Rev. Mat. Iberoam. **28** (2012), no. 2, 591–601
DOI 10.4171/RMI/686

Abelian varieties with many endomorphisms and their absolutely simple factors

Xavier Guitart

Abstract. We characterize the abelian varieties arising as absolutely simple factors of GL_2-type varieties over a number field k. In order to obtain this result, we study a wider class of abelian varieties: the k-varieties A/k satisfying that $\mathrm{End}_k^0(A)$ is a maximal subfield of $\mathrm{End}_{\bar{k}}^0(A)$. We call them *Ribet–Pyle varieties* over k. We see that every Ribet–Pyle variety over k is isogenous over $\bar{k}$ to a power of an abelian k-variety and, conversely, that every abelian k-variety occurs as the absolutely simple factor of some Ribet–Pyle variety over k. We deduce from this correspondence a precise description of the absolutely simple factors of the varieties over k of GL_2-type.

1. Introduction

Let k be a number field. An abelian variety A over k is said to be of GL_2-type if its algebra of k-endomorphisms $\mathrm{End}_k^0(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_k(A)$ is a number field of degree equal to the dimension of A. The aim of this note is to characterize the abelian varieties over $\bar{k}$ that arise as absolutely simple factors of GL_2-type varieties over k.

The interest in abelian varieties over $\mathbb{Q}$ of GL_2-type arose in connection with the Shimura–Taniyama conjecture on the modularity of elliptic curves over $\mathbb{Q}$, and its generalization to higher dimensional modular abelian varieties over $\mathbb{Q}$. To be more precise, to each $A/\mathbb{Q}$ of GL_2-type is attached a compatible system of λ-adic representations $\rho_{A,\lambda}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$, where $E = \mathrm{End}_{\mathbb{Q}}^0(A)$ and the λ's are primes of E. As a consequence of Serre's conjecture on Galois representations these $\rho_{A,\lambda}$ are modular; that is, there exists a newform $f \in S_2(\Gamma_1(N))$ such that $\rho_{A,\lambda} \simeq \rho_{f,\lambda}$ for all primes λ of E, where $\rho_{f,\lambda}$ is the λ-adic representation attached to f (see [4] for the details).

The study of the $\overline{\mathbb{Q}}$-simple factors of GL_2-type varieties over $\mathbb{Q}$ was initiated by K. Ribet in [4], in which the one-dimensional factors where characterized: they are the elliptic curves $C/\overline{\mathbb{Q}}$ that are isogenous to all their Galois conjugates, also known

Mathematics Subject Classification (2010): 11G10, 14K15.
Keywords: Abelian varieties of GL_2-type, k-varieties, building blocks.

as elliptic $\mathbb{Q}$-curves. This result was completed by Ribet's student E. Pyle in her PhD thesis [3], where she characterized the higher dimensional $\overline{\mathbb{Q}}$-simple factors as a certain type of abelian $\mathbb{Q}$-varieties called building blocks. More concretely, an abelian variety $B/\overline{\mathbb{Q}}$ is an *abelian* $\mathbb{Q}$-*variety* if it is $\operatorname{End}_{\overline{\mathbb{Q}}}(B)$-equivariantly isogenous to all of its Galois conjugates; this means that for each $\sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_\sigma\colon {}^\sigma B \to B$ such that $\varphi \circ \mu_\sigma = \mu_\sigma \circ {}^\sigma\varphi$ for all $\varphi \in \operatorname{End}_{\overline{\mathbb{Q}}}(B)$. A *building block* is an abelian $\mathbb{Q}$-variety B whose endomorphism algebra is a central division algebra over a totally real field F, with Schur index $t \leqslant 2$ and reduced degree $t[F:\mathbb{Q}] = \dim B$. The following statement is Proposition 1.3 and Proposition 4.5 of [3].

Theorem 1.1 (Ribet–Pyle). *Let $A/\mathbb{Q}$ be an abelian variety of GL_2-type such that $A_{\overline{\mathbb{Q}}}$ does not have complex multiplication. Then $A_{\overline{\mathbb{Q}}}$ decomposes up to $\overline{\mathbb{Q}}$-isogeny as $A_{\overline{\mathbb{Q}}} \sim B^n$ for some building block $B/\overline{\mathbb{Q}}$. Conversely, if $B/\overline{\mathbb{Q}}$ is a building block then there exists a GL_2-type variety $A/\mathbb{Q}$ such that $A_{\overline{\mathbb{Q}}} \sim B^n$ for some n.*

Observe that this result establishes a correspondence between abelian varieties of GL_2-type over $\mathbb{Q}$ without CM and building blocks. In the last chapter of Pyle's thesis, a series of questions were posed about whether a similar correspondence holds for GL_2-type varieties over other fields k. The goal of this note is to establish such a correspondence when k is a number field. In this case, the analogue of a building block is an abelian k-variety (that is, a variety $B/\bar{k}$ equivariantly isogenous to ${}^\sigma B$ for all $\sigma \in G_k$) whose endomorphism algebra is a central division algebra over a field F with Schur index $t \leqslant 2$ and $t[F:\mathbb{Q}] = \dim B$. We call these varieties *building k-blocks*. We prove in Section 3 that every GL_2-type variety A/k such that $A_{\bar{k}}$ does not have CM is $\bar{k}$-isogenous to the power of a building k-block. Conversely, every building k-block arises as the $\bar{k}$-simple factor of some variety over k of GL_2-type. In other words, we construct a correspondence

$$\frac{\{A/k \text{ of } \mathrm{GL}_2\text{-type without CM}\}}{k\text{-isogeny}} \longleftrightarrow \frac{\{\text{building } k\text{-blocks } B/\bar{k}\}}{\bar{k}\text{-isogeny}}. \tag{1.1}$$

This can be seen as a natural generalization of the results of Ribet and Pyle to a wider class of abelian varieties. Moreover, it is worth noting that varieties over k of GL_2-type play a similar role as their counterparts over $\mathbb{Q}$ with respect to modularity: they are conjectured to be modular, at least when k is totally real, in a similar sense as they are known to be modular for $k=\mathbb{Q}$. Indeed, if A/k is of GL_2-type and k is a totally real number field, a generalization of the Shimura–Taniyama conjecture predicts the existence of a Hilbert modular form f such that $\rho_{A,\lambda} \simeq \rho_{f,\lambda}$ for all primes λ of $E = \operatorname{End}_k^0(A)$. See Conjecture 2.4 in [1] for a precise statement.

