(1-\zeta(x)^p)[u(x,t)-\psi(x,t)]\frac{\partial u}{\partial t}\Big)\,dx\,dt\\ &\ge 0.\end{aligned}$$

The last integral is non-negative, because

$$(1-\zeta(x)^p)[u(x,t)-\psi(x,t)]$$

will do as a test function in the equation (3.1). This observation is important here.

Aiming at difference quotients, we gave x the increment h. The translated function $u(x+h,t)$ solves the obstacle problem with the translated obstacle $\psi(x+h,t)$, all this with respect to the shifted domain $\Omega^h \times (0,T)$ where $\Omega^h = \{x|\, x+h \in \Omega\}$. For sufficiently small h we have

$$\int_0^T \int_{\Omega^h} \Big(\langle \mathbf{A}_\varepsilon(x+h,t), \nabla \eta(x,t) \rangle + \eta(x,t) \, \frac{\partial u(x+h,t)}{\partial t} \Big) \, dx \, dt \geq 0 \tag{3.2}$$

whenever $\eta(x,t) \geq \psi(x+h,t) - u(x+h,t)$ and $\eta = 0$ on the parabolic boundary of $\Omega^h \times (0,T)$. Here

$$\begin{aligned} \eta &= \psi(x+h,t) - u(x+h,t) + \zeta(x)^p [u(x,t) - \psi(x,t)] \\ &= \zeta(x)^p \overbrace{[u(x+h,t) - u(x,t)]}^{\mathbf{\Delta_h} u} - \zeta^p(x) \overbrace{[\psi(x+h,t) - \psi(x,t)]}^{\mathbf{\Delta_h} \psi} \\ &\quad - (1 - \zeta(x)^p)[u(x+h,t) - \psi(x+h,t)] \end{aligned}$$

will do. We obtain

$$\begin{aligned} &- \int_0^T \int_{\Omega^h} \Big(\langle \mathbf{A}_\varepsilon(x+h,t), \nabla(\zeta^p \mathbf{\Delta_h} u) \rangle + \zeta^p \mathbf{\Delta_h} u \, \frac{\partial u(x+h,t)}{\partial t} \Big) \, dx \, dt \\ &\qquad + \int_0^T \int_{\Omega^h} \Big(\langle \mathbf{A}_\varepsilon(x+h,t), \nabla(\zeta^p \mathbf{\Delta_h} \psi) \rangle + \zeta^p \mathbf{\Delta_h} \psi \, \frac{\partial u(x+h,t)}{\partial t} \Big) \, dx \, dt \\ &\quad \geq \int_0^T \int_{\Omega^h} \Big(\langle \mathbf{A}_\varepsilon(x+h,t), \nabla\big((1-\zeta(x)^p)[u(x+h,t) - \psi(x+h,t)] \big) \rangle \\ &\qquad + (1-\zeta(x)^p)[u(x+h,t) - \psi(x+h,t)] \, \frac{\partial u(x+h,t)}{\partial t} \Big) \, dx \, dt \\ &\quad \geq 0. \end{aligned}$$

The last integral is positive because

$$(1-\zeta(x)^p)[u(x+h,t) - \psi(x+h,t)]$$

will do as a test function in the translated equation (3.2). This observation is essential here. The integrals in the left-hand member of the inequality are, in fact, taken only over the support of the function $\zeta(x)$. Hence we have an inequality with integrals taken only over Ω_T, provided that $|h| < \operatorname{dist}(\operatorname{supp} \zeta, \partial\Omega)$. Thus Ω^h is no longer directly involved.

We add the two estimates, grouping the differences, and obtain

$$\begin{aligned} &+ \int_0^T \int_\Omega \langle \mathbf{A}_\varepsilon(x+h,t) - \mathbf{A}_\varepsilon(x,t), \nabla(\zeta^p \mathbf{\Delta_h} u) \rangle \, dx \, dt \\ &\qquad \leq \int_0^T \int_\Omega \langle \mathbf{A}_\varepsilon(x+h,t) - \mathbf{A}_\varepsilon(x,t), \nabla(\zeta^p \mathbf{\Delta_h} \psi) \rangle \, dx \, dt \\ &\qquad\quad - \int_0^T \int_\Omega \zeta^p \mathbf{\Delta_h} u \cdot \mathbf{\Delta_h} \Big(\frac{\partial u}{\partial t} \Big) \, dx \, dt + \int_0^T \int_\Omega \zeta^p \mathbf{\Delta_h} \psi \cdot \mathbf{\Delta_h} \Big(\frac{\partial u}{\partial t} \Big) \, dx \, dt. \end{aligned}$$

