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An elliptic variational problem involving level surface area on Riemannian manifolds

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Abstract. We study a variational problem involving a Dirichlet integral and the area of a level surface on arbitrary n-dimensional Riemannian manifolds. We prove optimal regularity results for minimizers and derive a jump condition along the level surface. We also obtain smoothness of the interface up to a small singular set of Hausdorff dimension less then or equal to $n-8$.

1. Introduction

Let (M,g) be an n-dimensional, smooth, complete Riemannian manifold, and let D be a bounded subset of M with smooth boundary. Suppose h is a given smooth function on ∂D. We denote by H^1_h the set of H^1 functions on D whose trace on ∂D is h. In this article we consider the variational free boundary problem of minimizing

$$\mathcal{E}(v) := \int_D |\nabla_g v|^2 dV_g + \text{ Area of } \{v=0\} \text{ in } D \tag{1.1}$$

over all functions $v \in H^1_h(D)$. The meaning of "Area of $\{v=0\}$ in D" will be specified later.

One key feature of the functional $\mathcal{E}$ in (1.1) is the competition between the Dirichlet energy and the area of the zero level surface. There are several motivations for the study of this class of free boundary problems. For instance, this problem can be obtained as the limit of a balanced scaling of the Landau–Ginzburg functional (see for instance [2]). Also, the evolutionary version of this problem can be used in the study of the motion of free surfaces governed by the mean curvature, see [10], [3]. The analysis of the minimization problem (1.1) is also a starting point for the study of optimal design problems with perimeter constraints modeled on Riemannian manifolds.

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The Euclidean version of this problem has been studied by Athanasopoulos *et al.* in [2]. It is proved in that article that the minimizer of $\mathcal{E}(v)$ exists. The minimizing function u is Lipschitz continuous, with u^+ and u^- separated by its 0-level surface Γ, the reduced part of which (denoted by Γ^*) is analytic, and such that $H_s[\Gamma \setminus \Gamma^*] = 0$ for each $s > n-8$, where H_s represents the s dimensional Hausdorff content. Besides, the following jump condition is satisfied on Γ^*:

$$|\nabla u^+|^2 - |\nabla u^-|^2 = (n-1)\,\kappa(\Gamma^*),$$

where $\kappa(\Gamma^*)$ is the mean curvature of Γ^*.

To explain the area of the level surface in (1.1) precisely, we recall that for $u \in L^1(M)$, the variation of u is defined by

$$|Du|(M) = \sup\Big\{ \int_M u \operatorname{div}_g \omega \, dV_g, \quad \forall \omega \in \Gamma_c(T^*M),\ |\omega| \le 1 \Big\}, \tag{1.2}$$

where $\Gamma_c(T^*M)$ is the space of 1-forms with compact support on M, and $|\omega|$ is the norm of ω (see [11]). A set $\Omega \subset M$ is called a set with finite perimeter if $|D\chi_\Omega| < \infty$, where χ_Ω is the characteristic function on Ω. Recall that, in local coordinates,

$$\operatorname{div}_g \omega = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_j} \Big(\sqrt{\det(g)} g^{ij} < \omega, \frac{\partial}{\partial x_i} > \Big).$$

We refer the reader to [11] for the definition and discussion of BV functions on Riemannian manifolds.

We will say that a pair (v, Ω) is *admissible* if Ω is a set of finite perimeter in D, i.e., $\operatorname{Per}(\Omega, D) = \operatorname{Per}(\Omega) < \infty$, $v \in H^1(D), v - g \in H_0^1(D)$, and

$$v|_{\Omega \cap D} \ge 0, \quad v|_{\Omega^c \cap D} \le 0, \quad \text{a.e.}$$

The boundary of Ω is understood as its *essential boundary* $\partial_M \Omega$, as in [2]. Throughout the paper we shall work on the following equivalent reformulation of problem (1.1):

$$E(v, \Omega) := \int_D |\nabla_g v|^2 dV_g + \operatorname{Per}(\Omega, D) \to \min,$$

where (v, Ω) is any admissible pair, and $v \in H_h^1(D)$.

Our main theorem, which can be seen as a precise analog of the theory developed in [2], reads as follows:

Theorem 1.1. *There exists an admissible pair (u, Ω) that minimizes E. The minimizer u is Lipschitz and the reduced part of the 0-level surface of u (denoted by Γ^*) is smooth and satisfies $H_s(\Gamma \setminus \Gamma^*) = 0$ for each $s > n-8$. Furthermore,*

$$|\nabla_g u^+|^2 - |\nabla_g u^-|^2 = (n-1)\kappa(\Gamma^*) \quad \text{on} \quad \Gamma^*,$$

where $\kappa(\Gamma^*)$ denotes the mean curvature of Γ^* in M.