Observe that in correspondence (1.1) the objects in the right hand side are k-varieties whose endomorphism algebras satisfy certain conditions. Instead of proving (1.1) directly, what we do is to construct as a previous step a more general correspondence, in which the right hand side is enlarged to all abelian k-varieties. As we will see, the varieties that correspond to those in the left hand side are varieties A/k characterized by the fact that $A_{\bar{k}}$ is a k-variety and $\operatorname{End}_k^0(A)$ is a maximal subfield of $\operatorname{End}^0_{\bar{k}}(A)$. We call the varieties satisfying these properties *Ribet–Pyle varieties*, because they arise naturally in this generalization of the results of Ribet and Pyle. Section 2 is devoted to the study of Ribet–Pyle varieties and their absolutely simple factors, and we obtain the following main result:

Theorem 1.2. *Let k be a number field and let A/k be a Ribet–Pyle variety. Then $A_{\bar{k}}$ decomposes up to $\bar{k}$-isogeny as $A_{\bar{k}} \sim B^n$ for some abelian k-variety $B/\bar{k}$. Conversely, if $B/\bar{k}$ is a k-variety then there exists a Ribet–Pyle variety A/k such that $A_{\bar{k}} \sim B^n$ for some n.*

This result gives some insight into the nature of the correspondences of Theorem 1.1 and its generalization (1.1). Indeed, what we do in Section 3 is to prove that varieties over k of GL_2-type without CM are Ribet–Pyle varieties, and then we obtain (1.1) by applying Theorem 1.2 to GL_2-type varieties.

Acknowledgements. I am grateful to my advisor, Professor Jordi Quer, for his help and guidance throughout this work. I would also like to thank Francesc Fité for carefully reading a previous version of this manuscript.

2. Ribet–Pyle varieties

Let k be a number field. In this section we establish and prove the correspondence between abelian k-varieties and Ribet–Pyle varieties of Theorem 1.2. We begin by giving the relevant definitions.

Definition 2.1. *An abelian variety $B/\bar{k}$ is an* abelian k-variety *if for each $\sigma \in G_k$ there exists an isogeny $\mu_\sigma\colon {}^\sigma B \to B$ compatible with the endomorphisms of B; i.e., such that for all $\varphi \in \operatorname{End}_{\bar{k}}(B)$ the following diagram is commutative:*

(2.1)
$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\mu_\sigma} & B \\ {\scriptstyle {}^\sigma\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ {}^\sigma B & \xrightarrow{\mu_\sigma} & B. \end{array}$$

Definition 2.2. *An abelian variety A defined over k is a* Ribet–Pyle variety *if $A_{\bar{k}}$ is an abelian k-variety and $\operatorname{End}^0_k(A)$ is a maximal subfield of $\operatorname{End}^0_{\bar{k}}(A)$.*

Remark 2.3. We remark that not all abelian varieties A defined over k satisfy that $A_{\bar{k}}$ is a k-variety. Indeed, although in this case the identity is an obvious isogeny between ${}^\sigma A$ and A, it is not necessarily compatible with $\operatorname{End}_{\bar{k}}(A)$ in general.

One of the directions of the correspondence that we aim to establish follows almost immediately from the definitions.

Proposition 2.4. *Let A/k be a Ribet–Pyle variety. Then it decomposes up to $\bar{k}$-isogeny as $A_{\bar{k}} \sim B^n$, for some simple abelian k-variety B and some n.*

Proof. Let F be the center of $\mathrm{End}^0_{\bar{k}}(A)$ and let φ be an element of F. Since $A_{\bar{k}}$ is a k-variety, for each $\sigma \in G_k$ we have that

$$ {}^{\sigma}\varphi = \mu_\sigma^{-1} \circ \varphi \circ \mu_\sigma, \tag{2.2} $$

for some isogeny $\mu_\sigma \colon {}^{\sigma}A_{\bar{k}} \to A_{\bar{k}}$. Since A is defined over k the isogeny μ_σ belongs to $\mathrm{End}^0_{\bar{k}}(A)$. Then ${}^{\sigma}\varphi = \varphi$ because φ belongs to the center of $\mathrm{End}^0_{\bar{k}}(A)$. This gives the inclusion $F \subseteq \mathrm{End}^0_k(A)$. By hypothesis $\mathrm{End}^0_k(A)$ is a field, so F is a field as well and this implies that $A_{\bar{k}} \sim B^n$ for some simple variety B and some n. Next, we show that B is a k-variety. By fixing an isogeny $A_{\bar{k}} \sim B^n$ the center of $\mathrm{End}^0_{\bar{k}}(B)$ can be identified with F, and each compatible isogeny $\mu_\sigma \colon {}^{\sigma}A_{\bar{k}} \to A_{\bar{k}}$ gives rise to an isogeny $\nu_\sigma \colon {}^{\sigma}B \to B$. The relation (2.2) implies that $\psi = \nu_\sigma \circ {}^{\sigma}\psi \circ \nu_\sigma^{-1}$ for all $\psi \in Z(\mathrm{End}^0_{\bar{k}}(B)) \simeq F$, so that the map

