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Rationally cubic connected manifolds II

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Abstract. We study smooth complex projective polarized varieties (X,H) of dimension $n \geq 2$ which admit a dominating family V of rational curves of H-degree 3, such that two general points of X may be joined by a curve parametrized by V and which do not admit a covering family of lines (i.e., rational curves of H-degree one). We prove that such manifolds are obtained from RCC manifolds of Picard number one by blow-ups along smooth centers.

If we further assume that X is a Fano manifold, we obtain a stronger result, classifying all Fano RCC manifolds of Picard number $\rho_X \geq 3$.

1. Introduction

In our recent paper [15], inspired by the classification of conic-connected manifolds given in [10], we started the study of *rationally cubic connected* manifolds (RCC-manifolds, for short), i.e., smooth complex projective polarized varieties (X, H) of dimension $n \geq 2$ which are rationally connected by rational curves of degree 3 with respect to a fixed ample line bundle H.

In [15] we considered manifolds covered by lines (i.e., rational curves of degree one with respect to H), proving that the Picard number of such manifolds is at most three and that if equality holds then the manifold has an adjunction scroll structure over a smooth variety.

In the present paper we will complete our task by dealing with RCC-manifolds not covered by lines. Our main result shows that such manifolds are obtained by RCC-manifolds of Picard number one by blow-ups along smooth centers. More precisely we have the following:

Theorem 1.1. *Let (X, H) be an RCC-manifold with respect to V, not covered by lines. Then there exists a polarized manifold (X', H') of Picard number one not covered by lines and a contraction $\varphi_\Sigma : X \to X'$ expressing X as a blow-up of X' along disjoint smooth irreducible centers T_i with exceptional divisors E_i such*

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that $H \simeq \varphi_\Sigma^* H' - \sum E_i$. *The pair* (X', H') *is RCC with respect to* V', *the family of deformations of the image of a general curve parametrized by* V.

Moreover either $(X', H') \simeq (\mathbb{P}^n, \mathcal{O}_\mathbb{P}(3))$ *or* $\mathrm{Pic}(X') = \langle H' \rangle$ *and* $-K_{X'} = \frac{n+1}{3} H'$. *In the latter case we have* $\dim T_i < (n+1)/3$ *for every* i.

First of all we deal with the case of RCC-surfaces, giving a complete classification of them in Section 3, then we move to the general case.

The main idea is the following: if the Picard number of X is greater than one, then for every point $x \in X$ the cubics parametrized by V passing through x must degenerate into reducible cycles whose components are not numerically proportional to V. Clearly these degenerations can be into cycles consisting either of three lines or of a line and a conic. Since we are assuming that X is not covered by lines, the latter happens for a general point, and moreover the irreducible component through the general point is the conic, so for each possible degeneration we get a dominating family of conics and a family of lines.

It turns out that the loci of the families of lines arising in this way are divisors, which we will call *divisors of* V*-lines*; the main point of the proof is to show that these divisors are (disjoint) exceptional divisors of smooth blow-ups and that they do not meet a general cubic, so that, after blowing them down, we still have a RCC-manifold.

The hardest case is that of manifolds of Picard number two, which is treated in Section 5, while the general one is settled in Section 6.

If we further assume that the manifold is Fano, the results are much stronger and for $\rho_X \geq 3$ we have a complete classification:

Theorem 1.2. *Let* (X, H) *be a polarized Fano manifold of dimension* $n > 2$ *and* $\rho_X \geq 3$. *Suppose that* X *is RCC with respect to a family* V *and doesn't admit a covering family of lines. Then* $\rho_X = 3$, X *has a contraction* $\pi : X \to \mathbb{P}^n$ *which is the blow-up of* $\mathbb{P}^n$ *along two disjoint centers* T_1 *and* T_2 *which can be:*

(1) *two linear spaces* Λ_1, Λ_2 *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim \Lambda_1 + \dim \Lambda_2 = n - 2,$$

(2) *a linear space* Λ_1 *and a smooth quadric* $Q_1 \subset \Lambda_2 \simeq \mathbb{P}^{\dim Q_1 + 1}$ *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim Q_1 \geq \tfrac{n}{2} - 1, \quad \dim \Lambda_1 + \dim Q_1 = n - 2,$$

(3) *two smooth quadrics* $Q_1 \subset \Lambda_1 \simeq \mathbb{P}^{\frac{n}{2}}$ *and* $Q_2 \subset \Lambda_2 \simeq \mathbb{P}^{\frac{n}{2}}$ *such that*

$$Q_1 \cap Q_2 = \emptyset, \quad \dim \Lambda_1 \cap \Lambda_2 = 0, \quad \dim Q_1 = \dim Q_2 = \tfrac{n}{2} - 1,$$

and, denoting by E_1 *and* E_2 *the exceptional divisors,* $H \simeq \pi^* \mathcal{O}_{\mathbb{P}^n}(3) - E_1 - E_2$.

Section 7 is devoted to the proof of this theorem, and to some other considerations on RCC Fano manifolds of Picard number two.

In the last section we examine in detail the manifolds appearing in Theorem 1.2, describing their cones of curves, their contractions, and their families of rational curves.

2. Background material

We gather in this section the basic definitions and results regarding families of rational curves and Mori theory that we are going to use.

2.1. Families of rational curves and of rational 1-cycles

Definition 2.1. A *family of rational curves* V on X is an irreducible component of the scheme $\mathrm{Ratcurves}^n(X)$ (see Definition II.2.11 in [13]).

Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{Ratcurves}^n(X)$ containing the point parametrizing the curve.

We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parametrized by V; we say that V is a *covering family* if $\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.

By abuse of notation, given a line bundle $H \in \mathrm{Pic}(X)$, we will denote by $H \cdot V$ the intersection number $H \cdot B$, with B any curve among those parametrized by V. We will say that V is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.

We denote by V_x the subscheme of V parametrizing rational curves passing through a point x and by $\mathrm{Locus}(V_x)$ the set of points of X through which there is a curve among those parametrized by V_x. If, for a general point $x \in \mathrm{Locus}(V)$, V_x is proper, then we will say that the family is *locally unsplit*. Moreover, we say that V is *generically unsplit* if, through a general $x \in \mathrm{Locus}(V)$ and a general $y \in \mathrm{Locus}(V_x)$ there is a finite number of curves parametrized by V.

Definition 2.2. We define a *Chow family of rational 1-cycles* $\mathcal{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles. If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated with* V. If V is proper, i.e., if the family is unsplit, then V is the normalization of the associated Chow family $\mathcal{V}$.

Definition 2.3. Let V be a family of rational curves and let $\mathcal{V}$ be the associated Chow family. We say that V (and also $\mathcal{V}$) is *quasi-unsplit* if every component of any reducible cycle parametrized by $\mathcal{V}$ has numerical class proportional to the numerical class of a curve parametrized by V.

Definition 2.4. We say that k quasi-unsplit families $V^1, \dots, V^k$ are numerically independent if in $\mathrm{N}_1(X)$ we have $\dim\langle[V^1], \dots, [V^k]\rangle = k$.

For special families of rational curves we have useful dimensional estimates. The basic one is the following:

Proposition 2.5. ([13, Corollary IV.2.6]) *Let V be a family of rational curves on X and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

(a) $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$*;*

(b) *every irreducible component of* $\mathrm{Locus}(V_x)$ *has dimension* $\geq -K_X \cdot V - 1$*.*

Definition 2.6. Let $V^1, \dots, V^k$ be families of rational curves on X and $Z \subset X$. We define $\mathrm{Locus}(V^1)_Z$ to be the set of points $x \in X$ such that there exists a curve C among those parametrized by V^1 with $C \cap Z \neq \emptyset$ and $x \in C$. We inductively define $\mathrm{Locus}(V^1, \dots, V^k)_Z := \mathrm{Locus}(V^k)_{\mathrm{Locus}(V^1,\dots,V^{k-1})_Z}$.

Notation: If Γ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $\mathrm{N}_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parametrized by V. A proper family will always be denoted by a calligraphic letter.

If $Z \subset X$, we will denote by $\mathrm{N}_1(Z, X) \subseteq \mathrm{N}_1(X)$ the vector subspace generated by numerical classes of curves of X contained in Z; moreover, we will denote by $\mathrm{NE}\,(Z, X) \subseteq \mathrm{NE}(X)$ the subcone generated by numerical classes of curves of X contained in Z. We will denote by $\langle \dots \rangle$ the linear span.