The proof of Theorem 1.1 is inspired by the seminal work of Athanasopoulos *et al.* [2]. There are two main tools used in [2]. One is the celebrated monotonicity formula of Alt–Caffarelli–Friedman [4]. The other is a perturbation technique related to minimality. We will show that these techniques can be adapted to the setting of Riemannian geometry. In particular, instead of using the original monotonicity formula of Alt–Caffarelli–Friedman, we will use a variant of it, [15]. As far as the perturbation technique is concerned, several methods will be employed to handle the technical difficulties that come from the difference between a generic Riemannian metric and the flat Euclidean one.

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2. The proof of Theorem 1.1

As in the introduction, we say that a pair (v, Ω) is *admissible* if Ω is set of finite perimeter in D, i.e., $\mathrm{Per}(\Omega, D) = \mathrm{Per}(\Omega) < \infty$, $v \in H^1(D)$, $v - h \in H^1_0(D)$, and

$$v|_{\Omega\cap D} \geq 0, \quad v|_{\Omega^c\cap D} \leq 0, \quad \text{a.e.}$$

Recall that the boundary of Ω is understood as its *essential boundary* $\partial_M \Omega$.

Proposition 2.1. *There exists a pair (u, Ω) that minimizes E.*

Proof. Even with the different definitions, the proof is still the same as that of Proposition 2.1 in [2]. Namely, let $\{(u_k, \Omega_k)\}$ be a minimizing sequence. Then, by passing to a subsequence, there is a pair (u, Ω) such that

$$\begin{aligned} \chi|_{\Omega_k} \to \chi|_{\Omega} \quad \text{strongly in } L^1(D) \text{ and weakly in } BV(D) \\ u_k \rightharpoonup u \quad \text{weakly in } H^1(D), \\ u_k\chi|_{\Omega_k\cap D} \to u\chi|_{\Omega\cap D} \quad \text{a.e. in } D, \\ u_k\chi|_{\Omega_k^c\cap D} \to u\chi|_{\Omega^c\cap D} \quad \text{a.e. in } D. \end{aligned}$$

Note that it is established in [11] that the mapping $u \to |Du|(M)$ is L^1-lower semi-continuous, and for each $u \in BV(M)$, there exists $(f_j)_{j=1}^{\infty} \in C_c^{\infty}(M)$ such that $f_j \to u$ in $L^1(M)$ and

$$|Du|(M) = \lim_{j\to\infty} \int_M |\nabla_g f_j| dV_g.$$

Moreover (see [8]), for any bounded set $\tilde{\Omega}$ with smooth boundary, the set of functions uniformly bounded in BV norm is relatively compact in $L^1(\tilde{\Omega})$. □

In [2] there is a notion of *harmonic replacement*. We use essentially the same notion: Consider a measurable subset K of D and a function $f \in H^1(D)$. If $f = 0$ a.e. in $D \setminus K$, f is said to be supported in K. The following set S is closed and convex in $H^1(D)$:

$$S = \{f \in H^1(D) \mid f \text{ is supported in } K\}.$$

For $f \in S$, its *harmonic replacement* is defined as:

Definition 2.2. Let $f \in S$. The function f_0 is the *harmonic replacement* of f in D if

1) $f_0 \in S$,
2) $f_0 - f \in H_0^1(D)$, and
3) f_0 minimizes the Dirichlet integral $\int_D |\nabla_g u|^2 dV_g$ for u in $S \cap \{f + H_0^1(D)\}$.

It is worthwhile to note that the definition of the harmonic replacement avoids the assumption on the regularity of ∂K. The next lemma is similar to Lemma 2.3 in [2].

Lemma 2.3. *The following holds:*

1) f_0 *is unique.*
2) *If f is nonnegative, then f_0 is nonnegative and subharmonic ($\Delta_g f_0 \leq 0$). In particular, f_0 can be defined everywhere by solid averages. Also,*

$$\Delta_g (f_0)^2 = 2|\nabla_g f_0|^2 \tag{2.1}$$

in the sense of measures.

Here we use the following definition of Δ_g in local coordinates:

$$\Delta_g = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x_i}\Big(\sqrt{\det(g)}g^{ij}\frac{\partial}{\partial x_j}\Big).$$

The proof of Lemma 2.3 is similar to that of Lemma 2.3 in [2] and it is therefore omitted.

An important tool is the following monotonicity formula proved in [15]. Let $B_1(p)$ be a ball in (M, g), and let R_m be the curvature tensor. Suppose that $|R_m| + |\nabla_g R_m| \leq \Lambda$ on $B_1(p)$. Then,