$$ \begin{array}{ccc} \mathrm{End}^0_{\bar{k}}(B) & \longrightarrow & \mathrm{End}^0_{\bar{k}}(B) \\ \psi & \longmapsto & \nu_\sigma \circ {}^{\sigma}\psi \circ \nu_\sigma^{-1} \end{array} $$

is an F-algebra automorphism. By the Skolem–Noether Theorem it is inner, and there exists an element $\alpha_\sigma \in \mathrm{End}^0_{\bar{k}}(B)^*$ such that

$$ \nu_\sigma \circ {}^{\sigma}\psi \circ \nu_\sigma^{-1} = \alpha_\sigma^{-1} \circ \psi \circ \alpha_\sigma, $$

for all $\psi \in \mathrm{End}^0_{\bar{k}}(B)$. The isogeny $\alpha_\sigma \circ \nu_\sigma$ satisfies the compatibility condition (2.1) and we see that B is a k-variety. □

The following statement gives the other direction of the correspondence between k-varieties and Ribet–Pyle varieties in the number field case.

Theorem 2.5. *Let k be a number field, and let $B/\bar{k}$ be a simple abelian k-variety. Then there exists a Ribet–Pyle variety A/k such that $A_{\bar{k}} \sim B^n$ for some n.*

Before giving the proof of Theorem 2.5 we shall need some preliminary results.

Cohomology classes and splitting fields

Let k be a number field and let $B/\bar{k}$ be a simple abelian k-variety. Let $\mathcal{B}$ be its endomorphism algebra and let F be the center of $\mathcal{B}$. Since B has a model over a finite extension of k, we can choose for each $\sigma \in G_k$ a compatible isogeny $\mu_\sigma : {}^{\sigma}B \to B$ in such a way that the set $\{\mu_\sigma\}_{\sigma \in G_k}$ is locally constant; more precisely, such that $\mu_\sigma = \mu_\tau$ if ${}^{\sigma}B = {}^{\tau}B$. Then we can define a map $c_B : G_k \times G_k \to F^*$ by means of $c_B(\sigma,\tau) = \mu_\sigma \circ {}^{\sigma}\mu_\tau \circ \mu_{\sigma\tau}^{-1}$. It is easy to check that c_B is a continuous 2-cocycle of G_k with values in F^* (considering the trivial action of G_k in F^*). Its cohomology class $[c_B] \in H^2(G_k, F^*)$ is an invariant of the isogeny class of B and it is independent of the compatible isogenies used to define it.

The inclusion of G_k-modules with trivial action $F^* \hookrightarrow \overline{F}^*$ induces a homomorphism between the cohomology groups $H^2(G_k, F^*) \to H^2(G_k, \overline{F}^*)$. A theorem of Tate implies that $H^2(G_k, \overline{F}^*) = \{1\}$ (see Theorem 6.3 of [4]). Therefore, the image of $[c_B]$ in $H^2(G_k, \overline{F}^*)$ is trivial, which means that there exist continuous maps $\beta : G_k \to \overline{F}^*$ such that

$$c_B(\sigma,\tau) = \beta(\sigma)\beta(\tau)\beta(\sigma\tau)^{-1}. \tag{2.3}$$

We say that a map β satisfying (2.3) is a *splitting map* for the cocycle c_B. If $\chi\colon G_k \to \overline{F}^*$ is a character then $\beta' = \beta\chi$ is another splitting map for c_B. In fact, as we vary χ through all the characters from G_k to $\overline{F}^*$ we obtain all the splitting maps for c_B. For a splitting map β, we will denote by E_β the field $F(\{\beta(\sigma)\}_{\sigma\in G_k}) \subseteq \overline{F}$. The extension E_β/F is finite because β is continuous.

Let m be the order of $[c_B]$ in $H^2(G_k, F^*)$, and let d be a continuous map $d\colon G_k \to F^*$ expressing c_B^m as a coboundary:

$$c_B(\sigma,\tau)^m = d(\sigma)d(\tau)d(\sigma\tau)^{-1}. \tag{2.4}$$

We define a map

$$\begin{array}{rccc} \varepsilon_\beta\colon & G_k & \longrightarrow & \overline{F}^* \\ & \sigma & \longmapsto & \beta(\sigma)^m/d(\sigma). \end{array}$$

By (2.3) and (2.4) we see that $\varepsilon_\beta\colon G_k \to \overline{F}^*$ is a continuous character.

Lemma 2.6. *For each nonnegative integer n there exists a splitting map β such that $F(\zeta_n) \subseteq E_\beta$, where ζ_n is a primitive n-th root of unity in $\overline{F}$.*

Proof. Let β' be a splitting map for c_B, and let r be the order of $\varepsilon_{\beta'}$. Let $e = \gcd(n, r)$ and let $\chi\colon G_k \to \overline{F}^*$ be a character of order mn/e, where m is the order of $[c_B]$ in $H^2(G_k, F^*)$. Then the character $\chi^m\varepsilon_{\beta'}$ is the character that corresponds to the splitting map $\beta = \chi\beta'$ and its order is nr/e, which is a multiple of n. Therefore E_β contains a primitive n-th root of unity ζ_n. □

Cyclic splitting fields of simple algebras

Let $\mathcal{A}$ be a central simple algebra over a number field F. A well-known result of central simple algebras over number fields guarantees the existence of fields L cyclic over F that split $\mathcal{A}$ (i.e. with $\mathcal{A} \otimes_F L \simeq \mathrm{M}_n(L)$ for some n). In order to prove Theorem 2.5 we use a similar result, but with the extension L being cyclic over $\mathbb{Q}$ and such that LF splits $\mathcal{A}$. Although this is probably also well-known, for lack of reference we include a proof based on the Grunwald–Wang Theorem.

