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On the vector-valued Littlewood–Paley–Rubio de Francia inequality

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Abstract. The paper studies Banach spaces satisfying the Littlewood–Paley–Rubio de Francia property LPR_p, $2 \le p < \infty$. The paper shows that every Banach lattice whose 2-concavification is a UMD Banach lattice has this property. The paper also shows that every space having LPR_q also has LPR_p with $q \le p < \infty$.

1. Introduction

Let X be a Banach space and let $L^p(\mathbb{R}; X)$ be the Bochner space of p-integrable X-valued functions on $\mathbb{R}$. If $X = \mathbb{C}$, we abbreviate $L^p(\mathbb{R}; X) = L^p(\mathbb{R})$, $1 \le p < \infty$. For every $f \in L^1(\mathbb{R}; X)$, $\hat{f}$ stands for the Fourier transform. If $I \subseteq \mathbb{R}$ is an interval, then S_I is the Riesz projection adjusted to the interval I, i.e.,

$$S_I f(t) = \int_I \hat{f}(s)\, e^{2\pi i s t}\, ds.$$

The following remarkable inequality was proved by J. L. Rubio de Francia in [9]. For every $2 \le p < \infty$, there is a constant c_p such that for every collection of pairwise disjoint intervals $(I_j)_{j=1}^{\infty}$, the following estimate holds:

$$\Big\| \Big(\sum_{j=1}^{\infty} \big| S_{I_j} f \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R})} \le c_p \, \|f\|_{L^p(\mathbb{R})} , \quad \forall\, f \in L^p(\mathbb{R}). \tag{1.1}$$

In this note, we shall discuss a version of the theorem above when functions take values in a Banach space X. Let $(\varepsilon_k)_{k\ge 1}$ be the system of Rademacher functions on $[0, 1]$. The space $\mathrm{Rad}(X)$ is the closure in $L^p([0,1]; X)$, $1 \le p < \infty$, of all X-valued functions of the form

$$g(\omega) = \sum_{k=1}^{n} \varepsilon_k(\omega)\, x_k, \quad x_k \in X, \quad n \ge 1.$$

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The above definition is independent of $1 \leq p < \infty$. It follows from the Khintchine–Kahane inequality (see [6]). In fact, the above fact is a consequence of the so-called *contraction principle.* It states that, for every sequence of elements $\{x_j\}_{j=1}^{\infty} \subseteq X$ and sequence of complex numbers $\{\alpha_j\}_{j=1}^{\infty}$ such that $|\alpha_j| \leq 1$ for $j \geq 1$, the following inequality holds:

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \epsilon_j \, x_j \right\|_{L^p(\mathbb{R},\mathrm{Rad}(X))} \leq c_p \left\| \sum_{j=1}^{\infty} \epsilon_j \, x_j \right\|_{L^p(\mathbb{R},\mathrm{Rad}(X))}.$$

We shall employ this principle on numerous occasions in this paper.

Following [1], we shall call X *a space with the* LPR_p *property* with $2 \leq p < \infty$, if there exists a constant $c > 0$ such that for any collection of pairwise disjoint intervals $\{I_j\}_{j=1}^{\infty}$ we have that

$$\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} \leq c \, \|f\|_{L^p(\mathbb{R};X)} , \quad \forall \, f \in L^p(\mathbb{R}; X). \tag{1.2}$$

It was proved in [5] that every space with the LPR_p property is necessarily UMD and of type 2. It is an open problem whether the converse is true. It is also unknown whether LPR_p is independent of p. Note that Rubio de Francia's inequality says that $\mathbb{C}$ has the LPR_p property for every $2 \leq p < \infty$. By the Khintchine inequality and the Fubini theorem we see that any L^p-space with $2 \leq p < \infty$ has the LPR_p property. Using interpolation, we deduce that a Lorentz space $L^{p,r}$ has the LPR_q property for some indices p, r and q. However, until recently there were no non-trivial examples of spaces with LPR_p found.

If X is a Banach lattice, the estimate (1.2) admits a pleasant form, as in the scalar case:

$$\left\| \Big(\sum_{j=1}^{\infty} |S_{I_j} f|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R};X)} \leq c \, \|f\|_{L^p(\mathbb{R};X)} , \quad \forall \, f \in L^p(\mathbb{R}; X). \tag{1.3}$$

We shall show that if the 2-concavification $X_{(2)}$ of X is a UMD Banach lattice, then (1.3) holds for all $2 < p < \infty$, so X is a space with the LPR_p property. Recall that $X_{(2)}$ is the lattice defined by the following quasi-norm

$$\|f\|_{X_{(2)}} = \| \, |f|^{\frac{1}{2}} \, \|_X^2.$$

The space $X_{(2)}$ is a Banach lattice if and only if X is 2-convex, i.e.,

$$\left\| \Big(\sum_{j=1}^{n} |f_j|^2 \Big)^{\frac{1}{2}} \right\|_X \leq \Big(\sum_{j=1}^{n} \|f_j\|_X^2 \Big)^{\frac{1}{2}}.$$

We refer to [6] for more information on Banach lattices.

We shall also show that if X is a Banach space (not necessarily a lattice) with the LPR_q property for some q, then X has the LPR_p property for every $p \geq q$.

2. Dyadic decomposition

For every interval $I \subseteq \mathbb{R}$, let $2I$ be the interval of double length and the same centre as I. Let $\mathcal{I} = (I_j)_{j=1}^{\infty}$ be a collection of pairwise disjoint intervals. We set $2\mathcal{I} = (2I_j)_{j=1}^{\infty}$. The collection $\mathcal{I}$ is called *well-distributed* if there is a number d such that each element of $2\mathcal{I}$ intersects at most d other elements of $2\mathcal{I}$.

In this section, we fix a pairwise disjoint collection of intervals $(I_j)_{j=1}^{\infty}$ and we break each interval I_j, $j \geq 1$, into a number of smaller dyadic subintervals such that the new collection is well-distributed. This construction was employed in a number of earlier papers.