Theorem A. *Let $n \geq 2$ and $u_1, u_2 \in H^1(B_1(p))$ be nonnegative functions that satisfy*

$$\Delta_g u_i \geq -1, \quad \text{in} \quad B_1(p), \quad i = 1, 2$$

in distributional sense. Suppose, in addition, that $u_1 \cdot u_2 = 0$. Then $u_1, u_2 \in H^1_{\text{loc}}(B_1(p))$ and there exist $C(n, \Lambda)$ and $\delta(n, \Lambda)$ such that for $0 < r < \delta$,

$$\phi(r) \leq C(n,\Lambda)\left(1 + \int_{B_\delta(p)} \frac{|\nabla_g u_1|^2}{d(x,p)^{n-2}}\, dV_g + \int_{B_\delta(p)} \frac{|\nabla_g u_2|^2}{d(x,p)^{n-2}}\, dV_g\right)^2,$$

where

$$\phi(r) = \frac{1}{r^4}\int_{B_r(p)} \frac{|\nabla_g u_1|^2}{d(x,p)^{n-2}}\, dV_g \int_{B_r(p)} \frac{|\nabla_g u_2|^2}{d(x,p)^{n-2}}\, dV_g\, .$$

Moreover,

$$\phi(r) \leq c(n,\Lambda)(1 + \|u_1\|_{L^\infty(B_r)} + \|u_2\|_{L^\infty(B_r)})^4. \tag{2.2}$$

The following two estimates correspond to Lemmas 2.5 and 2.6 in [2].

Lemma 2.4. *Let f_0 be the harmonic replacement of $f \geq 0$ in D. Assume $B_\delta \subset\subset D$ (with δ less than the injectivity radius at 0) and*

$$0 = f_0^2(0) = \lim_{r\to 0} \frac{1}{\mathrm{Vol}(B_r)} \int_{B_r} (f_0)^2 dV_g.$$

There exists a positive constant $C(n, \Lambda)$ such that

$$\sup_{B_{(1-h)r}} (f_0^2) \leq \frac{C}{h^n} \int_{B_r} \frac{|\nabla_g f_0|^2}{d(x,0)^{n-2}}\, dV_g \tag{2.3}$$

for any $0 < h < 1$ and $0 < r < \delta$.

Proof. First we claim that

$$\sup_{B_{(1-h)r}} (f_0)^2 \leq \frac{C}{h^n r^n} \int_{B_r} (f_0)^2 dV_g. \tag{2.4}$$

Let $x_0 \in B_{(1-h)r}$ be such that $f_0^2(x_0) = \sup_{B_{(1-h)r}} f_0^2$. We use the following Green's function representation formula (see Theorem 4.17 in [5]):

$$f_0^2(x_0) = -\int_{B_t} G_t(x_0,q)\Delta_g(f_0^2)\, dV_g(q) - \int_{\partial B_t} \nu^i \nabla_{g,iq} G_t(x_0,q) f_0^2(q)\, ds(q)$$

for any $(1-\frac{h}{2})r \leq t \leq r$. Here ν is the unit outwards oriented normal vector and ds is the volume element on ∂B_t corresponding to the Riemannian metric i^*g ($i : \partial B_t \to \bar{B}_t$ is the canonical imbedding). G_t is the Green's function with respect to the Dirichlet boundary condition.

$$|\nabla_{g,q} G_t(x_0,q)| \leq C(hr)^{n-1}, \quad \forall q \in B_r \setminus B_{(1-\frac{h}{2})r}.$$

Since f_0^2 is subharmonic, we have

$$\sup_{B_{(1-h)r}} f_0^2 \leq \frac{C}{(hr)^{n-1}} \int_{\partial B_t} f_0^2(q)\, ds(q), \quad \forall t \in \left(\left(1-\frac{h}{2}\right)r, r\right).$$

For both sides of the above integrate t from $(1-h)r$ to r. Then (2.4) is established.

Next we prove (2.3) using (2.4). We will divide our analysis in two cases.

Case 1: $n \geq 3$.

Given $\delta_1 \in (0, \delta/2)$, we define $\hat{\psi}_{\delta_1}$ by

$$\begin{cases} \Delta_g \hat{\psi}_{\delta_1} = \frac{1}{\mathrm{Vol}_g(B_\delta)} \chi_{B_\delta} - \frac{N}{\mathrm{Vol}(B_{\delta_1})} \chi_{B_{\delta_1}}, & \text{in } B_\delta \\ \hat{\psi}_{\delta_1} = 0 \quad \text{on} \quad \partial B_\delta. \end{cases}$$

where χ_{B_δ} is the characteristic function on B_δ, and $\chi_{B_{\delta_1}}$ is understood similarly. With N chosen sufficiently large (independently of δ_1), we can arrange $\partial_{\nu_g} \hat{\psi}_{\delta_1} < 0$

on ∂B_δ. This fact can be easily verified by the Green's representation of $\hat{\psi}_{\delta_1}$ and is therefore omitted. Similarly we define $\tilde{\psi}_{\delta_1}$ by

$$\begin{cases} \Delta_g \tilde{\psi}_{\delta_1} = \frac{1}{\mathrm{Vol}_g(B_\delta)}\chi_{B_\delta} - \frac{1}{N\mathrm{Vol}(B_{\delta_1})}\chi_{B_{\delta_1}}, & \text{in } B_\delta \\ \tilde{\psi}_{\delta_1} = 0 \quad \text{on} \quad \partial B_\delta. \end{cases}$$