Theorem 2.7 (Grunwald–Wang Theorem). *Let M be a number field, and let $\{(v_1, n_1), \dots, (v_r, n_r)\}$ be a finite set of pairs, where each v_i is a place of M and each n_i is a positive integer such that $n_i \leqslant 2$ if v_i is a real place, and $n_i = 1$ if v_i is a complex place. Let m be the least common multiple of the n_i's, and let n be a positive integer divisible by m. Then there exists a cyclic extension L/M of degree n such that for each i the degree $[L_{v_i} : M_{v_i}]$ is divisible by n_i.*

Proposition 2.8. *Let F be a number field and let $\mathcal{D}$ be a central division algebra over F. There exists a cyclic extension $L/\mathbb{Q}$ such that LF is a splitting field for $\mathcal{D}$.*

Proof. Let F' be the Galois closure of F. Let $n = [F' : \mathbb{Q}]$ and let t be the Schur index of $\mathcal{D}$. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of primes of F where $\mathcal{D}$ ramifies, and let $\{p_1, \dots, p_l\}$ be the set of primes of $\mathbb{Q}$ below $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. The Grunwald–Wang Theorem, when applied to the primes p_i with $n_i = tn$, and to the infinite place of $\mathbb{Q}$ with $n_\infty = 2$, guarantees the existence of a cyclic extension $L/\mathbb{Q}$ of degree $2tn$ such that $[L_p : \mathbb{Q}_p] = tn$ for all p belonging to $\{p_1, \dots, p_l\}$ and $L_v = \mathbb{C}$ for all archimedean places v of L. Let $K = LF$.

If $\mathfrak{p}$ is a prime of F dividing p, and $\mathfrak{P}$ is a prime of K dividing $\mathfrak{p}$, the fields L_p and $F_\mathfrak{p}$ can be seen as subfields of $K_\mathfrak{P}$. Then the degree $g = [L_p \cap F_\mathfrak{p} : \mathbb{Q}_p]$ divides n, so $[L_p \colon L_p \cap F_\mathfrak{p}] = t\frac{n}{g} = [F_\mathfrak{p} L_p \colon F_\mathfrak{p}]$ and we see that t divides $[K_\mathfrak{P} : F_\mathfrak{p}]$. Therefore, K is a totally imaginary extension of F such that, for every prime $\mathfrak{p}$ of F ramifying in $\mathcal{D}$ and for every prime $\mathfrak{P}$ of K dividing $\mathfrak{p}$, the index $[K_\mathfrak{P} : F_\mathfrak{p}]$ is a multiple of the Schur index of $\mathcal{D}$. This implies that K is a splitting field for $\mathcal{D}$ (see Corollary 18.4-b and Corollary 17.10-a in [2]). □

Corollary 2.9. *Every central division F-algebra is split by an extension of the form $F(\zeta_m)$ for some m.*

Proof. By the previous proposition there exists a cyclic extension $L/\mathbb{Q}$ such that LF splits $\mathcal{D}$. The field L is contained in a field of the form $\mathbb{Q}(\zeta_m)$ by the Kronecker-Weber Theorem, and then $F(\zeta_m)$ splits $\mathcal{D}$. □

Construction of Ribet–Pyle varieties

In this paragraph we construct Ribet–Pyle varieties having a k-variety B as simple factor. Recall that $\mathcal{B}$ denotes $\operatorname{End}^0_{\bar{k}}(B)$, F is the center of $\mathcal{B}$ and t denotes the Schur index of $\mathcal{B}$. Fix also a locally constant set of isogenies $\{\mu_\sigma \colon {}^\sigma B \to B\}_{\sigma \in G_k}$, let c_B be the cocycle constructed with these isogenies and let β be a splitting map for c_B.

Let n be the degree $[E_\beta : F]$, and fix an injective F-algebra homomorphism

$$\phi \colon E_\beta \longrightarrow \mathrm{M}_n(F) \subseteq \mathrm{M}_n(\mathcal{B}) \simeq \operatorname{End}^0_{\bar{k}}(B^n).$$

The elements of E_β act as endomorphisms of B^n up to isogeny by means of ϕ. Let $\hat{\mu}_\sigma$ be the diagonal isogeny $\hat{\mu}_\sigma \colon {}^\sigma B^n \to B^n$ consisting in μ_σ in each factor.

Proposition 2.10. *There exists an abelian variety X_β over k and a $\bar{k}$-isogeny $\kappa \colon B^n \to X_\beta$ such that $\kappa^{-1} \circ {}^\sigma\kappa = \phi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma$ for all $\sigma \in G_k$. Moreover, the k-isogeny class of X_β is independent of the chosen injection ϕ.*

Proof. Let ν_σ be the isogeny defined as $\nu_\sigma = \phi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma$. In order to prove the existence of X_β, by Theorem 8.1 of [4], we need to check that

$$\nu_\sigma \circ {}^\sigma\nu_\tau \circ \nu_{\sigma\tau}^{-1} = 1.$$

By the compatibility of μ_σ we have that:

$$\begin{aligned}\nu_\sigma \circ {}^\sigma\nu_\tau \circ \nu_{\sigma\tau}^{-1} &= \phi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma \circ {}^\sigma\phi(\beta(\tau))^{-1} \circ {}^\sigma\hat{\mu}_\tau \circ \hat{\mu}_{\sigma\tau}^{-1} \circ \phi(\beta(\sigma\tau))\\ &= \phi(\beta(\sigma))^{-1} \circ \phi(\beta(\tau))^{-1} \circ \hat{\mu}_\sigma \circ {}^\sigma\hat{\mu}_\tau \circ \hat{\mu}_{\sigma\tau}^{-1} \circ \phi(\beta(\sigma\tau))\\ &= \phi(\beta(\sigma))^{-1} \circ \phi(\beta(\tau))^{-1} \circ c_B(\sigma,\tau) \circ \phi(\beta(\sigma\tau))\\ &= \phi(\beta(\sigma)^{-1} \circ \beta(\tau)^{-1} \circ \beta(\sigma\tau)) \circ c_B(\sigma,\tau)\\ &= \phi(c_B(\sigma,\tau)^{-1}) \circ c_B(\sigma,\tau) = c_B(\sigma,\tau)^{-1} \circ c_B(\sigma,\tau) = 1.\end{aligned}$$