We will use some properties of $\mathrm{Locus}(V)_Z$, summarized in the following

Lemma 2.7 (Section 5 of [1], proof of Lemma 1.4.5 in [4]). *Let $Z \subset X$ be an irreducible closed subset and $\mathcal{V}$ an unsplit family. Then every curve contained in $\mathrm{Locus}(\mathcal{V})_Z$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_Z + \mu C_{\mathcal{V}},$$

where C_Z is a curve in Z, $C_{\mathcal{V}}$ is a curve among those parametrized by $\mathcal{V}$ and $\lambda \geq 0$. If moreover curves contained in Z are numerically independent from curves in $\mathcal{V}$ and $Z \cap \mathrm{Locus}(\mathcal{V}) \neq \emptyset$ then

$$\dim \mathrm{Locus}(\mathcal{V})_Z \geq \dim Z - K_X \cdot \mathcal{V} - 1.$$

We will also need the following lemma, which is based on Proposition II.4.19 in [13]:

Lemma 2.8. *Let $Y \subset X$ be a closed subset, and let $\mathcal{V}$ be a Chow family of rational 1-cycles. Then every curve contained in $\mathrm{Locus}(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and of irreducible components of cycles parametrized by $\mathcal{V}$ which meet Y.*

2.2. Contractions and fibrations

Definition 2.9. Let X be a manifold such that K_X is not nef. Denote by $\overline{\mathrm{NE}}(X) \subset \mathrm{N}_1(X)$ the closure of the cone of effective 1-cycles in the $\mathbb{R}$-vector space of 1-cycles modulo numerical equivalence, and by $\overline{\mathrm{NE}}(X)_{K_X<0}$ the set $\{z \in \mathrm{N}_1(X) : K_X \cdot z < 0\}$. An *extremal face* is a face σ of $\overline{\mathrm{NE}}(X)$, associated with some nef line bundle L, contained in the negative part of the cone with respect to K_X, i.e., $\sigma = \overline{\mathrm{NE}}(X) \cap L^\perp \subset \overline{\mathrm{NE}}(X)_{K_X<0}$; an extremal face of dimension one is called an *extremal ray.*

With an extremal face $\sigma \subset \mathrm{NE}(X)$ is associated a morphism with connected fibers $\varphi_\sigma : X \to Z$ onto a normal variety, that contracts the curves whose numerical class is in σ; φ_σ is called an *extremal contraction* or a *Fano–Mori contraction*, while a Cartier divisor H such that $H = \varphi_\sigma^* A$ for an ample divisor A on Z is called a *supporting divisor* of the map φ_σ (or of the face σ). We denote by $\mathrm{Exc}(\varphi_\sigma) := \{x \in X \,|\, \dim \varphi_\sigma^{-1}(\varphi_\sigma(x)) > 0\}$ the *exceptional locus* of φ_σ.

An extremal contraction associated with an extremal ray is called an *elementary contraction*; an elementary contraction is said to be of *fiber type* if $\dim X > \dim Z$, otherwise the contraction is *birational*. Moreover, if the codimension of the exceptional locus of an elementary birational contraction is equal to one, then the contraction is called *divisorial*; otherwise it is called *small*.

Proposition 2.5, in the case $\mathcal{V}$ is the unsplit family of deformations of a minimal extremal rational curve, gives the *fiber locus inequality*:

Proposition 2.10 ([9], [17]). *Let φ be a Fano–Mori contraction of X and let $E = \mathrm{Exc}(\varphi)$ be its exceptional locus; let S be an irreducible component of a (non-trivial) fiber of φ. Then*

$$\dim E + \dim S \geq \dim X + l - 1,$$

where $l = \min\{-K_X \cdot C \mid C$ is a rational curve in $S\}$. If φ is the contraction of a ray R, then $l(R) := l$ is called the length of the ray.

The next theorem, which will be frequently used, gives us conditions to ensure that a birational contraction is a smooth blow-up:

Theorem 2.11 (Theorem 4.1 (iii) in [2]). *Let $\varphi : X \to Z$ be an extremal contraction of a smooth variety X. Assume that φ is birational and supported by $K_X + rH$, with H a φ-ample line bundle on X, and that, for each non-trivial fiber F of φ we have $\dim F = r$. Then Z is smooth and φ is a blow down of a smooth divisor $E \subset X$ to a smooth subvariety of Z.*

3. Preliminaries

Definition 3.1. Let (X, H) be a polarized manifold of dimension n; if there exists a dominating family V of rational curves such that $H \cdot V = 3$ and through two general points of X there is a curve parametrized by V we will say that X is *Rationally Cubic Connected* – RCC for short – with respect to V.

In Proposition 4.1 of [15] we proved the following result concerning RCC manifolds of Picard number one:

Proposition 3.2. *Let (X, H) be a RCC-manifold with respect to a family V. Then,*

1) *there exists $x \in X$ such that V_x is proper if and only if $(X, H) \simeq (\mathbb{P}^n, \mathcal{O}(3))$;*
2) *there exists $x \in X$ such that V_x is quasi-unsplit but not proper if and only if X is a Fano manifold of Picard number one and index $r(X) \geq \frac{n+1}{3}$ with fundamental divisor H.*

As a consequence, if the Picard number of X is greater than one, through a general point there exists at least one reducible cycle parametrized by the Chow family $\mathcal{V}$ whose components are not all numerically proportional to V.

Since $H \cdot V = 3$, such a cycle in $\mathcal{V}$ can have two or three irreducible rational components; we will call a component of H-degree one a *line* and a component of H-degree two a *conic*.

Throughout the present paper, after dealing with the case of surfaces in the next subsection, we will assume that X does not admit a covering family of lines; this assumption yields that every dominating family of conics is locally unsplit. Moreover, from Formula (1), Corollary 5.6 and Formula (4) in [15] we have that V is generically unsplit and so

$$-K_X \cdot V = n + 1. \tag{3.1}$$

3.1. RCC surfaces

We will now give a complete classification of RCC-surfaces. As a consequence, we will see that Theorem 1.1 holds for $n = 2$.

Proposition 3.3. *Let (S, H) be a polarized surface, which is RCC with respect to a family of rational curves V. Then (S, H) and V are one of the following:*

1. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, *the family of lines in* $\mathbb{P}^2$;
2. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, *the family of rational plane cubics;*
3. $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1, 2)$ *(or* $\mathcal{O}_{\mathbb{Q}^2}(2, 1)))$, *the family of curves of type* $(1, 1)$;
4. $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1, 1))$, *the family of curves of type* $(2, 1)$ *(or* $(1, 2)$*);*
5. $(\mathbb{F}_1, C_0 + 3f)$, *the family of curves of type* $C_0 + f$;
6. $(S_k, -K_{S_k})$ *with* S_k *a blow-up of* $\mathbb{P}^2$ *in* k *general points, with* $k = 1, \ldots, 8$, *the family of strict transforms of lines in* $\mathbb{P}^2$.

Proof. By the first part of Proposition 3.2, if V_x is proper for some $x \in S$ then $(S, H) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, while, by the second part, if V_x is quasi-unsplit for some $x \in S$ then $(S, H) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

We can assume from now on that the Picard number of S is at least two.

If V is not generically unsplit then either $(S, H) = (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1, 2))$ (and curves of V are curves of type $(1, 1)$) or through every pair of points of S there is a reducible cycle in V consisting of three lines by Proposition 5.5 in [15]. In this last case S admits two dominating families of lines, hence $(S, H) \simeq (\mathbb{Q}^2, \mathcal{O}(1, 1))$ and V is, up to exchanging the rulings, the family of curves of type $(2, 1)$.

We are thus left with the case of V generically unsplit. By formula (3.1) we then have $-K_S \cdot V = 3$.

We consider first the case in which S admits a covering family of lines $\mathcal{L}$. Recalling that $\rho_S \geq 2$, we have that S is a ruled surface $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-e) \oplus \mathcal{O})$, and the lines are the fibers of the projection to $\mathbb{P}^1$.

Denote by C_0 a minimal section and by f a fiber; since the fibers are lines with respect to H we can write $H \equiv C_0 + hf$. Let $B \equiv aC_0 + bf$ be a curve parametrized by V. By the genus formula we get

$$1 = B^2 = a(2b - ae),$$

hence $a = 1$ and $e = 2b - 1$. Since B is an effective curve, this is possible just for $b = e = 1$. Now, from $H \cdot B = 3$, we get $h = 3$.

Finally we treat the case of a surface not covered by lines.

Consider the set $\mathcal{B}' = \{(\mathcal{L}^i, C^i)\}$ of pairs of families $(\mathcal{L}^i, C^i)$ such that through a general point $x \in S$ there is a reducible cycle $\ell + \gamma$ belonging to $\mathcal{V}$, with ℓ and γ parametrized respectively by $\mathcal{L}^i$ and C^i.

The families of conics are locally unsplit and dominating, hence $-K_S \cdot C^i = 2$ for every i; this implies that $-K_S \cdot \mathcal{L}^i = 1$. The numerical classes of $\mathcal{L}^i$ and C^i are a system of generators for $N_1(S)$ (by Lemma 2.8), and $K_S + H$ is trivial on each of them.

It follows that H and $-K_S$ are numerically equivalent; being S rational they are also linearly equivalent. In particular $-K_S$ is ample, and S is a del Pezzo surface. We are now assuming that the Picard number of S is greater than one, and that $-K_S \cdot V = 3$, so S is not a projective space or a quadric. □

4. Divisors of V-lines

Having settled the case of surfaces in Proposition 3.3, we will assume from now on that $n = \dim X \geq 3$.

Consider the set $\mathcal{B}' = \{(\mathcal{L}^i, C^i)\}$ of pairs of families $(\mathcal{L}^i, C^i)$ such that through a general point $x \in X$ there is a reducible cycle $\ell + \gamma$, parametrized by $\mathcal{V}$, with ℓ and γ parametrized respectively by $\mathcal{L}^i$ and C^i.