For $\tilde{\psi}_{\delta_1}$ we have $\partial_{\nu_g}\tilde{\psi}_{\delta_1} > 0$ on ∂B_δ. Therefore there exists a smooth function $\beta_{\delta_1} \in (\frac{1}{N}, N)$ such that $\partial_{\nu_g}\psi_{\delta_1} = 0$ on ∂B_δ where ψ_{δ_1} is defined by

$$\begin{cases} \Delta_g \psi_{\delta_1} = \frac{1}{\mathrm{Vol}_g(B_\delta)}\chi_{B_\delta} - \frac{\beta_{\delta_1}}{\mathrm{Vol}(B_{\delta_1})}\chi_{B_{\delta_1}}, & \text{in } B_\delta \\ \psi_{\delta_1} = 0 \quad \text{on} \quad \partial B_\delta. \end{cases}$$

From the definition of ψ_{δ_1}, we see that $\psi_{\delta_1} \in W_0^{2,p}(B_\delta)$ for any $p > 1$. Therefore,

$$\int_{B_\delta} f_0^2 \Delta_g \psi_{\delta_1} dV_g = \frac{1}{\mathrm{Vol}_g(B_\delta)}\int_{B_\delta} f_0^2 dV_g - \frac{\beta_{\delta_1}}{\mathrm{Vol}_g(B_{\delta_1})}\int_{B_{\delta_1}} f_0^2\, dV_g. \tag{2.5}$$

The second term tends to 0 as $\delta_1 \to 0$ (recall that $\beta_{\delta_1} \in (\frac{1}{N}, N)$). Finally, elementary estimation gives

$$|\psi_{\delta_1}(x)| \le C|x|^{2-n}, \quad x \in B_\delta \tag{2.6}$$

with C independent of δ_1. Therefore

$$\int_{B_\delta} |\nabla_g f_0|^2 \psi_{\delta_1}\, dV_g \le C\int_{B_\delta} \frac{|\nabla_g f_0|^2}{d(x,0)^{n-2}}\, dV_g$$

Lemma 2.4 is established for $n \ge 3$.

Case 2: $n = 2$.

First, for $r = \delta$ (recall that δ depends on Λ only), we claim that there exists $C(\Lambda) > 0$ such that

$$\frac{1}{\delta^2}\int_{B_\delta} f_0^2\, dV_g \le C(\Lambda)\int_{B_\delta} |\nabla_g f_0|^2 dV_g\,. \tag{2.7}$$

If no such C can be found, there exists a sequence $f_k \in H^1(B_\delta)$ such that

$$\Delta_g(f_k^2) = 2|\nabla_g f_k|^2 \quad \text{weakly in } B_\delta, \tag{2.8}$$

$$f_k^2(0) := \lim_{\delta_1 \to 0} \frac{1}{\mathrm{Vol}(B_{\delta_1})}\int_{B_{\delta_1}} f_0^2\, dV_g = 0, \quad \text{and} \tag{2.9}$$

$$\frac{1}{\delta^2}\int_{B_\delta} f_k^2\, dV_g > k\int_{B_\delta} |\nabla_g f_k|^2\, dV_g. \tag{2.10}$$

Let

$$\bar{f}_k = \frac{f_k}{(\int_{B_\delta} f_k^2\, dV_g)^{\frac{1}{2}}}.$$

Clearly

$$\int_{B_\delta} \bar{f}_k^2 \, dV_g = 1. \tag{2.11}$$

Diving both sides of (2.10) by $\int_{B_\delta} f_k^2 dV_g$, we have

$$\frac{1}{\delta^2 k} > \int_{B_\delta} |\nabla_g \bar{f}_k|^2 \, dV_g.$$

Consequently, along a subsequence $\bar{f}_k$ converges weakly to a constant in $H^1(B_\delta)$, and then strongly to this constant in L^2 norm. By (2.9) this constant is 0, a contradiction to (2.11).

In general, for $r < \delta$, we define

$$f_1(y) = f_0\Big(\frac{r}{\delta}\, y\Big), \quad \tilde{g}_{ij}(y) = g_{ij}\Big(\frac{r}{\delta}\, y\Big), \quad |y| < \delta.$$

Then f_1 has the same properties as f_0. In particular,

$$\Delta_{\tilde{g}} f_1^2(y) = 2|\nabla_{\tilde{g}} f_1(y)|^2$$

in the weak sense. Thus the previous argument can be applied. Lemma 2.4 is established in all cases. □

Lemma 2.5. *Let f_0 be as in Lemma 2.4. There exists a positive constant $C(n, \Lambda)$ such that, for $0 < r < \frac{1}{2}\delta$,*

$$\int_{B_r} \frac{|\nabla_g f_0|^2}{d(x,0)^{n-2}} \, dV_g \le C r^{-n} \int_{B_{2r}\setminus B_r} f_0^2 \, dV_g. \tag{2.12}$$