The integrals with the time derivatives can be integrated by parts:

$$
\begin{aligned}
-\int_0^T\int_\Omega \zeta^p \frac{\partial}{\partial t}\frac{(\mathbf{\Delta_h}u)^2}{2}\,dx\,dt + \int_0^T\int_\Omega \zeta^p \mathbf{\Delta_h}\psi\cdot\mathbf{\Delta_h}\Big(\frac{\partial u}{\partial t}\Big)\,dx\,dt \\
= -\int_\Omega \zeta^p(x)\frac{(\mathbf{\Delta_h}u)^2}{2}\Big|_0^T\,dx + \int_\Omega \zeta^p(x)\mathbf{\Delta_h}\psi\cdot\mathbf{\Delta_h}u\Big|_0^T\,dx \\
-\int_0^T\int_\Omega \zeta^p\mathbf{\Delta_h}u\cdot\mathbf{\Delta_h}\Big(\frac{\partial\psi}{\partial t}\Big)\,dx\,dt.
\end{aligned}
$$

Since $\mathbf{\Delta_h}u = \mathbf{\Delta_h}\psi$ when $t=0$, the above expression is majorized by

$$
\frac{1}{2}\int_\Omega \zeta^p((\mathbf{\Delta_h}\psi)_T^2 - (\mathbf{\Delta_h}\psi)_0^2)\,dx + \frac{1}{2}\int_0^T\int_\Omega \zeta^p\Big((\mathbf{\Delta_h}u)^2 + \big(\mathbf{\Delta_h}\frac{\partial\psi}{\partial t}\big)^2\Big)\,dx\,dt,
$$

where the inequality $2\mathbf{\Delta_h}u\,\mathbf{\Delta_h}\psi \le (\mathbf{\Delta_h}u)^2 + (\mathbf{\Delta_h}\psi)^2$ was used at time T.

At this stage there are no "forbidden" time derivatives left and so we may safely let ε go to zero. By Lemma 4 we may pass to the limit under the integral sign and hence the estimate for the limit u (no longer u^ε) becomes

$$
\begin{aligned}
&\int_0^T\int_\Omega (\langle\mathbf{\Delta_h}\mathbf{A},\nabla(\zeta^p\mathbf{\Delta_h}u)\rangle\,dx\,dt \\
&\quad\le \int_0^T\int_\Omega (\langle\mathbf{\Delta_h}\mathbf{A},\nabla(\zeta^p\mathbf{\Delta_h}\psi)\rangle\,dx\,dt \\
&\qquad + \frac{1}{2}\int_0^T\int_\Omega \zeta^p\Big((\mathbf{\Delta_h}u)^2 + \big(\mathbf{\Delta_h}\frac{\partial\psi}{\partial t}\big)^2\Big)\,dx\,dt + \frac{1}{2}\int_\Omega \zeta^p(\mathbf{\Delta_h}\psi)_T^2\,dx,
\end{aligned}
$$

where

$$
\mathbf{\Delta_h}\mathbf{A} = \mathbf{A}(x+h,t) - \mathbf{A}(x,t).
$$

We write this more conveniently as

$$
\begin{aligned}
\int_0^T\int_\Omega \zeta^p(\langle\mathbf{\Delta_h}\mathbf{A},\nabla\mathbf{\Delta_h}u\rangle)\,dx\,dt \le \overbrace{\int_0^T\int_\Omega p\zeta^{p-1}|\mathbf{\Delta_h}\mathbf{A}||\mathbf{\Delta_h}u||\nabla\zeta|\,dx\,dt}^{\mathrm{I}} \\
+\overbrace{\int_0^T\int_\Omega p\zeta^{p-1}|\mathbf{\Delta_h}\mathbf{A}||\mathbf{\Delta_h}\psi||\nabla\zeta|\,dx\,dt}^{\mathrm{II}} + \overbrace{\int_0^T\int_\Omega \zeta^p|\mathbf{\Delta_h}\mathbf{A}||\nabla\mathbf{\Delta_h}\psi|\,dx\,dt}^{\mathrm{III}} \\
+\frac{1}{2}\int_0^T\int_\Omega \zeta^p\Big((\mathbf{\Delta_h}u)^2\big(\mathbf{\Delta_h}\frac{\partial\psi}{\partial t}\big)^2\Big)\,dx\,dt + \frac{1}{2}\int_\Omega \zeta^p(\mathbf{\Delta_h}\psi)_T^2\,dx.
\end{aligned}
\tag{3.3}
$$