We start with two elementary remarks on estimate (1.2) or (1.3). Firstly, it suffices to consider a finite sequence $(I_j)_j$ of disjoint finite intervals. Secondly, by dilation, we may assume $|I_j| \geq 4$ for all j. Thus all sums on j and k in what follows are finite. Fix $j \geq 1$. Let $I_j = (a_j, b_j]$. Let $n_j = \max\{n \in \mathbb{N} : \ 2^{n+1} \leq b_j - a_j + 4\}$. We first split I_j into two subintervals with equal lengths:

$$I_j^a = \left(a_j,\ \frac{a_j + b_j}{2}\right] \quad \text{and} \quad I_j^b = \left(\frac{a_j + b_j}{2},\ b_j\right].$$

Then we decompose I_j^a and I_j^b into relative dyadic subintervals as follows:

$$I_j^a = \bigcup_{k=1}^{n_j} (a_{j,k},\ a_{j,k+1}] \quad \text{and} \quad I_j^b = \bigcup_{k=1}^{n_j} (b_{j,k+1},\ b_{j,k}],$$

where

$$\begin{aligned} a_{j,k} &= a_j - 2 + 2^k, \quad 1 \leq k \leq n_j, \quad \text{and} \quad a_{j,n_j+1} = \frac{a_j + b_j}{2}; \\ b_{j,k} &= b_j + 2 - 2^k, \quad 1 \leq k \leq n_j, \quad \text{and} \quad b_{j,n_j+1} = \frac{a_j + b_j}{2}. \end{aligned}$$

Let

$$I_{j,k}^a = (a_{j,k},\ a_{j,k+1}], \quad I_{j,k}^b = (b_{j,k+1},\ b_{j,k}]$$

for $1 \leq k \leq n_j$ and let $I_{j,k}^a$, $I_{j,k}^b$ be the empty set for the other k's. Also put

$$\tilde{I}_{j,n_j}^a = (a_j - 2 + 2^{n_j},\ a_j - 2 + 2^{n_j+1}] \quad \text{and} \quad \tilde{I}_{j,n_j}^b = (b_j + 2 - 2^{n_j+1},\ b_j + 2 - 2^{n_j}].$$

Lemma 2.1. *A Banach space X has the* LPR_p *property if there is a constant $c > 0$ such that*

$$\max_{u=a,b} \left\| \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{n_j} \varepsilon'_k S_{I_{j,k}^u} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))} \leq c \, \|f\|_{L^p(\mathbb{R};X)}, \quad \forall\, f \in L^p(\mathbb{R};X), \tag{2.1}$$

where $\mathrm{Rad}_2(X) = \mathrm{Rad}(\mathrm{Rad}'(X))$ *and* $\mathrm{Rad}'(X)$ *is the space with respect to another copy of the Rademacher system* $(\varepsilon'_k)_{k\geq 1}$.

Observe that if (2.1) holds for every family of intervals $(I_j)_{j=1}^{\infty}$, then X is a UMD space. Indeed, (2.1) implies that

$$\left\|S_{I^u_{j,k}} f\right\|_{L^p(\mathbb{R},X)} \le c\,\|f\|_{L^p(\mathbb{R},X)}, \quad u = a, b,\ j \ge 1,\ 1 \le k \le n_j.$$

That is, by adjusting the choice of intervals, it implies that every projection S_I is bounded on $L^p(\mathbb{R}, X)$ and

$$\sup_{I \subseteq \mathbb{R}} \|S_I\|_{L^p(\mathbb{R},X) \mapsto L^p(\mathbb{R},X)} < +\infty.$$

The latter is equivalent to the fact that X is UMD (see [3]).

Proof. Let $f \in L^p(\mathbb{R}; X)$. Then

$$\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} \le \left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j^a} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} + \left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j^b} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))}.$$

Using the subintervals $I^a_{j,k}$ and the contraction principle, we write

$$\begin{aligned}\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j^a} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} &= \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} \\ &\sim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \exp(-2\pi \mathrm{i} a_j\,\cdot) S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))}.\end{aligned}$$

Note that

$$\exp(-2\pi \mathrm{i} a_j\,\cdot) S_{I^a_{j,k}} f = S_{I^a_{j,k} - a_j} [\exp(-2\pi \mathrm{i} a_j\,\cdot) f]$$

and

$$I^a_{j,k} - a_j = (2^k - 2,\ 2^{k+1} - 2],\ k < n_j; \quad I^a_{j,n_j} - a_j \subseteq (2^{n_j} - 2,\ 2^{n_j+1} - 2].$$

Recall that X is a UMD space. Therefore, applying Bourgain's Fourier multiplier theorem (see [3]) to the function

$$\sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \exp(-2\pi \mathrm{i} a_j\,\cdot) S_{I^a_{j,k}} f \in L^p(\mathbb{R}; \mathrm{Rad}(X))),$$

we obtain (the contraction principle being used in the last step)

$$\begin{aligned}&\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \exp(-2\pi \mathrm{i} a_j\,\cdot) S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} \\ &\qquad \sim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k \exp(-2\pi \mathrm{i} a_j\,\cdot) S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))} \\ &\qquad \sim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))}.\end{aligned}$$

Similarly,

$$\Big\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j^b} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}X)} \sim \Big\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k' S_{I_{j,k}^b} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))}.$$

Combining the preceding estimates, we get

$$\begin{aligned}\Big\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}X)} \\ \leq c_p \Big[\Big\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k' S_{I_{j,k}^a} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))} + \Big\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k' S_{I_{j,k}^b} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))} \Big].\end{aligned}$$

□

Let us observe that, if X is a UMD space, the argument in the proof above shows that

$$\Big\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}X)} \lesssim \max_{u=a,b} \Big\| \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{n_j} \varepsilon_k' S_{I_{j,k}^u} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))}.$$

Moreover, the argument can be reversed to show the opposite estimate (see the proof of (4.1) below). This observation is summarised in the following remark.

Remark 2.2. i) If X is a UMD space, then

$$\Big\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}X)} \lesssim \max_{u=a,b} \Big\| \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{n_j} \varepsilon_k' S_{I_{j,k}^u} f \Big\|_{L^p(\mathbb{R};\mathrm{Rad}_2(X))}.$$

ii) If $\mathcal{I} = (I_j)_{j\geq 1}$ is a collection of pairwise disjoint intervals and, for $u = a, b$, $\mathcal{I}_u = \big(I_{j,k}^u\big)_{j\geq 1, 1\leq k\leq n_j}$, then both collections $\mathcal{I}_a$ and $\mathcal{I}_b$ are well-distributed.

iii) If X is a Banach lattice then it has the α-property (see [7]). That is,

$$\Big\| \sum_{j,k=1}^{\infty} \varepsilon_j \varepsilon_k' x_{jk} \Big\|_{\mathrm{Rad}_2(X)} \sim \Big\| \sum_{j,k=1}^{\infty} \varepsilon_{jk} x_{jk} \Big\|_{\mathrm{Rad}(X)},$$

where (ε_{jk}) is an independent family of Rademacher functions.

iv) The above two observations imply that if X is a Banach lattice, then it has the LPR_p property if and only if estimate (1.2) holds for every well-distributed collection of intervals $\mathcal{I}$.