Now suppose that ϕ and ψ are F-algebra homomorphisms $E_\beta \to \mathrm{M}_n(F)$, and let $X_{\beta,\phi}$ and $X_{\beta,\psi}$ denote the varieties constructed by the above procedure using ϕ and ψ respectively to define the action of E_β on B^n. We aim to see that $X_{\beta,\phi}$ and $X_{\beta,\psi}$ are k-isogenous.

Let C denote the image of ϕ. The map $\phi(x) \mapsto \psi(x)\colon C \to \mathrm{M}_n(F)$ is an F-algebra homomorphism. Since C is simple and $\mathrm{M}_n(F)$ is central simple over F, by the Skolem–Noether Theorem there exists an element b in $\mathrm{M}_n(F)$ such that $\phi(x) = b\psi(x)b^{-1}$ for all x in E_β. By the defining property of $X_{\beta,\phi}$ and $X_{\beta,\psi}$ there exist $\bar{k}$-isogenies $\kappa\colon B^n \longrightarrow X_{\beta,\phi}$ and $\lambda\colon B^n \longrightarrow X_{\beta,\psi}$ such that

$$\kappa^{-1} \circ {}^\sigma\kappa = \phi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma = b \circ \psi(\beta(\sigma))^{-1} \circ b^{-1} \circ \hat{\mu}_\sigma, \tag{2.5}$$

$$\lambda^{-1} \circ {}^\sigma\lambda = \psi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma. \tag{2.6}$$

The $\bar{k}$-isogeny $\nu = \kappa \circ b \circ \lambda^{-1}\colon X_{\beta,\psi} \longrightarrow X_{\beta,\phi}$ is in fact defined over k, since for each σ of G_k we have that

$$\begin{aligned}\nu^{-1} \circ {}^\sigma\nu &= \lambda \circ b^{-1} \circ \kappa^{-1} \circ {}^\sigma\kappa \circ {}^\sigma b \circ {}^\sigma\lambda^{-1}\\ &= \lambda \circ b^{-1} \circ b \circ \psi(\beta(\sigma))^{-1} \circ b^{-1} \circ \hat{\mu}_\sigma \circ {}^\sigma b \circ {}^\sigma\lambda^{-1}\\ &= \lambda \circ \psi(\beta(\sigma))^{-1} \circ \hat{\mu}_\sigma \circ {}^\sigma b^{-1} \circ {}^\sigma b \circ {}^\sigma\lambda^{-1}\\ &= \lambda \circ \lambda^{-1} \circ {}^\sigma\lambda \circ {}^\sigma\lambda^{-1} = 1,\end{aligned}$$

where we used the compatibility of $\hat{\mu}_\sigma$ with the endomorphisms of B^n in the third equality, and the expressions (2.5) and (2.6) in the second and fourth equalities respectively. □

Proposition 2.11. *The algebra* $\mathrm{End}_k^0(X_\beta)$ *is isomorphic to the centralizer of* E_β *in* $\mathrm{M}_n(\mathcal{B})$.

Proof. $\mathrm{End}_{\bar{k}}^0(X_\beta)$ is isomorphic to $\mathrm{M}_n(\mathcal{B})$ and every endomorphism of X_β up to $\bar{k}$-isogeny is of the form $\kappa \circ \psi \circ \kappa^{-1}$, for some $\psi \in \mathrm{End}_{\bar{k}}^0(B^n)$. For σ in G_k we have:

$$\begin{aligned}{}^\sigma(\kappa \circ \psi \circ \kappa^{-1}) = \kappa \circ \psi \circ \kappa^{-1} \quad &\Longleftrightarrow \quad {}^\sigma\kappa \circ {}^\sigma\psi \circ {}^\sigma\kappa^{-1} = \kappa \circ \psi \circ \kappa^{-1}\\ &\Longleftrightarrow \quad \kappa^{-1} \circ {}^\sigma\kappa \circ {}^\sigma\psi \circ (\kappa^{-1} \circ {}^\sigma\kappa)^{-1} = \psi\\ &\Longleftrightarrow \quad \beta(\sigma) \circ \hat{\mu}_\sigma \circ {}^\sigma\psi \circ \hat{\mu}_\sigma^{-1} \circ \beta(\sigma)^{-1} = \psi\\ &\Longleftrightarrow \quad \beta(\sigma) \circ \psi \circ \beta(\sigma)^{-1} = \psi.\end{aligned}$$

Thus the endomorphisms of X_β defined over k are exactly the ones coming from endomorphisms ψ that commute with $\beta(\sigma)$, for all σ in G_k. Now the proposition is clear, since the $\beta(\sigma)$'s generate E_β. □

Corollary 2.12. *The algebra* $\mathrm{End}_k^0(X_\beta)$ *is isomorphic to* $E_\beta \otimes_F \mathcal{B}$.