Let us consider two pairs $(\mathcal{L}^i, C^i)$ and $(\mathcal{L}^j, C^j)$ such that $[\mathcal{L}^i] \neq [\mathcal{L}^j]$. Since no family of lines is covering, by the generality of x all the families of conics are dominating and locally unsplit; therefore $\dim \operatorname{Locus}(C^i)_x \cap \operatorname{Locus}(C^j)_x = 0$ for every $i \neq j$. It follows that

$$-K_X \cdot (C^i + C^j) \leq \dim \operatorname{Locus}(C^i)_x + \operatorname{Locus}(C^j)_x + 2 \leq n + 2, \tag{4.1}$$

so, recalling that $-K_X \cdot (C^i + \mathcal{L}^i) = -K_X \cdot V = n + 1$, we also have

$$-K_X \cdot (\mathcal{L}^i + \mathcal{L}^j) \geq n. \tag{4.2}$$

Let now $\mathcal{B} = \{(\mathcal{L}^i, C^i)\}_{i=1}^k$ be a maximal set of pairs in $\mathcal{B}'$ with the property that $[V], [\mathcal{L}^1], \ldots, [\mathcal{L}^k]$ are numerically independent. Denote by Π_i the two-dimensional vector subspace of $\mathrm{N}_1(X)$ spanned by $[V]$ and $[\mathcal{L}^i]$. By Lemma 2.8,

$$\mathrm{N}_1(X) = \langle [V], [\mathcal{L}^1], [C^1], \ldots, [\mathcal{L}^k], [C^k] \rangle = \langle [V], [\mathcal{L}^1], [\mathcal{L}^2], \ldots, [\mathcal{L}^k] \rangle,$$

hence the Picard number of X is $k + 1$.

For every $i = 1, \dots, k$ denote by E_i the set $\mathrm{Locus}(C^i, \mathcal{L}^i)_x$; by Lemma 2.7 and by Proposition 2.5 it has dimension $\dim E_i \geq n-1$; since $E_i \subset \mathrm{Locus}(\mathcal{L}^i)$, the inclusion is an equality and E_i is an irreducible divisor.

We will call the divisor E_i the *divisor of V-lines* associated with the pair $(\mathcal{L}^i, C^i)$. We will distinguish the divisors of V-lines in the following way:

1. E_i is of the first kind if $-K_X \cdot \mathcal{L}^i = n-1$;
2. E_i is of the second kind if $-K_X \cdot \mathcal{L}^i = 1$;
3. E_i is of the third kind in all the other cases.

The next lemma gives the description of the relative space of 1-cycles of the divisors of V-lines.

Lemma 4.1. *Let E_i be a divisor of V-lines, associated with a pair $(\mathcal{L}^i, C^i)$. If E_i is of the first kind then $\mathrm{N}_1(E_i, X) = \langle [\mathcal{L}^i] \rangle$. In the other cases $\mathrm{N}_1(E_i, X) = \langle [C^i], [\mathcal{L}^i] \rangle$ and $[\mathcal{L}^i]$ is extremal in $\mathrm{NE}\,(E_i, X)$.*

Proof. If E_i is of the first kind, then $E_i = \mathrm{Locus}(\mathcal{L}^i)_x$ for any $x \in \mathrm{Locus}(\mathcal{L}^i)$, while if E_i is either of the second or of the third kind then $E_i = \mathrm{Locus}(C^i, \mathcal{L}^i)_x$ for a general $x \in X$. The statement now follows from Lemma 2.7. □

As a consequence, we can prove that these divisors are disjoint:

Lemma 4.2. *Let E_i for $i = 1, \dots, k$ be divisors of V-lines associated with pairs in $\mathcal{B}$. Then the E_i's are pairwise disjoint. Moreover, if E_k is of the third kind then $E_i \cdot \mathcal{L}^k = E_i \cdot C^k = 0$ for every $i \neq k$.*

Proof. Since we are assuming that $n \geq 3$, if two divisors met, their intersection would be positive dimensional. Therefore, by the description of the relative space of cycles $\mathrm{N}_1(E_i, X)$, it is clear that any divisor of the first kind is disjoint from any other divisor. Moreover, if a divisor of the second kind exists, then, by equation (4.1), all the other divisors are of the first kind.

We will now show that, if E_k is of the third kind then $E_i \cdot \mathcal{L}^k = E_i \cdot C^k = 0$ for every $i \neq k$. This implies also that two divisors of the third kind are disjoint.

Both $\dim \mathrm{Locus}(C^k)_x$ and $\dim \mathrm{Locus}(\mathcal{L}^k)_x$ are greater than one, so if $E_i \cdot C^k > 0$ (respectively $E_i \cdot \mathcal{L}^k > 0$) then E_i would contain a curve whose numerical class is proportional to $[C^k]$ (resp. $[\mathcal{L}^k]$), a contradiction, since neither $[C^k]$ nor $[\mathcal{L}^k]$ is contained in $\mathrm{N}_1(E_i, X)$. □

Theorem 1.1 will follow if we prove that all the divisors of V-lines have intersection number zero with V. In fact we have the following:

Proposition 4.3. *Let $\mathcal{F} = \{E_1, \dots, E_k\}$ be a collection of pairwise disjoint divisors of V-lines such that $E_i \cdot V = 0$ for every $i = 1, \dots, k$.*

Then there exist a polarized manifold (X', H') not covered by lines and a contraction $\varphi_\sigma : X \to X'$ expressing X as a blow-up of X' along k disjoint centers T_i, with exceptional divisors $E_1, \dots, E_k$ and such that $H = \varphi_\sigma^* H' - \sum E_i$.

Moreover (X', H') is RCC with respect to V', the family of deformations of the image of a general curve parametrized by V.

Proof. The effective divisor E_i cannot be trivial on the whole $\mathrm{NE}(X)$; since it vanishes on $[V]$, which lies in the interior of $\mathrm{NE}(X)$, it must be negative on some effective curve B. This curve is therefore contained in E_i.

By Lemma 4.1, the numerical class of B is contained in the two-dimensional vector subspace of $\mathrm{N}_1(X)$ spanned by $[C^i]$ and $[\mathcal{L}^i]$. Since $E_i \cdot C^i \geq 0$, being C^i a dominating family, and $E_i \cdot V = 0$ we have $E_i \cdot \mathcal{L}^i < 0$.

Consider the divisor $H_i = -(E_i \cdot \mathcal{L}^i)H + E_i$. We will show that this divisor is nef and trivial only on $R^i = \mathbb{R}_+[\mathcal{L}^i]$. Assume that, for some curve B we have $H_i \cdot B \leq 0$. This implies $E_i \cdot B < 0$, so $B \subset E_i$, hence $[B] \subset \mathrm{NE}\,(E_i, X)$.

Recalling that $[\mathcal{L}^i]$ is extremal in $\mathrm{NE}\,(E_i, X)$, it is clear that for every curve whose numerical class is in $\mathrm{NE}\,(E_i, X) \subset \Pi_i$ the intersection number with H_i is nonnegative, and it is zero if and only if $[B] \in R^i$; hence H_i is nef and R^i is an extremal ray of $\mathrm{NE}(X)$.

Denote by φ_i the contraction associated with R^i. Since $\mathrm{Locus}(\mathcal{L}^i) = E_i$ and $E_i \cdot R^i < 0$ then $\mathrm{Exc}(\varphi_i) = E_i$. Moreover, since $E_i = \mathrm{Locus}(\mathcal{L}^i)_{\mathrm{Locus}(C^i)_x}$ for a general $x \in X$, any fiber F_i of φ_i meets $\mathrm{Locus}(C^i)_x$, hence

$$n \geq \dim F_i + \dim \mathrm{Locus}(C^i)_x \geq -K_X \cdot \mathcal{L}^i - K_X \cdot C^i - 1 = n.$$

Equality must then hold. In particular for any fiber of φ_i we have $\dim F_i = -K_X \cdot \mathcal{L}^i$. Thus, φ_i is a smooth blow-up by Theorem 2.11. Notice that from this it follows that $E_i \cdot \mathcal{L}^i = -1$.

Consider now the divisor $H + \sum E_i$. Arguing as we did for H_i, we prove that it is nef and it vanishes only on curves whose numerical classes belong to one of the R^i's. Therefore there is a k-dimensional face σ of $\mathrm{NE}(X)$ generated by the R^i's and the associated contraction $\varphi_\sigma : X \to X'$ contracts exactly the curves whose numerical class belongs to R^i for some i. Since the E_i's are disjoint, φ_σ is the blow-up of X' along smooth disjoint centers T_i's.

Let V' be a family of deformations of the image of a general curve parametrized by V. Clearly, through two general points of X' there is a curve parametrized by V'. The divisor $H + \sum E_i$ is nef and supports the face contracted by φ_σ, hence there exists an ample divisor H' on X' such that $\varphi_\sigma^* H' = H + \sum E_i$. From the projection formula we get $H' \cdot V' = 3$.

Assume by contradiction that X' is covered by lines, i.e., there exists a dominating family of rational curves $\mathcal{L}'$ of degree one with respect to H'. The family $\mathcal{L}$ of deformations of the strict transform of a general line ℓ' parametrized by $\mathcal{L}'$ will be a covering family of lines for X; in fact, being general, ℓ' is disjoint from T_i for every i, hence $H \cdot \mathcal{L} = 1$. □

5. Manifolds of Picard number two

In this section we are going to prove the first part of Theorem 1.1 under the assumption that the Picard number of X is two. This is the hardest case and represents a crucial step in the proof.