3. LPR-estimate for Banach lattices

Theorem 3.1. *If X is a Banach lattice such that $X_{(2)}$ is a* UMD *Banach space, then X has the* LPR_p *property for every $2 < p < \infty$.*

We shall need the following remark for the proof.

Remark 3.2. If X is UMD and $1 < p < \infty$, then the family $\{S_I\}_{I \subseteq \mathcal{I}}$ is R-bounded (see [4]), i.e.,

$$\Big\| \sum_{I \subseteq \mathcal{I}} \epsilon_I S_I f_I \Big\|_{L^p(\mathbb{R};\mathrm{Rad}(X))} \leq c_X \Big\| \sum_{I \subseteq \mathcal{I}} \epsilon_I f_I \Big\|_{L^p(\mathbb{R};\mathrm{Rad}(X))}.$$

Proof of Theorem 3.1. The proof directly employs the pointwise estimate of [9]. We assume that X is a Köthe function space on a measure space (Ω, μ).

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ and let $M(f)$ be the Hardy–Littlewood maximal function of f, i.e.,

$$M(f)(t) = \sup_{\substack{I \subseteq \mathbb{R} \\ t \in I}} \frac{1}{|I|} \int_I |f(s)| \, ds$$

and

$$M_2(f) = \left[M \, |f|^2 \right]^{\frac{1}{2}}.$$

Let

$$f^\sharp(t) = \sup_{\substack{I \subseteq \mathbb{R} \\ t \in I}} \frac{1}{|I|} \int_I |f(s) - f_I| \, ds, \quad f_I = \frac{1}{|I|} \int_I f(s) \, ds.$$

Note that $M(f)$ is a function of two variables (t, ω): for each fixed ω, $M(f)(\cdot, \omega)$ is the usual Hardy–Littlewood maximal function of $f(\cdot, \omega)$. The same remark applies to $M_2(f)$ and $f^\sharp$. For f sufficiently nice (which will be assumed in the sequel), all these functions are well-defined.

Observe that due to Remark 2.2 we have only to show estimate (1.2) for a well-distributed family of intervals. Let us fix a family of pairwise disjoint intervals $\mathcal{I}$ and let us assume that $\mathcal{I}$ is well-distributed. Fix a Schwartz function $\psi(t)$ whose Fourier transform satisfies

$$\chi_{[-1/2,1/2]} \leq \hat{\psi} \leq \chi_{[-1,1]}.$$

If $I \in \mathcal{I}$, then we set

$$\psi_I(t) = |I| \exp(2\pi \mathrm{i} c_I t) \, \psi(|I| \, t),$$

where c_I is the centre of I. The Fourier transform of ψ_I is adapted to I, i.e.,

$$\chi_I \leq \hat{\psi}_I \leq \chi_{2I}.$$

In particular,

$$S_I(f) = \psi_I * S_I(f).$$

Consequently, from the Khintchine inequality and Remark 3.2,

$$\Big\| \Big(\sum_{I \in \mathcal{I}} |S_I(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R},X)} \leq c_p \, \|G(f)\|_{L^p(\mathbb{R},X)} \,, \quad 1 < p < \infty,$$

where

$$G(f) = \Big(\sum_{I \in \mathcal{I}} |\psi_I * f|^2 \Big)^{\frac{1}{2}}, \quad f \in L^1(\mathbb{R}; X).$$

Thus, to finish the proof, we need to show that

$$\|G(f)\|_{L^p(\mathbb{R},X)} \leq c_p \, \|f\|_{L^p(\mathbb{R},X)} \,, \quad 2 < p < \infty.$$

It was shown in [9] that $G(f(\cdot,\omega))^\sharp$ is almost everywhere dominated by $M_2(f(\cdot,\omega))$, i.e.,

$$G(f(\cdot,\omega))^\sharp \leq c\, M_2(f(\cdot,\omega)), \quad \text{a.e. } \omega \in \Omega,$$

for some universal $c > 0$. Since

$$G(f)(t,\omega) = G(f(\cdot,\omega))(t) \quad \text{and} \quad M_2(f)(t,\omega) = M_2(f(\cdot,\omega))(t), \quad t \in \mathbb{R},\ \omega \in \Omega,$$

we clearly have that

$$G(f)^\sharp \leq c\, M_2(f).$$

Therefore,

$$\big\|G(f)^\sharp\big\|_{L^p(\mathbb{R};X)} \leq c \big\|M_2(f)\big\|_{L^p(\mathbb{R};X)} \,.$$

It remains to prove

$$\big\|G(f)\big\|_{L^p(\mathbb{R};X)} \leq C \big\|G(f)^\sharp\big\|_{L^p(\mathbb{R};X)} \quad \text{and} \quad \big\|M_2(f)\big\|_{L^p(\mathbb{R};X)} \leq C \big\|f\big\|_{L^p(\mathbb{R};X)} \,.$$

The second inequality above immediately follows from Bourgain's maximal inequality for UMD lattices (applied to $X_{(2)}$ here, see Theorem 3 in [10]):

$$\big\|M_2(f)\big\|^2_{L^p(\mathbb{R};X)} = \big\|M(|f|^2)\big\|_{L^{\frac{p}{2}}(\mathbb{R};X_{(2)})} \leq C \big\||f|^2\big\|_{L^{\frac{p}{2}}(\mathbb{R};X_{(2)})} = C \big\|f\big\|^2_{L^p(\mathbb{R};X)} \,.$$

It remains to show the first one. To this end we shall prove the following inequality (for a general f instead of $G(f)$):

$$\big\|f\big\|_{L^p(\mathbb{R};X)} \leq C \big\|f^\sharp\big\|_{L^p(\mathbb{R};X)} \,.$$

This is again an immediate consequence of the following classical duality inequality (see page 146 of [12]):

$$\Big| \int_{\mathbb{R}} uv \Big| \leq C \int_{\mathbb{R}} u^\sharp \mathcal{M}(v)$$

for any $u \in L^p(\mathbb{R})$ and $v \in L^{p'}(\mathbb{R})$, where $\mathcal{M}(v)$ denotes the grand maximal function of v. Note that $\mathcal{M}(v) \leq CM(v)$. Now let $g \in L^{p'}(\mathbb{R}; X^*)$ be a nice function. We then have

$$\begin{aligned} \Big| \int_{\mathbb{R}\times\Omega} fg \Big| &\leq C \int_{\mathbb{R}\times\Omega} f^\sharp M(g) \leq C \big\|f^\sharp\big\|_{L^p(\mathbb{R};X)} \big\|M(g)\big\|_{L^{p'}(\mathbb{R};X^*)} \\ &\leq C \big\|f^\sharp\big\|_{L^p(\mathbb{R};X)} \big\|g\big\|_{L^{p'}(\mathbb{R};X^*)} \,, \end{aligned}$$

where we have used again Bourgain's maximal inequality for g (noting that X^* is also a UMD lattice). Therefore, taking the supremum over all g in the unit ball of $L^{p'}(\mathbb{R}; X^*)$, we deduce the desired inequality, so prove the theorem.