Proof. For $n = 2$, this is a standard argument using a cut-off function. Thus we only prove the case $n \ge 3$. The idea of the proof is similar to the proof of Lemma 2.6 in [2]. We mainly address the difference. Let G satisfy

$$\begin{cases} -\Delta_g G(y) = \delta_0 & \text{in } B_\delta, \\ G(y) \ge 1 & \text{in } B_\delta. \end{cases}$$

Standard estimates show that

$$|D^j G(y)| \le C|y|^{2-n-j}, \quad j = 0, 1, 2, \quad y \in B_\delta. \tag{2.13}$$

Now, for $\epsilon > 0$ small, we define $G_\epsilon(y) \in C^2(B_\delta)$ as

$$G_\epsilon(y) = \begin{cases} G(y), & \epsilon \le |y| \le \delta, \\ |\Delta_g G_\epsilon(y)| \le C\epsilon^n & \text{in } B_\epsilon, \\ G_\epsilon \in C^2(B_\delta). \end{cases} \tag{2.14}$$

Note that the estimate of $\Delta_g G_\epsilon$ in B_ϵ can be obtained using (2.13). For a nonnegative L^1 function ϕ, (2.14) implies

$$\int_{B_\delta} \Delta_g G_\epsilon \phi \, dV_g \le \frac{C(n,\Lambda)}{\epsilon^n} \int_{B_\epsilon} \phi \, dV_g. \tag{2.15}$$

From (2.15) we see that if the average of ψ tends to 0 at 0 we have

$$\lim_{\epsilon\to 0}\int_{B_\delta}\Delta_g G_\epsilon \psi\, dV_g = 0. \tag{2.16}$$

The remaining part of the proof is similar to that of Lemma 2.6 in [2]. We include it for the reader's convenience. Let $\phi\equiv 1$ in B_r, $\phi\equiv 0$ in $B_{2r}\setminus B_r$. For ϕ we have

$$D\phi = O\Big(\frac{1}{r}\Big),\quad D^2\phi = O\Big(\frac{1}{r^2}\Big)\quad \text{in}\quad B_{2r}\setminus B_r.$$

Using the equation for f_0 and integration by parts we have

$$\begin{aligned}2\int_{B_\delta}\phi|\nabla_g f_0|^2 G_\epsilon\, dV_g &= \int_{B_\delta}\Delta_g(\phi G_\epsilon) f_0^2\, dV_g\\ &= \int_{B_\delta}(\Delta_g\phi G_\epsilon + 2\nabla_g\phi\nabla_g G_\epsilon + \phi\Delta_g G_\epsilon) f_0^2\, dV_g.\end{aligned}$$

The first two terms of the right hand side are less than

$$\frac{C}{r^n}\int_{B_{2r}\setminus B_r} f_0^2\, dV_g.$$

The third term tends to 0 as $\epsilon\to 0$, in consequence of (2.16) and $f_0^2(0)=0$ (in the sense of averages). Finally, by (2.13), (2.12) follows. Lemma 2.5 is established. □

3. Hölder continuity and uniform density

Let x_0 be a point on the free boundary Γ and let us consider $B(x_0,\delta)$, where $\delta < \min\{\mathrm{inj}_{x_0}, 1\}$. We also assume that $B(x_0,\delta)\subset D$. If (v,Ω) is an admissible pair, we set

$$\Omega^+ = \Omega\cap B(x_0,\delta),\quad \Omega^- = B(x_0,\delta)\setminus\bar\Omega^+,\quad \Gamma_{x_0} = \partial\Omega^+\cap B(x_0,\delta).$$

The new functional will be

$$E(v,\Omega^+) = \int_{B(x_0,\delta)}|\nabla_g v|^2 dV_g + \mathrm{Per}(\Omega^+, B(x_0,\delta)).$$

Then the following proposition holds:

Proposition 3.1. *Let (u,Ω^+) be a minimizer in $B(x_0,\delta)$. If u is bounded, then u is $C^{\frac12}$ Hölder continuous in $B(x_0,\delta/2)$,*

$$\|u\|_{C^{\frac12}(B(x_0,\delta/2))}\le C(n,\Lambda,\|u\|_{L^\infty(B(x_0,\delta))}),$$

and for each $x\in\Gamma_{x_0}\cap B(x_0,\delta/2)$, if $r\le\delta/8$,

$$|B_r(x)\cap\Omega_0^+|\ge c_0(n,\Lambda)r^n.$$

Moreover, $u^\pm$ are harmonic in their positivity sets.

The following proposition allows us to use the isoperimetric inequality on Riemannian manifolds.