Let $\mathcal{B}' = \{(\mathcal{L}^i, C^i)\}$ be as in the previous section; as we saw, to each pair in $\mathcal{B}'$ is associated a divisor $E_i = \mathrm{Locus}(\mathcal{L}^i)$; we need to show that one of them is the exceptional divisor of a smooth blow-up and does not meet a general curve of V.

We deal first with a particular case, namely the case in which a divisor of V-lines of the second kind exists.

Proposition 5.1. *Let (X, H) be an RCC-manifold with respect to V, not covered by lines and of Picard number two. Assume that there exists a divisor E of V-lines which is of the second kind. Then there exists a contraction $\varphi : X \to \mathbb{P}^n$ expressing X as a blow-up of $\mathbb{P}^n$ along a codimension two linear subspace or along a codimension two smooth quadric. Moreover $H = \varphi^*\mathcal{O}_{\mathbb{P}^n}(3) - E$ and V is the family of deformations of the strict transform of a general line in $\mathbb{P}^n$.*

Proof. Let $(\mathcal{L}, C)$ be the pair in $\mathcal{B}'$ whose associated divisor of V-lines is E. The two cases appearing in the statement differ by the position in $\mathrm{NE}(X)$ of the numerical class $[C]$. Assume first that $[C]$ spans an extremal ray of $\mathrm{NE}(X)$.

The associated contraction $\psi : X \to B$ is then of fiber type with fibers of dimension $n-1$. In fact a fiber F contains $\mathrm{Locus}(C_x)$ for every $x \in F$ and, for a general $x \in X$ we have $\dim \mathrm{Locus}(C_x) \geq n-1$, since E is a divisor of the second kind. A general fiber of ψ is a projective space by Theorem 3.6 in [12]. Since the contraction is elementary, by standard arguments we get that X is a projective bundle over B; since X is rationally connected we have that B is rational, and thus $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ with $\mathcal{E} = \oplus \mathcal{O}(a_i)$ and $0 = a_0 \leq a_1 \leq \cdots \leq a_n$.

The family C is the family of lines in the fibers of ψ. Recalling that $H \cdot C = 2$ and denoting by $\xi_{\mathcal{E}}$ the tautological line bundle of $\mathcal{E}$, we can write $H = 2\xi_{\mathcal{E}} + \psi^*\mathcal{O}(b)$ for some b. The ampleness of H yields $b \geq 1$; equality holds, since $H \cdot \mathcal{L} = 1$. Moreover from the last formula we get that a curve of $\mathcal{L}$ is a section corresponding to a surjection $\mathcal{E} \to \mathcal{O}$. The locus of curves in $\mathcal{L}$ is a divisor, hence we have $a_0 = a_1 = \cdots = a_{n-1} = 0$. Finally, from $-K_X \cdot \mathcal{L} = 1$ we get $a_n = 1$.

Assume now that $[C]$ is not extremal in $\mathrm{NE}(X)$. Since for a general $x \in X$ we have $\dim \mathrm{Locus}(C_x) = n-1$ by Proposition 2.5, in view of Theorem 2 of [5] this implies that C is not a quasi-unsplit family.

Let $\ell^1 + \ell^2$ be a reducible cycle in $\mathcal{C}$ whose components are not numerically proportional to C. For a general $x \in X$ we have seen that $\mathrm{Locus}(C_x)$ is a divisor D_x; D_x cannot contain curves numerically proportional to ℓ^i, hence, if $D_x \cdot \ell^i \neq 0$ then, for every point y in the locus of the corresponding family $\mathcal{L}^i$, we have $\dim \mathrm{Locus}(\mathcal{L}^i_y) = 1$.

Assume, up to exchanging indexes, that $D_x \cdot \ell^1 \neq 0$. Then, since $\mathcal{L}^1$ is not a covering family, by Proposition 2.5 we have $-K_X \cdot \mathcal{L}^1 = 1$, and thus $-K_X \cdot \mathcal{L}^2 = n-1$.

By Lemma 2.7, $\mathrm{N}_1(D_x, X) = \langle [C] \rangle$; in particular, since C is a dominating family, $D_{x|D_x}$ is nef. Since D_x is effective, it follows that it is nef.

The nef divisor D_x is trivial on $E_2 = \mathrm{Locus}(\mathcal{L}^2_x)$, hence $[\mathcal{L}^2]$ generates an extremal ray, which is birational and of length $n-1$, so it corresponds to the blow-up of a smooth point in a smooth X'.

Let W be a minimal dominating family of rational curves for X', and let W^* be the family of deformations of the strict transform of a curve in W. We have that $[W^*] = [C]$, since E_2 is trivial on both and X does not carry a covering family of lines. Therefore

$$-K_{X'} \cdot W = -K_X \cdot W^* = -K_X \cdot C = n \tag{5.1}$$

and X' is a smooth quadric by Theorem 0.1 (3) in [14]; in particular X has another contraction whose exceptional locus is E, which is the blow-up of $\mathbb{P}^n$ along a smooth quadric of codimension 2. □

Now we will show that, up to numerical equivalence, $\mathcal{B}'$ contains only one pair.

Proposition 5.2. *Let (X, H) be an RCC-manifold with respect to V, not covered by lines, and of Picard number two. Then, up to numerical equivalence, $\mathcal{B}'$ contains only one pair $(\mathcal{L}, C)$.*

Proof. We will prove the proposition by contradiction.

By Proposition 5.1, we can assume that there are no divisors of V-lines of the second kind. We choose $(\mathcal{L}^1, C^1)$ to be a pair such that $m := -K_X \cdot \mathcal{L}^1$ is maximum among the anticanonical degrees of families belonging to pairs in $\mathcal{B}'$; since there are no divisors of the second kind we have $m > 1$.

Since we are assuming that $\mathcal{B}'$ contains a pair $(\mathcal{L}^2, C^2)$ with $[\mathcal{L}^2] \neq [\mathcal{L}^1]$ we have that $m \geq n/2$ by formula (4.2).

Step 1. $E_1 \cdot C^2 = 0$.

From the maximality of m it follows that $-K_X \cdot \mathcal{L}^2 < -K_X \cdot \mathcal{L}^1$, hence that $-K_X \cdot C^2 > -K_X \cdot C^1$. Notice that the numerical class of C^2 cannot be proportional to $[\mathcal{L}^1]$, otherwise $-K_X \cdot C^2 = 2m \geq n$, and the divisor of V-lines associated with the pair $(\mathcal{L}^2, C^2)$ would be of the second kind.

Therefore, if $E_1 \cdot C^2 > 0$, then for a general $x \in X$ we have, by Lemma 2.7 and by Proposition 2.5, that

$$\dim \operatorname{Locus}(\mathcal{L}^1)_{\operatorname{Locus}(C^2)_x} \geq -K_X \cdot C^2 - K_X \cdot \mathcal{L}^1 - 2 \geq n,$$

a contradiction, since $\mathcal{L}^1$ is not a covering family.

Step 2. The adjoint divisor $D := K_X + mH$ is nef.

If this is not the case, since $D \cdot \mathcal{L}^1 = 0$ and $D \cdot V > 0$, there is an extremal ray R on the side of $[\mathcal{L}^1]$ with respect to $[V]$ on which D is negative. Denote by φ the associated contraction and let W be a family of rational curves such that $\overline{\operatorname{Locus}(W)} = \operatorname{Exc}(\varphi)$ whose degree with respect to H is minimal.

Every fiber of the contraction φ has dimension greater than m. If φ is birational then this follows from Proposition 2.10, since $l(R) > m$. If else φ is of fiber type then a general fiber F contains $\operatorname{Locus}(W)_x$ for some x, hence we have

$$\dim F \geq \dim \operatorname{Locus}(W)_x \geq mH \cdot W - 1 > m,$$

where the last inequality follows from the fact that X is not covered by lines and that $m > 1$.

It follows that $E_1 \cap \mathrm{Exc}(\varphi) = \emptyset$. In fact, if this were not the case, then E_1 would meet a fiber of φ, and in this case their intersection would contain a curve, contradicting the fact that $R \notin \mathrm{NE}\,(E_1, X)$ (cf. Lemma 4.1).

Therefore $E_1 \cdot R = 0$. Hence, by Step 1, $[C^2] \in R$. Since C^2 is a dominating family, φ is a fiber type contraction, contradicting $E_1 \cap \mathrm{Exc}(\varphi) = \emptyset$.

Step 3. The contraction associated with a multiple of D is a smooth blow-up.

Let $\varphi : X \to X'$ be the contraction associated with $R := \mathbb{R}_+[\mathcal{L}^1]$; let W be a family of rational curves such that $\overline{\mathrm{Locus}(W)} = \mathrm{Exc}(\varphi)$ whose degree with respect to H is minimal.

If φ is of fiber type, then $H \cdot W \geq 2$, since X is not covered by lines. Therefore, a general fiber of φ has dimension $\geq 2m-1$. Let x be a general point. Then

$$\dim F + \dim \mathrm{Locus}(C^1)_x \geq 2m - 1 + (n-m) = n + m - 1 > n,$$

a contradiction.