Finally, observe that the proof above operates with individual functions. This, coupled with the UMD property of X, implies that X can always be assumed separable and it can always be equipped with a weak unit. □

4. LPR property for general Banach spaces

Let X be a Banach space (not necessarily a lattice). We shall prove the following theorem:

Theorem 4.1. *If X has the* LPR_q *property for some $2 \leq q < \infty$, then X has the* LPR_p *property for any $q \leq p < \infty$.*

The proof of the theorem requires some lemmas.

Lemma 4.2. *Assume that X has the* LPR_q *property. Let $(I_j)_{j\geq 1}$ be a finite sequence of mutually disjoint intervals of $\mathbb{R}$ and $(I_{j,k})_{k=1}^{n_j}$ be a finite family of mutually disjoint subintervals of I_j for each $j \geq 1$. Assume that the relative position of $I_{j,k}$ in I_j is independent of j, i.e., $I_{j,k} - a_j = I_{j',k} - a'_j$ whenever both $I_{j,k}$ and $I_{j',k}$ are present (i.e., $k \leq \min\{n_j, n_{j'}\}$), where a_j is the left endpoint of I_j. Then*

$$\Big\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k S_{I_{j,k}} f \Big\|_{L^q(\mathbb{R};\mathrm{Rad}_2(X))} \leq c \left\| f \right\|_{L^q(\mathbb{R};X)}, \quad \forall\, f \in L^q(\mathbb{R}; X).$$

Proof. We first assume that $\bigcup_{k=1}^{n_j} I_{j,k} = I_j$ for each $j \geq 1$. Note that

$$S_{I_{j,k}} f = \exp(2\pi \mathrm{i} a_j \cdot) S_{I_{j,k}-a_j}(\exp(-2\pi \mathrm{i} a_j \cdot) f).$$

Thus, by the contraction principle,

$$\Big\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k S_{I_{j,k}} f \Big\|_q \sim \Big\| \sum_{k=1}^{\infty} \varepsilon'_k \sum_{j:\ n_j \geq k} \varepsilon_j S_{I_{j,k}-a_j}(\exp(-2\pi \mathrm{i} a_j \cdot) f) \Big\|_q .$$

Since X has the LPR_q property, so does $\mathrm{Rad}(X)$. Let us apply this property of $\mathrm{Rad}(X)$ to the intervals $\big(\tilde{I}_k\big)_{k\geq 1}$ where $\tilde{I}_k = I_{j,k} - a_j$, for some j such that $n_j \geq k$ (for any such j the interval $I_{j,k} - a_j$ is independent of j by the assumptions of the lemma). We apply this property to the function

$$\sum_{k=1}^{\infty} \sum_{j:\ n_j \geq k} \varepsilon_j S_{I_{j,k}-a_j}(\exp(-2\pi \mathrm{i} a_j \cdot) f) = \sum_{k=1}^{\infty} S_{\tilde{I}_k} \Big[\sum_{j:\ n_j \geq k} \epsilon_j \left(\exp(-2\pi \mathrm{i} a_j \cdot) f\right) \Big].$$

We obtain

$$\Big\|\sum_{k=1}^{\infty}\varepsilon'_k\sum_{j\colon\, n_j\ge k}\varepsilon_j S_{I_{j,k}-a_j}(\exp(-2\pi i a_j\cdot)f)\Big\|_q \tag{4.1}$$
$$\le c\,\Big\|\sum_{k=1}^{\infty}\sum_{j\colon\, n_j\ge k}\varepsilon_j S_{I_{j,k}-a_j}(\exp(-2\pi i a_j\cdot)f)\Big\|_q$$
$$\sim c\,\Big\|\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}\varepsilon_j S_{I_{j,k}}f\Big\|_q = c\,\Big\|\sum_{j=1}^{\infty}\varepsilon_j S_{I_j}f\Big\|_q \le c\,\|f\|_q.$$

Assume now that $\bigcup_{k=1}^{n_j} I_{j,k}\ne I_j$ for some j. In this case, consider the family of intervals $\big(\tilde I_k\big)_{k=1}^{\infty}$ introduced above. Observe that every $\tilde I_k\subseteq[0,+\infty)$. Observe also that the the right ends of the intervals $(I_j-a_j)_{j\ge1}$, that is the points b_j-a_j, do not belong to the union $\cup_{k=1}^{\infty}\tilde I_k$. Let $\big(\tilde I_\ell\big)_{\ell=1}^{\infty}$ be the family of disjoint intervals such that

$$\bigcup_{\ell=1}^{\infty}\tilde I_\ell=[0,+\infty)\setminus\bigcup_{k=1}^{\infty}\tilde I_k$$

and such that neither of the points $(b_j-a_j)_{j=1}^{\infty}$ is inner for some $\tilde I_\ell$. Let also m_j be the maximum number such that the intervals $\tilde I_\ell$ with $\ell\le m_j$ are all to the left of the point b_j-a_j. Set $I_{j,\ell}=\tilde I_\ell+a_j$. Then,

$$I_j=\bigcup_{k=1}^{n_j}I_{j,k}+\bigcup_{\ell=1}^{m_j}I_{j,\ell}.$$

It is clear that the relative position of $(I_{j,k})_{k=1}^{n_j}\cup(I_{j,\ell})_{\ell=1}^{m_j}$ in I_j is again independent of j.