Proof. Let C be the centralizer of E_β in $\mathrm{M}_n(\mathcal{B})$. In view of Proposition 2.11 we have to prove that $C \simeq E_\beta \otimes_F \mathcal{B}$. It is clear that E_β is contained in C. Moreover, $\mathcal{B}$ is contained in C because the elements of E_β can be seen as $n \times n$ matrices with entries in F, and these matrices commute with $\mathcal{B}$ (which is identified with the diagonal matrices in $\mathrm{M}_n(\mathcal{B})$). Since E_β and $\mathcal{B}$ commute there exists a subalgebra of C isomorphic to $E_\beta \otimes_F \mathcal{B}$, which has dimension nt^2 over F. By the Double Centralizer Theorem we know that

$$[C : F][E_\beta : F] = [\mathrm{M}_n(\mathcal{B}) : F] = n^2 t^2,$$

and from this we obtain that $[C : F] = nt^2$, hence C is isomorphic to $E_\beta \otimes_F \mathcal{B}$. □

At this point we have at our disposal all the tools needed to prove Theorem 2.5.

Proof of Theorem 2.5. By Corollary 2.9 there exists an integer m such that $F(\zeta_m)$ splits $\mathcal{B}$. Let β be a splitting map for c_B with E_β containing $F(\zeta_m)$; the existence of such a β is guaranteed by Lemma 2.6. Consider the variety X_β defined as in Proposition 2.10. By Corollary 2.12 we have that $\mathrm{End}_k^0(X_\beta) \simeq E_\beta \otimes_F \mathcal{B}$, and this latter algebra is in turn isomorphic to $M_t(E_\beta)$ because E_β is a splitting field for $\mathcal{B}$. Therefore, there exists an abelian variety A_β defined over k such that $X_\beta \sim_k A_\beta^t$ and $\mathrm{End}_k^0(A_\beta) \simeq E_\beta$. Clearly A_β is $\bar{k}$-isogenous to $B^{n/t}$, where $n = [E_\beta : F]$, and we claim that it is a Ribet–Pyle variety. First of all, it is easily seen that the power of a k-variety is also a k-variety. This implies that $(A_\beta)_{\bar{k}}$ is a k-variety. Moreover, we have that $[\mathrm{End}_k^0(A_\beta) \colon F] = [E_\beta \colon F] = n$, and the dimension of the ambient algebra is $[\mathrm{End}_{\bar{k}}^0(A_\beta) \colon F] = (\frac{n}{t})^2[\mathcal{B} \colon F] = n^2$. This implies (cf. Proposition 13.1 in [2]) that $\mathrm{End}_k^0(A)$ is a maximal subfield of $\mathrm{End}_{\bar{k}}^0(A)$. □

Proposition 2.13. *Let B be a k-variety and let A/k be a Ribet–Pyle variety having B as $\bar{k}$-simple factor. Then A is k-isogenous to the variety A_β obtained by applying the above procedure to some cocycle c_B attached to B and some splitting map β for c_B.*

Proof. Let $\mathcal{B} = \mathrm{End}_{\bar{k}}^0(B)$, let F be the center of $\mathcal{B}$ and let t be the Schur index of $\mathcal{B}$. Let E be the maximal subfield $\mathrm{End}_k^0(A)$ of $\mathrm{End}_{\bar{k}}^0(A)$, and fix an embedding of E into $\overline{F}$. Let κ be an isogeny $\kappa \colon B^n \to A_{\bar{k}}$. We have the relation $[E : F] = nt$. Let $\{\mu_\sigma \colon {}^\sigma B \to B\}_{\sigma \in G_k}$ be a locally constant set of compatible isogenies and denote by $\hat{\mu}_\sigma \colon {}^\sigma B^n \to B^n$ the diagonal of μ_σ. Define $\beta(\sigma) = \kappa \circ \hat{\mu}_\sigma \circ {}^\sigma\kappa^{-1}$, which is a compatible isogeny $\beta(\sigma) \colon A_{\bar{k}} \to A_{\bar{k}}$. The fact that $\beta(\sigma)$ is compatible implies that

$$\beta(\sigma) \circ \varphi = {}^\sigma\varphi \circ \beta(\sigma) \tag{2.7}$$

for all σ in G_k and for all $\varphi \in \mathrm{End}_{\bar{k}}^0(A)$. In particular, when applied to elements φ of E this property says that $\beta(\sigma)$ lies in $C(E)$, the centralizer of E. But $C(E)$ is equal to E, because E is a maximal subfield. Thus $\beta(\sigma)$ belongs to E and it is an isogeny defined over k. Now we have that

$$\begin{aligned} c_B(\sigma,\tau) &= \mu_\sigma \circ {}^\sigma\mu_\tau \circ \mu_{\sigma\tau}^{-1} = \hat{\mu}_\sigma \circ {}^\sigma\hat{\mu}_\tau \circ \hat{\mu}_{\sigma\tau}^{-1} \\ &= \beta(\sigma)\circ {}^\sigma\beta(\tau)\circ\beta(\sigma\tau)^{-1} = \beta(\sigma)\circ\beta(\tau)\circ\beta(\sigma\tau)^{-1}, \end{aligned}$$

and we see that the map $\sigma \mapsto \beta(\sigma)$ is a splitting map for c_B. We have already seen the inclusion $E_\beta \subseteq E$. From (2.7) it is clear that $C(E_\beta) \subseteq E$, and taking centralizers and applying the Double Centralizer Theorem we have that $E = C(E) \subseteq C(C(E_\beta)) = E_\beta$. Thus $E = E_\beta$ and, in particular, $[E_\beta : F] = nt$.