The integrand on the left-hand side is

$$
\begin{aligned}
\langle\mathbf{\Delta_h}\mathbf{A},\nabla\mathbf{\Delta_h}u\rangle &= \langle|\nabla u(x+h,t)|^{\frac{p-2}{2}}\nabla u(x+h,t) \\
&\qquad - |\nabla u(x,t)|^{\frac{p-2}{2}}\nabla u(x,t), \nabla u(x+h,t) - \nabla u(x,t)\rangle \\
&\ge \frac{4}{p^2}\,|\mathbf{F}(x+h,t) - \mathbf{F}(x,t)|^2 = \frac{4}{p^2}|\mathbf{\Delta_h}\mathbf{F}|^2,
\end{aligned}
\tag{3.4}
$$

where the elementary inequality

$$\frac{4}{p^2}\Big||b|^{\frac{p-2}{2}}b-|a|^{\frac{p-2}{2}}a\Big|^2 \le \langle |b|^{p-2}b-|a|^{p-2}a, b-a\rangle$$

for vectors was used. We aim at an estimate for the integral of $\zeta^p|\mathbf{\Delta_h F}|$.

We divide the $\mathbf{\Delta_h}$-terms by $|h|$ so that the desired difference quotients appear. The estimate

$$\Big|\frac{\mathbf{\Delta_h A}}{h}\Big| \le (p-1)\Big|\frac{\mathbf{\Delta_h F}}{h}\Big|\big(|\nabla u(x+h,t)|^{\frac{p-2}{2}}+|\nabla u(x,t)|^{\frac{p-2}{2}}\big),$$

coming from the elementary vector inequality

$$\big||b|^{p-2}b-|a|^{p-2}a\big| \le (p-1)\Big(|b|^{\frac{p-2}{2}}+|a|^{\frac{p-2}{2}}\Big)\,\Big||b|^{\frac{p-2}{2}}b-|a|^{\frac{p-2}{2}}a\Big|,$$

is used in the integrands of I, II, and III. In I we split the factors so that

$$\begin{aligned}
&p\,\zeta^{p-1}\Big|\frac{\mathbf{\Delta_h A}}{h}\Big|\Big|\frac{\mathbf{\Delta_h} u}{h}\Big||\nabla\zeta|\\
&\quad\le p(p-1)\Big[\zeta^{\frac{p}{2}}\big|\frac{\mathbf{\Delta_h F}}{h}\big|\Big]\Big[\big|\frac{\mathbf{\Delta_h} u}{h}\big||\nabla\zeta|\Big]\Big[\zeta^{\frac{p-2}{2}}\big(|\nabla u(x,t)|^{\frac{p-2}{2}}+|\nabla u(x{+}h,t)|^{\frac{p-2}{2}}\big)\Big]
\end{aligned}$$

and use Young's inequality

$$abc \le \frac{\epsilon^2a^2}{2}+\frac{\epsilon^{-p}b^p}{p}+\frac{(p-2)c^{\frac{2p}{p-2}}}{2p}$$

to get the bound

$$\begin{aligned}
\frac{\mathrm{I}}{|h|^2} \le{}& \frac{p(p-1)\epsilon^2}{2}\int_0^T\!\!\int_\Omega \zeta^p\Big|\frac{\mathbf{\Delta_h F}}{h}\Big|^2\,dx\,dt\\
&+(p-1)\epsilon^{-p}\int_0^T\!\!\int_\Omega\Big|\frac{\mathbf{\Delta_h} u}{h}\Big|^p|\nabla\zeta|^p\,dx\,dt\\
&+c_p\int_0^T\!\!\int_\Omega\zeta^p\Big(|\nabla u(x,t)|^p+|\nabla u(x+h,t)|^p\Big)\,dx\,dt
\end{aligned}$$