Therefore φ is birational. In particular there exists an irreducible divisor which is negative on R and therefore contains $\mathrm{Exc}(\varphi)$. So this divisor is E_1 and $E_1 \cdot R < 0$. Recall that, by construction, we have $E_1 = \mathrm{Locus}(\mathcal{L}^1)_{\mathrm{Locus}(C^1)_x}$ for a general $x \in X$; it follows that any fiber F of φ meets $\mathrm{Locus}(C^1)_x$, hence, by Proposition 2.5, we have

$$n \geq \dim F + \dim \mathrm{Locus}(C^1)_x \geq -K_X \cdot \mathcal{L}^1 - K_X \cdot C^1 - 1 = n,$$

hence equality holds. In particular for any fiber of φ we have $\dim F = -K_X \cdot \mathcal{L}^1$, thus φ is a smooth blow-up by Theorem 2.11.

Step 4. Conclusion.

By Step 1, $E_1 \cdot C^2 = 0$, hence $[C^2]$ is in the interior of the cone $\langle [\mathcal{L}^1], [V] \rangle$, and so $E_1 \cdot V > 0$, which in turn implies $E_1 \cdot C^1 \geq 2$. By formula (4.1) we have

$$-K_X \cdot C^2 \leq n + 2 - (-K_X \cdot C^1) = -K_X \cdot \mathcal{L}^1 + 1 = m + 1. \tag{5.2}$$

Let $\varphi : X \to X'$ be the blow-up. The line bundle $H + E_1$ is trivial on R, hence there exists $H' \in \mathrm{Pic}(X')$ such that $H + E_1 = \varphi^* H'$. Since $\rho_{X'} = 1$ we can write $-K_{X'} = kH'$. Since X' is Fano and $H + E_1$ is nef, we have $k > 0$.

By the canonical bundle formula

$$-K_X = -\varphi^* K_{X'} - mE_1 = kH + (k-m)E_1$$

we have

$$m + 1 \geq -K_X \cdot C^2 = -\varphi^* K_{X'} \cdot C^2 = 2k \tag{5.3}$$

and

$$n + 1 - m = -K_X \cdot C^1 = 2k + (k-m)(E_1 \cdot C^1).$$

Recalling that $E_1 \cdot C^1 \geq 2$ we have

$$n + 1 - m \leq 2k - 2(m-k) \leq 2,$$

where the last inequality follows from (5.3).

Since $m \leq n-1$, the only possibility is that all the inequalities are equalities; in particular $m = n-1$ and E_2 is of the second kind, a contradiction. □

Now we will prove the first part of Theorem 1.1 under the assumption that the Picard number of X is two.

Theorem 5.3. *Let (X, H) be an RCC-manifold with respect to V, not covered by lines, and of Picard number two. Then there exists a polarized manifold (X', H') of Picard number one not covered by lines and an elementary contraction $\varphi_\sigma : X \to X'$ expressing X as a blow-up of X' along a smooth center T, with exceptional divisor E, such that $H \simeq \varphi_\sigma^* H' - E$. The pair (X', H') is RCC with respect to V', the family of deformations of the image of a general curve parametrized by V.*

Proof. By Proposition 5.2, we know that, up to numerical equivalence, there is only one pair $(\mathcal{L}, C)$ in $\mathcal{B}'$. Let E be the corresponding divisor of V-lines. We claim that $E \cdot V = 0$.

Assume by contradiction that $E \cdot V > 0$. Let $x \in X$ be a general point, and consider the following diagram:

$$\begin{array}{ccccc} \mathcal{U}_x & \longleftarrow & U_x & \xrightarrow{i} & X \\ \downarrow p & & \downarrow p & & \\ \mathcal{V}_x & \longleftarrow & V_x & & \end{array}$$

By our assumptions, the inverse image $i^{-1}(E)$ dominates V_x; moreover, since V is generically unsplit $i_{|i^{-1}(E)} : i^{-1}(E) \to E$ is a generically finite map, hence it is dominating, since $\dim V_x = \dim i^{-1}(E) = \dim E$.

Let $y \in E$ be a general point and let F be a component of $\mathrm{Locus}(\mathcal{L})_y$. We can find a (non-complete) curve Γ^0 in $i^{-1}(F)$. Let B^0 be $p(\Gamma^0)$ and $S^0 := p^{-1}(B^0)$; notice that every curve parametrized by B^0 meets Γ^0.

Let $\overline{B}$ be the closure of B^0 in $\mathcal{V}_x$, let $\overline{S}$ be $p^{-1}(\overline{B})$, let $\nu : B \to \overline{B}$ be the normalization, $S = B \times_{\overline{B}} \overline{S}$ and Γ the curve in S whose image in S^0 is Γ^0. Notice that, by construction, the image in X of Γ is a curve contained in F, hence it is numerically proportional to $[\mathcal{L}]$.

By Proposition II.4.19 in [13] every curve in S is algebraically equivalent to a linear combination with rational coefficients of a section C_0 such that $i(C_0) = x$, and of the irreducible components of fibers of $p_{|S}$ (in the quoted proposition, take $X = S$, $T = B$ and $Z = C_0$).

The images of irreducible components of fibers of p are irreducible curves whose numerical class is $[V]$, $[\mathcal{L}]$ or $[C]$. If all the fibers of $p : S \to B$ are irreducible, then in S we can write

$$\Gamma \equiv \alpha C_0 + \beta f,$$

with $[i(f)] = [V]$, and we get that $i(\Gamma)$ is numerically proportional to V, a contradiction. So we can write

$$\Gamma \equiv \alpha C_0 + \sum \gamma_i C_i + \sum \delta_j \ell_j,$$

with $[i(C_i)] = [C]$, $[i(\ell_j)] = [\mathcal{L}]$. Hence,

$$[i(\Gamma)] = \sum \gamma_i [C] + \sum \delta_j [\mathcal{L}],$$

and we get $\sum \gamma_i = 0$.

Since $i(\Gamma) \not\ni x$, then $\Gamma \cdot C_0 = 0$. Recalling that x is not contained in any line, we get

$$\alpha C_0^2 + \sum \gamma_i = 0;$$

since C_0 goes to a point then $C_0^2 < 0$, hence $\alpha = 0$. This implies that, for a general fiber f we have $f \cdot \Gamma = 0$, a contradiction.

The statement now follows applying Proposition 4.3. □

6. Proof of Theorem 1.1

Proof. Let $\mathcal{B}'$ be as in Section 4 and let $\mathcal{B} = \{(\mathcal{L}^i, C^i)\}_{i=1}^k$ be a maximal set of pairs in $\mathcal{B}'$ such that $[V], [\mathcal{L}^1], \dots, [\mathcal{L}^k]$ are numerically independent, and chosen in the following way: if there exists a pair in $\mathcal{B}'$ whose divisor of V-lines is of the second kind we choose it to be $(\mathcal{L}^1, C^1)$; if no such pair exists, but there is a pair whose divisor of V-lines is of the third kind we choose it to be $(\mathcal{L}^1, C^1)$.

Since we have already proved the first part of the Theorem for $\rho_X = 2$ we can assume that $k \geq 2$.

We start by showing that all the divisors of the first kind in $\mathcal{B}$ correspond to blow-ups at points, and can be simultaneously contracted; by Proposition 4.3 this will be the case if $E_i \cdot V = 0$ for every such divisor.

If E_1 is not of the first kind, then $E_i \cdot C^1 = 0$ for every divisor E_i of the first kind. We already know from Lemma 4.2 that $E_i \cdot \mathcal{L}^1 = 0$ for every divisor of V-lines in $\mathcal{B}$ with $i \neq 1$, hence we get that $E_i \cdot V = 0$ for every divisor of the first kind. So we are left with the case in which every divisor of V-lines in $\mathcal{B}$ (and in $\mathcal{B}'$, by our choice of $\mathcal{B}$) is of the first kind. Again we want to prove that $E_i \cdot V = 0$ for every i.

To this aim we will consider the reduction morphism associated with the adjoint divisor $D := K_X + (n-1)H$, which we claim is nef and big.

Assume first that D is not nef. Then there exists an extremal ray R of X which has length $l(R) \geq n$. The associated contraction φ_R is of fiber type and has fibers of dimension $\geq n-1$ by Proposition 2.10. Let E be a first kind divisor and pick $x \in E$; let F_x be the fiber of φ_R passing through x. Then

$$\dim(E \cap F_x) \geq \dim E + \dim F_x - n > 0,$$

a contradiction, since $[\mathcal{L}^1] \notin R$, being $D \cdot \mathcal{L}^1 = 0$ and $D \cdot R < 0$.

Therefore D is nef and defines an extremal face $\tau \subset \mathrm{NE}(X)$, with associated contraction $\varphi_\tau : X \to Y$.

Assume now by contradiction that D is not big, i.e., that φ_τ is of fiber type, and let W be a minimal dominating family of rational curves such that $[W] \in \tau$.

Then $-K_X \cdot W = (n-1)H \cdot W \leq n$, where the last inequality follows from the fact that W is locally unsplit and $\rho_X \geq 1$. Therefore $H \cdot W = 1$, contradicting our assumptions that X is not covered by lines.

Therefore D is nef and big and we can apply Theorem 7.3.2 in [3] to get that Y is smooth and φ_τ is the blow-up of Y along t distinct points. Since $D \cdot \mathcal{L}^i = 0$ for every i we have $t \geq k$. On the other hand, since $\rho_X = k+1$ we have $t \leq k$, so φ_τ is a blow-up of a smooth X' along k points, and the exceptional divisors are the divisors of V-lines.

Take a curve B in X' not containing the centers of the blow-up and not meeting the images of the conics parameterized by the families C^j belonging to pairs in $\mathcal{B}'$ passing through a fixed general point x. Since all the divisors of V-lines are of the first kind there is a finite number of these conics through x.