Before we proceed, let us re-index the intervals $(I_{j,k})_{k=1}^{n_j}$ and $(I_{j,\ell})_{\ell=1}^{m_j}$ into a family $(I_{j,s})_{s=1}^{m_j+n_j}$ as follows. We arrange these intervals from left to right within I_j and index them sequentially from 1 up to n_j+m_j. Moreover, let $K_j\subseteq[1,n_j+m_j]$ be the subset corresponding to the first family of intervals and $L_j\subseteq[1,n_j+m_j]$ be the subset of indices corresponding to the second family of intervals. Observe that, if $K=\cup_{j=1}^{\infty}K_j$ and $L=\cup_{j=1}^{\infty}L_j$, then, for every j, $K_j=K\cap[1,n_j+m_j]$ and, similarly, $L_j=L\cap[1,n_j+m_j]$. Thus by the previous part we get

$$\Big\|\sum_{j=1}^{\infty}\sum_{s=1}^{n_j+m_j}\epsilon_j\epsilon'_s S_{I_{j,s}}f\Big\|_q\le c_q\,\|f\|_q.$$

Observe also that

$$\sum_{j=1}^{\infty}\sum_{s=1}^{n_j+m_j}\epsilon_j\epsilon'_s S_{I_{j,s}}f=\sum_{s=1}^{\infty}\sum_{j\colon\, n_j+m_j\ge s}\epsilon_j\epsilon'_s S_{I_{j,s}}f$$
$$=\sum_{s\in K}\sum_{j\colon\, n_j+m_j\ge s}\epsilon_j\epsilon'_s S_{I_{j,s}}f+\sum_{s\in L}\sum_{j\colon\, n_j+m_j\ge s}\epsilon_j\epsilon'_s S_{I_{j,s}}f\,.$$

Thus, by taking the projection onto the subspace spanned by $\{\epsilon'_s\}_{s\in K}$, we obtain

$$\Big\| \sum_{s\in K} \sum_{j:\ n_j+m_j\geq s} \epsilon_j \epsilon'_s S_{I_{j,s}} f \Big\|_q \leq c_q \left\| f \right\|_q.$$

Finally, we observe that

$$\sum_{s\in K} \sum_{j:\ n_j+m_j\geq s} \epsilon_j \epsilon'_s S_{I_{j,s}} f = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \epsilon_j \epsilon'_k S_{I_{j,k}} f.$$

Hence the lemma is proved. □

Next lemma is interesting in its own right. We shall only need its first part.

Lemma 4.3. *Let Y be a Banach space. Let (Σ, ν) be a measure space and $(h_j) \subset L^2(\Sigma)$ a finite sequence.*

i) *If Y is of cotype 2 and there exists a constant c such that*

$$\Big\| \sum_j \alpha_j h_j \Big\|_2 \leq c \Big(\sum_j |\alpha_j|^2 \Big)^{1/2}, \quad \forall\, \alpha_j \in \mathbb{C},$$

then

$$\Big\| \sum_j h_j a_j \Big\|_{L^2(\Sigma;Y)} \leq c' \Big\| \sum \varepsilon_j a_j \Big\|_{\mathrm{Rad}(Y)}, \quad \forall\, a_j \in Y.$$

ii) *If Y is of type 2 and there exists a constant c such that*

$$\Big(\sum_j |\alpha_j|^2 \Big)^{1/2} \leq c \Big\| \sum_j \alpha_j h_j \Big\|_2, \quad \forall\, \alpha_j \in \mathbb{C},$$

then

$$\Big\| \sum \varepsilon_j a_j \Big\|_{\mathrm{Rad}(Y)} \leq c' \Big\| \sum_j h_j a_j \Big\|_{L^2(\Sigma;Y)}, \quad \forall\, a_j \in Y.$$

Proof. i) Let $(a_j) \subset Y$ be a finite sequence. Consider the operator $u : \ell^2 \to Y$ defined by

$$u(\alpha) = \sum_j \alpha_j a_j, \quad \forall\, \alpha = (\alpha_j) \in \ell^2.$$

It is well known (see Lemma 3.8 and Theorem 3.9 in [8]) that

$$\pi_2(u) \leq c_0 \Big\| \sum \varepsilon_j a_j \Big\|_{\mathrm{Rad}(Y)},$$

where c_0 is a constant depending only on the cotype 2 constant of Y. Let $h(\sigma) = (h_j(\sigma))_j$ for $\sigma \in \Sigma$. Then by the assumption on (h_j) we get

$$\begin{aligned}\Big\| \sum_j h_j a_j \Big\|_{L^2(\Sigma;Y)} &= \pi_2(u) \sup \Big\{ \Big(\int_\Sigma \Big| \sum_j \xi_j h_j(s) \Big|^2 ds \Big)^{1/2} \ :\ \xi \in \ell^2, \|\xi\|_2 \leq 1 \Big\} \\ &\leq c' \Big\| \sum \varepsilon_j a_j \Big\|_{\mathrm{Rad}(Y)}.\end{aligned}$$

ii) Let H be the linear span of (h_j) in $L^2(\Sigma)$. Let h_j^* be the functional on H such that $h_j^*(h_k) = \delta_{j,k}$. We extend h_j^* to all of $L^2(\Sigma)$ by setting $h_j^* = 0$ on $H^\perp$. Then $h_j^* \in L^2(\Sigma)$ and the assumption implies that

$$\Big\| \sum_j \beta_j h_j^* \Big\|_2 \le c \Big(\sum_j |\beta_j|^2 \Big)^{1/2}, \quad \forall\, \beta_j \in \mathbb{C}.$$

Now let $(a_j^*) \subset Y^*$ be a finite sequence. Applying i) to Y^* and (h_j^*) we obtain

$$\begin{aligned}\Big| \sum_j \langle a_j^*,\, a_j\rangle \Big| &= \Big| \Big\langle \sum_j h_j^* a_j^*,\, \sum_j h_j a_j \Big\rangle \Big| \le \Big\| \sum_j h_j^* a_j^* \Big\|_{L^2(\Sigma;Y^*)} \Big\| \sum_j h_j a_j \Big\|_{L^2(\Sigma;Y)} \\ &\le c' \Big\| \sum_j \varepsilon_j a_j^* \Big\|_{\mathrm{Rad}(Y^*)} \Big\| \sum_j h_j a_j \Big\|_{L^2(\Sigma;Y)} .\end{aligned}$$

Taking the supremum over $(a_j^*) \subset Y^*$ such that $\big\| \sum \varepsilon_j a_j^* \big\|_{\mathrm{Rad}(Y^*)} \le 1$, we get the assertion. □

Now we proceed to the proof of Theorem 4.1. It is divided into several steps.