Now we define a $\bar{k}$-isogeny $\hat{\kappa}\colon (B^n)^t \to A^t_{\bar{k}}$ as the diagonal isogeny associated to κ, and we make E_β act on B^{nt} by means of $\hat{\kappa}$. It is easy to check that $\hat{\kappa}^{-1}\circ{}^\sigma\hat{\kappa} = \hat{\kappa}^{-1}\circ\beta(\sigma)^{-1}\circ\hat{\kappa}\circ\hat{\mu}_\sigma$, so A^t satisfies the property defining X_β. By the uniqueness property of X_β we have that $A^t \sim_k X_\beta$, and so $A_\beta \sim_k A$. □

Remark 2.14. The hypothesis that k is a number field has been used only in order to guarantee the existence of splitting maps for c_B, by means of Tate's theorem on the triviality of $H^2(G_k, \overline{F}^\times)$. Since Tate's theorem is valid for any global or local field k, Theorem 1.2 is valid for any global or local field k as well.

3. Varieties over k of GL_2-type and k-varieties

Let k be a number field. In this section we characterize the absolutely simple factors of the varieties over k of GL_2-type, in the case where they do not have complex multiplication.

Proposition 3.1. *Let A/k be an abelian variety of* GL_2*-type such that $A_{\bar{k}}$ does not have complex multiplication. Then A is a Ribet–Pyle variety.*

Proof. By Proposition 1.5 in [6], we can suppose that $A_{\bar{k}}$ does not have any simple factor with CM. Let $A_{\bar{k}} \sim B_1^{n_1} \times \cdots \times B_r^{n_r}$ be the decomposition of $A_{\bar{k}}$ into simple abelian varieties up to isogeny. Since $E = \mathrm{End}_k^0(A)$ is a field it acts on each factor $B_i^{n_i}$, and so it acts on the homology with rational coefficients $H_1((B_i^{n_i})_{\mathbb{C}}, \mathbb{Q})$, which is a vector space of dimension $2\dim B_i^{n_i}$ over $\mathbb{Q}$. Thus $2\dim B_i^{n_i}$ is divisible by $[E:\mathbb{Q}] = \dim A$. But $\dim A \geqslant \dim B_i^{n_i}$, so either $[E:\mathbb{Q}] = \dim B_i^{n_i}$ or $2[E:\mathbb{Q}] = \dim B_i^{n_i}$. The latter is not possible, because it would mean that $B_i^{n_i}$ has CM by E. Thus $\dim A = \dim B_i^{n_i}$ and $A_{\bar{k}}$ has only one simple factor up to isogeny; say $A_{\bar{k}} \sim B^n$.

Next, we see that E is a maximal subfield of $\mathrm{End}_{\bar{k}}^0(A)$. Let C be the centralizer of E in $\mathrm{End}_{\bar{k}}^0(A)$, and let φ be an element in C. A priori $\varphi(A_{\bar{k}})$ is isogenous to B^r for some $r \leqslant n$. Since $\varphi \in C$, the field E acts on $\varphi(A_{\bar{k}})$; as before this implies that $[E:\mathbb{Q}]$ divides $2\dim B^r$. But $[E:\mathbb{Q}] = \dim A = \dim B^n$, therefore $r = n$ or $r = n/2$. Again $r = n/2$ is not possible, because then B^r would be a factor of $A_{\bar{k}}$ with CM by E. Thus $r = n$ and φ is invertible in $\mathrm{End}_{\bar{k}}^0(A)$. This implies that C is a field, and then E is a maximal subfield of $\mathrm{End}_{\bar{k}}^0(B)$.

Finally, we see that $A_{\bar{k}}$ is an abelian k-variety. For each $\sigma \in G_k$ the map

$$\begin{array}{ccc} \operatorname{End}^0_{\bar{k}}(A) & \longrightarrow & \operatorname{End}^0_{\bar{k}}(A) \\ \varphi & \longmapsto & {}^{\sigma}\varphi \end{array} \tag{3.1}$$

is the identity when restricted to E. Since E is a maximal subfield, it contains the center F of $\operatorname{End}^0_{\bar{k}}(A)$, so (3.1) is an F-algebra automorphism. By the Skolem–Noether Theorem there exists an element μ_σ in $\operatorname{End}^0_{\bar{k}}(A)^*$ such that ${}^{\sigma}\varphi = \mu_\sigma^{-1}\circ\varphi\circ\mu_\sigma$, and we see that μ_σ is a compatible isogeny in the sense of Definition 2.1. □

Definition 3.2. *A* building k-block *is an abelian k-variety $B/\bar{k}$ such that $\operatorname{End}^0_{\bar{k}}(B)$ is a central division algebra over a field F, with Schur index $t \leqslant 2$ and reduced degree $t[F:\mathbb{Q}] = \dim B$.*

Theorem 3.3. *Let k be a number field and let A/k be an abelian variety of GL_2-type such that $A_{\bar{k}}$ does not have CM. Then $A_{\bar{k}} \sim B^n$ for some building k-block B. Conversely, if B is a building k-block then there exists a variety A/k of GL_2-type such that $A_{\bar{k}} \sim B^n$ for some n.*

Proof. By Propostion 3.1, A is a Ribet–Pyle variety, and by Proposition 2.4 we have that $A_{\bar{k}} \sim B^n$ for some k-variety B. Let $\mathcal{B} = \operatorname{End}^0_{\bar{k}}(B)$, let F be the center of $\mathcal{B}$ and let t be its Schur index. Then $E = \operatorname{End}^0_k(A)$ is a maximal subfield of $\operatorname{End}^0_{\bar{k}}(A) \simeq \mathrm{M}_n(\mathcal{B})$, which has dimension n^2t^2 over F. Therefore $[E:F] = nt$, and multiplying both sides of this equality by $[F:\mathbb{Q}]$ we see that $[E:\mathbb{Q}] = \dim A = nt[F:\mathbb{Q}]$. The equality $t[F:\mathbb{Q}] = \dim B$ follows. Since $\mathcal{B}$ is a division algebra of $\mathbb{Q}$-dimension $t^2[F:\mathbb{Q}]$ that acts on $H_1(B_{\mathbb{C}},\mathbb{Q})$, which has $\mathbb{Q}$-dimension $2\dim B = 2t[F:\mathbb{Q}]$, we see that necessarily $t \leqslant 2$ and B is a building k-block.