By construction the strict transform $\widetilde{B}$ does not meet cycles in $\mathcal{V}_x$ whose components are not proportional to V, hence its numerical class is proportional to V. Since $\widetilde{B}$ does not meet the E_i's we have $E_i \cdot \widetilde{B} = 0$ for every i, hence $E_i \cdot V = 0$.

We can thus apply Proposition 4.3 to the set of the divisors of the the first kind to get a new pair (X'', H''). If $\rho_{X''} \leq 2$ then we are done, otherwise every divisor of V''-lines is of the third kind.

By Lemma 4.2 these divisors are disjoint and have intersection number zero with V, so we can apply again Proposition 4.3. In any case we finally get to a pair (X', H') with $\rho_{X'} = 1$ and to a family V' as in the statement.

We now come to the description of (X', H').

If V' is a minimal dominating family then $X' \simeq \mathbb{P}^n$ by Theorem 1.1 of [11]; if this is not the case, then there is a dominating family W of rational curves in X' such that $-K_{X'} \cdot W < n+1$.

Let W^* be the family of deformations in X of the strict transform of a curve in W. Since W is dominating, a general curve parametrized by W does not meet the union $T = \cup T_i$ of the centers of the blow-up. Hence $E_i \cdot W^* = 0$ for every i and W^* is numerically proportional to V.

Since X is not covered by lines we can assume that $H \cdot W^* = 2$. Using again the canonical bundle formula and the projection formula we get

$$-K_{X'} \cdot W = -K_X \cdot W^* = -\frac{2}{3}K_X \cdot V = \frac{2}{3}(n+1).$$

We know that $H = \varphi^* H' - \sum E_i$, hence $H' \cdot W = 2$ and $\varphi^* H' \cdot C^i = 3$. Therefore, H' is the fundamental divisor of X', and the index of X' is $\frac{n+1}{3}$.

The family W^* is locally unsplit since X is not covered by lines. For a general x we have

$$\dim \operatorname{Locus}(W^*)_x \geq -K_X \cdot W^* - 1 \geq \frac{2n-1}{3}.$$

Notice that $\operatorname{codim}(T_i) - 1 = -K_X \cdot \mathcal{L}^i$, hence

$$\dim \operatorname{Locus}(C^i)_x = -K_X \cdot C_i - 1 = n + K_X \cdot \mathcal{L}^i = \dim T_i + 1.$$

If for some i we have $\dim T_i \geq (n+1)/3$ then we get a contradiction considering the intersection $\operatorname{Locus}(C^i)_x \cap \operatorname{Locus}(W^*)_x$ for any i. □

7. RCC Fano manifolds

In this section we will show how restricting to RCC Fano manifolds leads to stronger results.

Proposition 7.1. *With the assumptions of Theorem 1.1, if X is a Fano manifold and $(X', H') \not\simeq (\mathbb{P}^n, \mathcal{O}(3))$, then $\dim T_i = \frac{n-2}{3}$ for every i.*

Proof. We keep the notation of the proof of Theorem 1.1. The rc($\mathcal{W}^*$)-fibration contracts curves parametrized by V, hence it goes to a point. Therefore W^* cannot be a quasi-unsplit family, otherwise $\rho_X = 1$. Hence, there is a reducible cycle $l^* + \bar{l}^*$ in $\mathcal{W}^*$ whose components are not numerically proportional.

For at least one i the divisor E_i is not trivial on both l^* and $\bar{l}^*$, hence, up to exchanging them, we can assume $E_i \cdot l^* < 0$; therefore $[l^*] \in R^i$ and $-K_X \cdot l^* = -K_X \cdot \mathcal{L}^i$, so

$$-K_X \cdot \mathcal{L}^i < -K_X \cdot W^* = \frac{2(n+1)}{3},$$

and we get $\operatorname{codim} T_i \leq 2/3(n+1)$, which, combined with the bound obtained in Theorem 1.1 gives $\dim T_i = \frac{n-2}{3}$. □

7.1. Higher Picard number

In this section we are going to prove that a Fano RCC manifold has Picard number at most three, and to classify those of Picard number three.

We will need the following result from basic projective geometry; the proof we give here was pointed out to us by Francesco Russo.

Lemma 7.2. *Let $T \subset \mathbb{P}^n$ be a projective manifold, and denote by $\mathcal{S}(T)$ its secant variety. Then,*

1) *if $\dim \mathcal{S}(T) = \dim T$ then T is a linear subspace of $\mathbb{P}^n$;*
2) *if $\dim \mathcal{S}(T) = \dim T + 1$ then T is a hypersurface of $\langle T \rangle \simeq \mathbb{P}^{\dim T+1}$.*

Proof. We denote by $\mathbb{T}_z Z$ the projective tangent space to a variety Z at a point z. If $t \in T$, we denote by $\mathcal{S}(t, T)$ the relative secant variety of T with respect to t.

First of all notice that, for every $t \in T$,

$$T \subseteq \mathcal{S}(t,T) \subseteq \mathbb{T}_t \mathcal{S}(t,T) \subseteq \mathbb{T}_t \mathcal{S}(T). \tag{7.1}$$

Assume that $\dim \mathcal{S}(T) = \dim T$. Clearly $\mathcal{S}(T) = T$. Let $t \in T$ be a point of T. By (7.1) and by our assumptions we have that

$$T \subseteq \mathbb{T}_t \mathcal{S}(T) = \mathbb{T}_t T = \mathbb{P}^{\dim T}$$

and hence $T = \mathbb{P}^{\dim T}$ since T and $\mathbb{T}_t T$ are irreducible varieties of the same dimension $\dim T$.

Now we suppose that $\dim \mathcal{S}(T) = \dim T + 1$. For a general point $t \in T$,

$$T \subsetneq \mathcal{S}(t,T) \subseteq \mathcal{S}(T),$$

and hence
$$\dim T < \dim \mathcal{S}(t,T) \le \dim \mathcal{S}(T) = \dim T + 1.$$
This implies that for a general point $t \in T$ we have $\mathcal{S}(t,T) = \mathcal{S}(T)$, hence for a general point $x \in \mathcal{S}(T) \setminus T$ there exists $t' \in T$ such that

$$x \in \langle t, t' \rangle \subset \mathcal{S}(t,T) = \mathcal{S}(T).$$

From this it follows that a general point $t \in T$ is contained in $\mathbb{T}_x\mathcal{S}(T)$ and that

$$\mathcal{S}(T) \subseteq \langle T \rangle \subseteq \mathbb{T}_x\mathcal{S}(T),$$

where $\langle T \rangle$ is the linear span of T in $\mathbb{P}^n$.
Now, by the generality of $x \in \mathcal{S}(T)$ we know that $\dim \mathbb{T}_x\mathcal{S}(T) = \dim \mathcal{S}(T)$ and hence
$$\mathcal{S}(T) = \mathbb{T}_x\mathcal{S}(T) = \mathbb{P}^{\dim \mathcal{S}(T)} = \mathbb{P}^{\dim T+1},$$
which concludes the proof. □

Proof of Theorem 1.2. By Theorem 1.1 and Proposition 7.1, we know that (X,H) is the blow-up of (X',H') along disjoint centers. Assume that $\rho_X > 3$ and let $(\mathcal{L}^i, C^i), (\mathcal{L}^k, C^k)$ be two independent pairs in $\mathcal{B}'$ with associated divisors of V-lines E_i and E_k.

Since E_i cannot contain curves of C^k, but C^k is dominating, it follows that there exists a reducible cycle $l_k + \bar{l}_k$ in $\mathcal{C}^k$ such that $E_i \cdot l_k < 0$. This implies that $[l_k] \in \mathrm{NE}\,(E_i, X) \subset \Pi_i$. Notice that $H + E_i$ is nef on Π_i, hence $E_i \cdot l_k = -1$. Since both H and E_i have the same intersection number with l_k and $\mathcal{L}^i$, then $[l_k] = [\mathcal{L}^i]$.

Recalling that X is Fano, this implies that $-K_X \cdot C^k \ge -K_X \cdot \mathcal{L}^i + 1$, and so we get that $-K_X \cdot (\mathcal{L}^k + \mathcal{L}^i) \le n$. Hence, by formula (4.2) that

$$-K_X \cdot (\mathcal{L}^k + \mathcal{L}^i) = n. \tag{7.2}$$

Notice that $-K_X \cdot \mathcal{L}^i$ is the length of the contraction of the extremal ray generated by $[\mathcal{L}^i]$. In particular, if T_i is the image of E_i via π, we have $\dim T_i = n - (-K_X \cdot \mathcal{L}^i) - 1$, hence, for $i \neq k$

$$\dim T_k + \dim T_i = n - 2. \tag{7.3}$$

By Proposition 7.1 it follows that $(X', H') \simeq (\mathbb{P}^n, \mathcal{O}(3))$.

If $\rho_X > 3$, combining with formula (4.1) we have that, for every i,

$$-K_X \cdot C^i = \frac{n+2}{2} \quad \text{and} \quad -K_X \cdot \mathcal{L}^i = \frac{n}{2},$$

and hence $\dim T_i = \frac{n-2}{2}$ for every i.