The singular integral operator *T*. Let $(I_j)_j$ be a family of disjoint finite intervals and ψ be a Schwartz function as in Sections 2 and 3. We keep the notation introduced there. We now set up an appropriate singular integral operator corresponding to (2.1). It suffices to consider the family $(I_{j,k}^a)_{j,k}$, $(I_{j,k}^b)_{j,k}$ being treated similarly. Henceforth, we shall denote $I_{j,k}^a$ simply by $I_{j,k}$. Let $c_{j,k} = a_{j,k} + 2^{k-1}$ for $1 \le k \le n_j$. Note that $c_{j,k}$ is the centre of $I_{j,k}$ if $k < n_j$ and of $\tilde I_{j,k}$ if $k = n_j$. Define

$$\psi_{j,k}(x) = 2^k \exp(2\pi \mathrm{i} c_{j,k}\, x)\, \psi(2^k x)$$

so that the Fourier transform of $\psi_{j,k}$ is adapted to $I_{j,k}$, i.e.,

$$\chi_{I_{j,k}} \le \widehat{\psi}_{j,k} \le \chi_{2I_{j,k}} \text{ for } k < n_j \quad\text{and}\quad \chi_{\tilde I_{j,n_j}} \le \widehat{\psi}_{j,n_j} \le \chi_{2\tilde I_{j,n_j}} . \tag{4.2}$$

We should emphasize that our choice of $c_{j,k}$ is different from that of Rubio de Francia in [9], which was $c_{j,k} = n_{j,k}\, 2^k$ for some integer $n_{j,k}$. Rubio de Francia's choice makes his calculations easier than ours in the scalar-valued case. The sole reason for our choice of $c_{j,k}$ is that $c_{j,k}$ splits into a sum of two terms depending on j and k separately. Namely, $c_{j,k} = a_j - 2 + 2^k + 2^{k-1}$. By (4.2),

$$S_{I_{j,k}} f = S_{I_{j,k}} \psi_{j,k} * f.$$

We then deduce, by the splitting property and Remark 3.2,

$$\Big\| \sum_{j,k} \varepsilon_j \varepsilon_k' S_{I_{j,k}} f \Big\|_p \le c_p \Big\| \sum_{j,k} \varepsilon_j \varepsilon_k' \psi_{j,k} * f \Big\|_p .$$

Now write

$$\begin{aligned}\psi_{j,k} * f(x) &= \int 2^k \psi(2^k(x-y)) \exp(2\pi \mathrm{i} c_{j,k}(x-y)) f(y) dy \\ &= \exp(2\pi \mathrm{i} c_{j,k}\, x) \int 2^k \psi(2^k(x-y)) \exp(-2\pi \mathrm{i} c_{j,k}\, y) f(y) dy \\ &= \exp(2\pi \mathrm{i} c_{j,k}\, x) \int K_{j,k}(x,\, y) f(y) dy,\end{aligned}$$

where

$$K_{j,k}(x,\, y) = 2^k \psi(2^k(x-y)) \exp(-2\pi \mathrm{i} c_{j,k}\, y). \tag{4.3}$$

Using the splitting property of the $c_{j,k}$ mentioned previously and the contraction principle, for every $x \in \mathbb{R}$ we have

$$\begin{aligned}\Big\| \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f(x) \Big\|_{\mathrm{Rad}_2(X)} &= \Big\| \sum_{j,k} \varepsilon_j \varepsilon'_k \exp(2\pi \mathrm{i} c_{j,k}\, x) \int K_{j,k}(x,\, y) f(y) dy \Big\|_{\mathrm{Rad}_2(X)} \\ &\sim \Big\| \sum_{j,k} \varepsilon_j \varepsilon'_k \int K_{j,k}(x,\, y) f(y) dy \Big\|_{\mathrm{Rad}_2(X)}.\end{aligned}$$

Thus we are led to introduce the vector-valued kernel K:

$$K(x,\, y) = \sum_{j,k} \varepsilon_j \varepsilon'_k K_{j,k}(x,\, y) \in L^2(\Omega), \quad x,\, y \in \mathbb{R}. \tag{4.4}$$

K is also viewed as a kernel taking values in $B(X, \mathrm{Rad}_2(X))$ by multiplication. Let T be the associated singular integral operator:

$$T(f)(x) = \int K(x,\, y) f(y) dy, \quad f \in L^p(\mathbb{R}; X).$$

By the discussion above, inequality (2.1) is reduced to the boundedness of T from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R}; \mathrm{Rad}_2(X))$:

$$\big\| T(f) \big\|_p \le c_p \big\| f \big\|_p, \quad \forall\, f \in L^p(\mathbb{R}; X). \tag{4.5}$$

The L^q boundedness of T. We have the following:

Lemma 4.4. *T is bounded from $L^q(\mathbb{R}; X)$ to $L^q(\mathbb{R}; \mathrm{Rad}_2(X))$.*

Proof. Let $f \in L^q(\mathbb{R}; X)$. By the previous discussion we have

$$\|Tf\|_q \sim \Big\| \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f \Big\|_q.$$

By (4.2),

$$\sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f = \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * (S_{2I_{j,k}} f).$$

Note that for each j the last interval I_{j,n_j} above should be the dyadic interval $\widetilde{I}_{j,n_j}$. We claim that

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\psi_{j,k}*g_{j,k}\Big\|_q\le c\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k g_{j,k}\Big\|_q,\quad \forall\, g_{j,k}\in L^q(\mathbb{R};X).$$

Indeed, using the splitting property of the $c_{j,k}$ we have

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\psi_{j,k}*g_{j,k}\Big\|_q\sim\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{\psi}_{j,k}*\widetilde{g}_{j,k}\Big\|_q,$$

where

$$\widetilde{\psi}_{j,k}(x)=2^k\psi(2^kx)\quad\text{and}\quad\widetilde{g}_{j,k}(x)=\exp(-2\pi\mathrm{i}c_{j,k}\,x)g_{j,k}(x).$$

For $x\in\mathbb{R}$ define the operator $N(x):\mathrm{Rad}_2(X)\to\mathrm{Rad}_2(X)$ by

$$N(x)\Big(\sum_{j,k}\varepsilon_j\varepsilon'_k a_{j,k}\Big)=\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{\psi}_{j,k}(x)a_{j,k}.$$

It is obvious that $N:\mathbb{R}\to B(\mathrm{Rad}_2(X))$ is a smooth function and

$$\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{\psi}_{j,k}*\widetilde{g}_{j,k}=N*\widetilde{g}\quad\text{with}\quad\widetilde{g}=\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{g}_{j,k}.$$