Proposition 3.2 (Croke's inequality, [8]). *Let $\tilde{M}$ be an arbitrary Riemannian manifold. Given any $o \in \tilde{M}$, $\rho > 0$, such that exp_o is defined on $B(o,\rho)$, then for*

$$r < \frac{1}{2}\min\Big\{\inf_{x\in B(o,\rho)} \mathrm{inj}_x, \rho\Big\}$$

we have

$$\text{Area of } \partial\Omega \geq C(n)\,\mathrm{Vol}(\Omega)^{\frac{n}{n-1}}$$

for all $\Omega \subset B(o,r)$, which implies that

$$V(B(o,r)) \geq C(n)r^n \quad \forall r < \frac{1}{2}\min\Big\{\inf_{x\in B(o,\rho)} \mathrm{inj}_x, \rho\Big\}.$$

Proof. The proof of Proposition 3.1 is similar to that of Theorem 3.1 in [2]. In the proof of Theorem 3.1 in [2], the authors mainly use the monotonicity formula of Alt–Caffarelli–Friedman, and Lemmas 2.5 and 2.6 in [2]. In the context of Riemannian geometry, Theorem A plays the role of the monotonicity formula of Alt–Caffarelli–Friedman, and Lemmas 2.4 and 2.5 are the analogues of Lemmas 2.5 and 2.6 in [2], respectively. Besides these main tools, Proposition 3.2 guarantees that the isoperimetric inequality can be used as in $\mathbb{R}^n$. Also, in the neighborhood of x_0 we can consider Δ_g as a uniformly elliptic operator in divergence form. Therefore, Harnack and Poincaré inequalities ([9]) both hold. Finally it is also possible to convert a minimizer of E in B_r to one in B_1 by rescaling: If (u,Ω) is a minimizer of E in $B_r(p)$ for $r < \delta$ and p on the free boundary, take local coordinates at p and let

$$u_r(y) = \frac{1}{\sqrt{r}}u(ry), \quad \Omega_r^+ = \{y \mid ry = x,\, x \in \Omega^+\}, \quad \tilde{g}_{ij}(y) = g_{ij}(ry).$$

Then (u_r, Ω_r^+) is a minimizer of $\int_{B_1} |\nabla_{\tilde{g}} v|^2 dV_{\tilde{g}} + \mathrm{Per}_{\tilde{g}}(\Omega_r^+, B_1)$. With these properties, Proposition 3.1 can be proved in a similar manner as Theorem 3.1 in [2], and we will omit the details. □

4. Lipschitz continuity

The main idea of the proof of Lipschitz continuity is the same as in [2], therefore we mainly address the differences.

Let $p_0 \in \Gamma$ and let $B(p_0, \delta_0)$ be a small neighborhood of p_0. For $\epsilon > 0$ small, we consider the function

$$w = (u - \epsilon)^+$$

and the maximum of the function $\frac{w\phi}{d(x)}$ in $B(p_0, \delta_0/2)$. Here $\phi \geq 0$ is a smooth function which is equal to 1 in $B(p_0, \delta_0/4)$, and equal to 0 outside $B(p_0, \delta_0/2)$.

Such a function can be chosen so that it also satisfies

$$\frac{|\nabla\phi(x)|^2}{\phi(x)} \le C \quad \text{where} \quad \phi(x) \neq 0. \tag{4.1}$$

Note that the constant C in (4.1) does not change as x tends to the boundary of the support of ϕ because ϕ is a variant of the following function:

$$\gamma(x) = \begin{cases} 0 & |x| \ge 1 \\ Ce^{1/(|x|^2-1)} & |x| < 1. \end{cases}$$

Let $\mathcal{M}$ be the maximum of $\frac{w\phi(x)}{d(x)}$ on $B(p_0, \delta_0/2)$ and suppose that $\mathcal{M}$ is attained at p_1, that is

$$\mathcal{M}d(p_1) = w(p_1)\phi(p_1) \quad \text{and} \quad \mathcal{M}d(x) \ge w(x)\phi(x), \quad x \in B(p_0, \delta_0).$$

We use d_0 to denote the distance from p_1 to the free boundary and the corresponding point on Γ will be labeled q_1. Take local coordinates at p_1 (denote p_1 as 0) and assume that $q_1 = \exp_{p_1}(-d_0 e_1)$. In the neighborhood of the origin we have

$$\begin{aligned} \mathcal{M}d(x) &\ge \mathcal{M}d_0 + w(x)\phi(x) - w(0)\phi(0) \\ &= \mathcal{M}d_0 + \sum_{i=1}^{n} \partial_i(w\phi)(0)x^i + Q(x) + O(|x|^3) \end{aligned}$$

or

$$d(x) \ge d_0 + \sum_{i=1}^{n} \frac{\partial_i(w\phi)(0)x^i}{\mathcal{M}} + \frac{Q(x)}{\mathcal{M}} + O\Big(\frac{|x|^3}{\mathcal{M}}\Big), \tag{4.2}$$

where Q satisfies $\Delta Q = \Delta(w\phi)(0)$. Since

$$\frac{w(x)\phi(x)}{\mathcal{M}} \le d(x) \le |x - y_1|, \quad y_1 = -d_0 e_1,$$

and these three functions "punch" at the origin, we have

$$\partial_i(w\phi)(0) = 0, \quad i = 2, \ldots, n, \quad \text{and} \quad \partial_{11}Q \le 0.$$