The integral $\mathrm{II}/|h|^2$ has a similar majorant, the only difference being that $\mathbf{\Delta_h}u$ be replaced by $\mathbf{\Delta_h}\psi$. The integrand of III is estimated in a similar way:

$$\begin{aligned}
&p\zeta^p\Big|\frac{\mathbf{\Delta_h A}}{h}\Big|\Big|\frac{\mathbf{\Delta_h}\psi}{h}\Big|\\
&\le p(p-1)\Big[\zeta^{\frac{p}{2}}\big|\frac{\mathbf{\Delta_h F}}{h}\big|\Big]\Big[\zeta\big|\nabla(\frac{\mathbf{\Delta_h} u}{h})\big|\Big]\Big[\zeta^{\frac{p-2}{2}}\big(|\nabla u(x,t)|^{\frac{p-2}{2}}+|\nabla u(x{+}h,t)|^{\frac{p-2}{2}}\big)\Big]\\
&\le\frac{p(p-1)\epsilon^2}{2}\zeta^p\Big|\frac{\mathbf{\Delta_h F}}{h}\Big|^2+(p-1)\epsilon^{-p}\zeta^p\Big|\nabla\Big(\frac{\mathbf{\Delta_h}\psi}{h}\Big)\Big|^p\\
&\quad+c_p\zeta^p\big(|\nabla u(x,t)|^p+|\nabla u(x+h,t)|^p\big).
\end{aligned}$$

Adding up the three integrated estimates, we arrive at

$$
\begin{aligned}
&\frac{\mathrm{I}+\mathrm{II}+\mathrm{III}}{|h|^2}\\
&\quad\le 3\,\frac{p(p-1)\epsilon^2}{2}\int_0^T\!\!\int_\Omega \zeta^p\Big|\frac{\boldsymbol{\Delta}_{\mathbf{h}}\mathbf{F}}{h}\Big|^2\,dx\,dt\\
&\qquad+(p-1)\epsilon^{-p}\int_0^T\!\!\int_\Omega\Big(\Big|\frac{\boldsymbol{\Delta}_{\mathbf{h}}u}{h}\Big|^p|\nabla\zeta|^p+\Big|\frac{\boldsymbol{\Delta}_{\mathbf{h}}\psi}{h}\Big|^p|\nabla\zeta|^p+\zeta^p\Big|\frac{\nabla\big(\boldsymbol{\Delta}_{\mathbf{h}}\psi}{h}\Big|^p\Big)\,dx\,dt\\
&\qquad+3c_p\int_0^T\!\!\int_\Omega\zeta^p\Big(|\nabla u(x,t)|^p+|\nabla u(x+h,t)|^p\Big)\,dx\,dt.
\end{aligned}
$$

This complements (3.3). Recall (3.4). The next step is to absorb the first integral above in the right-hand member into the minorant in (3.4) by fixing ϵ small enough, say

$$3\,\frac{p(p-1)\,\epsilon^2}{2}=\frac{2}{p^2}.$$

The resulting estimate, written out without abbreviations, is

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega\zeta^p\Big|\frac{\mathbf{F}(x+h,t)-\mathbf{F}(x,t)}{h}\Big|^2dx\,dt\\
&\qquad\le a_p\int_0^T\!\!\int_\Omega\Big|\frac{u(x+h,t)-u(x,t)}{h}\Big|^p|\nabla\zeta|^p dx\,dt\\
&\qquad\quad+a_p\int_0^T\!\!\int_\Omega\Big|\frac{\psi(x+h,t)-\psi(x,t)}{h}\Big|^p|\nabla\zeta|^p\,dx\,dt\\
&\qquad\quad+a_p\int_0^T\!\!\int_\Omega\zeta^p\Big|\frac{\nabla\psi(x+h,t)-\nabla\psi(x,t)}{h}\Big|^p\,dx\,dt\\
&\qquad\quad+b_p\int_0^T\!\!\int_\Omega\zeta^p\Big(|\nabla u(x,t)|^p+|\nabla u(x+h,t)|^p\Big)\,dx\,dt\\
&\qquad\quad+c_p\int_0^T\!\!\int_\Omega\zeta^p\Big|\frac{u(x+h,t)-u(x,t)}{h}\Big|^2\,dx\,dt\\
&\qquad\quad+c_p\int_0^T\!\!\int_\Omega\zeta^p\Big|\frac{\psi_t(x+h,t)-\psi_t(x,t)}{h}\Big|^2\,dx\,dt\\
&\qquad\quad+c_p\int_\Omega\zeta^p\Big|\frac{\psi(x+h,T)-\psi(x,T)}{h}\Big|^2\,dx,
\end{aligned}
$$

where the constants depend only on p.