It is also easy to check that N satisfies Theorem 3.4 in [11]. Since $\mathrm{Rad}_2(X)$ is a UMD space, it follows from [11] that the convolution operator with N is bounded on $L^q(\mathbb{R};\mathrm{Rad}_2(X))$. Thus,

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{\psi}_{j,k}*\widetilde{g}_{j,k}\Big\|_q\le c\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\widetilde{g}_{j,k}\Big\|_q.$$

Using again the splitting property of the $c_{j,k}$ and going back to the $g_{j,k}$, we prove the claim. Consequently, we have

$$\|T(f)\|_q\le c\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k S_{2I_{j,k}}f\Big\|_q.$$

We split the family $\{2I_{j,k}\}$ into three subfamilies $\{2I_{j,3k+\ell}\}$ of disjoint intervals with $\ell\in\{0,1,2\}$. Accordingly, we have

$$\|T(f)\|_q\le c\sum_{\ell=0}^{2}\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k S_{2I_{j,3k+\ell}}f\Big\|_q.$$

Each subfamily $\{2I_{j,3k+\ell}\}_{j,k}$ satisfies the condition of Lemma 4.2. Hence,

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k S_{2I_{j,3k+\ell}}f\Big\|_q\le c\,\|f\|_q.$$

Thus the lemma is proved. □

An estimate on the kernel K. This subsection contains the key estimate on the kernel K defined in (4.4). Fix $x,\, z\in\mathbb{R}$ and an integer $m\ge 1$. Let

$$I_m(x,z)=\big\{y\in\mathbb{R}\ :\ 2^m|x-z|<|y-z|\le 2^{m+1}|x-z|\big\}.$$

Lemma 4.5. *If X^* is of cotype 2 and if $(\lambda_{j,k})\subset X^*$, then*

$$\int_{I_m(x,z)}\Big\|\sum_{j,k}[K_{j,k}(x,\,y)-K_{j,k}(z,\,y)]\lambda_{j,k}\Big\|_{X^*}^2 dy\le c\,\frac{\big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\lambda_{j,k}\big\|^2_{\mathrm{Rad}_2(X^*)}}{2^{5m/3}|x-z|}\,.$$

Proof. Let $(\lambda_{j,k})\subset X^*$ such that

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\lambda_{j,k}\Big\|_{\mathrm{Rad}_2(X^*)}\le 1.$$

By the definition of $K_{j,k}$ in (4.3), we have

$$\sum_{j,k}\big[K_{j,k}(x,\,y)-K_{j,k}(z,\,y)\big]\lambda_{j,k}=\sum_k\mu_k 2^k\,\big[\psi(2^k(x-y))-\psi(2^k(z-y))\big]\,q_k(y)\,,$$

where

$$\mu_k=\Big\|\sum_j\varepsilon_j\lambda_{j,k}\Big\|_{\mathrm{Rad}(X^*)}\quad\text{ and }\quad q_k(y)=\mu_k^{-1}\sum_j\lambda_{j,k}\exp(-2\pi\mathrm{i}c_{j,k}\,y).$$

Since $\mathrm{Rad}(X^*)$ is of cotype 2,

$$\sum_k\mu_k^2\le c\,\Big\|\sum_k\varepsilon'_k\sum_j\varepsilon_j\lambda_{j,k}\Big\|^2_{\mathrm{Rad}(\mathrm{Rad}(X^*))}\le c.$$

Thus,

$$\begin{aligned}\int_{I_m(x,z)}&\Big\|\sum_{j,k}\big[K_{j,k}(x,\,y)-K_{j,k}(z,\,y)\big]\lambda_{j,k}\Big\|^2_{X^*}dy\\ &\le\sum_k 2^{2k}\sup_{y\in I_m(x,z)}\big|\psi(2^k(x-y))-\psi(2^k(z-y))\big|^2\int_{I_m(x,z)}\|q_k(y)\|^2_{X^*}dy.\end{aligned}$$

Note that for fixed k

$$|c_{j,k}-c_{j',k}|\ge 2^k,\quad\forall\, j\ne j'\,.\tag{4.6}$$

Now we appeal to the following classical inequality on Dirichlet series with small gaps. Let (γ_j) be a finite sequence of real numbers such that

$$\gamma_{j+1}-\gamma_j\ge 1,\quad\forall\, j\ge 1.$$

Then, by Theorem 9.9 in Chapter V of [13], for any interval $I\subset\mathbb{R}$ and any sequence $(\alpha_j)\subset\mathbb{C}$,

$$\int_I\Big|\sum_j\alpha_j\exp(2\pi\mathrm{i}\gamma_j\,y)\Big|^2dy\le c\max(|I|,\ 1)\sum_j|\alpha_j|^2\,,$$

where c is an absolute constant. Applying this to the function q_k, using Lemma 4.3 and (4.6), we find

$$\begin{aligned}\int_{I_m(x,z)} \|q_k\|_{X^*}^2\, dy &\le c\, 2^{-k} \max(2^k |I_m(x,z)|,\ 1)\, \mu_k^{-2} \Big\| \sum_j \varepsilon_j \lambda_{j,k} \Big\|_{\mathrm{Rad}(X^*)}^2 \\ &= c\, \max(2^m|x-z|,\ 2^{-k})\,.\end{aligned}$$

Let

$$\begin{aligned} k_0 &= \min\big\{k \in \mathbb{N} :\ 2^{-k} \le 2^m |x-z|\big\} \\ \text{and} \quad k_1 &= \min\big\{k \in \mathbb{N} :\ 2^{-k} \le 2^{2m/3} |x-z|\big\}. \end{aligned}$$

Note that $k_0 \le k_1$. For $k \le k_1$ we have

$$|\psi(2^k(x-y)) - \psi(2^k(z-y))| \le c\, 2^k |x-z|.$$

Recall that ψ is a Schwartz function, in particular $|x|^2\, |\psi(x)| \le c$. Thus, for $k \ge k_1$, we have

$$|\psi(2^k(x-y)) - \psi(2^k(z-y))| \le c\, 2^{-2k} |y-z|^{-2} \le c\, 2^{-2k-2m} |x-z|^{-2}\,,$$

where the second estimate comes from the fact that $y \in I_m(x,z)$. Let

$$\alpha_k = 2^{2k} \sup_{y \in I_m(x,z)} |\psi(2^k(x-y)) - \psi(2^k(z-y))|^2 \int_{I_m(x,z)} \|q_k(y)\|_X^2\, dy.$$