At 0, $\mathcal{M}$ is attained, therefore

$$\mathcal{M}\partial_i d(0) = \phi(0)\partial_i w(0) + w(0)\partial_i\phi(0), \quad i = 1, \ldots, n. \tag{4.3}$$

For Q we have

$$\sum_{i=1}^{n} \partial_{ii}Q(0) = \phi(0)\sum_i \partial_{ii}w(0) + 2\sum_i \partial_i w(0)\partial_i\phi(0) + w(0)\sum_i \partial_{ii}\phi(0).$$

From $\Delta_g w = 0$, using the expression of Δ_g we have $\Delta_g w(0) = \sum_i \partial_{ii}w(0)$. Therefore

$$\sum_{i=1}^{n} \partial_{ii}Q(0) = 2\sum_i \partial_i w(0)\partial_i\phi(0) + w(0)\sum_i \partial_{ii}\phi(0).$$

Thus (4.3) yields

$$\sum_i \partial_i w(0)\partial_i\phi(0) = \frac{\mathcal{M}\sum_i \partial_i d(0)\partial_i\phi(0)}{\phi(0)} - \frac{w(0)\sum_i(\partial_i\phi(0))^2}{\phi(0)}.$$

For w and ϕ we have

$$w(0) = \frac{\mathcal{M}d_0}{\phi(0)}, \quad |D^i\phi(0)| \le C\,(i=1,2), \quad \frac{|\nabla\phi(0)|^2}{\phi(0)} \le C.$$

Using the above information about $\partial_{ii}Q$, we have

$$\Delta Q(0) \ge -\frac{C\mathcal{M}}{\phi(0)}.$$

If we put $\bar{Q}(x') = Q(0,x')$, then near the origin

$$\Delta_{x'}\bar{Q}(x') \ge -\frac{CM}{\phi(0)}$$

because $\partial_{11}Q \le 0$. Since $\nabla(w\phi)(0)$ is parallel to e_1, from (4.2) we have

$$d(x) \ge d_0 + \frac{\bar{Q}(x')}{\mathcal{M}} + O\Big(\frac{|x'|^3}{\mathcal{M}}\Big). \tag{4.4}$$

on $x_1 = 0$ and near the origin.

Now we give an estimate of $\partial_{\nu_g}u^+(-d_0e_1)$, where ν_g is the inner normal unit vector to $\partial B(0,d_0)$ at $-d_0e^1$. Note that $u>0$ on $B(0,d_0)$. First, by the Harnack inequality we have

$$u(x) > C(n,\Lambda)\mathcal{M}d_0/\phi(0), \quad x\in B(0,d_0/2).$$

Next we shall define a harmonic function $\mathcal{H}$ on the ring $B(0,d_0)\setminus B(0,d_0/2)$ such that

$$\mathcal{H}=0 \text{ on } \partial B(0,d_0) \quad\text{and}\quad \mathcal{H} = C\frac{\mathcal{M}d_0}{\phi(0)} < u \text{ on } \partial B(0,d_0/2).$$

For $\mathcal{H}$ we claim

$$\partial_{\nu_g}\mathcal{H}(-d_0e_1) > C\mathcal{M}/\phi(0). \tag{4.5}$$

This is a simple fact following from scaling and the Hopf Lemma. Let $\tilde{g}_{ij}(y) = g_{ij}(d_0y)$ be the rescaled metric and let f satisfy $\Delta_{\tilde{g}}f = 0$ in $B(0,1)\setminus B(0,\frac{1}{2})$, with $f=1$ on $\partial B(0,\frac{1}{2})$ and $f=0$ on ∂B_1. By the Hopf Lemma, $\partial_\nu f(-e_1) > \epsilon_0 > 0$. We then observe that

$$\mathcal{H}(x) = C\frac{\mathcal{M}d_0}{\phi(0)}\,f\Big(\frac{x}{d_0}\Big).$$

So (4.5) holds and then

$$\partial_{\nu_g}u(-d_0e_1) > \frac{C\mathcal{M}}{\phi(0)}.$$

Next we recall that q_1 is the closest point to p_1 on the free boundary. Now we consider local coordinates around q_1 so that $p_1 = \exp_{q_1}(d_0 e_1)$. Then, by (4.4), near the origin the free boundary Γ is below the surface

$$x_1 = \psi(x') = -\frac{\bar{Q}(x')}{\mathcal{M}} + C\Big(\frac{|x'|^3}{\mathcal{M}}\Big).$$

Near the origin, according to the estimate on the directional derivative at 0, we have

$$u^+(x) \geq \frac{C\mathcal{M}}{\phi(p_1)} x_1 + \circ(|x|). \tag{4.6}$$

As a consequence of (4.6),

$$J_r^+ = \int_{B(0,r)} \frac{|\nabla_g u^+|^2}{|x|^{n-2}}\, dV_g \geq \Big(\frac{C\mathcal{M}}{\phi(p_1)}\Big)^2 r^2.$$