Finally, letting the increment $h\to 0$ in any desired direction, we arrive at the estimate in the theorem. Here we use the characterization of Sobolev spaces in terms of integrated differential quotients, cf. Chapter 8.1 of [10] or [11]. This concludes our proof of Theorem 5. □

Proposition 6. *If u is the solution to the obstacle problem with the obstacle ψ, then $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ belongs to the space $L^{\frac{p}{p-1}}_{\text{loc}}(\Omega_T)$ and*

$$\int_0^T \int_\Omega \varphi \, \Delta_p u \, dx \, dt = -\int_0^T \int_\Omega \langle \, |\nabla u|^{p-2}\nabla u, \nabla \varphi \rangle \, dx \, dt$$

for all test functions φ in $C_0^\infty(\Omega_T)$.

Proof. Since $\mathbf{F}$ is in Sobolev's space and $p > 2$, we can differentiate

$$|\nabla u|^{p-2}\nabla u = |\mathbf{F}|^{\frac{p-2}{p}} \mathbf{F}$$

and hence

$$\Big| \frac{\partial}{\partial x_j}(|\nabla u|^{p-2}\nabla u) \Big| \leq 2\Big(1 - \frac{1}{p}\Big) \; |\mathbf{F}|^{\frac{p-2}{p}} \, \Big| \frac{\partial \mathbf{F}}{\partial x_j} \Big|.$$

By Hölder's inequality,

$$\frac{\partial}{\partial x_j}(|\nabla u|^{p-2}\nabla u) \in L^{\frac{p}{p-1}}_{\text{loc}}(\Omega_T),$$

since $\mathbf{F} \in L^2(\Omega_T)$ and $D\mathbf{F} \in L^2(\Omega_T)$. □

4. The time derivative

For the proof of the theorem we notice that the contact set $\Xi = \{u = \psi\}$ is a closed subset of $\overline{\Omega_T}$ and that its complement $\Upsilon = \Omega_T \setminus \Xi$ is open. In the set Υ, where the obstacle does not hinder, u is a solution to the evolutionary p-Laplace equation $u_t = \Delta_p u$. In other words, whenever $\phi \in C_0^\infty(\Upsilon)$,

$$\int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_\Omega \langle \, |\nabla u|^{p-2}\nabla u, \nabla \phi \rangle \, dx \, dt = -\int_0^T \int_\Omega \phi \Delta_p u \, dx \, dt,$$

the actual set of integration being Υ. Here Proposition 6 was used. Thus u_t is available, but only in Υ to begin with. (See also [20].) Let ϕ denote an arbitrary test function in $C_0^\infty(\Omega_T)$. We need a specific test function with compact support in Υ. To construct it, define

$$\theta_k = \min\{1, k(u - \psi)\}, \qquad k = 1, 2, \ldots$$

Then $1 - \theta_k = 1$ in Ξ and pointwise the monotone convergence $1 - \theta_k \longrightarrow \chi_\Xi$ holds as $k \longrightarrow \infty$. Moreover, the support of θ_k is compact in Υ. The time derivative of θ_k is available!