Recall now that $H = \pi^*\mathcal{O}(3) - \sum E_i$ is ample. Take two of the centers T_1 and T_2 and consider their join. By Proposition 11.37 in [8], it has dimension $n-1$, hence it meets some other center T_3. Take a line ℓ meeting three centers. Then $(\pi^*\mathcal{O}(3) - \sum E_i) \cdot \ell \le 0$, a contradiction. Therefore $\rho_X \le 3$.

Assume now that $\rho_X = 3$. Let $\mathcal{S}(T_1)$ be the secant variety of T_1. Suppose that $\dim \mathcal{S}(T_1) \geq \dim T_1 + 2$. Then

$$\dim(\mathcal{S}(T_1) \cap T_2) \geq \dim \mathcal{S}(T_1) + \dim T_2 - n \geq \dim T_1 + 2 + \dim T_2 - n = 0,$$

i.e., there is a line l in $\mathbb{P}^n$ which meets T_1 in two points and T_2 in a point; as above we show that $H \cdot l \leq 0$.

It follows by Lemma 7.2 that either T_i is a linear space or a hypersurface in a linear space of dimension $\dim T_i + 1$.

Notice also that from the ampleness of H it follows that there cannot exist trisecant lines of T_i in $\mathbb{P}^n$, and hence, if T_i is not a linear space, then it is a hyperquadric, and we can prove that $\dim T_i \geq \frac{n}{2} - 1$.

In fact, considering the strict transform l of a secant line of T_i and recalling that X is Fano, by the canonical bundle formula, we get

$$1 \leq -K_X \cdot l = (n+1) - 2\,\mathrm{codim}(T_i) + 2 \quad \Longrightarrow \quad \dim T_i \geq \frac{n}{2} - 1.$$

Therefore:

1. If $\dim \mathcal{S}(T_1) = \dim T_1$ and $\dim \mathcal{S}(T_2) = \dim T_2$, then X is the blow-up of $\mathbb{P}^n$ along two disjoint linear subspaces.
2. If $\dim \mathcal{S}(T_1) = \dim T_1$ and $\dim \mathcal{S}(T_2) = \dim T_2 + 1$, then X is the blow-up of $\mathbb{P}^n$ along a linear subspace T_1 and along a quadric $T_2 \subset \Lambda_2 \simeq \mathbb{P}^{\dim T_2 + 1}$ such that $\dim T_1 \leq \frac{n}{2} - 1$. Moreover Λ_2 and T_1 must be disjoint, because there cannot exist lines in $\mathbb{P}^n$ which meet T_1 in a point and T_2 in two points.
3. If $\dim \mathcal{S}(T_1) = \dim T_1 + 1$ and $\dim \mathcal{S}(T_2) = \dim T_2 + 1$, then X is the blow-up of $\mathbb{P}^n$ along two quadrics T_1, T_2 such that $\dim T_1 = \dim T_2 = \frac{n}{2} - 1$ (clearly n is even). Notice also that $T_i \subset \Lambda_i \simeq \mathbb{P}^{\frac{n}{2}}$, and $\dim(\Lambda_1 \cap \Lambda_2) = 0$ because there cannot exist trisecant lines of $T_1 \cup T_2$. □

7.2. Picard number two: some examples

Let (X, H) be an RCC Fano manifold of Picard number two obtained as the blow-up of $\mathbb{P}^n$ along a smooth center T. Denote by $\varphi : X \to \mathbb{P}^n$ the blow-up contraction and by E the exceptional divisor. Then, $H = \varphi^*\mathcal{O}_{\mathbb{P}^n}(3) - E$.

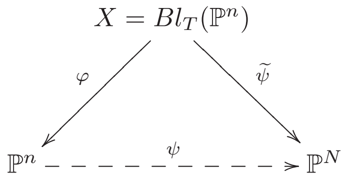
By the ampleness of the anticanonical bundle

$$-K_X = \varphi^*\mathcal{O}_{\mathbb{P}^n}(n+1) - (\mathrm{codim}\, T - 1)E,$$

we have that, if T is not a linear space, then $\dim T > (n-3)/2$. To see this, just compute the intersection of $-K_X$ with the strict transform of a secant line of T. Moreover, by the ampleness of H we get that T has no trisecants. A large class of examples is given by the following.

Proposition 7.3. *Let $T \subset \mathbb{P}^n$ be a smooth subvariety of dimension $t > (n-3)/2$ which is scheme theoretically defined by quadratic equations. Then the pair $(X, H) = (Bl_T(\mathbb{P}^n), 3\mathcal{H} - E)$ is a Fano RCC manifold. Moreover X is covered by H-lines if and only if $\mathcal{S}(T) = \mathbb{P}^n$.*

Proof. Consider the rational map $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ given by the system of quadrics through T and the resolution of this map:



The morphism $\widetilde{\psi}$ is given by the linear system $|\varphi^*\mathcal{O}_{\mathbb{P}^n}(2) - E|$. Hence, it contracts the strict transforms of the (bi)secants to T. Using the canonical bundle formula we see that the intersection number of $-K_X$ with these curves is positive, and therefore X is a Fano manifold.

The ampleness of $\varphi^*\mathcal{O}_{\mathbb{P}^n}(3) - E$ is now given by the Kleiman criterion. The family V is the family of deformations of the strict transform of a general line in $\mathbb{P}^n$. The last statement follow from the fact that an H-line not contained in E is the strict transform of a (bi)-secant line of T. □

Remark 7.4. Some examples of manifolds obtained as in Proposition 7.3 can be found in [6], Cases (b1)–(b6) and (c1)–(c2).

8. Examples

Example 8.1. $(X, H) \simeq (Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, where $Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n)$ is the blow-up of $\mathbb{P}^n$ along two disjoint linear spaces $\Lambda_1 \simeq \mathbb{P}^t$ and $\Lambda_2 \simeq \mathbb{P}^{n-2-t}$; E_1, E_2 are the exceptional divisors of π and $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Denote by

- R^i the extremal ray corresponding to the contraction of E_i;
- ε_i the contraction associated with R^i;
- ℓ_i a minimal curve whose numerical class is in R^i;
- ℓ a curve which is the strict transform of a line meeting both Λ_1 and Λ_2 in a point;
- $D_i = \mathcal{H} - E_i$.

The line bundles $\mathcal{H}, D_1$ and D_2 are nef on X. The cone of curves is therefore contained in the intersection of the positive halfspaces of $N_1(X)$ determined by them. By looking at the intersection numbers with the curves ℓ_1, ℓ_2, ℓ,

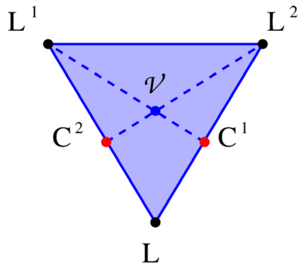
	ℓ_1	ℓ_2	ℓ
$\mathcal{H}$	0	0	1
D_1	1	0	0
D_2	0	1	0

we see that $NE(X)$ is the intersection of those halfspaces, and that is spanned by three rays, $R^1 = \mathbb{R}_+[\ell_1]$, $R^2 = \mathbb{R}_+[\ell_2]$, $R^3 = \mathbb{R}_+[\ell]$. Clearly the elementary contractions associated with R^1 and R^2 are the the blow-downs of E_1 and E_2.

The elementary contraction associated with R^3 is divisorial, and its exceptional locus is the strict transform of the join $J(\Lambda_1, \Lambda_2)$. It is possible to show that this contraction is the blow-up of $\mathbb{P}^{n-t-1} \times \mathbb{P}^{t+1}$ along a smooth subvariety $\mathbb{P}^{n-t-2} \times \mathbb{P}^t$.

Description of the families of rational curves. In this example the family V of cubics is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$; the set $\mathcal{B}'$ consists of two pairs, $(\mathcal{L}^1, C^1)$ and $(\mathcal{L}^2, C^2)$: the families $\mathcal{L}^i$ are the families of lines contracted by the blow-down, while the families C^i are the families of strict transforms of lines in $\mathbb{P}^n$ meeting one of the centers.

Curves in C^i degenerate into a line contracted by ε_j $(i \neq j)$ and the strict transform of a line meeting both Λ_1 and Λ_2.



Example 8.2. $(X, H) \simeq (Bl_{\Lambda_1, Q_1}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, where $Bl_{\Lambda_1, Q_1}(\mathbb{P}^n)$ is the blow-up of $\mathbb{P}^n$ along a linear space $\Lambda_1 \simeq \mathbb{P}^t$ and a smooth quadric $Q_1 \subset \Lambda_2 \simeq \mathbb{P}^{n-1-t}$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$; E_1, E_2 are the exceptional divisors of π and $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Denote by

- R^i the extremal ray corresponding to the contraction of E_i;
- ε_i the contraction associated with R^i;
- ℓ_i a minimal curve whose numerical class is in R^i;
- $\overline{\ell}_1$ a curve which is the strict transform of a line meeting both Λ_1 and Q_1 in a point;
- $\overline{\ell}_2$ a curve which is the strict transform of a general line contained in Λ_2;
- $D_1 = \mathcal{H} - E_1$;
- $D_2 = 2\mathcal{H} - E_2$;
- $D_3 = 2\mathcal{H} - E_1 - E_2$.

The line bundles $\mathcal{H}, D_1$ and D_2 are nef on X. We want to show that also D_3 is nef.