Then the monotonicity formula in Theorem A yields

$$J_r^- = \int_{B(0,r)} \frac{|\nabla_g u^-|^2}{|x|^{n-2}}\, dV_g \leq \Big(\frac{\phi(p_1)}{C\mathcal{M}}\Big)^2 r^2.$$

Using Lemma 2.4 we have

$$\sup_{B(y,r)} (u^-) \leq C\frac{\phi(p_1)}{\mathcal{M}} r, \quad |y| < r/2. \tag{4.7}$$

The remaining part of the proof of the Lipschitz continuity of u is almost exactly like the perturbation argument in [2]. The main reason that the difference between the Riemannian manifold and $\mathbb{R}^n$ does not cause major difficulty from this point on is that the perturbation argument is performed in a very small neighborhood of q_1. The size of the neighborhood can be arbitrarily small (independent of $\mathcal{M}$), and therefore all the error terms that come from the difference between (M, g) and $\mathbb{R}^n$ are easily controlled. Two differences in the notations should be mentioned. First, when we consider the Laplacian of the distance function to an $(n-1)$-dimensional sub-manifold, we have that $\Delta_g d$ equals $(n-1)$ times the mean curvature of the manifold when the point tends to the submanifold (Lemma 10.4 and equation (10.5) of [12]). Another important fact is that the Divergence Theorem holds for sets of finite perimeter under the definition in [11]. We omit the proof of this part. The Lipschitz continuity of u is established.

5. Higher regularity of the free boundary and the jump condition

5.1. $C^{1,\frac{1}{2}}$ regularity of the free boundary

The results of this section are similar to what is stated in [2], i.e., the reduced part of Γ is $C^{1,\frac{1}{2}}$ and the set of singular points has zero s-dimensional Hausdorff measure for any $s > n-8$. The proof is a modification of the one in [2], and the main idea is to "reduce it to the Euclidean case". Let $x_0 \in \Gamma$ and restrict the discussion to $B(x_0,\delta)$ for $\delta > 0$ small. Let $A \in B(x_0,\delta)$ and select Ω_1 such that $\Omega\Delta\Omega_1 \subset\subset B_r(x)$, $x \in A$, r small. Let u_r be any perturbation of u inside $B(x,r)$ with the same Lipschitz constant L and such that (u_r,Ω^+) is admissible. Then $E(u,\Omega^+) \le E(u_r,\Omega_1^+)$ yields

$$\mathrm{Per}(\Omega^+, B(x,r)) \le \mathrm{Per}(\Omega_1^+, B(x,r)) + CL^2 r^n.$$

Next we observe that $\mathrm{Per}(\Omega^+, B(x,r)) = O(r^{n-1})$ by the minimality of $E(u,\Omega^+)$, as $\mathrm{Per}(\Omega_1^+, B(x,r))$ can be chosen to be $O(r^{n-1})$. If we consider the area of these two quantities in the Euclidean metric (which we will denote by $\mathrm{Per}_{\mathrm{Euclid}}$) we have

$$\mathrm{Per}_{\mathrm{Euclid}}(\Omega^+, B(x,r)) \le \mathrm{Per}_{\mathrm{Euclid}}(\Omega_1^+, B(x,r)) + Cr^n. \tag{5.1}$$

Indeed, using $g_{ij} = \delta_{ij} + O(r^2)$, (5.1) can be obtained easily. With (5.1) we can define $\alpha(r) = Cr^2$ and the standard Almgren–Tamanini theory for almost minimal boundaries can be applied (see Theorem 5.1 in [2]) to obtain the regularity result mentioned at the beginning of this section.

5.2. Free boundary condition and higher regularity of the free boundary

We now derive the jump condition on Γ^*. Let $p \in \Gamma^*$ and let $B(p,\delta)$ be a small neighborhood of p. Suppose Γ^* is represented by $x_1 = f(x')$ in local coordinates at p such that $f(0) = 0$ and $f'(0) = 0$. Let S_Q be a quadratic surface touching Γ^* from the Ω^+ side. Using the same perturbation technique as in [2] we have

$$|\nabla_g u^+(0)|^2 - |\nabla_g u^-(0)|^2 \le (n-1)\kappa(S_Q)(0)\,, \tag{5.2}$$

where $\kappa(S_Q)$ is the mean curvature of S_Q in the metric g. According to Definition 6.1 in [2], (5.2) means that Γ^* is a weak sub-solution of

$$|\nabla_g u^+|^2 - |\nabla_g u^-|^2 = (n-1)\kappa(\Gamma^*). \tag{5.3}$$

Since the proof of Γ^* being a super-solution of (5.3) is similar, we conclude that (5.3) holds in the weak sense. Then the standard regularity estimate for viscosity solutions of elliptic equations yields $\Gamma^* \in C^{2,\frac{1}{2}}$ (see [2], [6]). Consequently, further regularity estimates can be obtained on u^+ and u^- over Ω^+ and Ω^- respectively. Then we can use a bootstrapping argument to obtain that Γ^* is smooth. Theorem 1.1 is established.

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