Combining the preceding inequalities, we deduce the following estimates on α_k:

$$\begin{aligned} &\alpha_k \le c\, 2^{2k} 2^{2k} |x-z|^2 2^{-k} = c\, 2^{3k} |x-z|^2 \quad \text{for} \quad k \le k_0; \\ &\alpha_k \le c\, 2^{2k} 2^{2k} |x-z|^2 2^m |x-z| = c\, 2^{4k} 2^m |x-z|^3 \quad \text{for} \quad k_0 < k < k_1; \\ &\alpha_k \le c\, 2^{2k} (2^{k+m}|x-z|)^{-4} 2^m |x-z| = c\, 2^{-2k} 2^{-3m} |x-z|^{-3} \quad \text{for} \quad k \ge k_1. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{I_m(x,z)} \Big\| \sum_{j,k} [K_{j,k}(x,\, y) - K_{j,k}(z,\, y)] \lambda_{j,k} \Big\|_{X^*}^2 dy \\ &\qquad \le \sum_{1 \le k \le k_0} \alpha_k + \sum_{k_0 < k < k_1} \alpha_k + \sum_{k \ge k_1} \alpha_k \\ &\qquad \le c\, \big[2^{3k_0} |x-z|^2 + 2^{4k_1} 2^m |x-z|^3 + 2^{-2k_1} 2^{-3m} |x-z|^{-3} \big] \\ &\qquad \le c\, 2^{-5m/3} |x-z|^{-1}\,. \end{aligned}$$

This is the desired estimate for the kernel K. □

The L^∞-BMO boundedness. Recall that T is the singular integral operator associated with the kernel K.

Lemma 4.6. *The operator T is bounded from $L^\infty(\mathbb{R};X)$ to* $\mathrm{BMO}(\mathbb{R};\mathrm{Rad}_2(X))$.

Proof. Recall that

$$\|g\|_{\mathrm{BMO}(\mathbb{R};\mathrm{X})} \leq 2 \sup_{I\subseteq\mathbb{R}} \frac{1}{|I|}\int_I \|g(x)-b_I\|_X \,dx,$$

where $\{b_I\}_{I\subseteq\mathbb{R}} \subseteq X$ is any family of elements of X assigned to each interval $I\subseteq\mathbb{R}$. Fix a function $f\in L^\infty(\mathbb{R};X)$ with $\|f\|_\infty \leq 1$ and an interval $I\subset\mathbb{R}$. Let z be the centre of I and let

$$b_I = \int_{(2I)^c} K(z,y)f(y)\,dy.$$

Then, for $x\in I$,

$$Tf(x)-b_I = \int_{(2I)^c}[K(x,y)-K(z,y)]f(y)\,dy + \int_{2I} K(x,y)f(y)\,dy.$$

Thus

$$\begin{aligned}\frac{1}{|I|}\int_I &\left\|Tf(x)-b_I\right\|_{\mathrm{Rad}_2(X)}dx\\ &\leq \frac{1}{|I|}\int_I \Big\|\int_{(2I)^c}[K(x,y)-K(z,y)]f(y)\,dy\Big\|_{\mathrm{Rad}_2(X)}dx\\ &\quad+\frac{1}{|I|}\int_I \Big\|\int_{2I} K(x,y)f(y)dy\Big\|_{\mathrm{Rad}_2(X)}dx\\ &\overset{\mathrm{def}}{=} A+B.\end{aligned}$$

By Lemma 4.4 we have

$$B \leq |I|^{-1/q}\left\|T(f\chi_{2I})\right\|_q \leq c.$$

To estimate A, fix $x\in I$. Choose $(\lambda_{j,k})\subset X^*$ such that

$$\Big\|\sum_{j,k}\varepsilon_j\varepsilon'_k\lambda_{j,k}\Big\|_{\mathrm{Rad}_2(X^*)} \leq 1.$$

and

$$\begin{aligned}\Big\|\int_{(2I)^c}&[K(x,y)-K(z,y)]f(y)\,dy\Big\|_{\mathrm{Rad}_2(X)}\\ &\sim \sum_{j,k}\Big\langle\lambda_{j,k},\ \int_{(2I)^c}[K_{j,k}(x,y)-K_{j,k}(z,y)]f(y)\,dy\Big\rangle\end{aligned}$$

Then by Lemma 4.5, we find

$$\begin{aligned}\Big\| \int_{(2I)^c} [K(x,y)-K(z,y)]f(y)\,dy\Big\|_{\mathrm{Rad}_2(X)} \\ \leq \int_{(2I)^c} \Big\| \sum_{j,k} [K_{j,k}(x,y)-K_{j,k}(z,y)]\lambda_{j,k}\Big\|_{X^*} dy \\ \leq \sum_{m=1}^{\infty} |I_m(x,z)|^{1/2}\Big(\int_{I_m(x,z)} \Big\| \sum_{j,k}[K_{j,k}(x,y)-K_{j,k}(z,y)]\lambda_{j,k}\Big\|_{X^*}^2 dy\Big)^{1/2} \\ \leq c\sum_{m=1}^{\infty} (2^m|x-z|)^{1/2}(2^{5m/3}|x-z|)^{-1/2} \leq \sum_{m=1}^{\infty} c\,2^{-m/3} \leq c.\end{aligned}$$

Therefore, $A \leq c$. Thus T is bounded from $L^\infty(\mathbb{R};X)$ to $\mathrm{BMO}(\mathbb{R};\mathrm{Rad}_2(X))$. □

Combining the results of Lemma 4.6 and Lemma 4.4 and applying interpolation (see [2]), we immediately see that the operator T is bounded from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};\mathrm{Rad}_2(X))$ for every $q < p < \infty$. Thus Theorem 4.1 is proved.

Remark 4.7. Let

$$T(f)^\sharp(x) = \sup_{x\in I} \frac{1}{|I|}\int_I \big\|T(f)(y)-T(f)_I\big\|_{\mathrm{Rad}_2(X)}\,dy$$

and

$$M_q(f)(x) = \sup_{x\in I}\Big(\frac{1}{|I|}\int_I \|f(y)\|_X^q\,dy\Big)^{\frac{1}{q}}.$$

Under the assumption of Theorem 4.1 one can show the following pointwise estimate:

$$T(f)^\sharp \leq c\,M_q(f).$$

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