Conversely, let B be a building k-block. In particular it is a k-variety, and by Theorem 2.5 there exists a Ribet–Pyle variety A/k such that $A_{\bar{k}} \sim B^n$ for some n. The field $E = \operatorname{End}^0_k(A)$ is a maximal subfield of $\operatorname{End}^0_{\bar{k}}(A) \simeq \mathrm{M}_n(\mathcal{B})$, which means that $[E:F] = nt$. Multiplying both sides of this equality by $[F:\mathbb{Q}]$ we see that $[E:\mathbb{Q}] = nt[F:\mathbb{Q}] = n\dim B = \dim A$, and so A is a variety of GL_2-type. □

In the case $k = \mathbb{Q}$ the center of the endomorphism algebra of a building k-block is necessarily totally real, but for arbitrary number fields k a priori it can be either totally real or CM. That is why in Definition 3.2 the field F is not required to be totally real. However, if k admits a real embedding then exactly the same argument of Theorem 1.2 in [3] shows that F is necessarily totally real. In addition, there are some extra restrictions on the endomorphism algebra.

Proposition 3.4. *Let k be a number field that admits a real embedding. Let B be a building k-block, let $\mathcal{B} = \operatorname{End}^0_{\bar{k}}(B)$ and let $F = Z(\mathcal{B})$. Then F is totally real and $\mathcal{B}$ is either isomorphic to F or to a totally indefinite division quaternion algebra over F.*

Proof. We view k as a subfield of $\mathbb{C}$ by means of a real embedding $k \hookrightarrow \mathbb{R}$. Let A/k be a GL_2-type variety such that $A_{\bar{k}} \sim B^n$. Let E be the maximal subfield

$\mathrm{End}_k^0(A)$ of $\mathrm{End}_{\bar{k}}^0(A)$, and identify F with $Z(\mathrm{End}_{\bar{k}}^0(A))$; under this identification F is contained in E. Let t be the Schur index of B and let $m = 2\dim B/[\mathcal{B}:\mathbb{Q}]$, for which we have that $mt = 2$.

The division algebra $\mathcal{B}$ belongs a priori to one of the four types of algebras with a positive involution, according to Albert's classification (see, for instance, Proposition 1 in [5]). However, type III is not possible; indeed, by Proposition 15 in [5], the variety B would then be isogenous to the square of a CM abelian variety.

To see that type IV is also not possible, suppose that F is a CM extension of a totally real field F_0. Let Φ denote the complex representation of $\mathcal{B}$ on the space of differential forms $H^0(B_{\mathbb{C}}, \Omega^1)$. For every real embedding ν of F_0 let $\chi_\nu, \overline{\chi}_\nu$ be the two complex-conjugate irreducible representations of $\mathcal{B}$ extending ν. Let r_ν and s_ν be the multiplicities of χ_ν and $\overline{\chi}_\nu$ in Φ. For each ν we have that $r_\nu + s_\nu = 2$; moreover, the equality $r_\nu = s_\nu = 1$ is not possible for all ν (cf. Propositions 18 and 19 in [5]). This implies that $\mathrm{Tr}(\Phi)_{|F} = \sum r_\nu \chi_{\nu|F} + s_\nu \overline{\chi}_{\nu|F}$ takes non-real values. On the other hand, if we denote by Ψ the complex representation of $\mathrm{End}_{\bar{k}}^0(A)$ on $H^0(A_{\mathbb{C}}, \Omega^1)$, then $\mathrm{Tr}(\Psi) = n\mathrm{Tr}(\Phi)$. Since A is defined over k we can take a basis of the differentials defined over k, and with respect to this basis the elements of E are represented by matrices with coefficients in k. Since $F \subseteq E$, the trace of Ψ restricted to F takes values in $k \subseteq \mathbb{R}$, giving a contradiction with the fact that $\mathrm{Tr}(\Phi)_{|F}$ takes non-real values. □

References

[1] Darmon, H.: Rigid local systems, Hilbert modular forms, and Fermat's last theorem. *Duke Math. J.* **102** (2000), no. 3, 413–449.

[2] Pierce, R. S.: *Associative algebras.* Graduate Texts in Mathematics 88. Studies in the History of Modern Science 9, Springer-Verlag, New York–Berlin, 1982.

[3] Pyle, E.: Abelian varieties over $\mathbb{Q}$ with large endomorphism algebras and their simple components over $\overline{\mathbb{Q}}$. In *Modular curves and abelian varieties,* 189–239. Progress in Math. 224, Birkhäuser, Basel, 2004.

[4] Ribet, K. A.: Abelian varieties over $\mathbb{Q}$ and modular forms. In *Algebra and topology (Taejŏn, 1992),* 53–79. Korea Adv. Inst. Sci. Tech., Taejŏn, 1992.

[5] Shimura, G.: On analytic families of polarized abelian varieties and automorphic functions. *Ann. of Math. (2)* **78** (1963), 149–192.

[6] Shimura, G.: Class fields over real quadratic fields and Hecke operators. *Ann. of Math. (2)* **95** (1972), 130–190.

Received July 19, 2010.

Xavier Guitart: Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya (ETSEIAT), C. Colom 11, 08222 Terrassa, Spain.
E-mail: xevi.guitart@gmail.com

Partially supported by Grants MTM2009-13060-C02-01 and 2009 SGR 1220.