Using

$$\int_0^T \int_\Omega u \frac{\partial}{\partial t}(\theta_k \phi) \, dx \, dt = \int_0^T \int_\Omega \langle \, |\nabla u|^{p-2}\nabla u, \nabla(\theta_k \phi) \rangle \, dx \, dt,$$

we write

$$\int_0^T\int_\Omega \phi\Delta_p u\,dx\,dt = -\int_0^T\int_\Omega \langle\,|\nabla u|^{p-2}\nabla u, \nabla\phi\rangle\,dx\,dt$$

$$= -\int_0^T\int_\Omega \langle\,|\nabla u|^{p-2}\nabla u, \nabla(\theta_k\phi + (1-\theta_k)\phi)\rangle\,dx\,dt$$

$$= -\int_0^T\int_\Omega \langle\,|\nabla u|^{p-2}\nabla u, \nabla(\theta_k\phi)\rangle\,dx\,dt - \int_0^T\int_\Omega \langle\,|\nabla u|^{p-2}\nabla u, \nabla\left((1-\theta_k)\,\phi\right)\rangle\,dx\,dt$$

$$= -\int_0^T\int_\Omega u\frac{\partial}{\partial t}(\theta_k\phi)\,dx\,dt + \int_0^T\int_\Omega (1-\theta_k)\phi\Delta_p u\,dx\,dt.$$

The last integral has the limit

$$\lim_{k\to\infty}\int_0^T\int_\Omega (1-\theta_k)\phi\,\Delta_p u\,dx\,dt = \iint\limits_{\Xi} \phi\,\Delta_p\psi\,dx\,dt.$$

In the integral with the time derivative we write

$$-u\frac{\partial}{\partial t}(\theta_k\phi) = -u\frac{\partial\phi}{\partial t} + (u-\psi)\frac{\partial}{\partial t}((1-\theta_k)\phi) + \psi\frac{\partial}{\partial t}((1-\theta_k)\phi)$$

and obtain

$$-\int_0^T\int_\Omega u\frac{\partial}{\partial t}(\theta_k\phi)\,dx\,dt = \int_0^T\int_\Omega -u\frac{\partial\phi}{\partial t}\,dx\,dt$$
$$+\int_0^T\int_\Omega (u-\psi)\frac{\partial}{\partial t}((1-\theta_k)\phi)\,dx\,dt - \int_0^T\int_\Omega (1-\theta_k)\phi\frac{\partial\psi}{\partial t}\,dx\,dt,$$

where an integration by parts has produced the last integral. It has the evident limit

$$\lim_{k\to\infty}\int_0^T\int_\Omega (1-\theta_k)\phi\frac{\partial\psi}{\partial t}\,dx\,dt = \iint\limits_{\Xi} \phi\,\frac{\partial\psi}{\partial t}\,dx\,dt.$$

The middle integral vanishes as $k\to\infty$:

$$\int_0^T\int_\Omega (u-\psi)\frac{\partial}{\partial t}((1-\theta_k)\phi)\,dx\,dt$$
$$= \int_0^T\int_\Omega (u-\psi)(1-\theta_k)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_0^T\int_\Omega \phi(u-\psi)\frac{\partial\theta_k}{\partial t}\,dx\,dt$$
$$= \int_0^T\int_\Omega (u-\psi)(1-\theta_k)\frac{\partial\phi}{\partial t}\,dx\,dt - \frac{1}{2k}\int_0^T\int_\Omega \phi\frac{\partial}{\partial t}\theta_k^2\,dx\,dt$$
$$= \int_0^T\int_\Omega (u-\psi)(1-\theta_k)\frac{\partial\phi}{\partial t}\,dx\,dt + \frac{1}{2k}\int_0^T\int_\Omega \theta_k^2\,\frac{\partial\phi}{\partial t}\,dx\,dt \quad \underset{k\to\infty}{\longrightarrow} \quad 0+0.$$

Collecting results,

$$\int_0^T\int_\Omega \phi\,\Delta_p u\,dx\,dt = -\int_0^T\int_\Omega u\phi_t\,dx\,dt - \iint\limits_{\Xi}(\psi_t - \Delta_p\psi)\phi\,dx\,dt.$$

In other words, the final formula

$$-\int_0^T\int_\Omega u\phi_t\,dx\,dt = \int_0^T\int_\Omega \phi\,[\Delta_p u + (\psi_t - \Delta_p\psi)\chi_\Xi]\,dx\,dt$$

holds for every ϕ in $C_0^\infty(\Omega_T)$. Therefore

$$u_t = \Delta_p u + (\psi_t - \Delta_p\psi)\chi_\Xi$$

and this is a *function* belonging to $L^{p/(p-1)}_{\mathrm{loc}}(\Omega_T)$. This concludes the proof of Theorem 2.

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