Suppose by contradiction that there is an irreducible curve $C \subset X$ such that $D_3 \cdot C < 0$. Then $(\mathcal{H} - E_1) \cdot C < 0$ or $(\mathcal{H} - E_2) \cdot C < 0$.

Assume that $(\mathcal{H} - E_2) \cdot C < 0$ (the other case is dealt with in a similar way). The map π factors as $\varepsilon_2 \circ \varepsilon_1$. Let $\widetilde{H}$ be a hyperplane of $\mathbb{P}^n$ which contains Λ_2 and let H' be the strict transform of $\widetilde{H}$ via ε_2. We have that

$$\mathcal{H} - E_2 = \varepsilon_1^* H'$$

and hence, by the projection formula, we get

$$(\mathcal{H} - E_2) \cdot C = H' \cdot \varepsilon_{1*} C < 0.$$

This implies that the curve C is not contracted by ε_1 and that $\varepsilon_1(C)$ is contained in H'. Since this holds for every hyperplane containing Λ_2 we have $\pi(C) \subset \Lambda_2$, so either C is contained in the strict transform of Λ_2 or $C \subset E_2$.

Since

$$E_2 = \mathbb{P}(\mathcal{N}^*_{Q_1/\mathbb{P}^n}) \simeq \mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus n-2-t}),$$

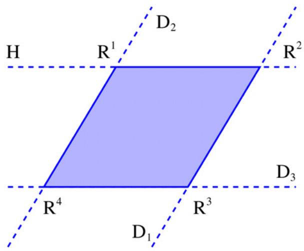
with the strict transform of Λ_2 cutting the section corresponding to the surjection $\mathcal{N}^*_{Q_1/\mathbb{P}^n} \to \mathcal{O}(-2)$ we have that $\mathrm{NE}\,(E_2, X) = \langle [\overline{\ell}_2], [\ell_2] \rangle$, while every curve contained in the strict transform of Λ_2 is numerically proportional to $[\overline{\ell}_2]$.

Since $D_3 \cdot \ell_2 = 1$ and $D_3 \cdot \overline{\ell}_2 = 0$, we get a contradiction which proves the nefness of D_3.

We have four nef line bundles: $\mathcal{H}, D_1, D_2$ and D_3. The cone of curves is therefore contained in the intersection of the positive halfspaces of $\mathrm{N}_1(X)$ determined by them. By looking at the intersection numbers with the four curves $\ell_1, \ell_2, \overline{\ell}_1, \overline{\ell}_2$,

	ℓ_1	ℓ_2	$\overline{\ell}_1$	$\overline{\ell}_2$
$\mathcal{H}$	0	0	1	1
D_1	1	0	0	1
D_2	0	1	1	0
D_3	1	1	0	0

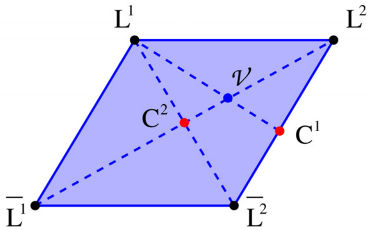
we see that $\mathrm{NE}(X)$ is the intersection of these halfspaces, that it is spanned by four rays, $R^1 = \mathbb{R}_+[\ell_1]$, $R^2 = \mathbb{R}_+[\ell_2]$, $R^3 = \mathbb{R}_+[\overline{\ell}_1]$, $R^4 = \mathbb{R}_+[\overline{\ell}_2]$ and that the position of these rays is as in the next figure, which shows a cross section of $\mathrm{NE}(X)$:



Clearly the elementary contractions associated with R^1 and R^2 are the the blow-downs of E_1 and E_2. The elementary contraction associated with R^3 is divisorial, and its exceptional locus is the strict transform of the join $J(\Lambda_1, Q_2)$, which is a divisor linearly equivalent to $2\mathcal{H} - 2E_1 - E_2$. Finally, the contraction associated with R^4 contracts the strict transform of Λ_2. Hence, if $\dim \Lambda_2 = n-1$ this contraction is divisorial, otherwise it is small.

Description of the families of rational curves. In this example the family V of cubics is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$; the set $\mathcal{B}'$ consists of two pairs, $(\mathcal{L}^1, C^1)$ and $(\mathcal{L}^2, C^2)$: the families $\mathcal{L}^i$ are the families of lines contracted by the blow-down, the family C^1 is the family of strict transforms of lines in $\mathbb{P}^n$ meeting Λ_1 at one point and the family C^2 is the family of strict transforms of lines in $\mathbb{P}^n$ meeting Q_1 at one point.

Curves parametrized by C^1 degenerate into the strict transform of a line meeting Λ_1 and Q_1 and a line contracted by ε_2, while curves parametrized by C^2 degenerate in two possible ways: either as a line contracted by ε_2 and the strict transform of a line contained in Λ_2 or as a line contracted by ε_1 and the strict transform of a line meeting both Λ_1 and Q_1.



Example 8.3. $(X, H) \simeq (Bl_{Q_1,Q_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, where $Bl_{Q_1,Q_2}(\mathbb{P}^n)$ is the blow-up of $\mathbb{P}^n$ along two smooth quadrics $Q_1 \subset \Lambda_1 \simeq \mathbb{P}^{\frac{n}{2}}$ and $Q_2 \subset \Lambda_2 \simeq \mathbb{P}^{\frac{n}{2}}$ such that

$$Q_1 \cap Q_2 = \emptyset, \quad \dim \Lambda_1 \cap \Lambda_2 = 0, \quad \dim Q_1 = \dim Q_2 = \tfrac{n}{2} - 1,$$

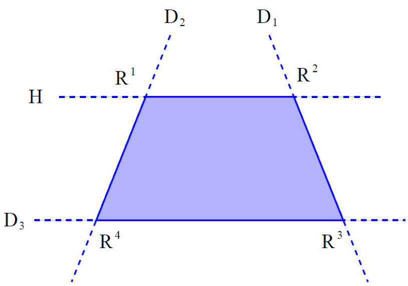
E_1, E_2 are the exceptional divisors of π and $\mathcal{H} = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$. Denote by

- R^i the extremal ray corresponding to the contraction of E_i;
- ε_i the contraction associated with R^i;
- ℓ_i a minimal curve whose numerical class is in R^i;
- $\overline{\ell}_i$ a curve which is the strict transform of a line contained in Λ_i;
- $D_i = 2\mathcal{H} - E_i$;
- $D_3 = 2\mathcal{H} - E_2 - E_1$.

As in Example 8.2 we can show that $\mathrm{NE}(X)$ is the intersection of the halfspaces determined by the nef divisors $\mathcal{H}, D_1, D_2, D_3$, that is spanned by four rays,

$$R^1 = \mathbb{R}_+[\ell_1], \quad R^2 = \mathbb{R}_+[\ell_2], \quad R^3 = \mathbb{R}_+[\overline{\ell}_1], \quad R^4 = \mathbb{R}_+[\overline{\ell}_2],$$

and that the position of these rays is as in the next figure, which shows a cross section of $\mathrm{NE}(X)$.



Clearly the elementary contractions associated with R^1 and R^2 are the the blow-downs of E_1 and E_2. The elementary contractions associated with R^3 and with R^4

are small with exceptional loci of dimension $\frac{n}{2}$ which are the strict transforms of the linear spaces Λ_i.

Let us devote a few words to the contraction of the face $\sigma = \langle R^3, R^4 \rangle$. Let P be the intersection point of Λ_1 and Λ_2, and let Σ be a 2-plane passing through P and meeting Λ_1 and Λ_2 in two lines, l_1 and l_2. It is not difficult to see that for a general point of $\mathbb{P}^n$ there is exactly one such 2-plane.

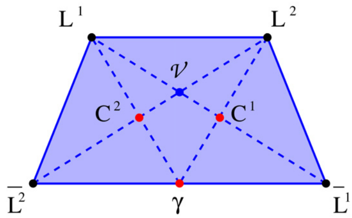
For a general point Q of Σ there is a conic passing through Q and through the (four) points of intersection of Σ with Q_1 and Q_2; denote by γ this conic. Since γ meets both Q_1 and Q_2 in two points we have $E_1 \cdot \gamma = E_2 \cdot \gamma = 2$.

The contraction $\varphi : X \to Y$ of σ is supported by $D_3 = 2\mathcal{H} - E_1 - E_2$. Hence, it contracts γ; it follows that φ is of fiber type. The restriction of D_3 to E_1 is big, hence $\dim Y = n - 1$.

Therefore φ is a conic bundle; the divisor of reducible conics is the strict transform of the join $J(Q_1, Q_2)$, and there is one special fiber of dimension $n-2$ consisting of two irreducible components which are projective spaces meeting at a point, namely the strict transforms of Λ_1 and Λ_2.

Description of the families of rational curves. In this example the family V of cubics is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$; the set $\mathcal{B}'$ consists of two pairs, $(\mathcal{L}^1, C^1)$ and $(\mathcal{L}^2, C^2)$: the families $\mathcal{L}^i$ are the families of lines contracted by the blow-down, while the families C^i are the families of strict transforms of lines in $\mathbb{P}^n$ meeting Q^i at one point.

Curves in C^i degenerate in two possible ways: either as a line contracted by ε_i and the strict transform of a line contained in Λ_i or as a line contracted by ε_j $(i \neq j)$ and the strict transform of a line meeting both Q_1 and Q_2. These last curves are numerically proportional to the conics meeting both Q_1 and Q_2 in two points.



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